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Maximal solvable extensions of m-dimensional some n-Lie algebras

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RESUME

This paper is devoted to the construction of maximal solvable extensions of m-dimensional n-Lie algebras. The study provides a systematic approach to identifying and classifying such extensions within the framework of higher-order Lie structures. The results contribute to the deeper understanding of the algebraic properties of n-Lie algebras and their solvable extensions, and may serve as a basis for further applications in both mathematics and theoretical physics.

Key words: n-Lie algebras, solvable algebras, nilpotent algebras, derivations, nil-independent derivations.

Introduction

The investigation of finite-dimensional n-Lie algebras has introduced a innovative perspective in the study of Lie algebras. Exploring these algebras is significant due to their potential applications in diverse areas, including dynamical systems, geometry, and physics.

Nambu extended the Poisson bracket while studying the classical dynamics of three particles as a foundation for quantum statistics in the quark model, introducing the trilinear product $[-,-,-]$ defined by: $\frac{dx}{dy} = [H_1, H_2, x]$ for Hamiltonians H_1 and H_2 [5]. Subsequently, in 1994, Takhtajan built upon Nambu's geometric framework and introduced the fundamental identity, analogous to the Jacobi identity [6]. This development allowed him to establish a connection between generalized Nambu mechanics and the theory of n-Lie algebras proposed by Filippov [3].

An analogue of Engel's theorem exists for finite-dimensional n-Lie algebras, but the Levi decomposition does not generally apply. This highlights the significance of studying finite-dimensional n-Lie algebras, which necessitates additional constraints. One such constraint is the imposition of a hyponilpotency condition on the maximal ideal when analyzing solvable n-Lie algebras, such as in the description of solvable n-Lie algebras or the extension of a given n-Lie algebra.

In 2009, the notion of a hyponilpotent ideal in n-Lie algebras was introduced. Certain characterizations of solvable ternary Filippov algebras featuring a specific m-dimensional maximal filiform hyponilpotent ideal were presented in [1]. Specifically, [1] described solvable 3-Lie algebras with an m-dimensional filiform 3-Lie algebra N, where $m \geq 5$, as a maximal hyponilpotent ideal. It was further demonstrated that this m-dimensional filiform 3-Lie algebra N cannot serve as the nilradical of any solvable 3-Lie algebra. Additionally, [1] identified several classes of solvable 3-Lie algebras obtained as one-dimensional extensions of given nilpotent n-Lie algebras. Furthermore, the studies in [4] and[7] also explore the properties of n-Lie algebras.

In this work we focus on the description of maximal solvable m-dimensional n-Lie algebras with maximal rank.

Preliminaries

A vector space $\mathcal{A}$ over a field $\mathbb{F}$ is an n-Lie algebra (sometimes called by Filippov algebra) provided that $\mathcal{A}$ is equipped with some n-ary multilinear operation $[-,-,\dots,-]$ satisfying the two identities

$$[x_1,x_2,\dots,x_n]=(-1)^{sign(\sigma)}[x_{\sigma(1)},x_{\sigma(2)},\dots,x_{\sigma(n)}],\ \sigma\in S_n,$$

$$[[x_1,x_2,\dots,x_n],y_2\dots,y_n]=\sum_{i=1}^{n}[x_1,\dots,x_{i-1},[x_i,y_2,\dots,y_n],x_{i+1},\dots,x_n].$$

We introduce the notation

$$Lie_n((x_1,x_2,\dots,x_n),y_2\dots,y_n)=$$

$$\sum_{i=1}^{n}[x_1,\dots,x_{i-1},[x_i,y_2,\dots,y_n],x_{i+1},\dots,x_n]-[[x_1,x_2,\dots,x_n],y_2\dots,y_n]$$

to use it later and $Lie_n((x_1,x_2,\dots,x_n),y_2\dots,y_n)=0$ is called as n-Lie identity.

Definition 1. Let $\mathcal{N}$ be an n-Lie algebra. A subspace $\mathcal{B}$ of $\mathcal{N}$ is an n-Lie subalgebra if

$$[\mathcal{B},\mathcal{B},...,\mathcal{B}]\subseteq\mathcal{B}.$$

A subspace $\mathcal{I}$ of $\mathcal{N}$ is an ideal if $[\mathcal{I},\mathcal{A},...,\mathcal{A}]\subseteq\mathcal{I}$.

Given an arbitrary ideal $\mathcal{I}$ of $\mathcal{N}$, we define the lower central series and the derived series as follows:

$$\mathcal{I}^1=\mathcal{I},\ \mathcal{I}^{k+1}=[\mathcal{I}^k,\mathcal{I},\mathcal{N},\dots,\mathcal{N}],\ k\geq 1,$$

$$\mathcal{I}^{(1)}=\mathcal{I},\ \mathcal{I}^{(s+1)}=[\mathcal{I}^{(s)},\mathcal{I}^{(s)},\mathcal{N},\dots,N],\ s\geq 1.$$

Definition 2. An ideal $\mathcal{I}$ is called solvable if there exists a natural r such that $\mathcal{I}^{(r)}=0$. An n-Lie algebra $\mathcal{N}$ is solvable if $\mathcal{N}^{(r)}=0$ for some $r\in\mathbb{N}$.

An ideal $\mathcal{I}$ is called *nilpotent* if there exists a natural r such that $\mathcal{I}^r=0$. An n-Lie algebra $\mathcal{N}$ is said to be *solvable* if $\mathcal{N}^r=0$ for some $r\in\mathbb{N}$.

Definition 3. A linear mapping $D:\mathcal{A}\to\mathcal{A}$ is said to be *a derivation* of an n-Lie algebra $\mathcal{A}$ provided that

$$D([x_1,x_2,\dots,x_n])=\sum_{i=1}^{n}[x_1,x_2,\dots,D(x_i),x_{i+1},\dots,x_n]$$

for all $x_1,x_2,\dots,x_n\in\mathcal{A}$. The vector space of all derivations is denoted by $Der(\mathcal{A})$.

One can check that $Der(\mathcal{A})$ is a subalgebra of the Lie algebra $gl(\mathcal{A})$ which is called the derivation algebra of $\mathcal{A}$.

Definition 4. A linear mapping $ad(x_2,x_3,\dots,x_n):\mathcal{A}\to\mathcal{A}$, defined as

$$ad(x_2,x_3,\dots,x_n)(y)=[y,x_2,x_3,\dots,x_n]\text{ for all }\ y\in\mathcal{A},$$

is the right multiplication operator. It is easy to verify $ad(x_2,x_3,\dots,x_n)$ is a derivation of $\mathcal{A}$.

The set of all finite linear combinations of the operators ad forms an ideal of the Lie algebra $Der(\mathcal{A})$ which is denoted by $Inder(\mathcal{A})$. Since derivations play crucial role in the variety of algebras, it is important to find out for which types of algebras $Der(\mathcal{A})=Inder(\mathcal{A})$.

Due to result of [2] (see Lemma 3.3) for a derivation d of n-Lie algebra $\mathcal{A}$ we have

$$d^k([x_1,x_2,\dots,x_n])=\sum_{i_1+i_2+\dots+i_n=k}\frac{k!}{i_1!i_2!\dots i_n!}[d^{i_1}(x_1),d^{i_2}(x_2),\dots,d^{i_n}(x_n)].\tag{5}$$

Let d be a nilpotent derivation. Then there exits $k\in\mathbb{N}$ such that $d^{k+1}=0$.

Then for the map

$$exp(d)=1+d+\frac{d^2}{2!}+\frac{d^3}{3!}+\dots+\frac{d^k}{k!}$$

applying (6) similar to the Lie algebras case one can prove:

$$[exp(d)(x_1), exp(d)(x_2), \ldots, exp(d)(x_n)] = exp(d)([x_1, \ldots, x_n]).$$

Consequently, $exp(d)$ is an automorphism of n-Lie algebra $\mathcal{A}$. Such kind of automorphisms and their products are called inner automorphisms.

Let $\mathcal{N}$ be an n-Lie algebra.

Definition 5. A torus on an n-Lie algebra $\mathcal{N}$ (denoted by $\mathcal{T}(\mathcal{N})$) is a commutative subalgebra of $Der(\mathcal{N})$ which consists of semisimple endomorphisms. A torus is said to be maximal, denoted by $\mathcal{T}_{max}(\mathcal{N})$, if it is not contained strictly in any other torus.

Let $\mathcal{N}$ be an m-dimensional n-Lie algebra, such that $[x_{i_1}, x_{i_2}, \ldots, x_{i_n}] = \sum\limits_{s=1}^{m} \gamma^k_{i_1,i_2,\ldots,i_n} x_k,\ 1 \leq i_1, i_2, \ldots i_n \leq m,\ \gamma^k_{i_1,i_2,\ldots,i_n} \in \mathbb{C}$. Let $D(x_i) = \alpha_i x_i,\ 1 \leq i \leq m$.

$$D([x_{i_1}, x_{i_2}, \ldots, x_{i_n}]) = \sum_{k=1}^{n} [x_{i_1}, x_{i_2}, \ldots, D(x_{i_k}), \ldots, x_{i_n}] = (\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_n})[x_{i_1}, x_{i_2}, \ldots, x_{i_n}]$$

Thus, $D \in Der(\mathcal{N})$.

Let

$$D([x_{i_1}, x_{i_2}, \ldots, x_{i_n}]) = \alpha_{i_1,i_2,\ldots i_n}[x_{i_1}, x_{i_2}, \ldots, x_{i_n}]$$

then

$$\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_n} = \alpha_{i_1,i_2,\ldots i_n},\ \text{if } \gamma^s_{i_1,i_2,\ldots,i_n} \neq 0.$$

For $(i_1, ..., i_n, k)$ which $\gamma^k_{i_1,i_2,\ldots,i_n} \neq 0$, we consider the system of the linear equations

$$S_e : \begin{cases} \sum\limits_{j=1}^{n} \alpha_{i_j} = \alpha_k, \end{cases}$$

in the variables $\alpha_{i_j}, \quad 1 \leq j \leq n$, as i_j, k run from 1 to m.

We denote by $r\{e_1, ..., e_m\}$ the rank of the system S_e. Setting $r\{\mathcal{N}\} = \min r\{x_1, ..., x_m\}$ as $\{x_1, ..., x_m\}$ runs over all bases of $\mathcal{N}$, similar to Lie algebras case, for a nilpotent n-Lie algebra $\mathcal{N}$ over an algebraically closed field one can establish the equality $dim\mathcal{T}_{max} = dim\mathcal{N} - r\{\mathcal{N}\}$.

Note that a diagonal transformation $d = diag(\alpha_1, ..., \alpha_m)$ is a derivation of $\mathcal{N}$ if and only if α_i are solutions of the system S_e.

Denote the free parameters in the solutions to the system S_e by $\alpha_1, ..., \alpha_s$. Then we get

$$\alpha_i = \sum_{j=1}^{s} \lambda_{i,j}\alpha_j, \quad s+1 \leq i \leq m.$$

Consider a basis $\{(\alpha_{1,i}, ..., \alpha_{m,i}) | 1 \leq i \leq s\}$ of fundamental solutions of the system S_e. Then the diagonal matrices $\{diag(\alpha_{1,i}, ..., \alpha_{m,i}) | 1 \leq i \leq s\}$ forms the basis of a maximal torus of $\mathcal{N}$.

Definition 6. A nilpotent n-Lie algebra $\mathcal{N}$ satisfying the condition $dim\mathcal{T}_{max} = dim(\mathcal{N}/\mathcal{N}^2)$ is called of maximal rank.

Let $\mathcal{N}$ be an n-Lie algebra, and let $\mathcal{I}$ be an ideal of $\mathcal{N}$. Note that the notions of nilpotency of $\mathcal{I}$ as a subalgebra and nilpotency of $\mathcal{I}$ as an ideal differ in general.

Definition 7. Let $\mathcal{N}$ be an n-Lie algebra, and let $\mathcal{I}$ be an ideal of $\mathcal{N}$. If $\mathcal{I}$ is a nilpotent subalgebra that is not nilpotent as an ideal, then $\mathcal{I}$ is a hyponilpotent ideal of $\mathcal{N}$. If $\mathcal{I}$ is not a proper subset of another hyponilpotent ideal then $\mathcal{I}$ is a maximal hyponilpotent ideal of $\mathcal{N}$.

Main part

Let $\mathcal{N}$ be an m-dimensional nilpotent n-Lie algebra satisfying the condition $n+1 = \dim \mathcal{T}_{\max} = \dim(\mathcal{N}/\mathcal{N}^2)$. The algebra $\mathcal{N}$ is defined by the following multiplications:

$$\mathcal{N}: \begin{cases} [e_1, e_2, e_3, ..., e_{n-1}, e_i] = e_{i+1}, & n+1 \le i \le m-2, \\ [e_1, e_2, e_3, ..., e_{n-1}, e_n] = e_m, \end{cases}$$

with $\mathcal{N} = \langle e_1, e_2, \dots, e_m \rangle$ and $\mathcal{N}/\mathcal{N}^2 = \langle e_1, e_2, \dots, e_{n+1} \rangle$.

Theorem. Let $\mathcal{R} = \mathcal{N} \rtimes \mathcal{T}_{max}$ be a maximal solvable n-Lie algebra with given hyponilpotent ideal $\mathcal{N}$ and let $\mathcal{T}_{max}$ be a maximal torus of $\mathcal{N}$. Then $\mathcal{R}$ is unique (up to isomorphism), and it is isomorphic to an algebra with the following multiplications table:

$$\begin{cases} [e_1, e_2, e_3, ..., e_{n-1}, e_i] = e_{i+1}, & n+1 \le i \le m-2, \\ [e_1, e_2, e_3, ..., e_{n-1}, e_n] = e_m, \\ [x, e_2, ..., e_{n-1}, e_i] = e_i, & i \in \{1, m\}, \\ [x, e_2, ..., e_{n-1}, e_i] = (i-n-1)e_i, & n+2 \le i \le m-1, \\ [y, e_2, ..., e_{n-1}, e_i] = e_i, & i \in \{n, m\}, \\ [z, e_2, ..., e_{n-1}, e_i] = e_i, & n+1 \le i \le m-1, \end{cases}$$

where $\{e_1, e_2, ..., e_m, x, y, z\}$ is a basis of $\mathcal{R}$.

Proof. Let consider the system S_e for $\mathcal{N}$.

$$S_e: \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_{n-1} + \alpha_i = \alpha_{i+1}, & n+1 \le i \le m-2, \\ \alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_{n-1} + \alpha_n = \alpha_m, \end{cases}$$

From this system, we derive that the maximal torus $\mathcal{T}_{\max}$ has the following matrix form: $\mathcal{T}_{\max} = Span\{d_1, d_2, ..., d_{n+1}\}$, where

$$\begin{cases} d_1 = diag(1, 0, 0, ..., 0, 0, 1, 2, 3, ..., m-n-2, 1), \\ d_2 = diag(0, 1, 0, ..., 0, 0, 1, 2, 3, ..., m-n-2, 1), \\ ... \\ d_s = diag(0, 0, ..., \underbrace{1}_{s-term} \dots 0, 0, 1, 2, 3, ..., m-n-2, 1), & 1 \le s \le n-1, \\ d_n = diag(0, 0, 0, ..., 0, \underbrace{1}_{n-term}, 0, 0, ..., 0, 1), \\ d_{n+1} = diag(0, 0, 0, ..., 0, \underbrace{1}_{n+1-term}, 1, 1, ..., 1, 0), \end{cases}$$

Let suppose $\dim(\mathcal{R}) = m+3$. We need to select three nil-independent derivations. Below, we consider all possible choices:

Case 1. Let us choose the following derivations:

$$d_1 := ad(x, e_2, ..., e_{n-1}), \quad d_n := ad(y, e_2, ..., e_{n-1}), \quad d_{n+1} := ad(z, e_2, ..., e_{n-1}).$$

Then, we obtain the following solvable algebra, denoted by $\mathcal{R}^1_{n,n+1}$:

$$\mathcal{R}^1_{n,n+1}: \begin{cases} [x, e_2, ..., e_{n-1}, e_1] = e_1, \\ [x, e_2, ..., e_{n-1}, e_i] = (i-n-1)e_i, & n+2 \le i \le m-1, \\ [x, e_2, ..., e_{n-1}, e_m] = e_m, \\ [y, e_2, ..., e_{n-1}, e_n] = e_n, \\ [y, e_2, ..., e_{n-1}, e_m] = e_m, \\ [z, e_2, ..., e_{n-1}, e_i] = e_i, & n+1 \le i \le m-1, \\ [\mathcal{N}, \mathcal{N}]. \end{cases}$$

This is an algebra provided in the theorem.

Case 2. Let us choose the following derivations for $2 \leq s \leq n-1$:

$$d_s := \mathrm{ad}(x, e_1, \dots, \widehat{e}_s, \dots, e_{n-1}), \quad d_n := \mathrm{ad}(y, e_1, \dots, \widehat{e}_s, \dots, e_{n-1}), \quad d_{n+1} := \mathrm{ad}(z, e_1, \dots, \widehat{e}_s, \dots, e_{n-1}),$$

where $\widehat{e}_s$ denotes the omitted element. Then, we obtain the following solvable algebra, denoted by $\mathcal{R}^s_{n,n+1}$:

$$\mathcal{R}^s_{n,n+1}: \begin{cases} [x, e_1, ..., \widehat{e}_s, \dots e_{n-1}, e_s] = e_s, \\ [x, e_1, ..., \widehat{e}_s, \dots e_{n-1}, e_i] = (i-n-1)e_i, \quad n+2 \leq i \leq m-1, \\ [x, e_1, ..., \widehat{e}_s, \dots e_{n-1}, e_m] = e_m, \\ [y, e_1, ..., \widehat{e}_s, \dots e_{n-1}, e_n] = e_n, \\ [y, e_1, ..., \widehat{e}_s, \dots e_{n-1}, e_m] = e_m, \\ [z, e_1, ..., \widehat{e}_s, \dots e_{n-1}, e_i] = e_i, \quad n+1 \leq i \leq m-1, \\ [\mathcal{N}, \mathcal{N}]. \end{cases}$$

By applying the basis transformation:

$$e'_1 = (-1)^s e_s, \quad e'_s = e_1, \quad e'_i = e_i \text{ for } 2 \leq i \neq s \leq m,$$

we obtain the algebra described in the theorem.

In both of the above cases, we selected the derivations d_n and d_{n+1} together. The cases below address what happens when only one of them is chosen.

Case 3. Let us choose the following derivations for $1 \leq s < t \leq n-1$:

$$d_s := ad(x, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_n), \quad d_t := ad(y, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_n),$$

$$d_{n+1} := ad(z, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_n),$$

where $\widehat{e}_s$ and $\widehat{e}_t$ denote the omitted basis elements. Then, we obtain the following solvable algebra, denoted by $\mathcal{R}^s_{t,n+1}$:

$$\mathcal{R}^s_{t,n+1}: \begin{cases} [x, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_n, e_s] = e_s, \\ [x, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_n, e_i] = (i-n-1)e_i, \quad n+2 \leq i \leq m-1, \\ [x, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_n, e_m] = e_m, \\ [y, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_n, e_t] = e_t, \\ [y, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_n, e_i] = (i-n-1)e_i, \quad n+2 \leq i \leq m-1, \\ [y, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_n, e_m] = e_m, \\ [z, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_n, e_i] = e_i, \quad n+1 \leq i \leq m-1, \\ [\mathcal{N}, \mathcal{N}]. \end{cases}$$

However,

$$[[x, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_n, e_s], e_1, e_2, ..., \widehat{e}_s, ..., e_{n-1}, e_i] =$$
$$= [x, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_n, [e_s, e_1, e_2, ..., \widehat{e}_s, ..., e_{n-1}, e_i]] =$$
$$= [x, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_n, (-1)^s e_{i+1}] = (-1)^s (i-n) e_{i+1},$$

On the other hand

$$[[x, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_n, e_s], e_1, e_2, ..., \widehat{e}_s, ..., e_{n-1}, e_i] = (-1)^s e_{i+1}.$$

This implies that $e_{i+1} = 0$ for $n+2 \leq i \leq m-1$, which is a contradiction. Therefore, $\mathcal{R}^s_{t,n+1}$ is not an $n-$Lie algebra.

Case 4. Let us choose the following derivations for $1 \leq s < t \leq n-1$:

$$d_s := ad(x, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_{n+1}), \quad d_t := ad(y, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_{n+1}),$$

$$d_n := ad(z, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_{n+1}),$$

where $\widehat{e}_s$ and $\widehat{e}_t$ denote the omitted basis elements. Then, we obtain the following solvable algebra, denoted by $\mathcal{R}^s_{t,n}$:

$$\mathcal{R}^s_{t,n} : \begin{cases} [x, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_{n+1}, e_s] = e_s, \\ [x, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_{n+1}, e_i] = (i-n-1)e_i, \quad n+2 \leq i \leq m-1, \\ [x, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_{n+1}, e_m] = e_m, \\ [y, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_{n+1}, e_t] = e_t, \\ [y, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_{n+1}, e_i] = (i-n-1)e_i, \quad n+2 \leq i \leq m-1, \\ [y, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_{n+1}, e_m] = e_m, \\ [z, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_{n+1}, e_n] = e_n, \\ [z, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_{n+1}, e_m] = e_m, \\ [\mathcal{N}, \mathcal{N}]. \end{cases}$$

However,

$$[[x, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_{n+1}, e_s], e_1, e_2, ..., \widehat{e}_s, ..., e_{n-1}, e_i] =$$
$$= [x, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_{n+1}, [e_s, e_1, e_2, ..., \widehat{e}_s, ..., e_{n-1}, e_i]] =$$
$$= [x, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_{n+1}, (-1)^s e_{i+1}] = (-1)^s (i-n) e_{i+1},$$

On the other hand

$$[[x, e_1, ..., \widehat{e}_s, \dots \widehat{e}_t, \dots e_{n-1}, e_{n+1}, e_s], e_1, e_2, ..., \widehat{e}_s, ..., e_{n-1}, e_i] = (-1)^s e_{i+1}.$$

This implies that $e_{i+1} = 0$ for $n+2 \leq i \leq m-1$, which is a contradiction. Therefore, $\mathcal{R}^s_{t,n}$ is not an $n-$Lie algebra.

Suppose that $\dim(\mathcal{R}) > m+3$. Assume $\dim(\mathcal{R}) = m+4$. In Case 1, we define:

$$d_1 := \operatorname{ad}(x, e_2, \dots, e_{n-1}), \quad d_n := \operatorname{ad}(y, e_2, \dots, e_{n-1}), \quad d_{n+1} := \operatorname{ad}(z, e_2, \dots, e_{n-1}),$$

and obtain an n-Lie algebra. In this case, we introduce a fourth element as follows:

$$d_i := ad(t, e_1, e_2, ..., \widehat{e}_i, \dots, \widehat{e}_j, \dots, \widehat{e}_k, \dots, e_{n+1}), \quad 2 \leq i \leq n-1.$$

Next, consider the following change of basis:

$$e'_i = e_1, \ e'_1 = e_i, \ e'_j = e_n, \ e'_n = e_j, e'_k = e_{n+1}, \ e'_{n+1} = e_k.$$

This change of basis leads to $x = t$, which contradicts the assumption that $\dim(\mathcal{R}) = m+4$. Therefore, the maximal solvable n-Lie algebra with a hyponilpotent ideal $\mathcal{N}$ is unique up to isomorphism.

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АННОТАЦИЯ

Данная работа посвящена построению максимальных разрешимых расширений m-мерных n-Лиевых алгебр. Исследование предлагает систематический подход к выявлению и классификации таких расширений в рамках n-арных структур Ли. Полученные результаты способствуют более глубокому пониманию алгебраических свойств n- Лиевых алгебр и их разрешимых расширений, а также могут служить основой для дальнейших приложений как в математике, так и в теоретической физике.

Ключевые слова: n-алгебры Ли, разрешимые алгебры, нильпотентные алгебры, дифференцирование, ниль-независимые дифференцирование.

ANNOTATSIYA

Ushbu maqola m-o'lchamli n-Li algebralarning maksimal yechiluvchi kengaytmalarini qurishga bag'ishlangan. Tadqiqot bunday kengaytmalarni aniqlash va tasniflashning tizimli yondashuvini taklif etadi hamda yuqori tartibli Li tuzilmalarining doirasida olib boriladi. Olingan natijalar n-Li algebralarning hamda ularning yechiluvchi kengaytmalarining algebraik xossalarini chuqurroq anglashga xizmat qiladi va matematika hamda nazariy fizikaning keyingi qo'llanmalariga asos bo'lishi mumkin.

Kalit so'zlar: n-Li algebralar, yechiluvchi algebralar, nilpotent algebralar, differensiallashlar, nilerkli differsiallashlar.