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On periodic ground states for the Chui-Weeks model

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RESUME

In this paper, we consider the three-state Chui–Weeks model on the Cayley tree of arbitrary order not less than four. For this model, all translation-invariant and all periodic ground states with respect to any normal divisor of index two are described.

Key words: Cayley tree, Chui-Weeks model, ground state, translation-invariant ground state, periodic ground state.

1. Introduction

The study of statistical mechanics models on non-Euclidean structures, particularly on the Cayley tree, has attracted considerable attention during the last decades. Unlike the traditional lattice $\mathbb{Z}^d$, the Cayley tree possesses a hierarchical and cycle-free structure that provides a convenient framework for the rigorous analysis of various physical phenomena. Many classical models of statistical physics, such as the Ising model, Potts model, SOS model, and Chui-Weeks model, have been extensively investigated on Cayley trees [1-11], revealing rich behaviors that differ significantly from those observed on Euclidean lattices.

One of the central problems in this area is the description of ground states and Gibbs measures associated with these models. Ground states represent configurations that minimize the system's Hamiltonian and play a crucial role in understanding the structure of Gibbs measures, especially at low temperatures [8-11]. In particular, the classification of translation-invariant and periodic ground states provides valuable insights into the symmetry and long-range order of the system.

The Chui-Weeks model, originally introduced to describe surface phenomena and wetting transitions [2], is of particular interest when studied on the Cayley tree. This model exhibits non-trivial ground state structures depending on the number of spin states, the interaction parameters, and the geometry of the underlying tree. While the two-state and three-state versions of the Chui-Weeks model on Cayley trees of lower orders have been studied in detail, the investigation of higher-order trees remains an open and challenging direction.

In this paper, we investigate all translation-invariant and periodic ground states with respect to arbitrary two-index normal subgroups of the Cayley tree group of the three-state Chui–Weeks model. The ground states on the Cayley tree of order two were described in [6], while those on the Cayley tree of order three were presented in [7]. In this paper, we solve these problems for the Chui–Weeks model on a Cayley tree of arbitrary order greater than or equal to four.

2. Preliminaries and Model Description

The Cayley tree Γ^k (see, e.g., [3,8]) of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly $k+1$ edges issue. Let $\Gamma^k = (V, L, i)$, where V is the set of vertices of Γ^k, L is the set of edges of Γ^k and i is the incidence function associating each edge $l \in L$ with its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then x and y are called *nearest neighboring vertices*, and we write $l = \langle x, y\rangle$.

It is known (see [3]) that there exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k \geq 1$ and the group G_k of the free products of $k+1$ cyclic groups $\{e, a_i\}$, $i = 1, ..., k+1$ of the second order (i.e. $a_i^2 = e$, $a_i^{-1} = a_i$) with generators $a_1, a_2, ..., a_{k+1}$.

We consider model where the spin takes values in the set $\Phi = \{0, 1, 2\}$. For $A \subseteq V$ a spin *configuration* σ_A on A is defined as a function $x \in A \mapsto \sigma_A(x) \in \Phi$; the set of all configurations coincides with $\Omega_A = \Phi^A$. Denote $\Omega = \Omega_V$ and $\sigma = \sigma_V$.

Definition 2.1. *A configuration $\sigma \in \Omega$ is called G_k^* -periodic, if $\sigma(yx) = \sigma(x)$ for any $x \in G_k$ and $y \in G_k^* \subset G_k$.*

For a given periodic configuration the index of the subgroup is called the *period of the configuration.*

Definition 2.2. *A configuration that is invariant with respect to all shifts is called translation-invariant.*

The Chui-Weeks model (see [2]) is defined by the following Hamiltonian

$$H(\sigma) = J \sum_{\langle x,y\rangle \in L} \mid \sigma(x) - \sigma(y) \mid + \alpha \sum_{x \in V} \delta_{\sigma(x),0}, \tag{1}$$

where $J, \alpha \in \mathbb{R}$, α is an external field and $\sigma \in \Omega$.

Remark 2.1. *Recall that model (1) coincides with the SOS model under the condition $\alpha = 0$ (see, e.g., [1,2,8]).*

Let M be the set of all unit balls with vertices in V and $S_1(x)$ be the set of all nearest neighboring vertices of $x \in V$.

We call the restriction of a configuration σ to the ball $b \in M$ a *bounded configuration* σ_b. The energy of configuration σ_b on b is defined by the formula

$$U(\sigma_b) = \frac{J}{2} \sum_{x \in S_1(c_b)} \mid \sigma(x) - \sigma(c_b) \mid + \frac{\alpha}{k+2} \sum_{x \in b} \delta_{\sigma(x),0}, \tag{2}$$

where $J = (J, \alpha) \in \mathbb{R}^2$ and c_b is the center of the unit ball b.

The Hamiltonian (1) can be written as

$$H(\sigma) = \sum_{b \in M} U(\sigma_b).$$

3. Ground states

In this section, we study the ground states of the three-state Chui-Weeks model on the Cayley tree of arbitrary order not less than four. These ground states were described on the Cayley tree of order two in [6], and on the Cayley tree of order three in [7].

We have the following lemma.

Lemma 3.1. *Let $k \geq 4$. Then for each configuration φ_b, we have the following*

$$U(\varphi_b) \in \{U_i : i \in I\},$$

where I denotes the set of natural numbers from 1 to $\mid L \mid$, and L represents the collection of all distinct $U(\sigma_b)$. In particular, for $k = 4$ the following relations hold:

$$I = \{U_i : i = 1, 2, 3, ..., 40\}; \quad \mid L \mid = 40;$$

$$U_1 = 0;\; U_2 = \alpha;\; U_3 = \frac{J}{2};\; U_4 = \frac{J}{2} + \frac{\alpha}{6};\; U_5 = \frac{J}{2} + \frac{5\alpha}{6};\; U_6 = J;\; U_7 = J + \frac{\alpha}{6};$$

$$U_8 = J + \frac{\alpha}{3};\; U_9 = J + \frac{2\alpha}{3};\; U_{10} = J + \frac{5\alpha}{6};\; U_{11} = \frac{3J}{2};\; U_{12} = \frac{3J}{2} + \frac{\alpha}{6};$$

$$U_{13} = \frac{3J}{2} + \frac{\alpha}{3};\; U_{14} = \frac{3J}{2} + \frac{\alpha}{2};\; U_{15} = \frac{3J}{2} + \frac{2\alpha}{3};\; U_{16} = 2J;\; U_{17} = 2J + \frac{\alpha}{6};$$

$$U_{18} = 2J + \frac{\alpha}{3};\; U_{19} = 2J + \frac{\alpha}{2};\; U_{20} = 2J + \frac{2\alpha}{3};\; U_{21} = \frac{5J}{2};\; U_{22} = \frac{5J}{2} + \frac{\alpha}{6};$$

$$U_{23} = \frac{5J}{2} + \frac{\alpha}{3};\; U_{24} = \frac{5J}{2} + \frac{\alpha}{2};\; U_{25} = \frac{5J}{2} + \frac{2\alpha}{3};\; U_{26} = \frac{5J}{2} + \frac{5\alpha}{6};\; U_{27} = 3J + \frac{\alpha}{6};$$

$$U_{28}=3J+\frac{\alpha}{3};\ U_{29}=3J+\frac{\alpha}{2};\ U_{30}=\frac{7J}{2}+\frac{\alpha}{6};\ U_{31}=\frac{7J}{2}+\frac{\alpha}{3};\ U_{32}=\frac{7J}{2}+\frac{\alpha}{2};$$
$$U_{33}=4J+\frac{\alpha}{6};\ U_{34}=4J+\frac{\alpha}{3};\ U_{35}=4J+\frac{\alpha}{2};\ U_{36}=4J+\frac{2\alpha}{3};\ U_{37}=\frac{9J}{2}+\frac{\alpha}{6};$$
$$U_{38}=\frac{9J}{2}+\frac{2\alpha}{3};\ U_{39}=5J+\frac{\alpha}{6};\ U_{40}=5J+\frac{5\alpha}{6}.$$

For the case $k>4$, we have

$$U_1=0;\ U_2=\alpha;\ U_3=\frac{J}{2};\ \dots;\ U_{|I|}=(k+1)J+\frac{(k+1)\alpha}{k+2},$$

here $\mid I\mid$ denotes the cardinality of the set I.

Definition 3.1. *A configuration φ is called a ground state for the Hamiltonian (1), if*

$$U(\varphi_b)=\min\{U_i : i\in I\}$$

for any $b\in M$.

We denote $A_\xi=\{(J,\alpha)\in\mathbb{R}^2 : U_\xi=\min\{U_i : i\in I\}\}$.

For $k=4$, calculations show that:

$$A_1=\{(J,\alpha)\in\mathbb{R}^2 : J\geq 0,\alpha\geq 0\};$$
$$A_2=\{(J,\alpha)\in\mathbb{R}^2 : 30J\geq\alpha,\alpha\leq 0\};$$
$$A_3=A_6=A_{11}=A_{16}=\{(J,\alpha)\in\mathbb{R}^2 : J=0,\alpha\geq 0\};$$
$$A_4=A_5=A_7=\dots=A_{10}=A_{12}=\dots=A_{15}=A_{17}=\dots=A_{20}$$
$$=A_{22}=\dots=A_{38}=\{(J,\alpha)\in\mathbb{R}^2 : J=0,\alpha=0\};$$
$$A_{21}=\{(J,\alpha)\in\mathbb{R}^2 : -\alpha\leq 15J\leq 0\};$$
$$A_{39}=\{(J,\alpha)\in\mathbb{R}^2 : 0\leq\alpha\leq -15J\};$$
$$A_{40}=\{(J,\alpha)\in\mathbb{R}^2 : 30J\leq\alpha\leq 0\}$$

and $\bigcup_{i=1}^{40}A_i=\mathbb{R}^2$.

For the case $k>4$, we obtain the following sets:

$$A_1=\{(J,\alpha)\in\mathbb{R}^2 : J\geq 0,\alpha\geq 0\};$$
$$A_2=\{(J,\alpha)\in\mathbb{R}^2 : (k+1)(k+2)J\geq\alpha,\ \alpha\leq 0\};$$
$$A_3=\{(J,\alpha)\in\mathbb{R}^2 : J=0,\alpha\geq 0\};$$
$$\dots$$
$$A_{|I|}=\{(J,\alpha)\in\mathbb{R}^2 : (k+1)(k+2)J\leq\alpha\leq 0\}$$

and $\bigcup_{i=1}^{|I|}A_i=\mathbb{R}^2$.

3.1. Translation-invariant ground states

In this subsection we study all translation-invariant ground states for the Chui-Weeks model on the Cayley tree of arbitrary order not less than four. The following theorem describes all translation-invariant ground states for the three-state Chui-Weeks model.

Theorem 3.1. *For the Chui-Weeks model on the Cayley tree of order $k\geq 4$, the following assertions hold*

i) the configurations $\sigma(x)=1\ \forall x\in V$ and $\sigma(x)=2\ \forall x\in V$ are translation-invariant ground states iff $(J,\alpha)\in A_1$;

ii) the configuration $\sigma(x)=0\ \forall x\in V$ is a translation-invariant ground state iff $(J,\alpha)\in A_2$;

iii) if $(J,\alpha)\in\mathbb{R}^2\setminus\{A_1\cup A_2\}$, then no translation-invariant ground state exists.

Proof. *i)* Let $k \geq 4$. We consider the configuration $\sigma(x) = i, i \in \{1,2\}, \quad \forall x \in V$. For any $b \in M$ by Lemma 3.1 we have $U(\sigma_b) = U_1 = 0$. Thus the configuration $\sigma(x) = i, i \in \{1,2\}, \quad \forall x \in V$ is a ground state iff $(J, \alpha) \in A_1$;

ii) Let $k \geq 4$. We consider the configuration $\sigma(x) = 0 \quad \forall x \in V$. For any $b \in M$ by Lemma 3.1 we have $U(\sigma_b) = U_2 = \alpha$. Thus the configuration $\sigma(x) = 0 \quad \forall x \in V$ is a ground state iff $(J, \alpha) \in A_2$;

iii) It is obvious. Theorem 3.1 is proved.

Remark 3.1. *It is known from [1] that for the SOS model with a non-zero external field, the configuration $\sigma(x) = 1 \quad \forall x \in V$ is not a translation-invariant ground state. From Theorem 3.1, we can see that this configuration is a translation-invariant ground state for the Chui-Weeks model.*

3.2. $G_k^{(2)}$-periodic ground states

In this subsection we study all $G_k^{(2)}$-periodic ground states for the Chui-Weeks model on the Cayley tree of arbitrary order not less than four, where

$$G_k^{(2)} = \{x \in G_k : \mid x \mid \text{ is even}\},$$

where $\mid x \mid$ means length of the word x.

All $G_k^{(2)}$-periodic configurations have the following form:

$$\sigma(x) = \begin{cases} \sigma_0, \text{ if } x \in G_k^{(2)}, \\ \sigma_1, \text{ if } x \in G_k \setminus G_k^{(2)}, \end{cases}$$

where $\sigma_0, \sigma_1 \in \Phi$.

The following theorem describes all $G_k^{(2)}$-periodic ground states for the three-state Chui-Weeks model.

Theorem 3.2. *Let $k \geq 4$. Then for the Chui-Weeks model the following assertions hold*

I. *i) if $(J, \alpha) \in \{(J, \alpha) \in \mathbb{R}^2 : -2\alpha \leq (k+1)(k+2)J \leq 0\}$, then $G_k^{(2)}$-periodic configurations*

$$\sigma(x) = \begin{cases} 1, \text{ if } x \in G_k^{(2)}, \\ 2, \text{ if } x \in G_k \setminus G_k^{(2)}, \end{cases} \quad \sigma(x) = \begin{cases} 2, \text{ if } x \in G_k^{(2)}, \\ 1, \text{ if } x \in G_k \setminus G_k^{(2)} \end{cases}$$

are $G_k^{(2)}$-periodic ground states;

ii) if $(J, \alpha) \in \{(J, \alpha) \in \mathbb{R}^2 : J \leq 0, \alpha = 0\}$, then $G_k^{(2)}$-periodic configurations

$$\sigma(x) = \begin{cases} 0, \text{ if } x \in G_k^{(2)}, \\ 2, \text{ if } x \in G_k \setminus G_k^{(2)}, \end{cases} \quad \sigma(x) = \begin{cases} 2, \text{ if } x \in G_k^{(2)}, \\ 0, \text{ if } x \in G_k \setminus G_k^{(2)} \end{cases}$$

are $G_k^{(2)}$-periodic ground states.

II. *All $G_k^{(2)}$-periodic ground states, except for points i) and ii) above, are translation-invariant.*

Proof. **I.** *i)* Let $k = 4$. We consider the following $G_4^{(2)}$-periodic configuration

$$\sigma(x) = \begin{cases} 1, \text{ if } x \in G_4^{(2)}, \\ 2, \text{ if } x \in G_4 \setminus G_4^{(2)}. \end{cases}$$

Then we have $\sigma(c_b) = 1$ or $\sigma(c_b) = 2$, $\forall b \in M$. If $\sigma(c_b) = 1$ then $\forall x \in S_1(c_b)$ we have $\sigma(x) = 2$. In this case by Lemma 3.1 we get $U(\sigma_b) = \frac{5J}{2} = U_{21}$. If $\sigma(c_b) = 2$ then $\forall x \in S_1(c_b)$ we have $\sigma(x) = 1$. In this case by Lemma 3.1 we get $U(\sigma_b) = \frac{5J}{2} = U_{21}$. From these cases, it follows that the $G_4^{(2)}$-periodic configuration we have considered is a ground state, if $(J, \alpha) \in A_{21}$.

ii) Let $k = 4$. We consider the following $G_4^{(2)}$-periodic configuration

$$\sigma(x) = \begin{cases} 0, \text{ if } x \in G_4^{(2)}, \\ 2, \text{ if } x \in G_4 \setminus G_4^{(2)}. \end{cases}$$

Then we have $\sigma(c_b) = 0$ or $\sigma(c_b) = 2$, $\forall b \in M$. If $\sigma(c_b) = 0$ then $\forall x \in S_1(c_b)$ we have $\sigma(x) = 2$. In this case by Lemma 3.1 we get $U(\sigma_b) = 5J + \frac{\alpha}{6} = U_{39}$. If $\sigma(c_b) = 2$ then $\forall x \in S_1(c_b)$ we have $\sigma(x) = 0$. In this case by Lemma 3.1 we get $U(\sigma_b) = 5J + \frac{5\alpha}{6} = U_{40}$. From these cases, it follows that the $G_4^{(2)}$-periodic configuration we have considered is a ground state, if

$$(J, \alpha) \in A_{39} \cap A_{40} = \{(J, \alpha) \in \mathbb{R}^2 : J \leq 0, \alpha = 0\}.$$

The remaining cases are proved as above. Theorem 3.2 is proved.

3.3. H_A-periodic ground states

In this subsection we study all H_A-periodic ground states for the Chui-Weeks model on the Cayley tree of order $k \geq 4$, where

$$H_A = \{x \in G_k : \sum_{i \in A} \omega_x(a_i) \text{ is an even number}\},$$

where $\emptyset \neq A \subseteq N_k = \{1, 2, 3, ..., k+1\}$, and $\omega_x(a_i)$ is the number of letters a_i in a word $x \in G_k$. Note that $| \ x \ | = \sum_{j=1}^{k+1} w_j(x)$. It is known that the sets H_A and $G_k^{(2)}$ are normal groups of index two of G_k, and also any normal group of index two in G_k is of the form H_A (see [3, 8]). If $A = \{1, 2, 3, ..., k+1\}$, then the normal subgroup H_A coincides with the group $G_k^{(2)}$.

All H_A-periodic configurations have the following form:

$$\sigma(x) = \begin{cases} \sigma_0, \text{ if } \ x \in H_A, \\ \sigma_1, \text{ if } \ x \in G_k \setminus H_A, \end{cases}$$

where $\sigma_0, \sigma_1 \in \Phi$.

The following theorem describes all H_A-periodic ground states for the three-state Chui-Weeks model.

Theorem 3.3. *For the Chui-Weeks model on the Cayley tree of order $k \geq 4$, the following assertions hold*
i) if $(J, \alpha) \in A_3$, then the following H_A-periodic configurations

$$\sigma(x) = \begin{cases} i, \text{ if } \ x \in H_A, \\ j, \text{ if } \ x \in G_k \setminus H_A, \end{cases} \quad \text{where } \ i \neq j, \ \ i, j \in \Phi \setminus \{0\}, | \ A \ | = 1, 2, 3, ..., k$$

are H_A-periodic ground states;
ii) if $(J, \alpha) \in \mathbb{R}^2 \setminus A_3$, then all H_A-periodic ground states are either translation-invariant or $G_k^{(2)}$-periodic.

Proof. Let $B_i = \{x \in S_1(c_b) : \sigma_b(x) = i\}, i \in \Phi$.

i) Let $k = 4$ and $| \ A \ | = 1$. We consider the following H_A-periodic configuration

$$\sigma(x) = \begin{cases} 1, \text{ if } \ x \in H_A, \\ 2, \text{ if } \ x \in G_4 \setminus H_A. \end{cases}$$

If $c_b \in H_A$, then we have

$$\sigma(c_b) = 1, \quad | \ B_1 \ | = 4, \quad | \ B_2 \ | = 1,$$

thus $U(\sigma_b) = U_3$.
If $c_b \in G_4 \setminus H_A$, then we have

$$\sigma(c_b) = 2, \quad | \ B_1 \ | = 1, \quad | \ B_2 \ | = 4,$$

thus $U(\sigma_b) = U_3$.
Consequently, the configuration σ we have considered is an H_A-periodic ground state if $(J, \alpha) \in A_3$.

The remaining cases are proved as above. Theorem 3.3 is proved.

Remark 3.2. *Recall that when $k \geq 4$ and $| \ A \ | = k+1$, all H_A-periodic ground states are identical to the $G_k^{(2)}$-periodic ground states described in Theorem 3.2.*

Remark 3.3. *In this work, we have provided a complete characterization of all translation-invariant and periodic ground states with respect to any two-index normal subgroup for the three-state Chui–Weeks model on the*

Cayley tree of arbitrary order greater than or equal to four. An important observation is that, by employing the contour method [5,7-11], one can rigorously prove the existence of Gibbs measures associated with the translation-invariant ground states described here. This connection highlights the fundamental role of ground state analysis in the study of phase transitions and equilibrium properties of lattice models on non-amenable graphs such as the Cayley tree.

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REZYUME

Bu ishda tartibi to'rtdan kichik bo'lmagan ixtiyoriy tartibli Keli daraxtida uch holatli Chui-Weeks modeli qaraladi. Ushbu model uchun barcha translatsion-invariant va ixtiyoriy ikki indeksli normal bo'luvchiga nisbatan barcha davriy asosiy holatlar tavsiflanadi.

Kalit so'zlar: Keli daraxti, Chui-Weeks modeli, asosiy holat, translatsion-invariant asosiy holat, davriy asosiy holat.

РЕЗЮМЕ

В данной работе рассматривается трехсостоянийная модель Чуи–Уикса на дереве Кэли произвольного порядка не меньше четырех. Для этой модели описаны все трансляционно-инвариантные и все периодические основные состояния относительно любого нормального делителя индекса два.

Ключевые слова: Дерево Кэли, модель Чуи-Уикса, основное состояние, трансляционно-инвариантное основное состояние, периодическое основное состояние.