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SOLVING A BOUNDARY VALUE PROBLEM FOR THE FIRST ORDER DIFFERENTIAL EQUATION INVOLVING THE PRABHAKAR OPERATOR

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RESUME

In this study, a first order partial differential equation involving the Prabhakar fractional derivative is examined within a rectangular domain. A boundary value problem associated with this equation is investigated, and the existence and uniqueness of its solution are established. To construct the solution, the Riemann method is employed. An auxiliary problem is formulated in terms of the Riemann function. By applying the Laplace transform, the auxiliary problem is reduced to a Cauchy problem for an ordinary differential equation. Subsequently, the inverse Laplace transform is utilized to derive an explicit expression for the solution of the auxiliary problem, which corresponds to the Riemann function of the original equation. The solution to the initial boundary value problem is then obtained using the Riemann method. Furthermore, sufficient conditions are derived for the given functions to ensure that the obtained solution satisfies the problem's constraints.

Key words: Prabhakar fractional derivative, Prabhakar fractional integral, three-parameter Mittag-Leffler function, regular solution, Laplace transform, two-variable Mittag-Leffler function.

1. Introduction

Fractional differential equations (FDEs) have emerged as a powerful mathematical tool to describe systems with memory and hereditary properties, extending classical integer-order differential equations. Their ability to capture anomalous dynamics and nonlocal behaviors has been widely recognized across various scientific disciplines, including mechanics, finance, biology, and engineering [1-5]. The fractional operators, by incorporating non-integer order derivatives, offer a richer framework for modeling complex phenomena where classical models fail to provide adequate accuracy.

Boundary value problems (BVPs) for first order partial differential equations (PDEs) form a foundational topic in applied mathematics, underpinning models in heat transfer, wave propagation, and transport phenomena. Consequently, the investigation of existence, uniqueness, and regularity of solutions for fractional PDEs has garnered increasing interest [6-9]. For instance, in [8], the solution of a BVP involving a fractional PDE was thoroughly analyzed. In this problem, the existence and uniqueness of the solution were established, and the solution was expressed in closed form using a special function of the Wright type. Similarly, in [9], the author addressed a BVP for a first order partial differential equation within a rectangular domain, where the differentiation involved a discretely distributed fractional operator defined by the Dzhrbashyan-Nersesyan framework. They derived an explicit representation of the solution and established theorems concerning its existence and uniqueness.

In addition, the Prabhakar fractional derivative has attracted significant attention in recent years, owing to its applicability in modeling complex relaxation and diffusion processes in various media [10-14]. Despite growing literature on fractional differential equations involving Prabhakar derivatives, the study of BVPs for first order PDEs with such operators remains relatively unexplored and mathematically intriguing. Therefore, there is a need for more comprehensive and in-depth research in this area, which would open up new opportunities for both theoretical investigations and practical applications.

In this paper, we consider a boundary value problem for a first order PDE involving the Prabhakar fractional derivative. We aim to investigate the existence and uniqueness of the regular solution under given boundary conditions, thereby contributing to the theoretical understanding of fractional PDEs with generalized fractional operators.

2. Formulation of the problem

Let us consider the following equation

$$ {}^{PRL}D_{0t}^{\alpha,\,\beta,\,\gamma,\,\delta}u\left(t,x\right)-u_{x}\left(t,x\right)=f\left(t,x\right) \tag{1}$$

in a domain $\Omega=\{(t,x):\ 0<t<T,\ 0<x<a\}$, $0<a,T\leq\infty$. Here

$$ {}^{PRL}D_{0t}^{\alpha,\,\beta,\,\gamma,\,\delta}y\left(t\right)=\frac{d^{m}}{dt^{m}}\,{}^{P}I_{0t}^{\alpha,\,m-\beta,\,-\gamma,\,\delta}y\left(t\right)$$

represents Prabhakar fractional derivative [15] and

$$ {}^{P}I_{0t}^{\alpha,\,\beta,\,\gamma,\,\delta}y\left(t\right)=\int\limits_{0}^{t}\left(t-s\right)^{\beta-1}E_{\alpha,\beta}^{\gamma}\left[\delta(t-s)^{\alpha}\right]y\left(s\right)ds,\quad t>0$$

represents Prabhakar fractional integral, also

$$E_{\alpha,\beta}^{\gamma}\left[z\right]=\sum_{k=0}^{+\infty}\frac{\left(\gamma\right)_{k}z^{k}}{\Gamma\left(\alpha k+\beta\right)k!}$$

symbolizes the three-parameter Mittag-Leffler function. We note that above-given definitions are valid for $\alpha,\ \beta,\ \gamma,\ \delta\in\mathbb{R}$ such that $\alpha>0$ and $m-1<\beta<m$, $m\in\mathbb{N}$. We see in the particular case $0<\beta<1$.

Problem. *Find the regular solution $u\left(t,x\right)$ of the equation (1) that satisfies the following boundary conditions:*

$$\lim_{t\to0}{}^{P}I_{0t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta}u\left(t,x\right)=\tau\left(x\right),\quad 0<x<a, \tag{2}$$

$$u\left(t,0\right)=\varphi\left(t\right),\quad 0<t<T. \tag{3}$$

Definition. *A regular solution of the equation (1) in the domain Ω is called a function $u\left(t,x\right)$ with the regularity $t^{1-\beta}u\left(t,x\right)\in C\left(\overline{\Omega}\right)$, $u_{x}\left(t,x\right)$, ${}^{PRL}D_{0t}^{\alpha,\,\beta,\,\gamma,\,\delta}u\left(t,x\right)\in C\left(\Omega\right)$ that satisfies the equation (1) at all points $(t,x)\in\Omega$.*

3. Existence and Uniqueness Theorem

Theorem. *Let $\tau\left(x\right)\in C\left[0;a\right]$, $t^{1-\beta}\varphi\left(t\right)\in C\left[0,T\right]$, $t^{1-\beta}f\left(t,x\right)\in C\left(\overline{\Omega}\right)$, $f\left(t,x\right)$ satisfies the Hölder condition with respect to at least one of its variables and the following compatibility condition is satisfied:*

$$\lim_{t\to0}{}^{P}I_{0t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta}\varphi\left(t\right)=\tau\left(0\right).$$

Then there exists a unique regular solution of the equation (1) in the domain Ω, satisfying the boundary conditions (2) and (3). Moreover, this solution has the form

$$u\left(t,x\right)=\int\limits_{0}^{t}\varphi\left(\eta\right)\left(t-\eta\right)^{-1}E_{12}\left(\begin{matrix}-\gamma,1,0;\\-\beta,\alpha,0;-\gamma,0;1,1;1,1\end{matrix}\middle|\begin{matrix}x(t-\eta)^{-\beta}\\\delta(t-\eta)^{\alpha}\end{matrix}\right)d\eta+$$

$$+\int\limits_{0}^{x}\tau\left(\xi\right)t^{-1}E_{12}\left(\begin{matrix}-\gamma,1,0;\\-\beta,\alpha,0;-\gamma,0;1,1;1,1\end{matrix}\middle|\begin{matrix}\left(x-\xi\right)t^{-\beta}\\\delta t^{\alpha}\end{matrix}\right)d\xi+$$

$$+\int\limits_{0}^{t}\int\limits_{0}^{x}f\left(\eta,\xi\right)\left(t-\eta\right)^{-1}E_{12}\left(\begin{matrix}-\gamma,1,0;\\-\beta,\alpha,0;-\gamma,0;1,1;1,1\end{matrix}\middle|\begin{matrix}\left(x-\xi\right)\left(t-\eta\right)^{-\beta}\\\delta(t-\eta)^{\alpha}\end{matrix}\right)d\xi d\eta.$$

Before proving the theorem, let us give some information about the function $E_{12}(x,y)$. $E_{12}(x,y)$ is the bivariate Mittag-Leffler type function [16]:

$$E_{12}\left(\begin{matrix}\alpha_1,\beta_1,\delta_1;\\ \alpha_2,\beta_2,\delta_2;\alpha_3,\delta_3;\alpha_4,\delta_4;\beta_3,\delta_5\end{matrix}\middle|\begin{matrix}x\\ y\end{matrix}\right)=$$

$$=\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\frac{\Gamma(\alpha_1 n+\beta_1 m+\delta_1)\,x^n y^m}{\Gamma(\alpha_2 n+\beta_2 m+\delta_2)\,\Gamma(\alpha_3 n+\delta_3)\,\Gamma(\alpha_4 n+\delta_4)\,\Gamma(\beta_3 m+\delta_5)},$$

$$(x,y,\alpha_l,\beta_i,\delta_j\in\mathbb{R};\min\{\alpha_l,\beta_i\}>0;(l=\{1,...,4\},i=\{1,2,3\},j=\{1,...,5\})),$$

in which the double series converges for $x,y\in\mathbb{R}$, if $\Delta_1>0$, and $\Delta_2>0$. Here $\Delta_1=\alpha_2+\alpha_3+\alpha_4-\alpha_1$, $\Delta_2=\beta_2+\beta_3-\beta_1$.

Proof.

Let $V(t,x;\eta,\xi)$ be a Riemann function of the variables η and ξ in a domain $\{(\eta,\xi):\ 0<\eta<t,\ 0<\xi<x\}$, which satisfies the following equation for any fixed (t,x):

$$V_\xi(t,x;\eta,\xi)-{}^{PC}D_{\eta t}^{\alpha,\beta,\gamma,\delta}V(t,x;\eta,\xi)=1, \tag{4}$$

where

$${}^{PC}D_{0t}^{\alpha,\,\beta,\,\gamma,\,\delta}y(t)={}^{P}I_{0t}^{\alpha,\,1-\beta,\,-\gamma,\,\delta}\frac{d}{dt}y(t)$$

represents regularized Prabhakar Caputo fractional derivative [17].
$V(t,x;\eta,\xi)$ also satisfies these conditions

$$V(t,x;\eta,\xi)|_{\eta=t}=0,\quad V(t,x;\eta,\xi)|_{\xi=x}=0. \tag{5}$$

First, we rewrite the equation (1) by replacing the variables x and t with ξ and η:

$${}^{PRL}D_{0\eta}^{\alpha,\,\beta,\,\gamma,\,\delta}u(\eta,\xi)-u_\xi(\eta,\xi)=f(\eta,\xi).$$

Then, we multiply both sides of the last equation by $V(t,x;\eta,\xi)$, integrate with respect to ξ over the interval $[0,x]$, and with respect to η over the interval $[0,t]$:

$$\int_0^t\int_0^x {}^{PRL}D_{0\eta}^{\alpha,\,\beta,\,\gamma,\,\delta}u(\eta,\xi)\,V(t,x;\eta,\xi)\,d\xi d\eta-\int_0^t\int_0^x u_\xi(\eta,\xi)\,V(t,x;\eta,\xi)\,d\xi d\eta=$$

$$=\int_0^t\int_0^x f(\eta,\xi)\,V(t,x;\eta,\xi)\,d\xi d\eta. \tag{6}$$

For convenience, let us denote the first integral in (4) by I_1:

$$I_1=\int_0^t\int_0^x {}^{PRL}D_{0\eta}^{\alpha,\,\beta,\,\gamma,\,\delta}u(\eta,\xi)\,V(t,x;\eta,\xi)\,d\xi d\eta=$$

$$=\int_0^t\int_0^x V(t,x;\eta,\xi)\,\frac{\partial}{\partial\eta}I_{0\eta}^{\alpha,\,1-\beta,\,-\gamma,\,\delta}u(\eta,\xi)\,d\xi d\eta.$$

Applying integration by parts with respect to η and considering the conditions (2) and (5), we obtain the following result:

$$I_1=-\int_0^x V(t,x;0,\xi)\,\tau(\xi)\,d\xi-\int_0^x d\xi\int_0^t V_\eta(t,x;\eta,\xi)\,I_{0\eta}^{\alpha,\,1-\beta,\,-\gamma,\,\delta}u(\eta,\xi)\,d\eta.$$

Now, we express the Prabhakar fractional integral and change the order of integration:

$$I_1 = -\int_0^x V(t,x;0,\xi)\,\tau(\xi)\,d\xi -$$

$$-\int_0^x d\xi \int_0^t V_\eta(t,x;\eta,\xi)\,d\eta \int_0^\eta (\eta-s)^{-\beta} E^{-\gamma}_{\alpha,\,1-\beta}\left[\delta(\eta-s)^\alpha\right] u(s,\xi)\,ds =$$

$$= -\int_0^x V(t,x;0,\xi)\,\tau(\xi)\,d\xi -$$

$$-\int_0^x d\xi \int_0^t u(s,\xi)\,ds \int_s^t (\eta-s)^{-\beta} E^{-\gamma}_{\alpha,\,1-\beta}\left[\delta(\eta-s)^\alpha\right] V_\eta(t,x;\eta,\xi)\,d\eta =$$

$$= -\int_0^x V(t,x;0,\xi)\,\tau(\xi)\,d\xi - \int_0^x\int_0^t u(\eta,\xi)\, I^{\alpha,\,1-\beta,\,-\gamma,\,\delta}_{\eta t} V_\eta(t,x;\eta,\xi)\,d\eta d\xi =$$

$$= -\int_0^x V(t,x;0,\xi)\,\tau(\xi)\,d\xi - \int_0^x\int_0^t u(\eta,\xi)\, {}^{PC}D^{\alpha,\beta,\gamma,\delta}_{\eta t} V(t,x;\eta,\xi)\,d\eta d\xi.$$

Let us evaluate I_2 that is the second integral of (6). We integrate by parts with respect to ξ and consider the conditions (3) and (5), then we get the following result:

$$I_2 = \int_0^t\int_0^x u_\xi(\eta,\xi)\,V(t,x;\eta,\xi)\,d\xi d\eta =$$

$$= -\int_0^t V(t,x;\eta,0)\,\varphi(\eta)\,d\eta - \int_0^t\int_0^x u(\eta,\xi)\,V_\xi(t,x;\eta,\xi)\,d\xi d\eta.$$

Substituting the results of the integrals I_1 and I_2 into the equation (6), we obtain the following expression:

$$\int_0^t\int_0^x u(\eta,\xi)\left[V_\xi(t,x;\eta,\xi) - {}^{PC}D^{\alpha,\beta,\gamma,\delta}_{\eta t} V(t,x;\eta,\xi)\right] d\xi d\eta =$$

$$= \int_0^t\int_0^x f(\eta,\xi)\,V(t,x;\eta,\xi)\,d\xi d\eta + \int_0^x V(t,x;0,\xi)\,\tau(\xi)\,d\xi - \int_0^t V(t,x;\eta,0)\,\varphi(\eta)\,d\eta.$$

According to (4), we get the following result:

$$\int_0^t\int_0^x u(\eta,\xi)\,d\xi d\eta = \int_0^t\int_0^x f(\eta,\xi)\,V(t,x;\eta,\xi)\,d\xi d\eta +$$

$$+\int_0^x V(t,x;0,\xi)\,\tau(\xi)\,d\xi - \int_0^t V(t,x;\eta,0)\,\varphi(\eta)\,d\eta.$$

To find the unknown function $u(t,x)$ from the final expression, we first differentiate with respect to x, then with respect to t, and arrive at the following result:

$$u(t,x)=\int_0^t\int_0^x V_{xt}(t,x;\eta,\xi)\,f(\eta,\xi)\,d\xi d\eta+$$

$$+\int_0^x V_{xt}(t,x;0,\xi)\,\tau(\xi)\,d\xi+\int_0^t V_{xt}(t,x;\eta,0)\,\varphi(\eta)\,d\eta. \tag{7}$$

Our goal is to find the function $V(t,x;\eta,\xi)$. Next, we will assume that the function $V(t,x;\eta,\xi)$ can be represented as a function of the difference of the arguments $t,\ \eta$ and $x,\ \xi$: $V(t,x;\eta,\xi)=V(t-\eta,x-\xi)$. In that case, it follows from (4) and (5) that $V=V(t,x)$ is the solution of the following problem:

$$V_x(t,x)-{}^{PC}D_{0t}^{\alpha,\beta,\gamma,\delta}V(t,x)=1, \tag{8}$$

$$V(0,x)=0,\quad V(t,0)=0 \tag{9}$$

We apply the Laplace transform to the both sides of the equation (8) with respect to t :

$$L_t\left[V_x(t,x)-{}^{PC}D_{0t}^{\alpha,\beta,\gamma,\delta}V(t,x)\right]=L_t\left[V_x(t,x)\right]-L_t\left[{}^{PC}D_{0t}^{\alpha,\beta,\gamma,\delta}V(t,x)\right]=L_t[1].$$

If we denote $L_t[V(t,x)]=\omega(x;p)$, then we get

$$\omega_x(x;p)-L_t\left[{}^{PC}D_{0t}^{\alpha,\beta,\gamma,\delta}V(t,x)\right]=\frac{1}{p}. \tag{10}$$

Now, we apply the Laplace transform to the operator:

$$L_t\left[{}^{PC}D_{0t}^{\alpha,\beta,\gamma,\delta}V(t,x)\right]=\int_0^{+\infty}e^{-pt}dt\int_0^t(t-z)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left[\delta(t-z)^{\alpha}\right]V_z(z,x)\,dz=$$

$$=\int_0^{+\infty}V_z(z,x)\,dz\int_z^{+\infty}e^{-pt}(t-z)^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left[\delta(t-z)^{\alpha}\right]dt=\{t=s+z\}=$$

$$=\int_0^{+\infty}e^{-pz}V_z(z,x)\,dz\int_0^{+\infty}e^{-ps}s^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left[\delta s^{\alpha}\right]ds.$$

First, we evaluate the integral with respect to s :

$$\int_0^{+\infty}e^{-ps}s^{-\beta}E_{\alpha,1-\beta}^{-\gamma}\left[\delta s^{\alpha}\right]ds=\sum_{k=0}^{+\infty}\frac{(-\gamma)_k\delta^k}{\Gamma(\alpha k+1-\beta)\,k!}\int_0^{+\infty}e^{-ps}s^{\alpha k-\beta}ds=\{ps=y\}=$$

$$=p^{\beta-1}\sum_{k=0}^{+\infty}\frac{(-\gamma)_k\delta^kp^{-\alpha k}}{\Gamma(\alpha k+1-\beta)\,k!}\int_0^{+\infty}e^{-y}y^{\alpha k-\beta}dy=p^{\beta-1}\sum_{k=0}^{+\infty}\frac{(-\gamma)_k\left(\delta p^{-\alpha}\right)^k}{k!}=$$

$$=\left\{\sum_{k=0}^{+\infty}\frac{(a)_kx^k}{k!}=(1-x)^{-a}\right\}=p^{\beta-1}\left(1-\frac{\delta}{p^{\alpha}}\right)^{\gamma}.$$

Now, we integrate by parts with respect to z :

$$\int_0^{+\infty}e^{-pz}V_z(z,x)\,dz=p\int_0^{+\infty}e^{-pz}V(z,x)\,dz=pL_t\left[V(t,x)\right].$$

As a result, we have identified

$$L_t\left[{}^{PC}D_{0t}^{\alpha,\beta,\gamma,\delta}V\left(t,x\right)\right]=p^{\beta}\left(1-\frac{\delta}{p^{\alpha}}\right)^{\gamma}L_t\left[V\left(t,x\right)\right],$$

so we can rewrite (10) and get the linear differential equation:

$$\omega_x\left(x;p\right)-\lambda\omega\left(x;p\right)=\frac{1}{p},$$

where $\lambda=p^{\beta}\left(1-\frac{\delta}{p^{\alpha}}\right)^{\gamma}$.

We solve this linear differential equation and determine that $\omega\left(x;p\right)=-\frac{1}{p\lambda}+Ce^{\lambda x}$. Since $V\left(t,0\right)=0$, it follows that $\omega\left(0;p\right)=0$. By using $\omega\left(0;p\right)=0$, we can find that $C=\frac{1}{p\lambda}$. Consequently, we obtain the following

$$L_t\left[V\left(t,x\right)\right]=\omega\left(x;p\right)=\frac{e^{\lambda x}}{p\lambda}-\frac{1}{p\lambda}.$$

Now it is time for the inverse Laplace transform:

$$L_t^{-1}\left[\omega\right]=L_t^{-1}\left[\frac{e^{\lambda x}}{p\lambda}-\frac{1}{p\lambda}\right]=L_t^{-1}\left[\frac{e^{\lambda x}}{p\lambda}\right]-L_t^{-1}\left[\frac{1}{p\lambda}\right]. \tag{11}$$

Let us take a function of this form

$$V_1\left(t,x\right)=t^{\beta}\sum_{n=0}^{+\infty}\frac{x^n t^{-\beta n}}{n!}E_{\alpha,-\beta n+1+\beta}^{-\gamma n+\gamma}\left[\delta t^{\alpha}\right]$$

and apply the Laplace transform to it:

$$L_t\left[V_1\left(t,x\right)\right]=L_t\left[t^{\beta}\sum_{n=0}^{+\infty}\frac{x^n t^{-\beta n}}{n!}E_{\alpha,-\beta n+1+\beta}^{-\gamma n+\gamma}\left[\delta t^{\alpha}\right]\right]=$$

$$=\int_0^{+\infty}e^{-pt}t^{\beta}\sum_{n=0}^{+\infty}\frac{x^n t^{-\beta n}}{n!}E_{\alpha,-\beta n+1+\beta}^{-\gamma n+\gamma}\left[\delta t^{\alpha}\right]dt=$$

$$=\sum_{n=0}^{+\infty}\frac{x^n}{n!}\sum_{k=0}^{+\infty}\frac{\left(-\gamma n+\gamma\right)_k\delta^k}{k!\Gamma\left(\alpha k-\beta n+1+\beta\right)}\int_0^{+\infty}e^{-pt}t^{\beta-\beta n+\alpha k}\,dt=\left\{pt=s\right\}=$$

$$=\sum_{n=0}^{+\infty}\frac{x^n}{n!}\sum_{k=0}^{+\infty}\frac{\left(-\gamma n+\gamma\right)_k\delta^k p^{-\alpha k+\beta n+1-\beta}}{k!}=$$

$$=p^{-\beta-1}\sum_{n=0}^{+\infty}\frac{x^n p^{\beta n}}{n!}\sum_{k=0}^{+\infty}\frac{\left(-\gamma n+\gamma\right)_k\left[\delta p^{-\alpha}\right]^k}{k!}.$$

Using the formula $\sum\limits_{k=0}^{+\infty}\frac{(a)_k x^k}{k!}=\left(1-x\right)^{-a}$, we get

$$L_t\left[V_1\left(t,x\right)\right]=p^{-\beta-1}\sum_{n=0}^{+\infty}\frac{x^n p^{\beta n}}{n!}\left(1-\delta p^{-\alpha}\right)^{\gamma n-\gamma}=$$

$$=\frac{1}{p}\sum_{n=0}^{+\infty}\frac{x^n\left[p^{\beta}\left(1-\frac{\delta}{p^{\alpha}}\right)^{\gamma}\right]^n}{n!\left[p^{\beta}\left(1-\frac{\delta}{p^{\alpha}}\right)^{\gamma}\right]}=\frac{1}{p\lambda}\sum_{n=0}^{+\infty}\frac{\left(\lambda x\right)^n}{n!}=\frac{e^{\lambda x}}{p\lambda}.$$

For $x=0$, we define

$$L_t\left[V_1\left(t,0\right)\right]=L_t\left[t^{\beta}E^{\gamma}_{\alpha,1+\beta}\left[\delta t^{\alpha}\right]\right]=\frac{1}{p\lambda}.$$

According to (11) and the results $L_t\left[V_1\left(t,x\right)\right]$ and $L_t\left[V_1\left(t,0\right)\right]$, we conclude

$$V\left(t,x\right)=L_t^{-1}\left[\omega\right]=t^{\beta}\sum_{n=0}^{+\infty}\frac{x^n t^{-\beta n}}{n!}E^{-\gamma n+\gamma}_{\alpha,-\beta n+1+\beta}\left[\delta t^{\alpha}\right]-t^{\beta}E^{\gamma}_{\alpha,1+\beta}\left[\delta t^{\alpha}\right]=$$

$$=t^{\beta}\sum_{n=1}^{+\infty}\frac{x^n t^{-\beta n}}{n!}E^{-\gamma n+\gamma}_{\alpha,-\beta n+1+\beta}\left[\delta t^{\alpha}\right]=\sum_{n=0}^{+\infty}\frac{x^{n+1}t^{-\beta n}}{(n+1)!}E^{-\gamma n}_{\alpha,-\beta n+1}\left[\delta t^{\alpha}\right].$$

To find the solution (7), we take the derivatives of the function $V\left(t-\eta,x-\xi\right)$ with respect to x and t:

$$V_x=\sum_{n=0}^{+\infty}\frac{\left(x-\xi\right)^n}{n!}\left(t-\eta\right)^{-\beta n}E^{-\gamma n}_{\alpha,-\beta n+1}\left[\delta(t-\eta)^{\alpha}\right]=$$

$$=1+\sum_{n=1}^{+\infty}\frac{\left(x-\xi\right)^n}{n!}\left(t-\eta\right)^{-\beta n}E^{-\gamma n}_{\alpha,-\beta n+1}\left[\delta(t-\eta)^{\alpha}\right];$$

$$V_{xt}=\sum_{n=1}^{+\infty}\frac{\left(x-\xi\right)^n}{n!}\left(t-\eta\right)^{-\beta n-1}E^{-\gamma n}_{\alpha,-\beta n}\left[\delta(t-\eta)^{\alpha}\right]=$$

$$=\sum_{n=0}^{+\infty}\frac{\left(x-\xi\right)^n}{n!}\left(t-\eta\right)^{-\beta n-1}E^{-\gamma n}_{\alpha,-\beta n}\left[\delta(t-\eta)^{\alpha}\right].$$

We can write the function $V_{xt}\left(t,x\right)$ using $E_{12}\left(t,x\right)$:

$$V_{xt}=\left(t-\eta\right)^{-1}\sum_{n=0}^{+\infty}\sum_{k=0}^{+\infty}\frac{\Gamma\left(-\gamma n+k\right)\left[\left(x-\xi\right)\left(t-\eta\right)^{-\beta}\right]^n\left[\delta(t-\eta)^{\alpha}\right]^k}{\Gamma\left(-\beta n+\alpha k\right)\Gamma\left(-\gamma n\right)\Gamma\left(n+1\right)\Gamma\left(k+1\right)}=$$

$$=\left(t-\eta\right)^{-1}E_{12}\left(\begin{matrix}-\gamma,1,0;\\-\beta,\alpha,0;-\gamma,0;1,1;1,1\end{matrix}\middle|\begin{matrix}\left(x-\xi\right)\left(t-\eta\right)^{-\beta}\\\delta(t-\eta)^{\alpha}\end{matrix}\right).$$

Here $\Delta_1=\alpha_2+\alpha_3+\alpha_4-\alpha_1=1-\beta>0$ and $\Delta_2=\beta_2+\beta_3-\beta_1=\alpha>0$.

Finally, we obtain the following solution:

$$u\left(t,x\right)=\int\limits_0^t\varphi\left(\eta\right)\left(t-\eta\right)^{-1}E_{12}\left(\begin{matrix}-\gamma,1,0;\\-\beta,\alpha,0;-\gamma,0;1,1;1,1\end{matrix}\middle|\begin{matrix}x(t-\eta)^{-\beta}\\\delta(t-\eta)^{\alpha}\end{matrix}\right)d\eta+$$

$$+\int\limits_0^x\tau\left(\xi\right)t^{-1}E_{12}\left(\begin{matrix}-\gamma,1,0;\\-\beta,\alpha,0;-\gamma,0;1,1;1,1\end{matrix}\middle|\begin{matrix}\left(x-\xi\right)t^{-\beta}\\\delta t^{\alpha}\end{matrix}\right)d\xi+$$

$$+\int\limits_0^t\int\limits_0^x f\left(\eta,\xi\right)\left(t-\eta\right)^{-1}E_{12}\left(\begin{matrix}-\gamma,1,0;\\-\beta,\alpha,0;-\gamma,0;1,1;1,1\end{matrix}\middle|\begin{matrix}\left(x-\xi\right)\left(t-\eta\right)^{-\beta}\\\delta(t-\eta)^{\alpha}\end{matrix}\right)d\xi d\eta.$$

The theorem is proved.

Remark. If $\delta=0$ or $\gamma=0$ in the considered problem, these cases have been studied in [18].

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REZYUME

Ushbu ishda Prabhakar kasr tartibli hosilasi qatnashgan birinchi tartibli xususiy hosilali differensial tenglama to’g‘ri to‘rtburchakli sohada qaralgan. Bu tenglama uchun bir chegaraviy masalaning bir qiymatli yechilishi o‘rganilgan. Bu masalaning yechimini topish uchun Riman usulidan foydalanilgan. Riman funksiyasiga nisbatan yordamchi masala hosil qilingan. Yordamchi masalani Laplas almashtirishi yordamida oddiy differensial tenglama uchun Koshi masalasiga olib kelingan va teskari Laplas almashtirishini qo‘llab yordamchi masalaning yechim formulasi, ya’ni berilgan masalaning Riman funksiyasi topilgan. So’ngra Riman metodidan foydalanib dastlabki masalaning yechimi topilgan. Topilgan yechim masalaning shartlarini qanoatlantirishi uchun berilgan funksiyalarga yetarli shartlar topilgan.

Kalit so'zlar: Prabhakar kasr tartibli hosilasi, Prabhakar kasr tartibli integrali, uch parametrli Mittag-Leffler funksiyasi, regular yechim, Laplas almashtirishi, ikki o'zgaruvchili Mittag-Leffler funksiyasi.

РЕЗЮМЕ

В настоящей работе рассматривается краевая задача для дифференциального уравнения с частными производными первого порядка, включающего дробный оператор Прабхакара, в правильной прямоугольной области. Изучается существование и единственность решения данной задачи. Для построения решения применяется метод Римана. Формулируется вспомогательная задача относительно функции Римана. С использованием преобразования Лапласа вспомогательная задача сводится к задаче Коши для обыкновенного дифференциального уравнения. Затем, применяя обратное преобразование Лапласа, получена явная формула решения вспомогательной задачи, которая одновременно является функцией Римана исходной задачи. На основе метода Римана получено решение исходной краевой задачи. Кроме того, устанавливаются достаточные условия на заданные функции, при которых найденное решение удовлетворяет условиям поставленной задачи.

Ключевые слова: Производная дробного порядка Прабхакара, интеграл дробного порядка Прабхакара, функция Миттаг-Леффлера с тремя параметрами, регулярное решение, преобразование Лапласа, двухпеременная функция Миттаг-Леффлера.