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ON THE CAUCHY PROBLEMS FOR BARENBLATT-ZHELTOV-KOCHINA TYPE FRACTIONAL EQUATIONS

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RESUME

In this article, the Cauchy problem for a homogeneous fractional-order equation of the Barenblatt–Zheltov–Kochina type with the Caputo derivative is studied. The existence of a solution to the given Cauchy problem is demonstrated using the Fourier method, and the continuity of the obtained solution is proved by employing the properties of functional series. Furthermore, the uniqueness of the solution is established. The properties of the Mittag-Leffler function are extensively used in the process of proving the existence and uniqueness of the solution to the proposed problem.

Keywords: The Cauchy problem, the Caputo derivatives, Mittag-Liffler function, Parseval equality.

Let consider the following equation:

$$\frac{\partial^k u}{\partial t^k} + \Delta\left(\frac{\partial^k u}{\partial t^k}\right) - \nu^2 \Delta u = 0 \tag{1}$$

where $\Delta = \sum\limits_{k=1}^{N} \frac{\partial^2}{\partial x_k^2}$ –Laplas operator. If $k = 1$, then this equation is called a partial differential equation of the Barenblatt-Zheltov-Kochina type, if $k = 2$, then it is called a partial differential equation of the Boussinesq type. Equations of the form (1) are encountered in modeling various processes. For example, equations of the Barenblatt-Zheltov-Kochina type are the basic equations for the filtration of homogeneous liquids. This was first noticed in the work of G. I. Barenblatt, Yu. P. Zheltov and I. N. Kochina [1]. Later, such equations were studied in the papers [2,3,4,5,6,7].

Let $A : H \to H$ be a self-adjoint, positive, unbounded arbitrary operator defined in a separable Hilbert space H. Suppose that operator A has a complete orthonormal system of eigenfunctions $\{v_k\}$ in H and the corresponding set of positive eigenvalues $\{\lambda_k\}$. Using the renumbering of eigenvalues, we can number them non-decreasing, and write as $0 < \lambda_1 \le \lambda_2 \cdots \to +\infty$.

Let the function $h(t)$ be defined in the interval $[0, +\infty)$ with values in H. The Caputo fractional derivative of order $0 < \rho < 1$ is defined as formula (see, [8]):

$$D_t^\rho h(t) = \frac{1}{\Gamma(1-\rho)} \int\limits_0^t \frac{h'(\xi)}{(t-\xi)^\rho} d\xi, \qquad t > 0.$$

Let $C((a, b); H)$ denote the set of continuous functions $u(t)$ on the interval $t \in (a, b)$ with values in H. Now, recall the concept of the degree of an operator A in a Hilbert space H. Let τ is an arbitrary real number. We introduce the degree of the operator A in H as follows:

$$A^\tau h = \sum_{k=1}^{\infty} \lambda_k^\tau h_k v_k,$$

where $h_k = (h, v_k)$ Fourier coefficients of the element $h \in H$. Obviously, the domain of this operator has the form:

$$D(A^\tau) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^{2\tau} |h_k|^2 < \infty\}.$$

For element $h \in D(A^\tau)$ we introduce the norm as follows:

$$||h||_\tau^2 = \sum_{k=1}^{\infty} \lambda_k^{2\tau} |h_k|^2 = ||A^\tau h||^2.$$

Together with this norm $D(A^\tau)$ turns into a Hilbert space.
Recall, the Mittag-Leffler function $E_{\rho,\mu}(t)$ has the form

$$E_{\rho,\mu}(t) = \sum_{k=0}^{\infty} \frac{t^n}{\Gamma(\rho n + \mu)},$$

where $\rho > 0$ and μ complex number.

Let $AC[0, T]$ be the set of absolute continuous functions defined on $[0, T]$ and let $AC([0, T]; H)$ stand for a space of absolute continuous functions $u(t)$ with values in H (see, [9] p 339).

Let us present the following properties:

Lemma 1. *Let $0 < \rho < 1$. Then the following equality holds:*

$$D_t^\rho \left[t^\rho E_{\rho,\rho+1}(-\lambda t^\rho)\right] = E_{\rho,1}(-\lambda t^\rho). \tag{2}$$

Proof of Lemma 1 you can find, for example, [14].

Lemma 2. *Let $0 < \rho < 1$. Then the following equality holds:*

$$D_t^\rho [E_{\rho,1}(-\lambda t^\rho)] = -\lambda E_{\rho,1}(-\lambda t^\rho). \tag{3}$$

Proof of Lemma 2 you can find, for example, [14].
Consider the following problem:

$$\begin{cases} D_t^\rho u\,(t) + A^a\,(D_t^\rho u\,(t)) + A^b u\,(t) = 0\,, \quad 0 < t \le T\,; \\ \qquad\qquad u(+0) = \varphi, \end{cases} \tag{4}$$

where $\varphi \in H$.

This (4) problem is called the Cauchy problem for fractional equations of Barenblatt-Zheltov-Kochina type.

Definition 1. *A function $u\,(t) \in AC\,([0, T]; H)$ with the properties $D_t^\rho u\,(t)$, $A^a\,(D_t^\rho u\,(t))$, $A^b u\,(t) \in C\,((0, T]; H)$ and satisfying conditions [4] is called the solution of the Cauchy problem [4].*

We note that problems similar to [4], in the case $a = 1, b = 1$ were studied in [14] as the Cauchy problem and in [15],[16] as non-local problems.

When $a = 2, b = 1$, problem (4) is referred to as a Benney–Luke type problem. Problem of this type have been studied in the work [17].

When $a = 0, b = 1$, problem [4] was studied in the work [18] as a non-local problem.

Theorem 1. *Let $0 < \rho < 1$, If $c = \min\{a, b\}$, $\varphi \in D\,(A^c)$, then problem (4) has a unique solution and this solution has the form:*

$$u\,(t) = \sum_{k=1}^{\infty} \varphi_k E_{\rho,1}\,(-\mu_k t^\rho)\,v_k, \tag{5}$$

where are φ_k- the Fourier coe?cients of the function φ and $\mu_k = \frac{\lambda_k^b}{1+\lambda_k^a}$.

Proof. Existence of the solution. Due to the completeness of the system $\{v_k\}$ in H, the arbitrary solution of (4) can be written in the form:

$$u\,(t) = \sum_{k=1}^{\infty} T_k\,(t)\,v_k.$$

Substituting into the first equation of (4) we have the following equation:

$$D_t^{\rho} T_k(t)\left(1+\lambda_k^a\right)+T_k(t)\,\lambda_k^b=0,$$

Therefore,

$$D_t^{\rho} T_k(t)+\mu_k T_k(t)=0,\ \mu_k=\frac{\lambda_k^b}{1+\lambda_k^a}.$$

Using the second condition of (4), we obtain the following Cauchy problem:

$$\begin{cases} D_t^{\rho} T_k(t)+\mu_k T_k(t)=0, \\ T_k(+0)=\varphi_k. \end{cases} \tag{6}$$

This problem has a unique solution (see, [28] p. 231):

$$T_k(t)=\varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right). \tag{7}$$

Then we get the formal solution (5).

From this, in particular, it follows that if a solution to the forward problem exists, then it is unique. Indeed, for this it is sufficient to prove that the solution $u(t)$ to the forward problem with the homogeneous condition (4) is identically zero. But from (7)it follows that $T_k(t)\equiv 0$ for all $k\geq 1$. Taking into account the definition $T_k(t)$ and the completeness of the system $\{v_k\}$, we obtain $u(t)\equiv 0$.

It remains to prove that the constructed formal solution satisfies all the requirements of Definition 1 is indeed a solution to problem (4). We will analyze the proof of the theorem separately for the cases $b\leq a$ and $b>a$.

1) Let $b\leq a$.

Denote by $S_n(t)$ be the partial sums of (5):

$$S_n(t)=\sum_{k=1}^{n}\varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right)v_k.$$

Then by using the following estimate of the Mittag-Liffler function $0<|E_{\rho,\mu}(-t)|<1$ and Parseval equality:

$$\|S_n(t)\|^2=\left\|\sum_{k=1}^{n}\left[\varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right)\right]v_k\right\|^2=\sum_{k=1}^{n}|\varphi_k E_{\rho,1}(-\mu_k t^{\rho})|^2\leq C\sum_{k=1}^{n}|\varphi_k|^2.$$

Therefore, if $\varphi\in H$ then $u(t)\in C((0,T];H)$.

Now we estimate $A^b u(t)$:

$$\left\|A^b S_n(t)\right\|^2=\left\|\sum_{k=1}^{n}\left[\varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right)\right]\lambda_k^b v_k\right\|^2=\sum_{k=1}^{n}\lambda_k^{2b}|\varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right)|^2\leq C\sum_{k=1}^{n}\lambda_k^{2b}|\varphi_k|^2.$$

Thus, if $\varphi\in D(A^b)$, then $A^b u(t)\in C((0,T];H)$.

Then, we estimate $A^a\left(D_t^{\rho}u(t)\right)$. By using Lemma 2:

$$\left\|A^a\left(D_t^{\rho}S_n(t)\right)\right\|^2=\left\|\sum_{k=1}^{n}\left[-\mu_k\varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right)\right]\lambda_k^a v_k\right\|^2=$$

$$=\sum_{k=1}^{n}\lambda_k^{2a}|-\mu_k\varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right)|^2\leq C\sum_{k=1}^{n}\lambda_k^{2a}|-\mu_k\varphi_k|^2=$$

$$=C\sum_{k=1}^{n}\lambda_k^{2a}\left|\frac{\lambda_k^b}{1+\lambda_k^a}\varphi_k\right|^2\leq C\sum_{k=1}^{n}\lambda_k^{2a}\cdot\frac{\lambda_k^{2b}}{\lambda_k^{2a}}|\varphi_k|^2=C\sum_{k=1}^{n}\lambda_k^{2b}|\varphi_k|^2.$$

From this, if $\varphi \in D(A^b)$, we obtain $A^a\left(D_t^{\rho} u\left(t\right)\right) \in C\left((0;T];H\right)$.

Finally, we estimate $D_t^{\rho} u\left(t\right)$:

$$\left\|D_t^{\rho} S_n\left(t\right)\right\|^2 = \left\|\sum_{k=1}^{n} [-\mu_k \varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right)] v_k\right\|^2 =$$

$$= \sum_{k=1}^{n} \left|-\mu_k \varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right)\right|^2 \leq C\sum_{k=1}^{n} \left|-\mu_k \varphi_k\right|^2 \leq C\sum_{k=1}^{n} \left|\varphi_k\right|^2.$$

If $\varphi \in H$, then $D_t^{\rho} u\left(t\right) \in C\left((0,T];H\right)$.

2) Let $b > a$.

Now we estimate $A^b u\left(t\right)$, for that by using the following estimate of the Mittag-Liffler function

$$0 < \left|E_{\rho,\mu}\left(-t\right)\right| \leq \frac{C}{1+t}$$

and Parseval equality:

$$\left\|A^b S_n\left(t\right)\right\|^2 = \left\|\sum_{k=1}^{n} [\varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right)] \lambda_k^b v_k\right\|^2 = \sum_{k=1}^{n} \lambda_k^{2b} \left|\varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right)\right|^2 \leq$$

$$\leq \sum_{k=1}^{n} \lambda_k^{2b} \left|\frac{C}{1+\mu_k t^{\rho}}\right|^2 \left|\varphi_k\right|^2 \leq \frac{C}{t^{2\rho}} \sum_{k=1}^{n} \lambda_k^{2a} \left|\varphi_k\right|^2.$$

From this, if $\varphi \in D\left(A^a\right)$, then $A^b u\left(t\right) \in C\left((0,T];H\right)$.

Then we estimate $A^a\left(D_t^{\rho} u\left(t\right)\right)$. By using Lemma 2:

$$\left\|A^a\left(D_t^{\rho} S_n\left(t\right)\right)\right\|^2 = \left\|\sum_{k=1}^{n} [-\mu_k \varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right)] \lambda_k^a v_k\right\|^2 =$$

$$= \sum_{k=1}^{n} \lambda_k^{2a} \left|-\mu_k \varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right)\right|^2 \leq \sum_{k=1}^{n} \lambda_k^{2a} \left|\mu_k \varphi_k \cdot \frac{C}{1+\mu_k t^{\rho}}\right|^2 \leq \frac{C}{t^{2\rho}} \sum_{k=1}^{n} \lambda_k^{2a} \left|\varphi_k\right|^2.$$

Thus, if $\varphi \in D(A^a)$, we obtain $A^a\left(D_t^{\rho} u\left(t\right)\right) \in C\left((0,T];H\right)$.

Finally, we estimate $D_t^{\rho} u\left(t\right)$:

$$\left\|D_t^{\rho} S_n\left(t\right)\right\|^2 = \left\|\sum_{k=1}^{n} [-\mu_k \varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right)] v_k\right\|^2 =$$

$$= \sum_{k=1}^{n} \left|-\mu_k \varphi_k E_{\rho,1}\left(-\mu_k t^{\rho}\right)\right|^2 \leq \sum_{k=1}^{n} \left|\mu_k \varphi_k \cdot \frac{C}{1+\mu_k t^{\rho}}\right|^2 \leq \frac{C}{t^{2\rho}} \sum_{k=1}^{n} \left|\varphi_k\right|^2.$$

If $\varphi \in H$, then $D_t^{\rho} u\left(t\right) \in C\left((0,T];H\right)$. Existence of the solution of problem (4) is proved.

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REZYUME

Ushbu maqolada Caputo hosilasi bilan berilgan Barenblatt-Jeltov-Kochina tipidagi bir jinsli kars tartibli tenglama uchun Koshi masalasi o'rganiladi. Berilgan Koshi masalasining yechimi mavjudligi Furye usuli yordamida ko'rsatiladi, topilgan yechimning uzluksizligi esa funksional qatorlar xossalaridan foydalangan holda isbotlanadi. Shuningdek, yechimning yagona ekanligi ham isbotlanadi. Masala yechimining mavjudligi va yagonaligini ko'rsatish jarayonida Mittag-Leffler funksiyasining xossalaridan keng foydalaniladi.

Kalit so'zlar: Koshi masalasi, Caputo hosilalari, Mittag-Leffler funksiyasi, Parseval tengligi.

РЕЗЮМЕ

В данной статье исследуется задача Коши для однородного дробного уравнения типа Баренблатта-Жельтова-Кочина с производной Капуто. Существование решения поставленной задачи Коши показано с использованием метода Фурье, а непрерывность найденного решения доказывается на основе свойств функциональных рядов. Кроме того, устанавливается единственность решения. В процессе доказательства существования и единственности решения задачи широко используются свойства функции Миттага-Лефлера.

Ключевые слова: Задача Коши, производные Капуто, функция Миттага-Лефлера, равенство Парсеваля.