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On the n-Lie algebras of generalized Jacobian and Wronskian

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RESUME

In this paper, we introduce skew-symmetric n-ary brackets on an associative commutative algebra, constructed by extending the Jacobian and Wronskian determinants with two additional columns. We determine the necessary and sufficient conditions under which the resulting n-ary algebras are n-Lie algebras. Moreover, we present several new examples of n-Lie algebras.

Key words:n-Lie algebras, Jacobian, Wronskian, derivation.

Introduction.

Lie algebras play a significant role in various areas of physics, including general relativity, quantum field theory, quantum mechanics, and string theory. As a central topic in mathematics, Lie theory has been the subject of extensive research for many years. The study of Lie algebras has led to numerous notable results and plentiful generalizations, highlighting their fundamental importance and wide-ranging applicability.

Building on the general concept of Ω-algebras introduced by Kurosh in [9], V.T. Fillipov proposed a broad extension of Lie algebras known as n-Lie algebras [5]. As an infinite-dimensional example, he introduced the n-Lie algebras defined by Jacobians [6]. Later, L. Takhtajan in [14] observed that this construction had already appeared in Nambu's generalization of Hamiltonian mechanics. Specifically, it involves the space $\mathbb{C}^\infty(M)$ of $\mathbb{C}^\infty$-functions on a finite-dimensional manifold M, equipped with an n-ary bracket defined via the Jacobian determinant.

Another notable example of an infinite-dimensional n-Lie algebra was provided by Dzhumadildaev in [4]. Specifically, any commutative associative algebra equipped with $n-1$ mutually commuting derivations, and endowed with an n-ary bracket defined via the Wronskian, forms an n-Lie algebra. These two constructions–the Jacobian and the Wronskian–constitute the primary known methods for constructing of infinite-dimensional n-Lie algebras. For a comprehensive overview of the structural theory of finite-dimensional n-Lie algebras, we refer the reader to [1-3,11-13,15] and the references therein.

Subsequently, the concepts of n-Lie algebras defined by Poisson and contact brackets were introduced. Both constructions endow an associative commutative algebra with an n-Lie algebra structure, subject to certain additional conditions. A comprehensive survey of these types of brackets can be found in [7].

It is worth noting that the n-Lie algebra defined by Wronskians serves as an n-ary analogue of the contact bracket. Motivated by this observation, in the present paper we introduce a new n-ary bracket and establish necessary and sufficient conditions under which it defines an n-contact bracket. Using these criteria, we construct several examples; for one of them, we prove simplicity and describe its ideals. Additionally, we propose an n-ary bracket formed by appending an extra column to the Jacobian, and we determine the conditions under which it defines an n-Lie Poisson algebra. We also note that simple subalgebras of the n-Lie algebra of Jacobians were previously investigated in [13].

Preliminaries.

Definition 1. A vector space L equipped with skew-symmetric ternary bracket $[-,-,\dots,-]$ is called an n-Lie algebra if the following identity holds for any $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_{n-1} \in L:$

$$[[x_1, x_2, \dots, x_n], y_1, \dots, y_{n-1}] = \sum_{i=1}^{n} [x_1, \dots, [x_i, y_1, \dots, y_{n-1}], \dots, x_n].$$

Definition 2. A derivation of an n-Lie algebra is a linear transformation D of L into itself satisfying

$$D([x_1, x_2, \dots, x_n]) = \sum_{i=1}^{n} [x_1, \dots, D(x_i), \dots, x_n],$$

for any $x_1, \dots, x_n \in L$. All the derivations of L generate a subalgebra of Lie algebra $gl(L)$ which is called the derivation algebra of L and denoted by $Der(L)$.

Here is a result from [5].

Theorem 1. The algebra of Jacobians $\mathcal{A}^*(d_1, \dots, d_n)$ for any associative and communtative $\mathbb{F}$-algebra $\mathcal{A}$ with its commuting derivations $d_1, \dots, d_n$ is an n-Lie algebra.

It is easy to see that the following equality holds true

$$[ab, x_2, \dots, x_n]_J = a[b, x_2, \dots, x_n]_J + b[a, x_2, \dots, x_n]_J.$$

This gives an example of n-Lie-Poisson algebras (see [4]).

Similarly, if one considers an associative commutative $\mathbb{F}$-algebra $\mathcal{A}$ and its commuting derivations $d_1, \dots, d_{n-1}$, then due to the result of [4] the vector space $\mathcal{A}$ with the following n-ary bracket $[x_1, \dots, x_n]_W := Wr(x_1, \dots, x_n)$, where

$$Wr(x_1, \dots, x_n) = \begin{vmatrix} x_1 & x_2 & \dots & x_n \\ d_1(x_1) & d_1(x_2) & \dots & d_1(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ d_{n-1}(x_1) & d_{n-1}(x_2) & \dots & d_{n-1}(x_n) \end{vmatrix}$$

forms an n-Lie algebra, which is called the n-Lie algebra of Wronskians.

Note that the n-ary bracket $[-,-]_W$ satisfies the following equality

$$[ab, x_2, \dots, x_n]_W = a[b, x_2, \dots, x_n]_W + b[a, x_2, \dots, x_n]_W - ab[\mathbf{1}, x_2, \dots, x_n]_W.$$

Let A be a commutative associative algebra over $\mathbb{F}$ and g be a Lie algebra of derivations of A, such that A contains no non-trivial g-invariant ideals.

Example 1. $S(A, g) = A$, where g is an n-dimensional Lie algebra with a basis $D_1, \dots, D_n$, being $[f_1, \dots, f_n]_J$ the n-ary Lie bracket.

Example 2. $W(A, g) = A$, where g is an $(n-1)$-dimensional Lie algebra with a basis $D_1, \dots, D_{n-1}$, being $[f_1, \dots, f_n]_W$ the n-ary Lie bracket.

Example 3. $SW(A, D) = A^{<1>} \oplus \dots \oplus A^{<n-1>}$ is the sum of $n-1$ copies of A and $g = \mathbb{F}D$, the n-ary Lie bracket is defined as follows: let $h \in A$ with $h^{<k>} \in A^{<k>}$. Define

$$[f_1^{<j_1>}, \dots, f_n^{<j_n>}] = 0, \text{ unless } \{j_1, \dots, j_n\} \supset \{1, \dots, n-1\};$$
$$[f_1^{<1>}, \dots, f_{k-1}^{<k-1>}, f_k^{<k>}, f_{k+1}^{<k>}, f_{k+2}^{<k+1>}, \dots, f_n^{<n-1>}] =$$
$$(-1)^{k+n-1}(f_1 \dots f_{k-1}(D(f_k)f_{k+1} - f_k D(f_{k+1}))f_{k+2} \dots f_n)^{<k>}.$$

In [10] it was claimed that Examples (1) - (3) above are only known infinite-dimensional simple n-Lie algebras over an algebraically closed field $\mathbb{F}$ of characteristic 0 for $n \geq 3$.

Generalization of Jacobian.

Let $\mathcal{A}$ be an associative commutative $\mathbb{F}$-algebra and $d_1, d_2, \dots, d_{n+2}$ be pairwise commuting derivations of $\mathcal{A}$. Then, define the following n-ary bracket on $\mathcal{A}$:

$$[x_1, x_2, \dots, x_n]_{\alpha\beta} = \begin{vmatrix} d_1(x_1) & \dots & d_1(x_n) & \alpha_1 & \beta_1 \\ d_2(x_1) & \dots & d_2(x_n) & \alpha_2 & \beta_2 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{n+1}(x_1) & \dots & d_{n+1}(x_n) & \alpha_{n+1} & \beta_{n+1} \\ d_{n+2}(x_1) & \dots & d_{n+2}(x_n) & \alpha_{n+2} & \beta_{n+2} \end{vmatrix} \quad (1)$$

where $\alpha_i, \beta_i \in \mathcal{A},\ i \in \{1, \dots, n+2\}$.

Our aim is to define necessary and sufficient conditions, under which $\langle \mathcal{A}, [-, -, \dots, -]_{\alpha\beta} \rangle$ forms an n-Lie algebra. Assume that $\beta_i \in \mathbb{F},\ i \in \{1, 2, \dots, n+2\}$. Then at least one parameter β_i is supposed to be nonzero. Without loss of generality, assume that $\beta_{n+2} \neq 0$. Then we make some elementary operations on (1) and obtain the following:

$$[x_1, x_2, \dots, x_n]_{\alpha\beta} = c \begin{vmatrix} \beta_{n+2}d_1(x_1) - \beta_1 d_{n+2}(x_1) & \beta_{n+2}d_1(x_2) - \beta_1 d_{n+2}(x_2) & \dots & \beta_{n+2}\alpha_1 - \beta_1\alpha_{n+2} & 0 \\ \beta_{n+2}d_2(x_1) - \beta_2 d_{n+2}(x_1) & \beta_{n+2}d_2(x_2) - \beta_2 d_{n+2}(x_2) & \dots & \beta_{n+2}\alpha_2 - \beta_2\alpha_{n+2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_{n+2}d_{n+1}(x_1) - \beta_{n+1} d_{n+2}(x_1) & \beta_{n+2}d_{n+1}(x_2) - \beta_{n+1} d_{n+2}(x_2) & \dots & \beta_{n+2}\alpha_{n+1} - \beta_{n+1}\alpha_{n+2} & 0 \\ d_{n+2}(x_1) & d_{n+2}(x_2) & \dots & \alpha_{n+2} & \beta_{n+2} \end{vmatrix}.$$

Set $D_k = \beta_{n+2}d_k - \beta_k d_{n+2}$ and $\alpha_k^* = \alpha_k\beta_{n+2} - \beta_k\alpha_{n+2}$, $(k = \overline{1, n+1})$, then we obtain the following generalized Jacobian:

$$[x_1, x_2, \dots, x_n]_{\alpha^*} = c \begin{vmatrix} D_1(x_1) & D_1(x_2) & \dots & D_1(x_n) & \alpha_1^* \\ D_2(x_1) & D_2(x_2) & \dots & D_2(x_n) & \alpha_2^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ D_{n+1}(x_1) & D_{n+1}(x_2) & \dots & D_{n+1}(x_n) & \alpha_{n+1}^* \end{vmatrix},$$

where $c = (-1)^{n+1}\frac{1}{\beta_{n+2}^{n+1}}$. In [8], it is presented the criterion for n-ary brackets defined as above to be an n-Lie algebra. Applying that, we obtain the following equality:

$$\begin{vmatrix} \alpha_i^* & \alpha_j^* \\ D_k(\alpha_i^*) & D_k(\alpha_j^*) \end{vmatrix} + \begin{vmatrix} \alpha_j^* & \alpha_k^* \\ D_i(\alpha_j^*) & D_i(\alpha_k^*) \end{vmatrix} + \begin{vmatrix} \alpha_k^* & \alpha_i^* \\ D_j(\alpha_k^*) & D_j(\alpha_i^*) \end{vmatrix} = 0$$

Expanding the equality above, we get the following relation:

$$\begin{aligned} &(\alpha_i\beta_{n+2} - \alpha_{n+2}\beta_i)\, d_k(\alpha_j) + (\alpha_j\beta_i - \alpha_i\beta_j)\, d_k(\alpha_{n+2}) + (\alpha_{n+2}\beta_j - \alpha_j\beta_{n+2})\, d_k(\alpha_i) \\ &+ (\alpha_k\beta_i - \alpha_i\beta_k)\, d_{n+2}(\alpha_j) + (\alpha_j\beta_{n+2} - \alpha_{n+2}\beta_j)\, d_i(\alpha_k) + (\alpha_k\beta_j - \alpha_j\beta_k)\, d_i(\alpha_{n+2}) \\ &+ (\alpha_{n+2}\beta_k - \alpha_k\beta_{n+2})\, d_i(\alpha_j) + (\alpha_j\beta_k - \alpha_k\beta_j)\, d_{n+2}(\alpha_i) + (\alpha_k\beta_{n+2} - \alpha_{n+2}\beta_k)\, d_j(\alpha_i) \\ &+ (\alpha_i\beta_k - \alpha_k\beta_i)\, d_j(\alpha_{n+2}) + (\alpha_{n+2}\beta_i - \alpha_i\beta_{n+2})\, d_j(\alpha_k) + (\alpha_i\beta_j - \alpha_j\beta_i)\, d_{n+2}(\alpha_k) = 0 \end{aligned}$$

With language of determinants, the relation above can be written in the following form:

$$\begin{vmatrix} \alpha_i & \alpha_j & \alpha_k \\ \beta_i & \beta_j & \beta_k \\ d_{n+2}(\alpha_i) & d_{n+2}(\alpha_j) & d_{n+2}(\alpha_k) \end{vmatrix} + \begin{vmatrix} \alpha_i & \alpha_{n+2} & \alpha_k \\ \beta_i & \beta_{n+2} & \beta_k \\ d_j(\alpha_i) & d_j(\alpha_{n+2}) & d_j(\alpha_k) \end{vmatrix} + \begin{vmatrix} \alpha_i & \alpha_{n+2} & \alpha_j \\ \beta_i & \beta_{n+2} & \beta_j \\ d_k(\alpha_i) & d_k(\alpha_{n+2}) & d_k(\alpha_j) \end{vmatrix} + \begin{vmatrix} \alpha_j & \alpha_{n+2} & \alpha_k \\ \beta_j & \beta_{n+2} & \beta_k \\ d_i(\alpha_j) & d_i(\alpha_{n+2}) & d_i(\alpha_k) \end{vmatrix} = 0$$

This proves the following theorem for generalized Jacobians:

Theorem 2. Let $\beta_i \in \mathbb{F}$ for $i \in \{1, 2, \dots, n+2\}$. Then the n-algebra $\langle L, [-, -, \dots, -]_{\alpha,\beta} \rangle$ is an n-Lie algebra if and only if

$$\begin{vmatrix} \alpha_i & \alpha_j & \alpha_k \\ \beta_i & \beta_j & \beta_k \\ d_{n+2}(\alpha_i) & d_{n+2}(\alpha_j) & d_{n+2}(\alpha_k) \end{vmatrix} + \begin{vmatrix} \alpha_i & \alpha_{n+2} & \alpha_k \\ \beta_i & \beta_{n+2} & \beta_k \\ d_j(\alpha_i) & d_j(\alpha_{n+2}) & d_{n+2}(\alpha_k) \end{vmatrix} + \begin{vmatrix} \alpha_i & \alpha_{n+2} & \alpha_j \\ \beta_i & \beta_{n+2} & \beta_j \\ d_k(\alpha_i) & d_k(\alpha_{n+2}) & d_k(\alpha_j) \end{vmatrix} + \begin{vmatrix} \alpha_j & \alpha_{n+2} & \alpha_k \\ \beta_j & \beta_{n+2} & \beta_k \\ d_i(\alpha_j) & d_i(\alpha_{n+2}) & d_i(\alpha_k) \end{vmatrix} = 0$$

for all $\alpha_i \in \mathcal{A},\ i \in \{1, 2, \dots, n+2\}$.

Here, we provide some examples for generalized n-Lie algebra.

Example 4. Consider a unitary associative commutative algebra A. Let d_i, $1 \leq i \leq n+2$ be pairwise commuting derivations of A and $\alpha_i \in \mathbb{F}$ for all $1 \leq i \leq n+2$. Then, $\langle A, [-, \dots, -]_{\alpha,\beta} \rangle$ is an n-Lie algebra.

Example 5. Consider a unitary associative commutative algebra A with mutually permuting derivations d_i, $1 \leq i \leq n+2$. Let $\alpha_i = d_i(\alpha),\ \alpha \in A$, $1 \leq i \leq n+2$, then, $\langle A, [-, \dots, -]_{\alpha,\beta} \rangle$ is an n-Lie algebra.

Generalization of Wronskian.

In this section we construct another n-ary bracket on associative commutative algebra with given mutually commuting derivations of the algebra. Furthermore, we provide conditions for an n-algebra with the n-ary bracket constructed to be an n-Lie algebra.

Let $\mathcal{A}$ be an associative commutative $\mathbb{F}$-algebra and $d_1, \ldots, d_{n+1}$ be pairwise commuting derivations of $\mathcal{A}$. Fix $x_1, \ldots, x_n, \alpha_0, \alpha_1, \ldots, \alpha_{n+1}, \beta_0, \beta_1, \ldots, \beta_{n+1} \in \mathcal{A}$ define the following n-ary bracket on $\mathcal{A}$ as follows:

$$\{x_1, \ldots, x_n\}_{\alpha,\beta} = Wr(x)_{\alpha,\beta} = \begin{vmatrix} x_1 & \ldots & x_n & \alpha_0 & \beta_0 \\ d_1(x_1) & \ldots & d_1(x_n) & \alpha_1 & \beta_1 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ d_n(x_1) & \ldots & d_n(x_n) & \alpha_n & \beta_n \\ d_{n+1}(x_1) & \ldots & d_{n+1}(x_n) & \alpha_{n+1} & \beta_{n+1} \end{vmatrix}.$$

We present a condition under which the n-ary algebra defined as above forms an n-Lie algebra. For this, let us first define a relation between generalized Jacobian and Wronskian. Consider the tensor algebra $\tilde{\mathcal{A}} = \mathcal{A} \otimes \mathbb{F}[t^{\pm}]$. Define the following maps on $\tilde{\mathcal{A}}$:

$$\mathrm{d}_0(a \otimes f(t)) = a \otimes t\frac{\partial f}{\partial t}, \quad \mathrm{d}_i(a \otimes f(t)) = d_i(a) \otimes f(t), 1 \le i \le n+1.$$

Clearly $\mathrm{d}_0, \mathrm{d}_i$ are commuting derivations of $Der(\tilde{\mathcal{A}})$. Let us define generalized Jacobian on $\tilde{\mathcal{A}}$ as follows:

$$Jac(x_1 \otimes f_1(t), \ldots, x_n \otimes f_n(t))_{\tilde{\alpha},\tilde{\beta}} = \begin{vmatrix} \mathrm{d}_0(x_1 \otimes f_1(t)) & \ldots & \mathrm{d}_0(x_n \otimes f_n(t)) & \alpha_0 \otimes t & \beta_0 \otimes t^{-n} \\ \mathrm{d}_1(x_1 \otimes f_1(t)) & \ldots & \mathrm{d}_1(x_n \otimes f_n(t)) & \alpha_1 \otimes t & \beta_1 \otimes t^{-n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathrm{d}_n(x_1 \otimes f_1(t)) & \ldots & \mathrm{d}_n(x_n \otimes f_n(t)) & \alpha_n \otimes t & \beta_n \otimes t^{-n} \\ \mathrm{d}_{n+1}(x_1 \otimes f_1(t)) & \ldots & \mathrm{d}_{n+1}(x_n \otimes f_n(t)) & \alpha_{n+1} \otimes t & \beta_{n+1} \otimes t^{-n} \end{vmatrix}$$

where $\tilde{\alpha} = (\alpha_0 \otimes t, \ldots, \alpha_{n+1} \otimes t), \tilde{\beta} = (\beta_0 \otimes t^{-n}, \ldots, \beta_{n+1} \otimes t^{-n})$.

Then, for $x_1 \otimes t, \ldots, x_n \otimes t, y_1 \otimes t, \ldots, y_{n-1} \otimes t \in \tilde{\mathcal{A}}$ one has

$$Jac(x_1 \otimes t, \ldots, x_{s-1} \otimes t, Jac(x_s \otimes t, y_1 \otimes t \ldots, y_{n-1} \otimes t)_{\tilde{\alpha}\tilde{\beta}}, x_{s+1} \otimes t, \ldots, x_n \otimes t)_{\tilde{\alpha}\tilde{\beta}}$$

$$= Jac(x_1 \otimes t, \ldots, x_{s-1} \otimes t, Wr(x_s, y_1 \ldots, y_{n-1})_{\alpha\beta} \otimes t, x_{s+1} \otimes t, \ldots, x_n \otimes t)_{\tilde{\alpha}\tilde{\beta}}$$

$$= Wr(x_1, \ldots, x_{s-1}, Wr(x_s, y_1, \ldots, y_{n-1})_{\alpha\beta}, x_{s+1}, \ldots, x_n)_{\alpha\beta} \otimes t,$$

where $1 \le s \le n$. This gives the direct relation between the generalized Jacobian and Wronskian.

$$\sum_{s=1}^{n} Jac(x_1 \otimes t, \ldots, x_{s-1} \otimes t, Jac(x_s \otimes t, y_1 \otimes t, \ldots, y_{n-1} \otimes t)_{\tilde{\alpha}}, x_{s+1} \otimes t \ldots, x_n \otimes t)_{\tilde{\alpha}}$$

$$-Jac(Jac(x_1 \otimes t, \ldots, x_n \otimes t)_{\tilde{\alpha}}, y_1 \otimes t, \ldots, y_{n-1} \otimes t)_{\tilde{\alpha}} =$$

$$\left(\sum_{s=1}^{n} \{x_1, \ldots, x_{s-1}, \{x_s, y_1, \ldots, y_{n-1}\}_\alpha, x_{s+1}, \ldots, x_n\}_\alpha - \{\{x_1, \ldots, x_n\}_\alpha, y_1, \ldots, y_{n-1}\}_\alpha\right) \otimes t$$

Therefore, we can state the following theorem:

Theorem 3. Let $\beta_i \in \mathbb{F}$ for $i \in \{0, 1, \ldots, n+1\}$. Then the n-algebra $\langle \mathcal{A}, \{-, \ldots, -\}_{\alpha,\beta} \rangle$ is an n-Lie algebra if and only if

$$\begin{vmatrix} \alpha_0 & \alpha_p & \alpha_q \\ \beta_0 & \beta_p & \beta_q \\ d_{n+1}(\alpha_0) & d_{n+1}(\alpha_p) & d_{n+1}(\alpha_q) \end{vmatrix} + \begin{vmatrix} \alpha_0 & \alpha_{n+1} & \alpha_q \\ \beta_0 & \beta_{n+1} & \beta_q \\ d_p(\alpha_0) & d_p(\alpha_{n+1}) & d_p(\alpha_q) \end{vmatrix} + \begin{vmatrix} \alpha_0 & \alpha_{n+1} & \alpha_p \\ \beta_0 & \beta_{n+1} & \beta_p \\ d_q(\alpha_0) & d_q(\alpha_{n+1}) & d_q(\alpha_p) \end{vmatrix} = 0$$

$$\begin{vmatrix} \alpha_i & \alpha_j & \alpha_k \\ \beta_i & \beta_j & \beta_k \\ d_{n+1}(\alpha_i) & d_{n+1}(\alpha_j) & d_{n+1}(\alpha_k) \end{vmatrix} + \begin{vmatrix} \alpha_i & \alpha_{n+1} & \alpha_k \\ \beta_i & \beta_{n+1} & \beta_k \\ d_j(\alpha_i) & d_j(\alpha_{n+1}) & d_{n+1}(\alpha_k) \end{vmatrix} + \begin{vmatrix} \alpha_i & \alpha_{n+1} & \alpha_j \\ \beta_i & \beta_{n+1} & \beta_j \\ d_k(\alpha_i) & d_k(\alpha_{n+1}) & d_k(\alpha_j) \end{vmatrix} + \begin{vmatrix} \alpha_j & \alpha_{n+1} & \alpha_k \\ \beta_j & \beta_{n+1} & \beta_k \\ d_i(\alpha_j) & d_i(\alpha_{n+1}) & d_i(\alpha_k) \end{vmatrix} = 0$$

for all $\alpha_i \in \mathcal{A}$, $1 \le p < q \le n+2$, $1 \le i < j < k \le n+2$.

Proof. By the equality above the statement of the theorem we can conclude that $\langle \mathcal{A}, \{-, \ldots, -\}_{\alpha,\beta} \rangle$ is an n-Lie algebra if and only if $\langle \tilde{\mathcal{A}}, [-, \ldots, -]_{\tilde{\alpha},\tilde{\beta}} \rangle$ is an n-Lie algebra. Therefore, by Theorem 2 the algebra $\tilde{\mathcal{A}}$ is n-Lie algebra if and only if

$$\begin{vmatrix} \alpha_i\otimes t & \alpha_j\otimes t & \alpha_k\otimes t \\ \beta_i\otimes t & \beta_j\otimes t & \beta_k\otimes t \\ d_{n+1}(\alpha_i\otimes t) & d_{n+1}(\alpha_j\otimes t) & d_{n+1}(\alpha_k\otimes t) \end{vmatrix} + \begin{vmatrix} \alpha_i\otimes t & \alpha_{n+1}\otimes t & \alpha_k\otimes t \\ \beta_i\otimes t & \beta_{n+1}\otimes t & \beta_k\otimes t \\ d_j(\alpha_i\otimes t) & d_j(\alpha_{n+1}\otimes t) & d_{n+1}(\alpha_k\otimes t) \end{vmatrix}$$

$$+ \begin{vmatrix} \alpha_i\otimes t & \alpha_{n+1}\otimes t & \alpha_j\otimes t \\ \beta_i\otimes t & \beta_{n+1}\otimes t & \beta_j\otimes t \\ d_k(\alpha_i\otimes t) & d_k(\alpha_{n+1}\otimes t) & d_k(\alpha_j\otimes t) \end{vmatrix} + \begin{vmatrix} \alpha_j\otimes t & \alpha_{n+1}\otimes t & \alpha_k\otimes t \\ \beta_j\otimes t & \beta_{n+1}\otimes t & \beta_k\otimes t \\ d_i(\alpha_j\otimes t) & d_i(\alpha_{n+1}\otimes t) & d_i(\alpha_k\otimes t) \end{vmatrix} = 0$$

But, this is equivalent to the following relations:

$$\begin{vmatrix} \alpha_0 & \alpha_p & \alpha_q \\ \beta_0 & \beta_p & \beta_q \\ d_{n+1}(\alpha_0) & d_{n+1}(\alpha_p) & d_{n+1}(\alpha_q) \end{vmatrix} + \begin{vmatrix} \alpha_0 & \alpha_{n+1} & \alpha_q \\ \beta_0 & \beta_{n+1} & \beta_q \\ d_p(\alpha_0) & d_p(\alpha_{n+1}) & d_p(\alpha_q) \end{vmatrix} + \begin{vmatrix} \alpha_0 & \alpha_{n+1} & \alpha_p \\ \beta_0 & \beta_{n+1} & \beta_p \\ d_q(\alpha_0) & d_q(\alpha_{n+1}) & d_q(\alpha_p) \end{vmatrix} = 0$$

$$\begin{vmatrix} \alpha_i & \alpha_j & \alpha_k \\ \beta_i & \beta_j & \beta_k \\ d_{n+1}(\alpha_i) & d_{n+1}(\alpha_j) & d_{n+1}(\alpha_k) \end{vmatrix} + \begin{vmatrix} \alpha_i & \alpha_{n+1} & \alpha_k \\ \beta_i & \beta_{n+1} & \beta_k \\ d_j(\alpha_i) & d_j(\alpha_{n+1}) & d_{n+1}(\alpha_k) \end{vmatrix}$$

$$+ \begin{vmatrix} \alpha_i & \alpha_{n+1} & \alpha_j \\ \beta_i & \beta_{n+1} & \beta_j \\ d_k(\alpha_i) & d_k(\alpha_{n+1}) & d_k(\alpha_j) \end{vmatrix} + \begin{vmatrix} \alpha_j & \alpha_{n+1} & \alpha_k \\ \beta_j & \beta_{n+1} & \beta_k \\ d_i(\alpha_j) & d_i(\alpha_{n+1}) & d_i(\alpha_k) \end{vmatrix} = 0$$

for all $\alpha_i \in \mathcal{A},\ \beta_i \in \mathbb{F},\ 1 \le p < q \le n+2,\ 1 \le i < j < k \le n+2.$

Below, we provide some examples of generalized Wronskian.

Example 6. Consider an associative commutative algebra $\mathcal{A}$ with mutually permuting derivations d_i, $1 \le i \le n+1$. Let $\alpha_i = d_i(\alpha_0)$ for $1 \le i \le n+1$. Then, $\langle \mathcal{A}, [-,\dots,-]_{\alpha,\beta}\rangle$ is an n-Lie algebra.

Remark A. It should be noted that if $\alpha_i, \beta_i \in \mathbb{F}$, then, $\langle \mathcal{A}, [-,\dots,-]_{\alpha,\beta}\rangle$ and $\langle \mathcal{A}, \{-,\dots,-\}_{\alpha,\beta}\rangle$ are n-Lie algebras.

Remark B. We should note that if the vectors $(\alpha_1, \alpha_2, \dots, \alpha_{n+2}$, and $(\beta_1, \beta_2, \dots, \beta_{n+2}) \in \mathbb{F}^{n+2}$ are collinear, then, the n-Lie algebra $\langle \mathcal{A}, [-,\dots,-]_{\alpha,\beta}\rangle$ is abelian. Similarly, we can claim the same statement for the n-Lie algebra $\langle \mathcal{A}, \{-,\dots,-\}_{\alpha,\beta}\rangle$.

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REZYUME

Ushbu maqolada biz assotsiativ kommutativ algebra orqali Yakobian va Vronskian determinantlariga ikkita qo'shimcha ustun qo'shish orqali tuzilgan koso-simmetrik n-ar qavslar bilan tanishamiz. Hosil bo'lgan n-ar algebralari n-Li algebralari bo'lishi uchun zarur va etarli shartlarni aniqlaymiz. Bundan tashqari, biz n-Li algebralariga bir nechta yangi misollarini keltiramiz.

Kalit so'zlar: n-Li algebralari, Yakobian, Vronskian, differensiallash.

РЕЗЮМЕ

В данной работе мы вводим кососимметричные n-арные скобки на ассоциативной коммутативной алгебре, построенные путем расширения определителей Якобиана и Вронского двумя дополнительными столбцами. Мы определяем необходимые и достаточные условия, при которых полученные n-арные алгебры являются n-Ли алгебрами. Кроме того, приводим несколько новых примеров n-Ли алгебр.

Ключевые слова: n-Лиевы алгебры, Якобиан, Вронскиан, дифференцирование.