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SOME RETRACTIONS OF n-FOLD SYMMETRIC PRODUCT OF THE SPACE X

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RESUME

In this paper, we study some retractions of the space n-fold symmetric product of the space X. We prove that if a set A is a retract of a topological space X, then the set $\mathcal{F}_n(A)$ is also a retract of the space $\mathcal{F}_n(X)$. Also shown that if a set A is a weak retract of a topological space X, then the set $\mathcal{F}_n(A)$ is also a weak retract of the space $\mathcal{F}_n(X)$. Besides proved that if a set A is a deformation retract of a topological space X, then the set $\mathcal{F}_n(A)$ is also a deformation retract of the space $\mathcal{F}_n(X)$.

Key words: Retract, n-fold symmetric product, weak retract, deformation retract.

Recently, the topological properties on hyperspaces with the Vietoris topology and the homotopy properties of the topological spaces have been studied by many authors ([1], [2], [3], [4], [5], [6]).

In [1] the connection between a finally compact, pceudocompact, extremely disconnected, $\aleph$-space and its hyperspace is studied. And in the work [2] have been studied the connection between a uniformly connected, uniformly pseudocompact, $P-$precompact and its hyperspace. In the works [3] and [4] have been studied some cardinal and homotopy properties of the superextension λX of a topological space X. And in [4] proved that the superextension functor λ preserves homotopy, i.e. that it is a homotopy functor. In [5] showed that the functor of Permutation Degree SP^n_G preserves the homotopy and the retraction of topological spaces. And in [6] have been studied some homotopy properties of the space of complete linked systems.

Recall that a covariant functor is a mapping $\mathcal{F}$ which assigns to a topological space X the space $\mathcal{F}(X)$, and to a continuous mapping $f : X \to Y$, the mapping $\mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{F}(Y)$ satisfying the following conditions:

1) $\mathcal{F}$ preserves identity, that is, if id_X is the identity mapping of X, then $\mathcal{F}(id_X) = id_{\mathcal{F}(X)}$;

2) $\mathcal{F}$ preserves composition, that is, if $f : X \to Y$ and $g : Y \to Z$ are continuous mappings, then we have

$$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f).$$

We refer the reader to the book [7] and the article [8] for more information about functors. Some metric properties of n-fold symmetric product of the space X is studied in the work [9]. In this paper we study some homotopy properties and retractions of n-fold symmetric product of the space X.

All of our space are Hausdorff unless otherwise indicated. The symbol N stands for the set of positive integers and R stands for the set of real numbers. Given a space X, we define its hyperspaces as the following sets:

1) $CL(X) = \{A \subset X \mid A \text{ is closed and nonempty } \}$;

2) $2^X = \{A \in CL(X) \mid A \text{ is compact } \}$;

3) $\mathcal{F}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ points } \}$, $n \in N$ (see [9, 10]).

$CL(X)$ is topologized by the Vietoris topology defined as the topology generated by

$$\beta = \{\langle U_1, ..., U_k\rangle \mid U_1, ..., U_k \text{ are open subsets of } X,\ k \in N\},$$

where $\langle U_1, ..., U_k\rangle = \{A \in CL(X) \mid A \subset \bigcup U_j \text{ and } A \cap U_j \neq \emptyset \text{ for each } j \in \{1, ..., k\}\}$.

Note that, by definition, 2^X, $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ are subsets of $CL(X)$. Hence, they are topologized with the appropriate restriction of the Vietoris topology. Moreover,

1) $CL(X)$ is called the *hyperspace of nonempty closed subsets of* X;

2) 2^X is called the *hyperspace of nonempty compact subsets of* X;

3) $\mathcal{F}_n(X)$ is called the $n-$*fold symmetric product of* X;

4) $\mathcal{F}(X)$ is called the *hyperspace of finite subsets of* X.

On the other hand, it is obvious that $\mathcal{F}(X) = \bigcup_{n=1}^{\infty} \mathcal{F}_n(X)$ and $\mathcal{F}_n(X) \subset \mathcal{F}_{n+1}(X)$ for each $n \in N$ (see [9, 10]).

Remark 1. Let X be a space and let $n \in N$.
1) $\mathcal{F}_n(X)$ is closed in $\mathcal{F}(X)$;
2) $f_1 : X \twoheadrightarrow \mathcal{F}_1(X)$, $(x \mapsto \{x\})$, is a homeomoerphism;
3) Every $\mathcal{F}_m(X)$ is a closed subset of $\mathcal{F}_n(X)$ for each $m, n \in N$, $m < n$ (see [11]).

Notation 1. If $U_1, U_2, ..., U_n$ are open subsets of a space X, then $\langle U_1, U_2, ..., U_n\rangle_{\mathcal{F}(X)}$ denotes the intersection of the open set $\langle U_1, U_2, ..., U_n\rangle$ of the Vietotis topology, with $\mathcal{F}(X)$ (see [12]).

Notation 2. Let X be a space. If $\{x_1, x_2, ..., x_r\}$ is a point of $\mathcal{F}(X)$ and $\{x_1, x_2, ..., x_r \in \langle U_1, U_2, ..., U_n\rangle_{\mathcal{F}(X)}\}$, then for each $j \leq r$, we let $U_{x_j} = \cap\{U \in \{U_1, U_2, ..., U_s\} : x_j \in U\}$ Observe that $\langle U_{x_1}, U_{x_2}, ..., U_{x_r}\rangle_{\mathcal{F}(X)} \subset \langle U_1, U_2, ..., U_s\rangle_{\mathcal{F}(X)}$ (see [13]).

For some undefined or related concepts, we refer the reader to [14], [15] and [16].

Now we will consider some retractions of n-fold symmetric product of the space X. We begin with definitions of notions that will be used in this section. We mainly follow terminology from [15] and [16].

A subset A of a topological space X is called a *retract* of X if there exists a continuous mapping $r : X \to A$ such that $r|A = id_A$. The mapping r is called a retraction [15].

For the functor of n-fold symmetric product $\mathcal{F}_n$ the following theorem holds.

Theorem 1. Let for a subset $A \subseteq X$ the relation $\mathcal{F}_n(A) \subseteq \mathcal{F}_n(X)$ is correct. If a set A is a retract of a topological space X, then the set $\mathcal{F}_n(A)$ is also a retract of the space $\mathcal{F}_n(X)$.

Proof. Suppose that A is a retract of X. Then there exists a continuous mapping $r : X \to A$ such that $r(a) = a$ for all $a \in A$. Now we consider the mapping $\mathcal{F}_n r : \mathcal{F}_n X \to \mathcal{F}_n A$. It is clear that for every $A^{'} \in \mathcal{F}_n A$ we have that $(\mathcal{F}_n r)(A^{'}) = r(A^{'}) = A^{'}$. It means that the mapping $\mathcal{F}_n r : \mathcal{F}_n X \to \mathcal{F}_n A$ is a retraction. Hence, the set $\mathcal{F}_n(A)$ is a retract of the space $\mathcal{F}_n(X)$. Theorem 1 is proved.

From the Theorem 1 we get the following corollary.

Corollary 1. If the mapping $r : X \to A$ is a retraction, then the mapping $\mathcal{F}_n r : \mathcal{F}_n X \to \mathcal{F}_n A$ is also a retraction.

A subset $A \subseteq X$ is said to be a *weak retract* of X if there exists a continuous map $r : X \to A$ such that $r \circ i \simeq id_A$ where $i : A \to X$ is the inclusion map (see [15]).

Proposition 1. Let for a subset $A \subseteq X$ the relation $\mathcal{F}_n(A) \subseteq \mathcal{F}_n(X)$ is correct. If a set A is a weak retract of a topological space X, then the set $\mathcal{F}_n(A)$ is also a weak retract of the space $\mathcal{F}_n(X)$.

Proof. Suppose that A is a weak retract of X, then there exists a continuous map $r : X \to A$ such that $r \circ i \simeq id_A$ where $i : A \to X$ is the inclusion map. Now we consider the mapping $\mathcal{F}_n r : \mathcal{F}_n X \to \mathcal{F}_n A$ such that $\mathcal{F}_n r \circ \mathcal{F}_n i \simeq \mathcal{F}_n id_{\mathcal{F}_n A}$ where $\mathcal{F}_n i : \mathcal{F}_n A \to \mathcal{F}_n X$ is the inclusion map. Proposition 1 is proved.

Corollary 2. If the mapping $r : X \to A$ is a weakly retraction, then the mapping $\mathcal{F}_n r : \mathcal{F}_n X \to \mathcal{F}_n A$ is also a weakly retraction.

A space X is said to be *contractible* if it is homotopy equivalent to a point [16]. In [4, 6], some propositions about homotopy properties of topological spaces were given. For instance, contractibility is a homotopy property of the spaces. We have the following.

Proposition 2. If a topological space X is contractible, then the space $\mathcal{F}_n X$ is also contractible.

Proof. Assume that X is a contractible space. It means that $X \simeq \{a\}$. By proposition 1 [17] it implies immediately that $\mathcal{F}_n X \simeq \mathcal{F}_n\{a\}$, which means that $\mathcal{F}_n X$ is homotopy equivalent to the point $\mathcal{F}_n\{a\}$. Proposition 2 is proved.

Corollary 3. If a topological space X is homotopy equivalent to a point, then the space $\mathcal{F}_n X$ is also homotopy equivalent to a point.

A subset A of a topological space X is called a *deformation retract* of X if there exists a retraction $r : X \to A$ such that $i \circ r \simeq id_X$ where $i : A \to X$ is the inclusion.

In other words A is a deformation retract of X if there is a homotopy $F : X \times I \to X$ such that $F(x, 0) = x$ for all $x \in X$ and $F(x, 1) \in A$ for all $x \in X$.

Theorem 2. Let for a subset $A \subseteq X$ the relation $\mathcal{F}_n(A) \subseteq \mathcal{F}_n(X)$ is correct. If a set A is a deformation retract of a topological space X, then the set $\mathcal{F}_n(A)$ is also a deformation retract of the space $\mathcal{F}_n(X)$.

Proof. Suppose that A is a deformation retract of X. Then there is a homotopy $F : X \times I \to X$ such that $F(x, 0) = x$ for all $x \in X$ and $F(x, 1) \in A$ for all $x \in X$. Now we consider the mapping $\mathcal{F}_n F : \mathcal{F}_n X \times I \to \mathcal{F}_n X$ is a homotopy such that $(\mathcal{F}_n F)(\{x\}, 0) = \{x\}$ for all $\{x\} \in \mathcal{F}_n X$ and $(\mathcal{F}_n F)(\{x\}, 1) \in \mathcal{F}_n A$ for all $\{x\} \in \mathcal{F}_n X$. By theorem 1 [17] if the mapping $F : X \times I \to X$ is a homotopy, then the mapping $\mathcal{F}_n F : \mathcal{F}_n X \times I \to \mathcal{F}_n X$ is also homotopy. Clearly that $(\mathcal{F}_n F)(\{x\}, 0) = F(\{x\}, 0) = \{x\}$ for all $\{x\} \in \mathcal{F}_n X$ and $(\mathcal{F}_n F)(\{x\}, 1) = F(\{x\}, 1) \in \mathcal{F}_n A$ for all $\{x\} \in \mathcal{F}_n X$. Hence, the set $\mathcal{F}_n(A)$ is a deformation retract of the space $\mathcal{F}_n(X)$. Theorem 2 is proved.

Corollary 4. If the mapping $r : X \to A$ is a deformation retraction, then the mapping $\mathcal{F}_n r : \mathcal{F}_n X \to \mathcal{F}_n A$ is also a deformation retraction.

A subset A of X is a *strong deformation retract* if there is a retraction $r : X \to A$ such that $i \circ r \simeq_{relA} id_X$.

In other words A is a strong deformation retract of X if there is a homotopy $F : X \times I \to X$ such that $F(x, 0) = x$ for all $x \in X$, $F(a, t) = a$ for all $a \in A$, $t \in I$ and $F(x, 1) \in A$ for all $x \in X$.

Proposition 3. Let for a subset $A \subseteq X$ the relation $\mathcal{F}_n(A) \subseteq \mathcal{F}_n(X)$ is correct. If a set A is a strong deformation retract of a topological space X, then the set $\mathcal{F}_n(A)$ is also a strong deformation retract of the space $\mathcal{F}_n(X)$.

Proof. Suppose that A is a strong deformation retract of X. Then there is a homotopy $F : X \times I \to X$ such that $F(x, 0) = x$ for all $x \in X$, $F(a, t) = a$ for all $a \in A$, $t \in I$ and $F(x, 1) \in A$ for all $x \in X$. Now we consider the mapping $\mathcal{F}_n F : \mathcal{F}_n X \times I \to \mathcal{F}_n X$ is a homotopy such that $(\mathcal{F}_n F)(\{x\}, 0) = \{x\}$ for all $\{x\} \in \mathcal{F}_n X$, $(\mathcal{F}_n F)(\{a\}, t) = \{a\}$ for all $\{a\} \in \mathcal{F}_n A$, $t \in I$ and $(\mathcal{F}_n F)(\{x\}, 1) \in \mathcal{F}_n(A)$ for all $\{x\} \in \mathcal{F}_n X$. Clearly that $(\mathcal{F}_n F)(\{x\}, 0) = F(\{x\}, 0) = \{x\}$ for all $\{x\} \in \mathcal{F}_n X$, $(\mathcal{F}_n F)(\{a\}, t) = F(\{a\}, t) = \{a\}$ for all $\{a\} \in \mathcal{F}_n A$, $t \in I$ and $(\mathcal{F}_n F)(\{x\}, 1) = F(\{x\}, 1) \in \mathcal{F}_n(A)$ for all $\{x\} \in \mathcal{F}_n X$. Hence, the set $\mathcal{F}_n(A)$ is a strong deformation retract of the space $\mathcal{F}_n(X)$. Proposition 3 is proved.

Corollary 4. If the mapping $r : X \to A$ is a strongly deformation retraction, then the mapping $\mathcal{F}_n r : \mathcal{F}_n X \to \mathcal{F}_n A$ is also a strongly deformation retraction.

A subset $A \subseteq X$ is said to be a *weak deformation retract* of X if the inclusion map $i : A \to X$ is a homotopy equivalence.

By Proposition 1 and Proposition 2 we will get the following result.

Corollary 5. Let for a subset $A \subseteq X$ the relation $\mathcal{F}_n(A) \subseteq \mathcal{F}_n(X)$ is correct. If a set A is a weak deformation retract of a topological space X, then the set $\mathcal{F}_n(A)$ is also a weak deformation retract of the space $\mathcal{F}_n(X)$.

A continuous mapping $f : [0, 1] \to X$ is called a *path* in X. The point $f(0)$ is called the initial point and $f(1)$ is called the final or terminal point of this path. If $x \in X$, then one defines $e_x : I \to X$ as the constant path, i.e. $e_x(t) = x$ for any $t \in I$. A topological space X is said to be *path-connected* if given any two points x_0, x_1 in X there is a path in X from x_0 to x_1.

Proposition 4. If the mapping $f : I \to X$ is path in X from the point x_0 to the point x_1, then a mapping $\mathcal{F}_n f : I \to \mathcal{F}_n X$ defined by $\mathcal{F}_n f(t) = \bigcap\{A \in \mathcal{F}_n X : f(t) \in A, \quad \forall t \in [0, 1]\}$ is a path from the point $A_0 = \bigcap\{A \in \mathcal{F}_n : f(0) \in A\}$ to the point $A_1 = \bigcap\{A \in \mathcal{F}_n : f(1) \in A\}$ in $\mathcal{F}_n X$.

Corollary 6. If a topological space X is path-connected, then the space $\mathcal{F}_n X$ is also path-connected.

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REZYUME

Ushbu maqolada biz X fazoning n-darajali simmetrik ko'paytma fazosining ba'zi retraktlik xossalarini o'rganamiz. Agar A to'plam X fazo uchun retrakt bo'lsa, u holda unga mos $\mathcal{F}_n(A)$ to'plam ham $\mathcal{F}_n(X)$ fazo uchun retrakt bo'lishi isbotlangan. Bundan tashqari, agar A to'plam X fazo uchun kuchsiz retrakt bo'lsa, u holda unga mos $\mathcal{F}_n(A)$ to'plam ham $\mathcal{F}_n(X)$ fazo uchun kuchsiz retrakt ekanligi ko'rsatilgan. Shu bilan birga, agar A to'plam X fazo uchun deformatsion retrakt bo'lsa, u holda unga mos $\mathcal{F}_n(A)$ to'plam ham $\mathcal{F}_n(X)$ fazo uchun deformatsion retrakt ekanligi isbotlangan.

Kalit so'zlar: Retrakt, n-darajali simmetrik ko'paytma, kuchsiz retrakt, deformatsion retrakt.

РЕЗЮМЕ

В данной работе изучаются некоторые ретракции пространства n-кратного симметрического произведения пространства X. Доказывается, что если множество A является ретрактом топологического пространства X, то множество $\mathcal{F}_n(A)$ также является ретрактом пространства $\mathcal{F}_n(X)$. Также показано, что если множество A является слабым ретрактом топологического пространства X, то множество $\mathcal{F}_n(A)$ также является слабым ретрактом пространства $\mathcal{F}_n(X)$. Кроме того, доказано, что если множество A является деформационным ретрактом топологического пространства X, то множество $\mathcal{F}_n(A)$ также является деформационным ретрактом пространства $\mathcal{F}_n(X)$.

Ключевые слова: Ретракт, n-кратное симметричное произведение, слабый ретракт, деформационный ретракт.