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LARGE ENTROPY MEASURES OF HÉNON-LIKE MAPS

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RESUME

We study the Lyapounov exponent of ergodic invariant measures for Henon-like maps under appropriate entropy conditions. Specifically, we consider an ergodic measure ν for a Henon-like map f satisfying $h_\nu(f) > \log d^+_{p-1}$ when $d^+_{p-1} < d$. We establish that ν has at least p strictly positive Lyapounov exponents bounded below by $\left(h_\nu(f) - \log d^+_{p-1}\right)/2k$. These results provide insight into the interplay between entropy, degree growth, and Lyapounov exponents in the dynamical behavior of Henon-like maps.

Key words: Entropy, Horizontal-like map, Lyapounov exponent.

Introduction

In this paper, we investigate invertible horizontal-like in higer dimension. A horizontal-like map is essentially a holomorphic map defined on a bounded convex domain $D \subset \mathbb{C}^k$, exhibiting an expanding behavior in p directions while contracting in the remaining $k-p$ directions. Precise information is formally provided in Definition 1. The dynamical degrees of horizontal-like maps are play important role in our work. Consider convex, bounded open subsets $M' \Subset M$ and $N' \Subset N$, and let $D := M \times N$ and $D' := M' \times N'$ such that $f^{-1}(D) \subset M' \times N$ and $f(D) \subset M \times N'$. For each $0 \le s \le p$, we define the dynamical degree d^+_s of f as follows:

$$d^+_s = d_s(f) := \limsup_{n\to\infty} \left\{ \sup_S \|(f^n)_* S\|_{M'\times N} \right\}^{1/n}$$

where the supremum is taken over all positive closed horizontal currents S of bidegree $(k-s, k-s)$ on $D' = M' \times N'$ with the mass $\|S\|_{D'} = 1$. For a precise behaviour of these dynamical degrees, see [1, 3] and [2] for the case of polynomial-like maps. In this work we show that when the main dynamical degree of Hènon-like map f is larger than other ones, the ergodic, $f-$ invariant measure ν with satisfying $h_\nu > \log d^+_{p-1}$ is hyperbolic: it admits p stricly positive and $k-p$ stricly negative Lyapounov exponents.

Preliminaries

In this section, we recall several fundamental definitions and preliminary results that will be used in the subsequent sections of the paper.

Let p and k be integers with $k \ge 2$ and $1 \le p < k-1$. Let $M \subset \mathbb{C}^p$ and $N \subset \mathbb{C}^{k-p}$ be two bounded, convex domains. Consider the product domain $D := M \times N \subset \mathbb{C}^k$. We define the *vertical boundary* of D as $\partial_v D := \partial M \times N$ and the *horizontal boundary* as $\partial_h D := M \times \partial N$. A subset $E \subset D$ is said to be *vertical* if its closure $\overline{E}$ does not intersect $\overline{\partial_v D}$, and *horizontal* if $\overline{E}$ does not intersect $\overline{\partial_h D}$.

We now proceed to introduce the concept of a *horizontal-like map* f on the domain D.

Let π_1 and π_2 be the canonical projections of the product space $D \times D$ onto its first and second factors, respectively.

Definition 1. *A map f in D is said to be a* horizontal-like map *if it satisfies the following properties:*

1. *The graph Γ of f forms an irreducible submanifold of $D \times D$;*

2. *The restriction of the first projection* $\pi_1|_\Gamma$ *is injective, while the second projection* $\pi_2|_\Gamma$ *has finite fibers;*

3. *The closure* $\overline{\Gamma}$ *does not intersect either* $\overline{\partial_v D} \times \overline{D}$ *or* $\overline{D} \times \overline{\partial_h D}$.

Generally, such a map is not defined on the whole of D, but rather on a vertical subset $f^{-1}(D) \subset D$. The map f then takes values within a horizontal subset $f(D) \subset D$.

An invertible horizontal-like map f is referred to as a *Henon-like map* when the restriction of π_2 to the graph Γ, i.e., $\pi_{2|\Gamma}$, is injective. In the following, we focus our analysis on Henon-like maps. Let us denote by $f^n := f \circ f \circ \cdots \circ f$ (applied n times) the n-th iterate of the map f, and similarly, by $f^{-n} := f^{-1} \circ \cdots \circ f^{-1}$ (applied n times) its inverse. Define the *filled Julia sets* $\mathcal{K}_+$ and $\mathcal{K}_-$ as

$$\mathcal{K}_+ := \bigcap_{n \geq 0} f^{-n}(D), \quad \mathcal{K}_- := \bigcap_{n \geq 0} f^{n}(D).$$

These sets describe the regions of points that remain confined within D under repeated iteration by f and f^{-1}, respectively. The boundaries of $\mathcal{K}_+$ and $\mathcal{K}_-$ are known as the *Julia sets* of f and f^{-1}. Additionally, define $\mathcal{K} := \mathcal{K}_+ \cap \mathcal{K}_-$, which is a compact subset of D. This set satisfies the invariance properties:

$$f^{-1}(\mathcal{K}_+) = \mathcal{K}_+, \quad f(\mathcal{K}_-) = \mathcal{K}_-, \quad f^{\pm}(\mathcal{K}) = \mathcal{K}.$$

The operator $f_* := (\pi_{2|\Gamma})_* \circ (\pi_{1|\Gamma})^*$ acts continuously on horizontal currents. According to [1, Proposition 3.2], there exists an integer $d \geq 1$ such that for any horizontal positive closed current S, we have the relation

$$||f_*(S)||_h = d||S||_h,$$

The integer d is referred to as the *main dynamical degree* of the map f.

We now define the other dynamical degrees of the map f in relation to currents. Consider convex, bounded open subsets $M' \Subset M$ and $N' \Subset N$, and let $D := M \times N$ and $D' := M' \times N'$ such that $f^{-1}(D) \subset M' \times N$ and $f(D) \subset M \times N'$. Thus, the restriction of f to $M' \times N'$ remains a horizontal-like map. A current on D is classified as *vertical* (resp. *horizontal*) if its support lies in a vertical (resp. horizontal) in D. For each $0 \leq s \leq p$, we define the dynamical degree d_s^+ of f as follows:

$$d_s^+ = d_s(f) := \limsup_{n \to \infty} \left\{ \sup_S \|(f^n)_* S\|_{M' \times N} \right\}^{1/n}$$

where the supremum is taken over all positive closed horizontal currents S of bidimension (s, s) on $D' = M' \times N'$ with the mass $\|S\|_{D'} = 1$. Similarly, for each $0 \leq s \leq k - p$, we define the dynamical degree d_s^- of f^{-1} as:

$$d_s^- = d_s(f) := \limsup_{n \to \infty} \left\{ \sup_R \left\|(f^n)^* R\right\|_{M \times N'} \right\}^{1/n}$$

where the supremum is taken over all positive closed vertical currents R of bidimension (s, s) on $D' = M' \times N'$ such that $\|R\|_{D'} = 1$.

Below are some properties of dynamical degrees identified above. It is proved in [1], Lemma 3.5 that the dynamical degrees $d_0^+ = d_0^- = 1$ and $d_p^+ = d_{k-p}^- = d$. Moreover, the $\limsup$ in the definition of d_s^+ and d_s^- can be replaced by $\lim$; see [3], Lemma 3.6. The monotonicity of the dynamical degrees d_s^+ and d_s^- is established in [3], that is, $d_s^+ \leq d_{s+1}^+ \quad \text{for} \quad 0 \leq s \leq p - 1$ and $\quad d_s^- \leq d_{s+1}^- \quad \text{for} \quad 0 \leq s \leq k - p - 1$. We fix integers $1 \leq p < k$, a bounded and convex domain $D = M \times N \subset \mathbb{C}^p \times \mathbb{C}^{k-p}$ and the convex open sets $M'' \Subset M' \Subset M$ and $N'' \Subset N' \Subset N$ are assumed to be sufficiently close to M and N. Use $\omega_{|_{M'' \times N}}$ denotes the restriction to $M'' \times N$ of the standart Kähler form ω on $\mathbb{C}^k$.

Lemma 2. (See [1], Lemma 5.5) *Let* f *be a Hènon-like map on* $D = M \times N \subset \mathbb{C}^p \times \mathbb{C}^{k-p}$, *and* M', M'', N', N'', d_s^+ *be as above. Let* $0 \leq s \leq p - 1$ *be arbitrary integer and* σ *be a constant such that* $\sigma > d_s^+$. *Then, there exists a constant* $A > 0$ *such that for any positive closed current* Φ *of bidemention* (s, s), *supported on* $M \times N'$, *and for all integers* $m_1 \geq m_2 \geq \cdots \geq m_s \geq 0$ *the following inequality holds:*

$$\int \Phi \wedge (f^{m_1})^* \omega_{|_{M'' \times N}} \wedge \cdots \wedge (f^{m_s})^* \omega_{|_{M'' \times N}} \leq A \sigma^{m_1} \|\Phi\|_D.$$

This result demonstrates the exponential growth control of iterates of the vertical forms $\omega_{|_{M''\times N}}$ under the pullback by f, where Φ is bounded by the constant A and the exponential factor σ^{m_1}, depending on the current's norm $\|\Phi\|_D$. We now introduce some concepts related to the entropy of Henon-like maps, which will play a central role in this part.

Definition 3. *Let f be a Hénon-like map on D, and let n be an integer.*

1. *A subset E of D is called (n,ε)-separated if the map f^j is well-defined on E such that $f^j(E) \subset D'' := M'' \times N''$ for $0 \le j \le n$ and for any two distinct points $x, y \in E$, $\mathrm{dist}(f^j(x), f^j(y)) \ge \varepsilon$ for at last one $0 \le j \le n$.*
2. *For $X \subset D$, the* topological entropy *of the map f restricted to X is defined as:*

$$h_{top}(f,X) := \sup_{\varepsilon>0} \limsup_{n\to\infty} \frac{\log \max \#\{E \subset X \mid E \text{ is } (n,\varepsilon)\text{-separated}\}}{n}.$$

We have the following version of the Gromov inequality; see [2, 4].

Proposition 4. ([1], Proposition 5.7) *Let f be a Hènon-like map on D, and let $0 \le s \le p-1$ be arbitrary integer. If σ is a constant satisfying $\sigma > d_s^+$ and X is a horizontal subvariety of D of dimension s, then for every $\varepsilon > 0$, there exists a constant $A_\varepsilon > 0$ such that every (n,ε)-separated subset in X contains at most $A_\varepsilon \sigma^n$ points. Consequently, it follows that the topological entropy of f restricted to X satisfies:*

$$h_{top}(f,X) \le \log d_s^+.$$

Let X be a complex manifold of dimension k. Consider a smooth dynamical system $T : X \to X$ and an invariant ergodic probality measure ν. The map T induces a linear map H from the tangent space at z to the tangent space of $T(z)$, i.e. $H : X \to GL(\mathbb{C}, k)$. For $n \ge 0$ define

$$H_n(z) := H(z) \cdot H(T(z)) \cdots H(T^{n-1}(z)).$$

We call H_n the multiplicative cocycle over X generated by H. For $n, m \ge 0$ it is satisfy the identity

$$H_{n+m}(z) = H_n(T^m(z))H_m(z).$$

Let us recall famous Oseledec theorem.

Theorem 5.(Oseledec) *Let $T : X \to X$, ν and the cocycle H_n be as above. Assume that ν is ergodic and that $\log^+ ||H^{\pm}(z)||$ are in $L^1(\nu)$, where $log^+ := \max\{\log, 0\}$. Then there is an integer l, real numbers $\Lambda_1 < \Lambda_2 < \cdots < \Lambda_l$, and for $\nu-$ almost every z, a unique decomposition of $\mathbb{C}^k$ into a direct sum of linear subspaces*

$$\mathbb{C}^k = \bigoplus_{i=1}^{l} \mathscr{E}_i(z)$$

such that

1. $\dim \mathscr{E}_i$ *does not depend on z.*
2. *The decomposition $H : \mathscr{E}_i \mapsto \mathscr{E}_i \circ T$ is invariant.*
3. *For any vector $v \in \mathscr{E}_i(z) \setminus \{0\}$ we have locally uniformly:*

$$\lim_{n\to\infty} \frac{1}{n} \log ||H_n(z) \cdot v|| = \Lambda_i.$$

4. *For $\mathscr{I} \subset \{1, 2, ..., l\}$, define $\mathscr{E}_{\mathscr{I}} := \oplus_{i\in\mathscr{I}} \mathscr{E}_i(z)$. The angle between $\mathscr{E}_{\mathscr{I}}(z)$ and $\mathscr{E}_{\mathscr{I}'}(z)$ is a tempered while $\mathscr{I}, \mathscr{I}'$ are disjoint, i.e*

$$\lim_{n\to\infty} \frac{1}{n} \log \sin |\angle(\mathscr{E}_{\mathscr{I}}(T^n(z)), \mathscr{E}_{\mathscr{I}'}(T^n(z)))| = 0.$$

If T is invertable, above decomposition is the same for T^{-1} and exponents Λ_i are replased by $-\Lambda_i$. The corresponding constants Λ_i are called *Lyapounov exponents* of T with respect to ν and $\dim\mathscr{E}_i$ is the multiplicity of Λ_i.

Now we construct s dimensional complex subspace $\mathscr{F}$ in the sequel. According to the Oseledec-Pesin theory, there is a decomposition $\mathscr{T}_z = \mathscr{E}_z \oplus \mathscr{F}_z$ for ν almost every z and there exists Borel set $\mathcal{M} \in \Omega$ that is $\nu(\mathcal{M}) \geq 1/2$ with satisfy

$$\left\|DT^{-1}(v)\right\| \geq e^{-\lambda_1}\|v\|, \left\|DT^{-1}(u)\right\| \leq e^{-\lambda_2}\|u\|, \angle\left(\mathscr{E}_{T^{-n}(z)}, \mathscr{F}_{T^{-n}(z)}\right) \geq \beta e^{-n\alpha}$$

for $v \in \mathscr{E}_z,\ u \in \mathscr{F}_z,\ z \in \mathcal{M},$ and $n \geq 0$. We will establish precise conditions in the next lemma for the positive parameters $\lambda_1, \lambda_2, \alpha,$ and β, defining λ_2 as $\lambda_1 + 7\alpha$. Let γ be a small positive constant such that $\gamma \ll \beta$ and $\gamma \ll \varepsilon$, where ε is the constant associated with α as mentioned earlier. Define $\mathcal{D}_{z_{-n}}$ as the small ball centered at $z_{-n} := T^{-n}(z)$ with radius $\gamma e^{-n\lambda_2}$ within $\mathscr{E}_{z_{-n}}$. Our focus is on the graphs in $\mathscr{T}_{z_{-n}} = \mathscr{E}_{z_{-n}} \oplus \mathscr{F}_{z_{-n}}$ of holomorphic maps over $\mathcal{D}_{z_{-n}}$.

Lemma 6. *For every $z \in \mathcal{M}$, there exist holomorphic maps $g_n : \mathcal{D}_{z_{-n}} \to \mathscr{F}_{z_{-n}}$ with graph $\Gamma_{z_{-n}}$ such that $g_n(0) = 0$, $\|Dg_n\| \leq e^{-4n\alpha}$, and T maps $\Gamma_{z_{-n-1}}$ into $\Gamma_{z_{-n}}$.*

Proof. The proof of this lemma proceeds by induction. For $n = 0$, it suffices to select $g_0 = 0$. The subspace $\Gamma_{z_{-n}}$ will be obtained as an open subset within $T^{-1}\left(\Gamma_{z_{-n+1}}\right)$. Consider the map T^{-1} defined on a small neighborhood of z_{-n+1}, which maps into a neighborhood of z_{-n}. In the dynamical coordinates associated with $\mathscr{T}_{z_{-n+1}}$ and $\mathscr{T}_{z_{-n}}$, the map T^{-1} can be expressed as:

$$T^{-1}(z) = \mathcal{L}(z) + \mathcal{R}(z) \quad \text{where} \quad \mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2) \quad \text{and} \quad \mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$$

Here, $\mathcal{L}(z)$ represents the linear component of T, specifically the differential DT^{-1} at z_{-n+1}, while $\mathcal{R}(z)$ denotes the remainder, which is of order ≥ 2 in z. Let $\mathcal{L}_1 : \mathscr{E}_{z_{-n+1}} \to \mathscr{E}_{z_{-n}}$ and $\mathcal{L}_2 : \mathscr{F}_{z_{-n+1}} \to \mathscr{F}_{z_{-n}}$. We have the following bounds:

$$\|\mathcal{L}_1(z')\| \geq e^{-\lambda_1}\|z'\|, \quad \|\mathcal{L}_2(z'')\| \leq e^{-\lambda_2}\|z''\|$$

for $z' \in \mathscr{E}_{z_{-n+1}}$, and for $z'' \in \mathscr{F}_{z_{-n+1}}$, respectively. The derivatives of T^{-1} are uniformly bounded in standard coordinates. Considering the distortions in dynamical coordinates, we obtain $\|\,\mathrm{D}\mathcal{R}(z)\| \leq Ae^{6n\alpha}\|z\|$, where $A > 0$ is independent of γ, n, and α. Let $z = (z', z'')$ and $w = (w', w'')$ be points in $\mathscr{E}_{z_{-n+1}} \oplus \mathscr{F}_{z_{-n+1}}$ contained in $\mathcal{D}_{z_{-n+1}}$. Hence, $\|z\|$ and $\|w\|$ are smaller than $2\gamma e^{-(n-1)\lambda_2}$. Set $\widetilde{z} = (\widetilde{z}', \widetilde{z}'') = T^{-1}(z)$ and $\widetilde{w} = (\widetilde{w}', \widetilde{w}'') = T^{-1}(w)$. Utilizing the estimates for $\mathcal{L}_1$, $D\mathcal{R}$, and Dg_{n-1}, we derive:

$$\begin{aligned} \|\widetilde{z}' - \widetilde{w}'\| &\geq \|\mathcal{L}_1(z') - \mathcal{L}_1(w')\| - \|\mathcal{R}_1(z) - \mathcal{R}_1(w)\| \geq \\ &\geq e^{-\lambda_1}\|z' - w'\| - 2\gamma Ae^{6n\alpha}e^{-(n-1)\lambda_2}\|z - w\| \geq \\ &\geq e^{-\lambda_1}\|z' - w'\| - 4\gamma Ae^{6n\alpha}e^{-(n-1)\lambda_2}\|z' - w'\|, \end{aligned}$$

leading to

$$\|\widetilde{z}' - \widetilde{w}'\| \geq e^{-(\lambda_1+\alpha)}\|z' - w'\|$$

since γ and α are small and $\alpha \ll \lambda_1$. Consequently, $T^{-1}(\mathcal{D}_{z_{-n+1}})$ is the graph of a holomorphic map g_n over an open subset $\mathcal{D}$ of $\mathscr{E}_{z_{-n}}$. The final estimate for $w' = 0$ implies that $\mathcal{D}$ includes the ball $\mathcal{D}_{z_{-n}}$. Furthermore, we have

$$\begin{aligned} \|\widetilde{z}'' - \widetilde{w}''\| &\leq \|\mathcal{L}_2(z'') - \mathcal{L}_2(w'')\| + \|\mathcal{R}_2(z) - \mathcal{R}_2(w)\| \\ &\leq e^{-\lambda_2}\|z'' - w''\| + 2\gamma Ae^{6n\alpha}e^{-(n-1)\lambda_2}\|z - w\| \\ &\leq e^{-\lambda_2}e^{-4(n-1)\alpha}\|z' - w'\| + 4\gamma Ae^{6n\alpha}e^{-(n-1)\lambda_2}\|z' - w'\|, \end{aligned}$$

implying that $\|\widetilde{z}'' - \widetilde{w}''\| \leq e^{-4n\alpha}\|\widetilde{z}' - \widetilde{w}'\|$ given that $\alpha \ll \lambda_1$ and γ is small. This concludes the proof of the lemma. □

Let $\mathscr{F}'_z$ denote the orthogonal complement of $\mathscr{E}_z$. We use coordinate systems on $\mathscr{F}'_z$ that induce the standard metric. Define $\mathcal{D}'_{z_{-n}}$ as the ball centered at 0 with radius $\gamma' e^{-n\lambda_3}$ in $\mathscr{E}_{z_{-n}}$, where $\gamma' > 0$ is suitably small and $\lambda_3 = \lambda_1 + 10\alpha$. We show that that $\Gamma_{z_{-n}}$ contains a flat graph $\Gamma'_{z_{-n}}$.

Corollary 7. *For every $z \in \mathcal{M}$, the set $\Gamma_{z_{-n}}$ contains the graph $\Gamma'_{z_{-n}}$ of a holomorphic map $g'_n : \mathcal{D}'_{z_{-n}} \to \mathscr{F}'_{z_{-n}}$ such that $g'_n(0) = 0$ and $\|Dg'_n\| \lesssim e^{-n\alpha}$.*

Proof. Using the coordinate systems on $\mathscr{E}_{z_{-n}}$, $\mathscr{F}_{z_{-n}}$, and $\mathscr{F}'_{z_{-n}}$, let $\tau : \mathscr{E}_{z_{-n}} \oplus \mathscr{F}_{z_{-n}} \to \mathscr{E}_{z_{-n}} \oplus \mathscr{F}'_{z_{-n}}$ denote the linear map of coordinate change. Given that the angle between $\mathscr{E}_{z_{-n}}$ and $\mathscr{F}_{z_{-n}}$ exceeds $\beta e^{-n\alpha}$, we can express τ as (τ', τ'') with $\|\tau'(z) - z'\| \lesssim e^{n\alpha} \|z''\|$ and $\|\tau''(z)\| \leq \|z''\|$ for $z = (z', z'')$ in $\mathscr{E}_{z_{-n}} \oplus \mathscr{F}_{z_{-n}}$. This corollary can be proved similarly to apply previous lemma, but by replacing T^{-1} with τ. We omit the details here. □

Main result

In this section we give our main result with is as following.

Main Theorem. *Let f be an Hénon-like map with $d^+_{p-1} < d$ and let ν be an ergodic f-invariant measure satisfying $h_\nu(f) > \log d^+_{p-1}$. Then ν admits p strictly positive Lyapounov exponents larger than or equal to $\left(h_\nu(f) - \log d^+_{p-1}\right)/2k$. In particularly, if $d^-_{k-p-1} < d$ and $h_\nu(f) > \log d^-_{k-p-1}$ then ν admits $k - p$ strictly negative ones with are smaller than or equal to $-(h_\nu(f) - \log d^-_{k-p-1})/2k$.*

Let $\mathcal{D}_{-n}(z_0, \varepsilon)$ denote the Bowen $(-n, \varepsilon)$-ball with center z_0, i.e. the set of the points z such that $f^{-j}(z)$ is defined and $\left\|f^{-j}(z) - f^{-j}(z_0)\right\| \leq \varepsilon$ for $0 \leq j \leq n$. The entropy $h(\nu)$ for f^{-1} can be obtained by the following Brin-Katok formula

$$h(\nu) := \sup_{\varepsilon>0} \liminf_{n\to\infty} -\frac{1}{n} \log \nu\left(\mathcal{D}_{-n}(z, \varepsilon)\right)$$

for ν-almost every z. So, for every $\alpha > 0$, there are positive constants A, ε and a Borel set $\mathcal{M}_0$ with $\nu\left(\mathcal{M}_0\right) > 3/4$ such that $\nu\left(\mathcal{D}_{-n}(z, 6\varepsilon)\right) \leq A e^{-n(\log d - \alpha)}$ for $z \in \mathcal{M}_0$ and $n \geq 0$.

Proof of Main Theorem . Assume, for the sake of contradiction, that the measure ν possesses at least $k-p+1$ Lyapounov exponents that are strictly less than $\frac{1}{2k}\left(h_\nu(f) - \log d^+_{p-1}\right)$. Let $s \leq p-1$ be an integer, and let λ be a positive constant where $\lambda < \frac{1}{2k}\left(h_\nu(f) - \log d^+_{p-1}\right)$. Assume that ν has exactly $k-s$ Lyapounov exponents strictly less than λ, with the remaining exponents being greater than or equal to $\frac{1}{2k}\left(h_\nu(f) - \log d^+_{p-1}\right)$. We will construct a complex subspace $\mathscr{F}$ of dimension s, which will contradict the estimate given in Proposition 4 by having too many (n, ε)-separated points. Fix a positive constant α such that $\alpha \ll \lambda$ and $\alpha \ll \frac{1}{2k}\left(h_\nu(f) - \log d^+_{p-1}\right) - \lambda$.

Note that all constructed graphs are localized within a compact neighborhood $\mathscr{U}$ surrounding the filled Julia set $\mathscr{K}$. Returning now to the standard metric on $\mathbb{C}^k$, let $\mathcal{N}$ be a subset of $\mathcal{M} \cap \mathcal{M}_0$ such that the balls $\mathcal{D}_{-n}(z, 3\varepsilon)$, centered at points $z \in \mathcal{N}$, are mutually disjoint. We take $\mathcal{N}$ to be maximal under this disjointness constraint. As a result, the balls $\mathcal{D}_{-n}(z, 6\varepsilon)$ with centers $z \in \mathcal{N}$ provide a covering of $\mathcal{M} \cap \mathcal{M}_0$. Given that $\nu(\mathcal{M} \cap \mathcal{M}_0) \geq \frac{1}{4}$ and $\nu(\mathcal{D}_{-n}(z, 6\varepsilon)) \leq A e^{-n(h_\nu(f) - \alpha)}$, it follows that $\mathcal{N}$ must contain at least $(4A)^{-1} e^{n(h_\nu(f) - \alpha)}$ points. Now consider the graphs $\Gamma_{z_{-n}}$ and $\Gamma'_{z_{-n}}$, previously constructed for each $z \in \mathcal{N}$. Since the balls $\mathcal{D}_{-n}(z, 3\varepsilon)$ are disjoint, the set $\{z_{-n}\}$ is $(n, 3\varepsilon)$-separated. By Lemma 6, we have that the diameter of each $\Gamma_{z_{-n}}$ is less than ε for $\lambda_1 = \lambda$. Consequently, replacing each z_{-n} with a point $z'_{-n} \in \Gamma_{z_{-n}}$ yields a set that remains (n, ε)-separated.

Let Π be an orthogonal projection of $\mathbb{C}^k = \mathbb{C}^p \times \mathbb{C}^{k-p}$ onto a subspace $\mathscr{E}$ of dimension $k - s$. If $\mathscr{E}$ is a product of a subspace of $\mathbb{C}^p$ with $\mathbb{C}^{k-p}$, then the fibers of Π that are sufficiently close to $\mathscr{K}$ (in particular, those intersecting $\mathscr{U}$) are horizontal in D. This property holds for the projection onto any sufficiently small perturbation of $\mathscr{E}$. Therefore, we can select a finite number of projections $\Pi_1, \ldots, \Pi_N$ onto $\mathscr{E}_1, \ldots, \mathscr{E}_N$ that satisfy this property, and a constant $\alpha_0 > 0$ such that any subspace $\mathscr{F}$ of dimension s in $\mathbb{C}^k$ makes an angle $\geq \alpha_0$ with at least one of $\mathscr{E}_i$. From Corollary 7, for each graph $\Gamma'_{z_{-n}}$, we find following estimate:

$$\mathrm{vol}(\Pi_i\left(\Gamma'_{z_{-n}}\right)) \geq \gamma'' e^{-2n(k-s)\lambda_3}$$

for some projection Π_i with a fixed constant $\gamma'' > 0$. Select an i such that this property holds for at least $N^{-1}\#\mathcal{N}$ graphs $\Gamma'_{z_{-n}}$. Since $\#\mathcal{N} \geq (4A)^{-1} e^{n(h_\nu(f) - \alpha)}$, we have

$$\sum_{i=1}^{N} \mathrm{vol}(\Pi_i\left(\Gamma'_{z_{-n}}\right)) \gtrsim e^{n(h_\nu(f)-\alpha) - 2n(k-s)\lambda_3}$$

Consequently, there exists a fiber $\mathscr{F}$ of Π_i that intersects $\gtrsim e^{n(h_\nu(f)-\alpha)-2n(k-s)\lambda_3}$ graphs $\Gamma_{z_{-n}}$. This implies that $\mathscr{F}$ contains an (n, ε)-separated subset of $\gtrsim e^{n(h_\nu(f)-\alpha)-2n(k-s)\lambda_3} \geq e^{n\left(\log d^+_{p-1} + \alpha\right)}$ points since $\alpha \ll \frac{1}{2k}\left(h_\nu(f) - \log d^+_{p-1}\right) - \lambda$. This contradicts Proposition 4 for $X = \mathscr{F}$ as $d^+_{p-1} \geq d^+_s$, thus concluding the proof of Theorem. □

Remark. The bound $\frac{1}{2k}\left(h_\nu(f) - \log d_{p-1}^+\right)$ can be replaced by the

$$\inf_{s \le p-1}\{\frac{1}{2(k-s)}\left(h_\nu(f) - \log d_s^+\right)\}.$$

REFERENCES

1. Tien-Cuong Dinh, Viet-Anh Nguyen and Nessim Sibony. *Dynamics of horizontal-like maps in higher dimension.* Advances in Mathematics, V.219, N.5, 2008, P.1689-1721.
2. Tien-Cuong Dinh and Nessim Sibony. *Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings.* Springer, Berlin, 2010, P.165-294.
3. Fabrizio Bianchi, Tien-Cuong Dinh and Karim Rakhimov. *Monotonicity of dynamical degrees for Hénon-like and polynomial-like maps.*Transactions of the American Mathematical Society, V.93, 2024, P.1-23.
4. Mikhael Gromov. *On the entropy of holomorphic maps.* Enseignement math, Manuscript, 2003, p.217-235.

REZYUME

Ushu maqolada Hénon akslantirishlari uchun berilgan invariant ergodik o'lchovlarning Lyapunov eksponentalari entropiyaga qo'yilagn ma'lum shart asosida o'rganildi. Boshqacha aytganda, $d_{p-1}^+ < d$ shartni qanoatlantiruvchi f Hénon akslantirishi berilgan bo'lsin. Agar ν ergodik o'lchov entropiyasi uchun $h_\nu(f) > \log d_{p-1}^+$ o'rinli bo'lsa, ν o'lchovga bog'liq kamida p ta musbat Lyapunov eksponentalari topilib, quyidan $(h_\nu(f) - \log d_{p-1}^+)/2k$ bilan chegaralangan ekanligi ko'rsatilgan.

Kalit so'zlar: Entropiya, Gorizontalsimon akslantirishlar, Lyapunov eksponentalari.

РЕЗЮМЕ

В данной статье исследуются показатели Ляпунова эргодических инвариантных мер для отображений Энона при выполнении определённого условия на энтропию. Иными словами, пусть f – отображение Энона, удовлетворяющее условию $d_{p-1}^+ < d$. Тогда для всякой эргодической меры ν, энтропия которой удовлетворяет неравенству $h_\nu(f) > \log d_{p-1}^+$, показано, что у меры ν имеется как минимум p положительных показателей Ляпунова. Более того, эти показатели снизу ограничены выражением $(h_\nu(f) - \log d_{p-1}^+)/2k$.

Ключевые слова: Энтропия, отображения горизонтального типа, показатели Ляпунова.