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# Exponentially weighted optimal quadrature formulas with complex exponential weights in the periodic Sobolev space $\widetilde{W_2^{(2,1,0)}}(0,1]$

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**RESUME**

This paper is devoted to the construction of optimal quadrature formulas for the approximate evaluation of integrals of periodic functions in the Sobolev space $\widetilde{W_2^{(2,1,0)}}(0,1]$. The quadrature formulas involve a complex exponential weight function $e^{2\pi i\omega x}$.The coefficients of the formulas are obtained by minimizing the norm of the corresponding error functional in the conjugate space. Using Fourier analysis and extremal function methods, explicit expressions for the optimal coefficients are derived. These results extend the classical theory of quadrature formulas to exponentially weighted and oscillatory cases, yielding efficient schemes for the numerical integration of periodic functions.

***Key words:*** Optimal quadrature formula, exponential weight, complex exponential function, periodic Sobolev space, Fourier transform.

**Introduction** Optimal quadrature formulas are fundamental tools in approximation theory and numerical integration. Classical quadrature formulas such as Gaussian or Newton-Cotes rules are effective for polynomial approximation in unweighted spaces. However, in many applications involving oscillatory phenomena or exponential modulation, weighted quadrature formulas provide better accuracy. $\widetilde{W_2^{(2,1,0)}}(0,1] = \{\varphi : (0,1] \to \mathbb{C}$, $\varphi$-absolutely continuous and $\varphi'' \in \widetilde{L_2}\}$ and for$\forall\varphi \in \widetilde{W_2^{(2,1,0)}}$ The function satisfies the 1-periodicity condition:$\varphi(x+\beta) = \varphi(x)$, $\forall x \in \mathbb{R}$ , $\beta \in \mathbb{Z}$.

The inner product in this space is defined as follows: .

$$(\ell,\varphi) = \langle \psi_\ell, \varphi \rangle_{\widetilde{W}_2^{(2,1,0)}(0,1)} = \int_0^1 \varphi''(x)\psi''(x)dx + 2\int_0^1 \varphi'(x)\psi'(x)dx + \int_0^1 \varphi(x)\psi(x)dx. \tag{1}$$

We consider the following quadrature formula

$$\int_0^1 e^{2\pi i\omega x}\varphi(x)dx \cong \sum_{k=1}^{N} C_k\varphi(hk), \tag{2}$$

with the error

$$(\ell,\varphi) = \int_0^1 e^{2\pi i\omega x}\varphi(x)dx - \sum_{k=1}^{N} C_k\varphi(hk), \tag{3}$$

and the corresponding error functional is:

$$\ell(x) = e^{2\pi i\omega x} - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} \delta(x - hk - \beta). \tag{4}$$

Here $C_k$ are the coecients of formula (2), $h = \frac{1}{N}$, $N \in \mathbb{N}$, $\omega \in \mathbb{Z}$.

**Statement of the problem** Our goal is to estimate the error of the considered quadrature formula (3) from above, for which it is sufficient to calculate the norm of the error functional (4) . This leads to the solution of the following two problems, and we will first consider this for $m = 2$. Now, we consider the following problem.

**Problem 1**: (4) Find the analytical representation of the error functional norm in $\widetilde{W_2^{(2,1,0)}}$ space.

**Problem 2**: Finding the optimal coefficients of $C_k = \overset{0}{C_k}$ that minimize

To solve problem 1, we use the concept of an extremal function introduced by Sobolev. Using Riesz's theorem for the space $\widetilde{W_2^{(2,1,0)}}(0,1)$, we can write the following

$$(\ell,\varphi) = \langle \psi_\ell, \varphi\rangle_{\widetilde{W}_2^{(2,1,0)}(0,1)} = \int_0^1 \varphi''(x)\psi''(x)dx + 2\int_0^1 \varphi'(x)\psi'(x)dx + \int_0^1 \varphi(x)\psi(x) =$$

$$= \int_0^1 (\psi^{(4)}(x) - 2\psi^{(2)}(x)dx + \psi(x))\varphi(x)dx$$

From the above equation we get the following equation

$$\bar{\psi}_\ell^{(4)}(x) - 2\bar{\psi}_\ell^{(2)}(x) + \bar{\psi}_\ell(x) = \ell(x) \tag{5}$$

**Theorem 1**. In the Sobolev space of $\widetilde{W_2^{(2,1,0)}}(0,1)$periodic functions, the extremal function of the quadratic formula (2) has the following form:

$$\psi_\ell(x) = \frac{e^{-2\pi i\omega x}}{(2\pi\omega)^4 + 2(2\pi\omega)^2 + 1} - \sum_{k=1}^{N} \bar{C}_k \sum_{\beta=-\infty}^{\infty} \frac{e^{2\pi i\beta(x-hk)}}{(2\pi\beta)^4 + 2(2\pi\beta)^2 + 1} \tag{6}$$

**Proof:** To find the generalized periodic solution of the differential equation (5), we apply the Fourier transform to both sides of the equation and use the following properties of the Fourier transform:

$$F[\varphi] = \int_{-\infty}^{\infty} \varphi(x)e^{2\pi ipx}dx.$$

$$F^{-1}[\varphi] = \int_{-\infty}^{\infty} \varphi(p)e^{-2\pi ipx}dp.$$

$$F\left[\varphi^{(\alpha)}\right] = (-2\pi ip)^\alpha F[\varphi],\ (\alpha \in \mathbb{N}).$$

$$F[\phi_0(x)] = \phi_0(p).$$

$$F^{-1}[F[\varphi(x)]] = \varphi(x).$$

Here $*$ is the convolution operation.

We apply Fourier Transform to both sides of equation (5).

$$F\left[\bar{\psi}_\ell^{(4)}(x) - 2\bar{\psi}_\ell^{(2)}(x) + \bar{\psi}_\ell(x)\right] = F[\ell(x)].$$

Since, the Fourier Transform is linear operator, we have

$$((2\pi p)^2 + 1)^2 F[\bar{\psi}_\ell] = F\left[e^{2\pi i\omega x} - \sum_{k=1}^{N} C_k \sum_{\beta=-\infty}^{\infty} \delta(x - hk - \beta)\right],$$

where

$$F\left[\delta(x-hk-\beta)\right]=\int\limits_{-\infty}^{\infty}\delta(x-hk-\beta)e^{2\pi ipx}dx=e^{2\pi ip(hk+\beta)}.$$

$$F\left[e^{2\pi i\omega x}\right]=\int\limits_{-\infty}^{\infty}e^{2\pi i\omega x}e^{2\pi ipx}dx=\delta(p+\omega).$$

$$F\left[\bar{\psi}_\ell\right]=\frac{\delta(p+\omega)}{\left((2\pi\omega)^2+1\right)^2}-\sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\frac{e^{2\pi i\beta hk}\delta(p-\beta)}{(2\pi\beta)^4+2(2\pi\beta)^2+1}.$$

Then, we apply the Inverse Fourier Transform to the above equality and we obtain the following

$$\bar{\psi}_\ell(x)=\frac{e^{2\pi i\omega x}}{(2\pi\omega)^4+2(2\pi\omega)^2+1}-\sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\frac{e^{-2\pi i\beta(x-hk)}}{(2\pi\beta)^4+2(2\pi\beta)^2+1}.$$

$$\psi_\ell(x)=\frac{e^{-2\pi i\omega x}}{(2\pi\omega)^4+2(2\pi\omega)^2+1}-\sum_{k=1}^{N}\bar{C}_k\sum_{\beta=-\infty}^{\infty}\frac{e^{2\pi i\beta(x-hk)}}{(2\pi\beta)^4+2(2\pi\beta)^2+1}.$$

And so, theorem 1 is proved from the last equality.

We calculate the analytical form of the error functional norm

$$\|\ell\|^2_{\widetilde{W_2^{(2,1,0)}}(0,1)^*}=\int\limits_0^1\ell(x)\psi_\ell(x)dx=\int\limits_0^1\left(e^{2\pi i\omega x}-\sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\delta(x-hk-\beta)\right)\times$$

$$\times\left(\frac{e^{-2\pi i\omega x}}{(2\pi\omega)^4+2(2\pi\omega)^2+1}-\sum_{k=1}^{N}\bar{C}_k\sum_{\beta=-\infty}^{\infty}\frac{e^{2\pi i\beta(x-hk)}}{(2\pi\beta)^4+2(2\pi\beta)^2+1}\right)dx.$$

From the above equality, we obtain the following:

$$\|\ell\|^2_{\widetilde{W}_2^{(2,1,0)}(0,1)^*}=\frac{1}{\left((2\pi\omega)^2+1\right)^2}-\sum_{k'=1}^{N}\bar{C}_{k'}\frac{e^{2\pi i\omega hk'}}{\left((2\pi\omega)^2+1\right)^2}-\sum_{k=1}^{N}C_k\frac{e^{-2\pi i\omega hk}}{\left((2\pi\omega)^2+1\right)^2}+$$

$$+\sum_{k=1}^{N}\sum_{k'=1}^{N}\bar{C}_{k'}C_k\sum_{\beta=-\infty}^{\infty}\frac{e^{-2\pi i\beta h(k-k')}}{((2\pi\beta)^2+1)^2}. \tag{8}$$

So problem 1 is solved. **Finding the coefficient of the quadrature formula (4) Theorem 2.** The coefficients of the quadrature formula in the form (2) that minimizes the norm of the error functional are as follows:

$$\overset{0}{C_k}=C(\omega,h)\cdot e^{2\pi i\omega hk}=\frac{8\cdot e^{2\pi i\omega hk}}{\left((2\pi\omega)^2+1\right)^2}\cdot\left[\frac{2h\lambda e^h+\lambda^2e^{2h}-1}{(\lambda e^h-1)^2}-\frac{\lambda^2-e^{2h}-2h\lambda e^h}{(\lambda-e^h)^2}\right]^{-1}. \tag{9}$$

Here $\lambda=e^{2\pi i\omega h}$.

**Proof:** Taking the first derivative of the coefficient from equality (8) and equating it to zero, we obtain the following equality

$$-\frac{e^{2\pi i\omega hk'}}{(2\pi\omega)^4+2(2\pi\omega)^2+1}+\sum_{k=1}^{N}C_k\sum_{\beta=-\infty}^{\infty}\frac{e^{2\pi i\beta h(k-k')}}{(2\pi\beta)^4+2(2\pi\beta)^2+1}=0. \tag{10}$$

Assume the optimal coefficients are as follows: $\overset{0}{C}_k = C(\omega, h) \cdot e^{2\pi i\omega hk}$

$$-\frac{e^{2\pi i\omega hk'}}{(2\pi\omega)^4 + 2(2\pi\omega)^2 + 1} + C(\omega, h)\sum_{k=1}^{N} e^{2\pi i\omega hk} \sum_{\beta=-\infty}^{\infty} \frac{e^{2\pi i\beta h(k-k')}}{(2\pi\beta)^4 + 2(2\pi\beta)^2 + 1} = 0 \tag{11}$$

$$A = C(\omega, h)\sum_{k=1}^{N} e^{2\pi i\omega hk} \sum_{\beta=-\infty}^{\infty} \frac{e^{2\pi i\beta h(k-k')}}{(2\pi\beta)^4 + 2(2\pi\beta)^2 + 1} = C(\omega, h) \sum_{\beta=-\infty}^{\infty} \frac{e^{-2\pi i\beta hk'}}{\left((2\pi\beta)^2 + 1\right)^2} \cdot \sum_{k=1}^{N} e^{2\pi i\omega hk} e^{2\pi i\beta hk} =$$

$$= C(\omega, h) \sum_{\beta=-\infty}^{\infty} \frac{e^{-2\pi i\beta hk'}}{\left((2\pi\beta)^2 + 1\right)^2} \cdot \sum_{k=1}^{N} e^{2\pi ihk(\beta+\omega)}$$

.

$$A == C(\omega, h) \sum_{\beta=-\infty}^{\infty} \frac{e^{-2\pi i\beta hk'}}{\left((2\pi\beta)^2 + 1\right)^2} \cdot \sum_{k=1}^{N} e^{2\pi ihk(\beta+\omega)} = C(\omega, h) \sum_{t=-\infty}^{\infty} \frac{e^{-2\pi i(tN-\omega)hk'}}{\left((2\pi(tN-\omega))^2 + 1\right)^2} \cdot N =$$

$$= \frac{1}{16\pi^4 h} C(\omega, h) e^{2\pi i\omega hk'} \sum_{t=-\infty}^{\infty} \frac{1}{\left((tN-\omega)^2 - \left(\frac{i}{2\pi}\right)^2\right)^2}$$

By substituting A from the previous equation into equation (11), we derive the following:

$$\frac{1}{16\pi^4 h} C(\omega, h) e^{2\pi i\omega hk'} \sum_{t=-\infty}^{\infty} \frac{1}{\left((tN-\omega)^2 - \left(\frac{i}{2\pi}\right)^2\right)^2} - \frac{e^{2\pi i\omega hk'}}{(2\pi\omega)^4 + 2(2\pi\omega)^2 + 1} = 0$$

$$C(\omega, h) = \frac{16\pi^4 h}{((2\pi\omega)^2 + 1)^2} \left( \sum_{t=-\infty}^{\infty} \frac{h^4}{\left((t-\omega h)^2 - \left(\frac{hi}{2\pi}\right)^2\right)^2} \right)^{-1}$$

In order to compute $C(\omega, h)$, it suffices to evaluate the infinite series

$$\sum_{t=-\infty}^{\infty} \frac{h^4}{\left((t-\omega h)^2 - \left(\frac{hi}{2\pi}\right)^2\right)^2}$$

This infinite series is computed by means of the residue theorem [5].

$$\sum_{t=-\infty}^{\infty} \frac{h^4}{\left((t-\omega h)^2 - \left(\frac{hi}{2\pi}\right)^2\right)^2} = 2\pi h^4 \left[ \frac{2h\lambda e^h + \lambda^2 e^{2h} - 1}{(\lambda e^h - 1)^2} - \frac{\lambda^2 - e^{2h} - 2h\lambda e^h}{(\lambda - e^h)^2} \right] \tag{12}$$

$$C(\omega, h) = \frac{16\pi^4 h}{((2\pi\omega)^2 + 1)^2} \left( \sum_{t=-\infty}^{\infty} \frac{h^4}{\left((t-\omega h)^2 - \left(\frac{hi}{2\pi}\right)^2\right)^2} \right)^{-1} = \frac{16\pi^4 h}{((2\pi\omega)^2 + 1)^2} \times$$

$$\times \frac{1}{2\pi^4 h} \left[ \frac{2h\lambda e^h + \lambda^2 e^{2h} - 1}{(\lambda e^h - 1)^2} - \frac{\lambda^2 - e^{2h} - 2h\lambda e^h}{(\lambda - e^h)^2} \right]^{-1} =$$

$$= \frac{8 \cdot e^{2\pi i\omega hk}}{\left((2\pi\omega)^2 + 1\right)^2} \cdot \left[ \frac{2h\lambda e^h + \lambda^2 e^{2h} - 1}{(\lambda e^h - 1)^2} - \frac{\lambda^2 - e^{2h} - 2h\lambda e^h}{(\lambda - e^h)^2} \right]^{-1}$$

The theorem has been proven.

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## REZYUME


Ushbu maqola Sobolev fazosida $\widetilde{W_2}^{(2,1,0)}(0,1]$ davriy funksiyalar integrallarini yaqin hisoblash uchun optimal kvadratur formulalarini qurishga bag'ishlangan. Kvadratur formulalar murakkab eksponensial og'irlik funksiyasini $e^{2\pi i\omega x}$ o'z ichiga oladi. Formulalarning koeffitsiyentlari mos keluvchi xatolik funksionalining konjugat fazodagi normasini minimallashtirish orqali aniqlanadi. Furye tahlili va ekstremal funksiya usullaridan foydalanib, optimal koeffitsiyentlarning aniq ifodalari keltirib chiqariladi. Ushbu natijalar kvadratur formulalarining klassik nazariyasini eksponensial og'irlikli va osillatsion hollarga kengaytiradi hamda davriy funksiyalarni sonli integrallash uchun samarali sxemalarni beradi.



***Kalit so'zlar:*** Optimal kvadratur formulasi, eksponensial og'irlik, murakkab eksponensial funksiya, davriy Sobolev fazosi, Furye o'zgartirish.


## РЕЗЮМЕ


Данная статья посвящена построению оптимальных квадратурных формул для приближённого вычисления интегралов периодических функций в пространстве Соболева $\widetilde{W_2}^{(2,1,0)}(0,1]$ Квадратурные формулы включают комплексную экспоненциальную весовую функцию $e^{2\pi i\omega x}$ Коэффициенты формул определяются путём минимизации нормы соответствующего функционала ошибки в сопряжённом пространстве. С использованием методов Фурье-анализа и экстремальной функции выводятся точные выражения для оптимальных коэффициентов. Полученные результаты расширяют классическую теорию квадратурных формул на случай экспоненциальных весов и осциллирующих функций, а также обеспечивают эффективные схемы для численного интегрирования периодических функций.



***Ключевые слова:***оптимальная квадратурная формула; экспоненциальный вес; комплексная экспоненциальная функция; периодическое пространство Соболева; преобразование Фурье.