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BOUNDARY VALUE PROBLEMS FOR MIXED-TYPE DIFFERENTIAL EQUATIONS OF THE FIRST AND SECOND ORDER WITH RESPECT TO THE TIME VARIABLE

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RESUME

This work is devoted to the study of boundary value problems for mixed-type differential equations of the first and second order with respect to the time variable. Boundary value problems for mixed-type equations arise in various fields of natural sciences, including laser physics, plasma modeling, and mathematical biology. In this paper, we establish theorems on the uniqueness and conditional stability of the solution to the problem under consideration within a set of well-posedness. An a priori estimate of the solution is obtained using the method of logarithmic convexity and spectral decomposition.

Key words: Boundary value problem, ill-posed problem, mixed-type equation, a priori estimate, estimate of conditional stability, uniqueness of solution, set of correctness.

Introduction

The paper studies boundary value problems for differential equations of mixed-type of the first and second order with respect to the time variable. The problem studied in this paper belongs to the class of ill-posed problems of mathematical physics, namely, in this problem there is no continuous dependence of the solution on the initial data.

Boundary value problems for equations of mixed-type have practical applications, they arise in solving problems of gas dynamics, momentless theory of shells with curvature of variable sign, in the theory of infinitesimal bending of surfaces, in magneto hydrodynamics, in the theory of electron scattering, in predicting groundwater levels and in other areas of physics and engineering (see [1], [2], [3])

Correct boundary value problems for equations of mixed-type with two degenerate lines were studied by such mathematicians as A.M. Nakhushev, M.M. Zainulabidov, V.F. Volkodavov, V.V. Azovsky, O.I. Marichev, A.M. Ezhov, N.I. Polivanov, He Kan Cher [4], S.I. Makarov, S.S. Isamukhamedov, Zh. Oramov, M.S. Salakhitdinov and his students, K.B. Sabitov, A.A. Gimaltdinova [5], O.A. Repin and others.

In the works of E. M. Landis, S. G. Krein, S. P. Shishatsky, H. A. Levine and others, ill-posed boundary value problems for a parabolic equation with backward time flow were studied. In these works, the inverse and non-characteristic Cauchy problem for conditional well-posedness were considered for a parabolic equation. S. G. Krein and H. A. Levine generalized these results for abstract evolution equations with self-adjoint operator coefficients.

The works of S.P. Shishatsky, K.S. Fayazov and M. Kh. Alaminov, in which ill-posed boundary value problems for degenerate parabolic and elliptic equations were investigated, also deserve special attention.

The subject of K.S. Fayazov's works were ill-posed boundary value problems for parabolic equations with changing time direction and mixed-type equations. Research close to our topic was conducted in the works of K. S. Fayazov [6], I. O. Khazhiev [7], Y. K. Khudayberganov [8].

Formulation of the problem

Let $Q=\Omega_0\times\Omega_1$, $\Omega_0=\left\{(x_1,x_2,...,x_n):-1<x_j<1, j=\overline{1,n}\right\}$, $\Omega_1=\{0<t<T,\ T<\infty\}$.

Let's consider the equation

$$\frac{\partial^i u(x,t)}{\partial t^i}+\sum_{j=1}^{n} sign(x_j)\frac{\partial^2 u(x,t)}{\partial x_j^2}=0, \tag{1}$$

in the domain $Q\backslash\{x_j\neq 0, j=\overline{1,n}\}$, where $i=1,2,\ x=(x_1,x_2,...,x_n)$.

Problem. Find a solution to equation (1) in the domain Q so that the following conditions: initial

$$\begin{aligned}&\textbf{a)}\ i=1,\quad u(x,t)|_{t=0}=\varphi_0(x),\ \ x\in\bar{\Omega}_0,\\ &\textbf{b)}\ i=2,\quad \left.\frac{\partial^k u(x,t)}{\partial t^k}\right|_{t=0}=\varphi_{k+1}(x),\ \ x\in\bar{\Omega}_0,\ \ k=0,1,\end{aligned} \tag{2}$$

boundary

$$u(x,t)|_{\partial\Omega_0}=0,\qquad t\in\bar{\Omega}_1, \tag{3}$$

and gluing

$$\left.\frac{\partial^k u(x,t)}{\partial x_j^k}\right|_{x_j=-0}=\left.\frac{\partial^k u(x,t)}{\partial x_j^k}\right|_{x_j=+0},\quad t\in\bar{\Omega}_1,\ \ k=0,1.,\ \ j=\overline{1,n}, \tag{4}$$

conditions, where - $\varphi_m(x)$, $m=\overline{1,3}$ is a given sufficiently smooth function, and $\varphi_m(x)|_{\partial\Omega_0}=0$.

In this paper, a priori estimates for the solution of equation (1) are established, and theorems on the uniqueness and conditional stability of the solution of the desired problems are proved.

Spectral problem. Find such values for which the following problem

$$\sum_{j=1}^{n} sign(x_j)\vartheta_{x_jx_j}(x)+\lambda\vartheta(x)=0,\ x\in\Omega_0\backslash\{x_j\neq 0\}, \tag{5}$$

$$\begin{aligned}&\vartheta(x)\,|_{\partial\Omega_0}=0,\\ &\left.\frac{\partial^k\vartheta(x)}{\partial x_j^k}\right|_{x_j=-0}=\left.\frac{\partial^k\vartheta(x)}{\partial x_j^k}\right|_{x_j=+0},\ j=\overline{1,n},\ \ \left(k=\overline{0,1}\right),\end{aligned} \tag{6}$$

has non-trivial solutions.

We will seek the solution to problem (5), (6) using the method of separation of variables, assuming

$$\vartheta(x)=\prod_{j=1}^{n}X_j(x_j). \tag{7}$$

From conditions (7) we obtain:

$$\begin{gathered}X_j(\pm 1)=0,\\ X_j(-0)=X_j(+0),\\ X'_j(-0)=X'_j(+0),\ j=\overline{1,n}\end{gathered} \tag{8}$$

We find the second-order partial derivatives of the function $\vartheta(x)$:

$$\frac{\partial^2\vartheta(x)}{\partial x_k^2}=\frac{\partial^2}{\partial x_k^2}\prod_{j=1}^{n}X_j(x_j),\ \ k=\overline{1,n}.$$

Substituting into (5) and separating the variables, we obtain:

$$\sum_{j=1}^{n}\frac{sign(x_j)X''(x_j)}{X_j(x_j)}=-\lambda.$$

Thus, we have

$$\frac{sign(x_j)X''_j(x_j)}{X_j(x_j)} = -\lambda_j,\ j = \overline{1,n}.$$

As a result, to find the functions $X_j(x_j)$ we obtain the equations:

$$sign(x_j)X''_j(x_j) = -\lambda_j X_j(x_j). \tag{9}$$

Let us consider equations (9) with the corresponding conditions (8)

$$\begin{gathered} sign(x_j)X''_j(x_j) = -\lambda_j X_j(x_j), \\ X_j(\pm 1) = 0, X_j(-0) = X_j(+0), \\ X'_j(-0) = X'_j(+0),\ j = \overline{1,n}. \end{gathered} \tag{10}$$

Thus, the solutions to problems (10) have the form:

if $\lambda_j > 0$,

$$X^{(1)}_{jl_j}(x_j) = \begin{cases} \sin\mu_{l_j}(x_j-1)/\cos\mu_{l_j} & ,0 \leqslant x_j \leqslant 1, \\ sh\mu_{l_j}(x_j+1)/ch\mu_{l_j}, & -1 \leqslant x_j \leqslant 0, \end{cases} \quad l_j \in N,$$

and also $\lambda_j < 0$,

$$X^{(2)}_{jl_j}(x_j) = \begin{cases} sh\mu_{l_j}(x_j-1)/ch\mu_{l_j}, & 0 \leqslant x_j \leqslant 1, \\ \sin\mu_{l_j}(x_j+1)/\cos\mu_{l_j}, & -1 \leqslant x_j \leqslant 0, \end{cases} \quad l_j \in N,$$

where $\lambda_j = \mu^2_{l_j} > 0$, $\lambda_j = -\mu^2_{l_j} < 0$, $j = \overline{1,n}$. In both cases, $\mu_{l_j}-$ are positive roots of the transcendental equation $tg\alpha = -th\alpha$.

Thus, the eigenvalues of the spectral problem (5), (6) have the form

$$\lambda^{(1)}_{k_1,k_2,\dots k_n} = \sum_{j=1}^{n} \mu^2_{k_j},$$

$$\lambda^{(2)}_{k_1,k_2,\dots k_n} = \sum_{j=1}^{n-1} \mu^2_{k_j} - \mu^2_{k_n},$$

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$$\lambda^{(2^n)}_{k_1,k_2,\dots k_n} = -\sum_{j=1}^{n} \mu^2_{k_j}$$

and the corresponding eigenfunctions

$$\vartheta^{(1)}_{l_1,l_2,\dots,l_n}(x) = \prod_{j=1}^{n} X^{(1)}_{l_j}(x_j),$$

$$\vartheta^{(2)}_{l_1,l_2,\dots,l_n}(x) = X^{(2)}_{l_n}(x_n)\prod_{j=1}^{n-1} X^{(1)}_{l_j}(x_j),$$

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$$\vartheta^{(2^n)}_{l_1,l_2,\dots,l_n}(x) = \prod_{j=1}^{n} X^{(2)}_{l_j}(x_j),$$

Let $\|u\|^2=(u,u)$, where the scalar product is $(u,v)=\int\limits_{\partial\Omega_0} uv d\Omega_0$. Besides,

$$\left(\vartheta^{(p)}_{l_1,l_2,...,l_n}(x),\vartheta^{(q)}_{k_1,k_2,...,k_n}(x)\prod_{j=1}^{n} sign(x_j)\right)=0, p\neq q,\ (p,q=\overline{1,2^n}), \forall\, l_j,k_j,$$

$$\left|\left(\vartheta^{(p)}_{l_1,l_2,...,l_n}(x),\vartheta^{(p)}_{k_1,k_2,...,k_n}(x)\prod_{j=1}^{n} sign(x_j)\right)\right|=\begin{cases}1, & l_1=k_1\wedge...\wedge l_n=k_n\\ 0, & l_1\neq k_1\vee...\vee l_n\neq k_n\end{cases},(p=\overline{1,2^n}),$$

where $l_j,k_j\in N$.

The norm

$$\|u(x,t)\|_0^2=\sum_{p=1}^{2^n}\left(\sum_{k_1,k_2,...,k_n=1}^{\infty}\left|\left(\prod_{j=1}^{n} sign(x_j)u(x,t),\vartheta^{(p)}_{k_1,k_2,...,k_n}(x)\right)\right|^2\right), \tag{11}$$

defined by the following formula is equivalent to the original norm in the space H_0. Let us denote by H_0 the closure of the linear shells of the systems of functions $\vartheta^{(p)}_{k_1,k_2,...,k_n}(x)$, $p=\overline{1,2^n}$ according to the norms $W_2^0(\Omega_0)$.

In [10], [11] it is proved that the eigenfunctions $\vartheta^{(p)}_{k_1,k_2,...,k_n}(x)$, $p=\overline{1,2^n}$ of problem (5) - (6) normalized in $L_2((-1;1)^n)$ form a Rissa basis in $L_2((-1;1)^n)$.

Main results

a) $i=1$.

Definition 1. By a generalized solution of the boundary value problem (1) - (4) we mean a function $u(x,t)$ such that $u(x,t)\in(L_2(-1,1)^n,\ [0;T])$ and

$$\begin{gathered}\int\limits_{\partial Q} u(x,t)\left(\prod_{j=1}^{n} sign(x_j)V_t(x,t)-\sum_{j=1}^{n}\prod_{\substack{i\neq j\\ i=1}}^{n} sign(x_j)V_{x_jx_j}(x,t)\right)dQ=\\ -\int\limits_{\partial\Omega_0}\prod_{j=1}^{n} sign(x_j)V(x,0)\varphi_0(x)d\Omega_0,\end{gathered} \tag{12}$$

for any function $V(x,t)\in W_2^{2,1}\left((-1;1)^n,\ \bar{\Omega}_0\right)$ satisfying conditions $V(x,T)=0,\ V(x,t)|_{\partial\Omega_0}=0$.

Let

$$M=\left\{u:\|u(x,T)\|_0\leq m,\ m<\infty\right\}.$$

Lemma 1. Let $u(x,t)$ satisfy equation (1) and conditions (2) - (4). Then for any solution $u(x,t)\in(L_2(-1,1)^n,\ \Omega_1)$ the inequality

$$\|u(x,t)\|_0\leqslant 2^{\frac{n}{2}}\|u(x,0)\|_0^{\frac{T-t}{T}}\cdot\|u(x,T)\|_0^{\frac{t}{T}} \tag{13}$$

holds.

Proof. The solution to problem (1) - (4), if it exists and $u(x,t)\in M$ we have the form

$$u(x,t)=\sum_{p=1}^{2^n}\sum_{k_1,...,k_n=1}^{\infty} u^{(p)}_{k_1,...,k_n}(t)\,\vartheta^{(p)}_{k_1,...,k_n}(x), \tag{14}$$

where $\vartheta^{(p)}_{k_1,k_2,...,k_n}(x)$, $p=\overline{1,2^n}$- are the eigenfunctions of problem (5)-(6). Let $V(x,t)=\mu_{k_1,k_2,...,k_n}(t)\vartheta^{(p)}_{k_1,k_2,...,k_n}(x)$, $\left(p=\overline{1,2^n}\right)$, and $\mu_{k_1,k_2,...,k_n}(T)=0$, $\mu_{k_1,k_2,...,k_n}(t)\in W_2^1\left(\bar{\Omega}_1\right)$. Then

$$0=\int\limits_{\partial Q}\prod_{j=1}^{n} sign(x_j)u(x,t)\left({\mu'}_{k_1,k_2,...,k_n}(t)\vartheta^{(p)}_{k_1,k_2,...,k_n}(x)+\lambda^{(p)}_{k_1,k_2,...,k_n}\mu_{k_1,k_2,...,k_n}(t)\vartheta^{(p)}_{k_1,k_2,...,k_n}(x)\right)dQ$$

$$+\mu_{k_1,k_2,\ldots,k_n}(0)\int\limits_{\partial\Omega_0}\prod_{j=1}^{n}sign(x_j)\vartheta^{(p)}_{k_1,k_2,\ldots,k_n}(x)\,\varphi_0(x)d\Omega_0.$$

From this we have

$$\int\limits_0^T u^{(p)}_{k_1,k_2,\ldots,k_n}(t)\left({\mu'}_{k_1,k_2,\ldots,k_n}(t)+\lambda^{(p)}_{k_1,k_2,\ldots,k_n}\mu_{k_1,k_2,\ldots,k_n}(t)\right)dt=-\mu_{k_1,k_2,\ldots,k_n}(0)\varphi^{(p)}_{0k_1,k_2,\ldots,k_n},$$

where

$$u^{(p)}_{k_1,k_2,\ldots,k_n}(t)=\pm\left(\prod_{j=1}^{n}sign(x_j)u(x,t),\vartheta^{(p)}_{k_1,k_2,\ldots,k_n}(x)\right),$$

$$\varphi^{(p)}_{0k_1,k_2,\ldots,k_n}=\pm\left(\prod_{j=1}^{n}sign(x_j)\varphi_0(x),\vartheta^{(p)}_{k_1,k_2,\ldots,k_n}(x)\right),\ \left(p=\overline{1,2^n}\right),\ k_j\in N.$$

Therefore, for $u^{(p)}_{k_1,k_2,\ldots,k_n}(t)$ the equalities are true

$$\left(u^{(p)}_{k_1,k_2,\ldots,k_n}(t)\right)_t=\lambda^{(p)}_{k_1,k_2,\ldots,k_n}u^{(p)}_{k_1,k_2,\ldots,k_n}(t)\,, \tag{15}$$

$$u^{(p)}_{k_1,k_2,\ldots,k_n}(0)=\varphi^{(p)}_{0k_1,k_2,\ldots,k_n}\,,\quad \left(p=\overline{1,2^n}\right),\ k_j\in N\,. \tag{16}$$

The solution to problem (15), (16) has the form

$$u^{(p)}_{k_1,k_2,\ldots,k_n}(t)=\varphi^{(p)}_{0k_1,k_2,\ldots,k_n}e^{\lambda^{(p)}_{k_1,k_2,\ldots,k_n}t},\ \left(p=\overline{1,2^n}\right),\ k_j\in N. \tag{17}$$

Taking into account (14), (17), from (11) we have

$$\|u(x,t)\|_0^2=\sum_{p=1}^{2^n}\sum_{k_1,\ldots,k_n=1}^{\infty}\left(u^{(p)}_{k_1,\ldots,k_n}(t)\right)^2.$$

Let's consider the function

$$\phi_{k_1,\ldots,k_n}(t)=\left(\varphi^{(1)}_{0k_1,\ldots,k_n}\right)^2e^{2\lambda^{(1)}_{k_1,\ldots,k_n}t}.$$

Let's calculate the derivatives of function $\phi_{k_1,\ldots,k_n}(t)$

$${\phi'}_{k_1,\ldots,k_n}(t)=2\lambda^{(1)}_{k_1,\ldots,k_n}\left(\varphi^{(1)}_{0k_1,\ldots,k_n}\right)^2e^{2\lambda^{(1)}_{k_1,\ldots,k_n}t},$$

$${\phi''}_{k_1,\ldots,k_n}(t)=4\left(\lambda^{(1)}_{k_1,\ldots,k_n}\right)^2\left(\varphi^{(1)}_{0k_1,\ldots,k_n}\right)^2e^{2\lambda^{(1)}_{k_1,\ldots,k_n}t}=4\left(\lambda^{(1)}_{k_1,\ldots,k_n}\right)^2\phi_{k_1,\ldots,k_n}(t).$$

Let's introduce function $\psi(t)=\ln(\phi_{k_1,\ldots,k_n}(t))$.

$$\psi''(t)=\frac{{\phi''}_{k_1,\ldots,k_n}(t)\phi_{k_1,\ldots,k_n}(t)-({\phi'}_{k_1,\ldots,k_n}(t))^2}{\phi^2_{k_1,\ldots,k_n}(t)}=$$

$$\frac{4\left(\lambda^{(1)}_{k_1,\ldots,k_n}\right)^2\phi_{k_1,\ldots,k_n}(t)\cdot\phi_{k_1,\ldots,k_n}(t)-4\left(\lambda^{(1)}_{k_1,\ldots,k_n}\right)^2\phi^2_{k_1,\ldots,k_n}(t)}{\phi^2_{k_1,\ldots,k_n}(t)}=0,$$

or

$$\psi''(\mathrm{t})\geq 0. \tag{18}$$

From (18) we have

$$\psi(t)\leq\left(1-\frac{t}{T}\right)\psi(0)+\frac{t}{T}\psi(T).$$

From this it easily follows that

$$\left|u^{(p)}_{k_1,\dots,k_n}(t)\right|^2 \leqslant \left(\left|u^{(p)}_{k_1,\dots,k_n}(0)\right|^2\right)^{1-\frac{t}{T}} \cdot \left(\left|u^{(p)}_{k_1,\dots,k_n}(T)\right|^2\right)^{\frac{t}{T}}, \left(p=\overline{1,2^n}\right). \tag{19}$$

Summing the inequalities (19) over $k_1, \dots, k_n \in N$ and using Holder's inequality, we obtain

$$\sum_{p=1}^{2^n}\sum_{k_1,\dots,k_n=1}^{\infty}\left(u^{(p)}_{k_1,\dots,k_n}(t)\right)^2 \leqslant 2^n\left(\sum_{p=1}^{2^n}\sum_{k_1,\dots,k_n=1}^{\infty}\left(u^{(p)}_{k_1,\dots,k_n}(0)\right)^2\right)^{1-\frac{t}{T}} \times$$
$$\times\left(\sum_{p=1}^{2^n}\sum_{k_1,\dots,k_n=1}^{\infty}\left(u^{(p)}_{k_1,\dots,k_n}(T)\right)^2\right)^{\frac{t}{T}}, t\in\Omega_0$$

or

$$\|u(x,t)\|_0 \leqslant 2^{\frac{n}{2}}\|u(x,0)\|_0^{1-\frac{t}{T}}\cdot\|u(x,T)\|_0^{\frac{t}{T}}.$$

Lemma 1 is proven.

Theorem 1. If a solution to problem (1) - (4) exists and$u(x,t)\in M$, then the solution to problem (1) - (4) is unique.

Proof. Let equation (1) with conditions (2)-(4) have solutions $u_1(x,t)$ and $u_2(x,t)$, i.e. 2 solutions. Then the function $U(x,t)=u_1(x,t)-u_2(x,t)$ is a solution with zero data. For the last function, estimate (13) is true. Using the results of lemma 1, we have $U(x,t)=0$ for all$(x,t)\in Q$, $u_1(x,t)\equiv u_2(x,t)$. Theorem 1 is proven.

Let $u(x,t)$ be the solution to problem (1) - (4) with exact data, and $u_\varepsilon(x,t)$ be the solution to problem (1) - (4) with approximate data.

Theorem 2. Let the solution of the original problem exist and $u(x,t)$, $u_\varepsilon(x,t) \in M$, in addition$\|\varphi(x)-\varphi_\varepsilon(x,)\|_0 \leq \varepsilon$. Then for the function $U(x,t)=u(x,t)-u_\varepsilon(x,t)$ at $t\in\Omega_1$ the following inequality

$$\|U(x,t)\|_0 \leqslant 2^{\frac{n}{2}}(\varepsilon)^{1-\frac{t}{T}}\cdot(2m)^{\frac{t}{T}}$$

is true.

Proof. Let the function $U(x,t)$ be the solution of the corresponding problem (1) - (4), and $U(x,0)=\varphi(x)-\varphi_\varepsilon(x)$. In addition, $\|U(x,T)\|_0^2\leq 4m^2$. For the function $U(x,t)$, using the results of lemma 1, we have

$$\|U(x,t)\|_0 \leqslant 2^{\frac{n}{2}}(\varepsilon)^{1-\frac{t}{T}}\cdot(2m)^{\frac{t}{T}}.$$

Theorem 2 is proven.

b) $i=2$.

Definition 2. By a generalized solution of problem (1) - (4) we mean a function $u(x,t), u_t(x,t)\in C\left(L_2(-1;1)^n;\ \bar{Q}\right)$, which for any arbitrary function $V(x,t)\in W_2^2\left((-1;1)^n,\bar{\Omega}_1\right)$, $\left.\frac{\partial^k V(x,t)}{\partial t^k}\right|_{t=T}=0$, $k=0,1.,.$ satisfies the following integral identity

$$\int_{\partial Q}u(x,t)\left(\prod_{j=1}^{n}\operatorname{sign}(x_j)V_{tt}(x,t)+\sum_{j=1}^{n}\prod_{\substack{i=1\\i\neq j}}^{n}\operatorname{sign}(x_j)V_{x_jx_j}(x,t)\right)dQ=$$
$$\int_{\partial\Omega_0}\prod_{j=1}^{n}\operatorname{sign}(x_j)V(x,0)\varphi_2(x)\,d\Omega_0-\int_{\partial\Omega_0}\prod_{j=1}^{n}\operatorname{sign}(x_j)V_t(x,0)\varphi_1(x)\,d\Omega_0.$$

Lemma 2. (See pp. 825-826, [9]) Let $v(t)$ be a solution to equation

$$v''(t)-\lambda v(t)=0$$

and satisfy conditions $v(0)=p_1, v'(0)=p_2$. Then for the solution of this equation f $t\in\Omega_1$ the inequality holds

$$v^2(t)\leq e^{2t(T-t)}\left(v^2(0)+|\alpha|\right)^{1-\frac{t}{T}}\left(v^2(T)+|\alpha|\right)^{\frac{t}{T}}-|\alpha|,$$

where $\lambda-$ is some constant, $\alpha=\frac{1}{2}\left(\lambda v^2(0)-v_t^2(0)\right)$.

Lemma 3. Let $u(x,t)$ be a solution to the equation

$$\frac{\partial^2 u(x,t)}{\partial t^2}+\sum_{j=1}^{n} sign(x_j)\frac{\partial^2 u(x,t)}{\partial x_j^2}=0,$$

and satisfy conditions (2) - (4). Then for the solution of this equation at $t\in\Omega_1$ the inequality

$$\|u(x,t)\|_0^2\le 2^n e^{2t(T-t)}\Big(\|u(x,0)\|_0^2+\alpha\Big)^{1-\frac{t}{T}}\Big(\|u(x,T)\|_0^2+\alpha\Big)^{\frac{t}{T}}-\alpha,$$

holds, where $\alpha=\frac{1}{2}\left(\|\varphi_1\|_1^2+\|\varphi_2\|_0^2\right)$.

Proof. If a solution to problem (1)-(4) exists and belongs to M, then it has the form (14), where $\vartheta^{(p)}_{k_1,k_2,\ldots,k_n}(x)$, $p=\overline{1,2^n}$ are eigenfunctions of problem (5)-(6). In addition,

$$u^{(p)}_{k_1,k_2,\ldots,k_n}(t)=\pm\left(\prod_{j=1}^{n} sign(x_j)u(x,t),\vartheta^{(p)}_{k_1,k_2,\ldots,k_n}(x)\right),$$

$$\varphi^{(p)}_{1k_1,k_2,\ldots,k_n}=\pm\left(\prod_{j=1}^{n} sign(x_j)\varphi_1(x),\vartheta^{(p)}_{k_1,k_2,\ldots,k_n}(x)\right),$$

$$\varphi^{(p)}_{2k_1,k_2,\ldots,k_n}=\pm\left(\prod_{j=1}^{n} sign(x_j)\varphi_2(x),\vartheta^{(p)}_{k_1,k_2,\ldots,k_n}(x)\right),\ \left(p=\overline{1,2^n}\right),\ k_j\in N.$$

Let in (20) $V(x,t)=\omega_{k_1,k_2,\ldots,k_n}(t)\vartheta^{(p)}_{k_1,k_2,\ldots,k_n}(x)$, $p=\overline{1,2^n}$,where $\omega_{k_1,k_2,\ldots,k_n}(T)=0$, ${\omega'}_{k_1,k_2,\ldots,k_n}(T)=0$, $\omega_{k_1,k_2,\ldots,k_n}(t)\in W_2^2\left(\bar\Omega_1\right)$. Then

$$\begin{gathered}\int\limits_{\partial Q} u(x,t)\prod_{j=1}^{n} sign(x_j)\left({\omega''}_{k_1,k_2,\ldots,k_n}(t)\vartheta^{(p)}_{k_1,k_2,\ldots,k_n}(x)-\omega(t)\lambda^{(p)}_{k_1,k_2,\ldots,k_n}\vartheta^{(p)}_{k_1,k_2,\ldots,k_n}(x)\right)dQ=\\ \omega_{k_1,k_2,\ldots,k_n}(0)\int\limits_{\partial\Omega_0}\prod_{j=1}^{n} sign(x_j)\vartheta^{(p)}_{k_1,k_2,\ldots,k_n}(x)\varphi_2(x)d\Omega_0\\ -{\omega'}_{k_1,k_2,\ldots,k_n}(0)\int\limits_{\partial\Omega_0}\prod_{j=1}^{n} sign(x_j)\vartheta^{(p)}_{k_1,k_2,\ldots,k_n}(x)\varphi_1(x)d\Omega_0\end{gathered}\tag{21}$$

From (21) we have

$$\begin{gathered}\int\limits_0^T u^{(p)}_{k_1,k_2,\ldots,k_n}(t)\left({\omega''}_{k_1,k_2,\ldots,k_n}(t)-\lambda^{(p)}_{k_1,k_2,\ldots,k_n}\omega_{k_1,k_2,\ldots,k_n}(t)\right)dt=\\ \omega_{k_1,k_2,\ldots,k_n}(0)\varphi^{(p)}_{2k_1,k_2,\ldots,k_n}-{\omega'}_{k_1,k_2,\ldots,k_n}(0)\varphi^{(p)}_{1k_1,k_2,\ldots,k_n},\ p=\overline{1,2^n},\ k_j\in N.\end{gathered}$$

Thus, for $u^{(p)}_{k_1,k_2,\ldots,k_n}(t)$, $\left(p=\overline{1,2^n}\right)$ we have the following sequence of solutions

$$\left(u^{(p)}_{k_1,k_2,\ldots,k_n}(t)\right)_{tt}=\lambda^{(p)}_{k_1,k_2,\ldots,k_n}u^{(p)}_{k_1,k_2,\ldots,k_n}(t)\tag{22}$$

$$u^{(p)}_{k_1,k_2,\ldots,k_n}(0)=\varphi^{(p)}_{1k_1,k_2,\ldots,k_n},\ \left(u^{(p)}_{k_1,k_2,\ldots,k_n}(0)\right)_t=\varphi^{(p)}_{2k_1,k_2,\ldots,k_n},\ k_j\in N.\tag{23}$$

$$u^{(p)}_{k_1,\dots,k_n}(t)=\begin{cases}\varphi^{(p)}_{1k_1,\dots,k_n} ch\sqrt{\lambda^{(p)}_{k_1,\dots,k_n}}t+\dfrac{\varphi^{(p)}_{2k_1,\dots,k_n} sh\sqrt{\lambda^{(p)}_{k_1,\dots,k_n}}t}{\sqrt{\lambda^{(p)}_{k_1,\dots,k_n}}},\ \lambda^{(p)}_{k_1,\dots,k_n}>0,\\ \varphi^{(p)}_{2k_1,\dots,k_n}t+\varphi^{(p)}_{1k_1,\dots,k_n},\ \lambda^{(p)}_{k_1,\dots,k_n}=0,\\ \varphi^{(p)}_{1k_1,\dots,k_n}\cos\sqrt{-\lambda^{(p)}_{k_1,\dots,k_n}}t+\dfrac{\varphi^{(p)}_{2k_1,\dots,k_n}\sin\sqrt{-\lambda^{(p)}_{k_1,\dots,k_n}}t}{\sqrt{-\lambda^{(p)}_{k_1,\dots,k_n}}},\ \lambda^{(p)}_{k_1,\dots,k_n}<0.\end{cases}$$

Let's introduce a norm

$$\|\varphi_1(x)\|_1^2=\sum_{p=1}^{2^n}\sum_{k_1,\dots,k_n}^{\infty}\lambda^{(p)}_{k_1,k_2,\dots,k_n}\left(\varphi^{(p)}_{1k,l,n}\right)^2, p=\overline{1,2^n},\ k_j\in N.$$

According to Lemma 2, for solutions of problems (22) - (23), for each fixed $k_j\in N$, the following inequalities are true:

$$\left(u^{(p)}_{k_1,k_2,\dots,k_n}(t)\right)^2\leqslant e^{2t(T-t)}\left(\left(u^{(p)}_{k_1,k_2,\dots,k_n}(0)\right)^2+\alpha^{(p)}_{k_1,k_2,\dots,k_n}\right)^{1-\frac{t}{T}}\times$$
$$\left(\left(u^{(p)}_{k_1,k_2,\dots,k_n}(T)\right)^2+\alpha^{(p)}_{k_1,k_2,\dots,k_n}\right)^{\frac{t}{T}}-\alpha^{(p)}_{k_1,k_2,\dots,k_n},\ t\in\Omega_1, \tag{24}$$

where

$$\alpha^{(p)}_{k_1,k_2,\dots,k_n}=\frac{1}{2}\left(\lambda^{(p)}_{k_1,k_2,\dots,k_n}\left(u^{(p)}_{k_1,k_2,\dots,k_n}(0)\right)^2-\left(\left(u^{(p)}_{k_1,k_2,\dots,k_n}(0)\right)_t\right)^2\right),\ \left(p=\overline{1,2^n}\right). \tag{25}$$

After an elementary transformation from (25) we can write

$$\alpha^{(p)}_{k_1,k_2,\dots,k_n}\leq\frac{1}{2}\left(\left|\lambda^{(p)}_{k_1,k_2,\dots,k_n}\right|\left(u^{(p)}_{k_1,k_2,\dots,k_n}(0)\right)^2+\left(\left(u^{(p)}_{k_1,k_2,\dots,k_n}(0)\right)_t\right)^2\right),\ \left(p=\overline{1,2^n}\right).$$

We sum up the inequalities (24) and (25) by $k_j\in N$ and taking into account the Holder inequality we get

$$\sum_{p=1}^{2^n}\sum_{k_1,\dots,k_n=1}^{\infty}\left(u^{(p)}_{k_1,k_2,\dots,k_n}(t)\right)^2\leqslant$$
$$2^ne^{2t(T-t)}\left(\sum_{p=1}^{2^n}\left(\sum_{k_1,\dots,k_n=1}^{\infty}\left(\left(u^{(p)}_{k_1,k_2,\dots,k_n}(0)\right)^2+\alpha^{(p)}_{k_1,k_2,\dots,k_n}\right)\right)\right)^{1-\frac{t}{T}}\times$$
$$\times\left(\sum_{p=1}^{2^n}\left(\sum_{k_1,\dots,k_n=1}^{\infty}\left(\left(u^{(p)}_{k_1,k_2,\dots,k_n}(T)\right)^2+\alpha^{(p)}_{k_1,k_2,\dots,k_n}\right)\right)\right)^{\frac{t}{T}}-\sum_{p=1}^{2^n}\left(\sum_{k_1,\dots,k_n=1}^{\infty}\alpha^{(p)}_{k_1,k_2,\dots,k_n}\right)$$

and summing up the above inequalities we finally get

$$\|u(x,t)\|_0^2\leqslant 2^ne^{2t(T-t)}\left(\|u(x,0)\|_0^2+\alpha\right)^{1-\frac{t}{T}}\left(\|u(x,T)\|_0^2+\alpha\right)^{\frac{t}{T}}-\alpha,$$

where $\alpha=\frac{1}{2}\left(\|\varphi_1\|_1^2+\|\varphi_2\|_0^2\right)$. Lemma 3 has been proved.

Theorem 3. If a solution to problem (1) - (4) exists and belongs to M, then it is unique.

Proof. Let $u_1(x,t)$ and $u_2(x,t)$ be solutions of problems (1) - (4). Then their difference $u(x,t)=u_1(x,t)-u_2(x,t)$ will be a solution of the homogeneous problem (1) - (4). Applying the estimates of Lemma 3, we obtain $\|u(x,t)\|_0=0$, and from this it follows that $u(x,t)=0$ for any for $\forall(x,t)\in\Omega$, $u_1(x,t)\equiv u_2(x,t)$. holds. Theorem 3 is proved.

Theorem 4. Let the solution of problem (1) - (4) exist and $u(x,t), u_\varepsilon(x,t) \in M$, in addition $\|\varphi_1(x) - \varphi_{1\varepsilon}(x)\|_1 \leq \varepsilon$, $\|\varphi_2(x) - \varphi_{2\varepsilon}(x)\|_0 \leq \varepsilon$. Then for the function $U(x,t) = u(x,t) - u_\varepsilon(x,t)$ at $t \in Q$ the following inequality is true

$$\|U(x,t)\|_0^2 \leqslant 2^n e^{2t(T-t)} \left(2\varepsilon^2\right)^{1-\frac{t}{T}} \left(4m^2 + \varepsilon^2\right)^{\frac{t}{T}} - \varepsilon^2.$$

Proof. Let the function $U(x,t)$ be a solution of equation (1) satisfying the boundary conditions and gluing conditions (3)-(4) with initial data $U(x,0) = \varphi_1(x) - \varphi_{1\varepsilon}(x)$, $U_t(x,0) = \varphi_2(x) - \varphi_{2\varepsilon}(x)$, where $\|\varphi_1(x) - \varphi_{1\varepsilon}(x)\|_1 \leq \varepsilon$, $\|\varphi_2(x) - \varphi_{2\varepsilon}(x)\|_0 \leq \varepsilon$. Then, using the estimates of Lemma 3 and elementary transformations for the norm of the function $U(x,t)$, we have

$$\|U(x,t)\|_0^2 \leqslant 2^n e^{2t(T-t)} \left(2\varepsilon^2\right)^{1-\frac{t}{T}} \left(4m^2 + \varepsilon^2\right)^{\frac{t}{T}} - \varepsilon^2.$$

Theorem 4 is proven.

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REZYUME

Ushbu ish yo‘nalishini vaqt bo‘yicha o‘zgartiruvchi birinchi va ikkinchi tartibli aralash tipdagi differensial tenglamalar uchun chegaraviy masalalarni o‘rganishga bag‘ishlangan. Aralash tipdagi tenglamalar uchun chegaraviy masalalar lazer fizikasi, plazmani modellashtirish va matematik biologiya kabi tabiiy fanlarning turli sohalarida uchraydi. Mazkur maqolada qaralgan masalaning yechimi uchun yagonalik va shartli turg'unlik teoremalari isbotlangan. Yechimning a priori bahosi logarifmik qavariqlik usuli va spektral yoyilma yordamida olingan hamda shartli turg'unlik teoremalari isbotlangan.

Kalit so'zlar: Chegaraviy masala, nokorrekt masala, aralash tipdagi tenglama, parabolik tipidagi tenglama, a priori baho, shartli turg'unlik bahosi, yechimning yagonaligi, korrektlik to'plami.

РЕЗЮМЕ

Данная работа посвящена изучению краевых задач для дифференциальных уравнений смешанного типа первого и второго порядка по временной переменной. Краевые задачи для уравнений смешанного типа возникают в различных областях естественных наук, включая лазерную физику, моделирование плазмы и математическую биологию. В данной статье доказаны теоремы об единственности и условной устойчивости решения рассматриваемой задачи в классе корректности. А приорная оценка решения получена с использованием метода логарифмической выпуклости и спектрального разложения.

Ключевые слова: Краевая задача, некорректная задача, уравнение смешанного типа, уравнение параболического типа, априорная оценка, оценка условной устойчивости, единственность решения, множество корректности.