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On the geodesics of smooth manifolds

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RESUME

This paper is devoted to the study of geodesics on smooth manifolds such as elliptical paraboloid and sphere in three-dimensional Euclidean space. The main result is finding equations of the geodesic on the $SO(3)$ group, which is the smooth three-dimensional manifold in $\mathbb{R}^9$.

Key words: Geodesics, Hamiltonian system, Hamiltonian vector field, 3D rotation group.

INTRODUCTION

Let M^n be a smooth Riemannian manifold of dimension n with a Riemannian metric $g_{ij}(x)$.

Definition. *The geodesics of the given metric* are defined as smooth parameterized curves

$$\gamma(t) = (x^1(t), \dots, x^n(t)),$$

that are solutions to the system of differential equations

$$\nabla_{\dot\gamma}\dot\gamma = 0,$$

where $\dot\gamma = \frac{d\gamma}{dt}$ is the velocity vector of the curve γ, and ∇ is the covariant derivation operator related to the symmetric connection associated with the metric g_{ij}. In local coordinates, these equations can be rewritten in the form

$$\frac{d^2x^i}{dt^2} + \sum \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0,$$

where $\Gamma^i_{jk}(x)$ are smooth functions called the Christoffel symbols of the connection ∇ and defined by the following explicit formulas [1]:

$$\Gamma^i_{jk}(x) = \frac{1}{2}\sum g^{is}\left(\frac{\partial g_{sj}}{\partial x^k} + \frac{\partial g_{sk}}{\partial x^j} - \frac{\partial g_{kj}}{\partial x^s}\right).$$

Geodesics can be interpreted as the trajectories of a single mass point that moves on the manifold without any external action, i.e. by inertia. Indeed, the equation of geodesics means exactly that the acceleration of the point is equal to zero.

The equation of geodesics can be considered as a Hamiltonian system [3] in the cotangent bundle T^*M, and the geodesics themselves can be regarded as projections of the trajectories of this Hamiltonian system on M. To this end, consider the natural coordinates x and p on the cotangent bundle T^*M, where $x = (x^1, \dots, x^n)$ are the coordinates of a point on M and $p = (p_1, \dots, p_n)$ are the coordinates of a covector from the cotangent space T^*_xM on the basis $dx^1, \dots, dx^n$. Take the standard symplectic structure $\omega = dx \wedge dp$ on T^*M and consider the following function as a Hamiltonian:

$$H(x,p) = \frac{1}{2}\sum g^{ij}(x)p_ip_j = \frac{1}{2}|p|^2. \tag{1}$$

It is well known the following proposition [1,4].

Proposition. a) Let $\gamma(t) = (x(t); p(t))$ be an integral trajectory of the Hamiltonian system $v = sgradH$ on T^*M. The curve $x(t)$ is then a geodesic, and its velocity vector $\dot{x}(t)$ is connected to $p(t)$ by the following relation:

$$\frac{dx^i(t)}{dt} = \sum g^{ij}(x)p_j(t).$$

b) Conversely, if a curve $x(t)$ is a geodesic on M, then the curve $(x(t); p(t))$, where $p_i(t) = \sum g_{ij}(x)p_j(t)$, is an integral trajectory of the Hamiltonian system $v = sgradH$.

I. Geodesics on the elliptical paraboloid

Let F be an elliptical paraboloid on $\mathbb{R}^3$ given with the equations

$$\begin{cases} x = \sqrt{z}cos\phi \\ y = \sqrt{z}sin\phi \\ z = z \end{cases} \tag{2}$$

To write the equation of a geodesic on the surface F according to Proposition, we first find the Hamitonian system on the cotangent bundle of the surface F. To do this, we should find the tangent vectors

$$r_z = \{\frac{1}{2\sqrt{z}cos\phi}, \frac{1}{2\sqrt{z}cos\phi}, 1\},$$

$$r_\phi = \{-\sqrt{z}sin\phi, \sqrt{z}cos\phi, 0\}.$$

We get the first quadratic form matrix and the inverse matrix of the first quadratic form, which are shown below, respectively.

$$(g_{ij}) = \begin{pmatrix} \frac{4z+1}{4z} & 0 \\ 0 & 4z \end{pmatrix} \text{ and } (g^{ij}) = \begin{pmatrix} \frac{4z}{4z+1} & 0 \\ 0 & \frac{1}{4z} \end{pmatrix}$$

Now, we are ready to write the Hamiltonian system on the T^*F

$$\begin{cases} p_1^{'} = \frac{dp_1}{dt} = -\frac{2p_1^2}{(4z+1)^2} + \frac{p_2^2}{2z^2} \\ p_2^{'} = \frac{dp_2}{dt} = 0 \\ z^{'} = \frac{dz}{dt} = \frac{4zp_1}{4z+1} \\ \phi^{'} = \frac{d\phi}{dt} = \frac{p_2}{z} \end{cases} \tag{3}$$

which corresponds to the Hamiltonian,

$$H = \frac{1}{2}(\frac{4z}{4z+1}p_1^2 + \frac{1}{z}p_2^2). \tag{4}$$

Theorem 1. The curve with the equation

$$\phi(z) = -\frac{2C}{C_1}ln\left|\frac{C_1\sqrt{4z+1} - 2\sqrt{C_1^2 z - C^2}}{C_1\sqrt{4z+1} + 2\sqrt{C_1^2 z - C^2}}\right| - \arctan C\frac{\sqrt{4z+1}}{\sqrt{C_1^2 z - C^2}} + C_2,$$

where $C_1^2 z - C^2 \neq 0$ and $C, C_1 \neq 0$ is a geodesic line on the elliptical paraboloid with equation (2).

Proof. Let us find an integral trajectory of the Hamiltonian system (3).

Using the equations of the system, we have the following.

$$p_1 = \frac{z^{'}(4z+1)}{4z}, \qquad p_2 = C, \qquad p_1^{'} = \frac{4z+1}{4z}z^{''} - \frac{1}{4z^2}z^{'2},$$

and

$$\frac{4z+1}{4z}z'' - \frac{1}{8z^2}z'^2 = \frac{C^2}{2z^2}.$$

As a result of the ODE we get

$$z' = \frac{dz}{dt} = \frac{2\sqrt{C_1^2 z - C^2}}{\sqrt{4z+1}}$$

and

$$\begin{cases} \dfrac{dz}{dt} = \dfrac{2\sqrt{C_1^2 z - C^2}}{\sqrt{4z+1}}, \\[2ex] \dfrac{d\phi}{dt} = \dfrac{C}{z}, \\[2ex] \dfrac{d^2 z}{dt^2} = \dfrac{2C_1^2}{4z+1} - \dfrac{8(C_1^2 z - C^2)}{(4z+1)^2}, \\[2ex] \dfrac{d^2\phi}{dt^2} = -\dfrac{C}{z^2} \cdot \dfrac{2\sqrt{C_1^2 z - C^2}}{\sqrt{4z+1}}. \end{cases} \tag{5}$$

Let us now check that the curve that satisfies the system (5) is geodesic.

We know that curve with the equations

$$\begin{cases} z = z(t) \\ \phi = \phi(t) \end{cases} \tag{6}$$

is geodesic if only if it satisfies the following system by Definition:

$$\begin{cases} \dfrac{d^2 z}{dt^2} - \dfrac{1}{2z(4z+1)}\Big(\dfrac{dz}{dt}\Big)^2 - \dfrac{2z}{4z+1}\Big(\dfrac{d\phi}{dt}\Big)^2 = 0, \\[2ex] \dfrac{d^2\phi}{dt^2} + \dfrac{1}{z}\dfrac{dz}{dt}\dfrac{d\phi}{dt} = 0. \end{cases} \tag{7}$$

It is easy to check that if curve (6) satisfies the system (5) then it satisfies the system (7) also.

We have the following ODE from the equations in the system (5):

$$\frac{d\phi}{dz} = \frac{d\phi}{dt} \cdot \frac{dt}{dz} = \frac{C}{2z} \cdot \frac{\sqrt{4z+1}}{\sqrt{C_1^2 z - C^2}}.$$

if we replace the variables as:

$$x = \frac{\sqrt{4z+1}}{\sqrt{C_1^2 z - C^2}}, \; z = \frac{C^2}{C_1^2}\frac{x^2 + \frac{1}{C^2}}{x^2 - \frac{4}{C_1^2}} \text{ and } dz = -\frac{2C^2}{C_1^2}\frac{\left(\frac{4}{C_1^2} + \frac{1}{C^2}\right)xdx}{\left(x^2 - \frac{4}{C_1^2}\right)^2},$$

where $C_1^2 z - C^2 \neq 0$ and $C, C_1 \neq 0$, we get following ODE

$$d\phi = -C\left(\frac{4}{C_1^2} + \frac{1}{C^2}\right)\frac{x^2 dx}{\left(x^2 - \frac{4}{C_1^2}\right)\left(x^2 + \frac{1}{C^2}\right)}$$

or

$$d\phi = -\frac{4C}{C_1^2}\frac{dx}{x^2 - \dfrac{4}{C_1^2}} - \frac{1}{C}\frac{dx}{x^2 + \dfrac{1}{C^2}},$$

then

$$d\phi = -\frac{2C}{C_1}\left(\frac{dx}{x - \dfrac{2}{C_1}} - \frac{dx}{x + \dfrac{2}{C_1}}\right) - \frac{1}{C}\frac{dx}{x^2 + \dfrac{1}{C^2}}.$$

By solving it, we find the following

$$\phi(x) = -\frac{2C}{C_1} ln\left|\frac{C_1 x - 2}{C_1 x + 2}\right| - \arctan Cx + C_3,$$

if we replace the old variables, we get a function

$$\phi(z) = -\frac{2C}{C_1} ln\left|\frac{C_1\sqrt{4z+1} - 2\sqrt{C_1^2 z - C^2}}{C_1\sqrt{4z+1} + 2\sqrt{C_1^2 z - C^2}}\right| - \arctan C\frac{\sqrt{4z+1}}{\sqrt{C_1^2 z - C^2}} + C_2.$$

This function represents the explicit equation of the geodesic we are looking for. Theorem 1 has been proven.

II. Geodesics on the Sphere

Let us now find a geodesics on the unit sphere $\mathbf{S}^2$ on $\mathbb{R}^3$.

Let us

$$\begin{cases} x = cosucosv \\ y = cosusinv \\ z = sinv \end{cases} \tag{8}$$

is a parametric equation of $\mathbf{S}^2$.

Theorem 2. The curve with the equation

$$\text{a)} \quad v(u) = -\frac{2}{\sqrt{C^2 - 2C_1^2}} \arctan\Big(\frac{C}{\sqrt{C^2 - 2C_1^2}}(\tan u)\Big) + C_2,$$

where $\dfrac{2C_1^2}{C^2} < 1$ and $C, C_1 \neq 0$;

$$\text{b)} \quad v(u) = -\frac{1}{\sqrt{C^2 - 2C_1^2}} \ln\Big|\frac{C\tan u + \sqrt{C^2 - 2C_1^2}}{C\tan u - \sqrt{C^2 - 2C_1^2}}\Big| + C_2,$$

where $\dfrac{2C_1^2}{C^2} > 1$ and $C \neq 0$. is a geodesic line on the sphere $\mathbf{S}^2$ with equation (8).

Proof. According to the proposition, a projection of the trajectories of this Hamiltonian system is geodesic on $\mathbf{S}^2$. Let us find it.

In this case, the tangent vectors are

$$r_u = \{-\sin u \cos v, -\sin u \sin v, \cos u\},$$

$$r_v = \{-\cos u \sin v, \cos u \cos v, 0\}.$$

We get the first quadratic form matrix and the inverse matrix of the first quadratic form, they are shown below, respectively:

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u \end{pmatrix} \text{ and } (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{\cos^2 u} \end{pmatrix}.$$

Hamiltonian system in the cotangent bundle $T^*\mathbf{S}^2$ is the following system:

$$\begin{cases} p_1' = -\dfrac{\sin u}{\cos^3 u} p_2^2, \\ p_2' = 0, \\ u' = p_1, \\ v' = \dfrac{p_2}{\cos^2 u}. \end{cases} \tag{9}$$

The corresponding Hamiltonian function is

$$H = \frac{1}{2}\Big(p_1^2 + \frac{1}{\cos^2 u} p_2^2\Big).$$

Using the equations of the system, we have the following

$$p_2 = C, \qquad u'' = p_1' = -\frac{\sin u}{\cos^3 u} C^2, \qquad u' = p_1 = -\frac{C^2}{2\cos^2 u} + C_1^2$$

and

$$\begin{cases} \dfrac{dp_1}{dt} = -\dfrac{\sin u}{\cos^3 u} C^2, \\ \dfrac{dp_2}{dt} = 0, \\ \dfrac{du}{dt} = \dfrac{2C_1^2 \cos^2 u - C^2}{2\cos^2 u}, \\ \dfrac{dv}{dt} = \dfrac{C}{\cos^2 u}. \end{cases} \tag{10}$$

Let us find the projection of the trajectories of the system (10).

If we consider the following equalities as

$$\frac{dt}{du} = \frac{2\cos^2 u}{2C_1^2 \cos^2 u - C^2}, \quad \frac{dv}{dt} = \frac{C}{\cos^2 u} \quad \textit{and} \quad \frac{dv}{du} = \frac{C}{C_1^2} \cdot \frac{1}{\cos^2 u - \dfrac{C^2}{2C_1^2}},$$

then we can write the explicit equation of the geodesic as $v = v(u)$.

If we replace the variables as: $x = \tan u$, $du = \dfrac{dx}{x^2+1}$ and $\cos^2 u = \dfrac{1}{x^2+1}$, we get

$$dv = -\frac{2}{C} \cdot \frac{dx}{x^2 + 1 - \dfrac{2C_1^2}{C^2}}. \tag{11}$$

Let us solve the differential equation (11).

When $\dfrac{2C_1^2}{C^2} < 1$ and $C, C_1 \neq 0$, we have

$$v(x) = -\frac{2}{\sqrt{C^2 - 2C_1^2}} \arctan\Big(\frac{C}{\sqrt{C^2 - 2C_1^2}} x\Big) + C_2,$$

or if $\dfrac{2C_1^2}{C^2} > 1$ and $C, C_1 \neq 0$, then we get

$$v(u) = -\frac{1}{\sqrt{C^2 - 2C_1^2}} \ln\left|\frac{Cx + \sqrt{C^2 - 2C_1^2}}{Cx - \sqrt{C^2 - 2C_1^2}}\right| + C_2.$$

Now we replace the old variables and we get equation of the geodesic with the following equation which we seek

$$\text{a)} \quad v(u) = -\frac{2}{\sqrt{C^2 - 2C_1^2}} \arctan\left(\frac{C}{\sqrt{C^2 - 2C_1^2}}(\tan u)\right) + C_2,$$

where $\dfrac{2C_1^2}{C^2} < 1$ and $C, C_1 \neq 0$;

$$\text{b)} \quad v(u) = -\frac{1}{\sqrt{C^2 - 2C_1^2}} \ln\left|\frac{C\tan u + \sqrt{C^2 - 2C_1^2}}{C\tan u - \sqrt{C^2 - 2C_1^2}}\right| + C_2,$$

where $\dfrac{2C_1^2}{C^2} > 1$ and $C \neq 0$.

Theorem 2 has been proven.

III. Geodesics of the SO(3)

Any rotation of a rigid heavy body around a fixed point in the case of Euler can be described by a rotation matrix belonging to SO(3). The group $SO(3)$ is a compact and smooth three-dimensional manifold $\mathbb{R}^9$.

The parametric equations of the matrix $A \in SO(3)$ and the Hamiltonian function on the cotangent bundle of the SO(3) smooth manifold are given in work [2]. We use them to find an equation of geodesic on SO(3).

The Hamiltonian in the cotangent bundle $T^*SO(3)$ is a function

$$H = \frac{1}{2}\left(\frac{1}{2\cos^2 v}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{2\cos^2 v}p_3^2 - \frac{\sin v}{\cos^2 v}p_1p_3\right). \tag{12}$$

Hamiltonian system corresponding to the Hamiltonian function (12) is

$$\begin{cases} \dfrac{dp_1}{dt} = 0 \\ \dfrac{dp_2}{dt} = -\dfrac{\sin v}{2\cos^3 v}(p_1^2 + p_3^2) + \dfrac{1+\sin^2 v}{2\cos^3 v}p_1p_3 \\ \dfrac{dp_3}{dt} = 0 \\ \dfrac{du}{dt} = \dfrac{1}{2\cos^2 v}p_1 - \dfrac{\sin v}{2\cos^2 v}p_3 \\ \dfrac{dv}{dt} = \dfrac{1}{2}p_2 \\ \dfrac{dw}{dt} = \dfrac{1}{2\cos^2 v}p_3 - \dfrac{\sin v}{2\cos^2 v}p_1 \end{cases} . \tag{13}$$

Consider the projection of the trajectory of this system (13). From the system, we know that

$$\frac{dv}{dt} = \frac{1}{2}p_2, \quad \frac{d^2v}{dt^2} = \frac{1}{2}\frac{dp_2}{dt} = -\frac{\sin v}{4\cos^3 v}(p_1^2 + p_3^2) + \frac{1+\sin^2 v}{4\cos^3 v}p_1p_3$$

when coefficients C, p_1, p_3 are non-zero. After solving this ODE, we have

$$\frac{dv}{dt} = -\frac{p_1^2 + p_3}{8}\frac{1}{\cos^2 v} + \frac{p_1 p_3}{4}\frac{\sin v}{\cos^2 v} + C^2 \tag{14}$$

or

$$\frac{dt}{dv} = \frac{8\cos^2 v}{8C^2\cos^2 v + 2p_1p_3\sin v - (p_1^2 + p_3^2)}.$$

As a result of ODE (14), we obtain the following ODEs.

$$du = \frac{4(p_1 - \sin v \cdot p_3)dv}{8C^2\cos^2 v + 2p_1p_3\sin v - (p_1^2 + p_3^2)},$$

$$dw = \frac{4(p_3 - \sin v \cdot p_1)dv}{8C^2\cos^2 v + 2p_1p_3\sin v - (p_1^2 + p_3^2)}.$$

or

$$du = \frac{p_3}{2C^2}\frac{\sin v dv}{\sin^2 v - \frac{p_1p_3}{4C^2}\sin v + \frac{p_1^2 + p_3^2 - 8C^2}{8C^2}} - \frac{p_1}{2C^2}\frac{dv}{\sin^2 v - \frac{p_1p_3}{4C^2}\sin v + \frac{p_1^2 + p_3^2 - 8C^2}{8C^2}},$$

$$dw = \frac{p_1}{2C^2}\frac{\sin v dv}{\sin^2 v - \frac{p_1p_3}{4C^2}\sin v + \frac{p_1^2 + p_3^2 - 8C^2}{8C^2}} - \frac{p_3}{2C^2}\frac{dv}{\sin^2 v - \frac{p_1p_3}{4C^2}\sin v + \frac{p_1^2 + p_3^2 - 8C^2}{8C^2}}.$$

We find the integral curve of the system when the new coefficients a, b satisfy the following conditions $a = \frac{p_1p_3}{4C^2} > 0$, $b = \frac{p_1^2 + p_3^2}{8C^2} - 1 > 0$, and $a^2 - 4b = \frac{p_1^2p_3^2}{16C^4} - 4 \cdot \frac{p_1^2 + p_3^2 - 8C^2}{8C^2} > 0$.

Our ODEs obtain the following view.

$$du = \frac{p_3}{2C^2}\frac{\sin v dv}{\sin^2 v - a\sin v + b} - \frac{p_1}{2C^2}\frac{dv}{\sin^2 v - a\sin v + b},$$

$$dw = \frac{p_1}{2C^2}\frac{\sin v dv}{\sin^2 v - a\sin v + b} - \frac{p_3}{2C^2}\frac{dv}{\sin^2 v - a\sin v + b}.$$

Leave the replacement as $x = \tan\frac{v}{2}$

$$du = \frac{2p_3}{bC^2} \cdot \frac{xdx}{(x^2 - \frac{a}{b}x + 1)^2 - \frac{a^2 - 4b}{b^2}x^2} - \frac{p_1}{bC^2} \cdot \frac{(1 + x^2)dx}{(x^2 - \frac{a}{b}x + 1)^2 - \frac{a^2 - 4b}{b^2}x^2},$$

$$dw = \frac{2p_1}{bC^2} \cdot \frac{xdx}{(x^2 - \frac{a}{b}x + 1)^2 - \frac{a^2 - 4b}{b^2}x^2} - \frac{p_3}{bC^2} \cdot \frac{(1 + x^2)dx}{(x^2 - \frac{a}{b}x + 1)^2 - \frac{a^2 - 4b}{b^2}x^2}.$$

Finally, we obtain the following geodesic equations with the parameter v;

$$
\begin{cases}
u(v) = \dfrac{2p_3b - p_1(\sqrt{a^2-4b}+a)}{C^2\sqrt{a^2-4b}\sqrt{4b^2-(a+\sqrt{a^2-4b})^2}} \cdot \arctan \dfrac{2b\tan\frac{v}{2} - a - \sqrt{a^2-4b}}{\sqrt{4b^2-(a+\sqrt{a^2-4b})^2}} - \\
- \dfrac{2p_3b + p_1(\sqrt{a^2-4b}-a)}{C^2\sqrt{a^2-4b}\sqrt{4b^2-(a-\sqrt{a^2-4b})^2}} \cdot \arctan \dfrac{2b\tan\frac{v}{2} - a + \sqrt{a^2-4b}}{\sqrt{4b^2-(a-\sqrt{a^2-4b})^2}} + C_2, \\
\\
w(v) = \dfrac{2p_1b - p_3(\sqrt{a^2-4b}+a)}{C^2\sqrt{a^2-4b}\sqrt{4b^2-(a+\sqrt{a^2-4b})^2}} \cdot \arctan \dfrac{2b\tan\frac{v}{2} - a - \sqrt{a^2-4b}}{\sqrt{4b^2-(a+\sqrt{a^2-4b})^2}} - \\
- \dfrac{2p_1b + p_3(\sqrt{a^2-4b}-a)}{C^2\sqrt{a^2-4b}\sqrt{4b^2-(a-\sqrt{a^2-4b})^2}} \cdot \arctan \dfrac{2b\tan\frac{v}{2} - a + \sqrt{a^2-4b}}{\sqrt{4b^2-(a-\sqrt{a^2-4b})^2}} + C_3
\end{cases}
\tag{15}
$$

After all calculations we obtained parametric equations of the curve on SO(3) and we have already proved the following theorem.

Theorem 4. The curve given with equations (15) is a geodesic on SO(3).

REFERENCES

1. Bolsinov A. and Fomenko A. Integrable Hamiltonian Systems. Geometry, Topology, Classification, Ijevsk: Chapman and Hall, 2004.
2. Narmanov A. and Ergashova Sh. On the geometry of SO(3). J. Geom. Symmetry Phys. **72** (2025) 83–94.
3. Narmanov A. and Ergashova Sh. Geometry of some completely integrable Hamiltonian systems. Uzbek Mathematical Journal **69** (2025) 110–119.
4. Trofimov V. and Fomenko A. Algebra and geometry of integrable Hamiltonian differential equations. Faktorial, Moscow 1995.

REZYUME

Bu maqolada uch o‘lchamli Yevklid fazosida joylashgan elliktik paraboloid va ikki o‘lchamli sfera kabi silliq ko‘pxilliklardagi geodezik chiziqlar o‘rganilgan. Asosiy natija $\mathbb{R}^9$ da joylashgan 3 o‘lchamli silliq ko‘pxillik bo‘lgan $SO(3)$ gruppaning geodezik chiziq tenglamasi topilgan.

Kalit sozlar: Geodezik chiziq, Hamilton sistemasi, Hamilton vektor maydoni, 3D aylanishlar gruppasi

РЕЗЮМЕ

Данная работа посвящена изучению геодезических на гладких многообразиях, таких как эллиптический параболоид и сфера, в трёхмерном евклидовом пространстве. Основным результатом является нахождение уравнений геодезических на группе $SO(3)$, представляющей собой гладкое трёхмерное многообразие в $\mathbb{R}^9$.

Ключевые слова: геодезические, Гамильтонова система, Гамильтонова векторные поля, 3D группа вращения.