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Global existence results for coupled nonlinear parabolic equations with weighted coefficients

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RESUME

In this paper, we investigate a class of nonlinear weighted parabolic systems describing the coupled dynamics of two interacting scalar fields. We establish sufficient conditions for the global existence of weak solutions in appropriate weighted Sobolev spaces by employing energy estimates and integral inequalities. Furthermore, we develop a numerical scheme based on the Peaceman-Rachford splitting method combined with the Thomas algorithm to approximate the solutions efficiently. The proposed computational framework is implemented and illustrated with two and three-dimensional numerical simulations, including dynamic surface plots and animated profiles. The results demonstrate the qualitative features of the global solution, confirm the analytical findings, and provide additional insight into the interplay between nonlinear diffusion, weighted heterogeneity, and inter-component coupling.

Key words: Nonlinear parabolic system, variable coefficients, global existence, finite difference method, inhomogeneous medium, numerical simulation.

We focus on a class of nonlinear parabolic systems in divergence form that describe the coupled dynamics of two interdependent scalar fields, denoted by $u_1(x,t)$ and $u_2(x,t)$. Such models are relevant, for instance, in describing dust concentration in the atmosphere and on surfaces, or in the interaction of chemical or biological species in a heterogeneous medium.

The governing equations are formulated as a weighted system with nonlinear diffusion and mutual coupling effects. Specifically, the model incorporates spatially dependent density functions, nonlinear exponents in the diffusion terms, and time-dependent coefficients that regulate the transfer between the two components. This framework generalizes a wide spectrum of classical equations such as the porous medium equation [26,27], the p-Laplacian [6], the heat equation [16], and Boussinesq-type models arising in fluid filtration and transport theory. Applications extend to resistive diffusion phenomena in force-free magnetic fields, population dynamics, and the spread of airborne particles, where diffusion interacts with nonlinear growth and coupling effects [11,22].

A broad range of studies has addressed related nonlinear diffusion problems. For instance, Martynenko, Tedeev, and Shramenko [15] studied the Cauchy problem for a degenerate parabolic equation with inhomogeneous density of the type

$$\rho(x)\,\partial_t u = \nabla\cdot(|u|^{m-1}\nabla u) + f(u),$$

under slowly decaying initial data. Their results highlight the delicate interplay between density inhomogeneity and global solvability. Similarly, DiBenedetto [6] analyzed local behavior for degenerate parabolic equations with measurable coefficients of p-Laplacian type

$$\partial_t u = \nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right),$$

which serve as prototypes for nonlinear diffusion operators.

Nicolosi, Skrypnik, and Skrypnik [18], as well as Hui [9], examined removable singularities for quasilinear parabolic equations of the form

$$\partial_t u - \Delta_p u = 0,$$

while Nurumova [19] focused on blow-up behavior for nonlinear differential inequalities related to parabolic equations. Matyakubov and Raupov [17] obtained explicit estimates for blow-up in nonlinear systems with variable density, typically modeled by

$$\rho(x)\,\partial_t u = \Delta(u^m) + g(u).$$

Giachetti and Porzio [8] considered global existence for nonlinear parabolic equations with damping, such as

$$\partial_t u - \Delta(u^m) + \lambda u = 0,$$

and Punzo [20] analyzed the well-posedness of degenerate elliptic and parabolic problems of porous medium type.

Closer to our setting, Mamatov [16] studied qualitative properties of solutions for a reaction–diffusion system with variable density

$$\begin{cases} \rho_1(x)\,\partial_t u_1 = \nabla\cdot(D_1(u_2)\nabla u_1) + f_1(u_1,u_2), \\ \rho_2(x)\,\partial_t u_2 = \nabla\cdot(D_2(u_1)\nabla u_2) + f_2(u_1,u_2), \end{cases}$$

highlighting how density heterogeneity affects the dynamics. Abdugapor and Mamatov [1] investigated double nonlinear thermal conductivity problems, based on approximately self-similar reductions of PDEs of the type

$$\partial_t u = \nabla\cdot\left(u^{m-1}|\nabla u|^{p-2}\nabla u\right).$$

Aripov, Bobokandov, and Uralov [3] analyzed cross-diffusion systems with time-dependent nonlinear absorption, e.g.,

$$\partial_t u_i = \Delta(u_i^{m_i}) - \alpha_i(t)u_i^{q_i}, \quad i = 1,2,$$

which share structural similarities with our coupled model.

From the computational side, LeVeque [13] developed classical finite difference methods for PDEs, and Gander–Stuart [7] provided an analysis of the Peaceman–Rachford splitting scheme applied to nonlinear evolution equations of the form

$$\partial_t u = A(u) + B(u).$$

Smith and Zhao [24] proposed numerical methods for nonlinear degenerate diffusion equations, while Ketcheson 10 introduced fully implicit Runge–Kutta relaxation methods suitable for hyperbolic and parabolic problems. Although these studies provide powerful tools, they rarely address the additional complexity of heterogeneous weights and mutual coupling.

The contribution of this paper is twofold. First, we establish sufficient conditions for the global existence of weak solutions to the considered weighted nonlinear parabolic system with mutual coupling and time-dependent coefficients, thereby extending previous existence results [16-25] to a broader heterogeneous setting. Second, we develop a numerical scheme based on the Peaceman–Rachford splitting method combined with the Thomas algorithm for spatial discretization. Unlike earlier computational studies [26-28], our approach is specifically adapted to weighted nonlinearities and coupled dynamics. Numerical simulations, including two- and three-dimensional visualizations, confirm the theoretical predictions and illustrate the qualitative behavior of solutions.

Thus, this work provides both theoretical and computational insights into the study of nonlinear weighted parabolic systems with heterogeneous diffusion and mutual coupling effects, filling important gaps left by prior research on uncoupled or homogeneous models.

Problem statement. The problem under consideration is formulated as we investigate a coupled nonlinear parabolic system of the form

$$\rho_1(x)\frac{\partial u_i}{\partial t} = \nabla\cdot\left(\rho_2(x)\,u_i^{m_i-1}\left|\nabla u_i^k\right|^{p-2}\nabla u_i^{l_i}\right) + \gamma(t)\,e^{\alpha_i t}\,\varepsilon_i\,\rho_3(x)\,u_{3-i}^{k_i}, \quad i = 1,2, \tag{14}$$

with initial condition

$$u_i(x,0) = u_{i,0}(x), \quad x \in \mathbb{R}^N,\ i = 1,2. \tag{15}$$

Where, the unknown functions $u_1(x,t)$ and $u_2(x,t)$ describe the spatio-temporal evolution of two interacting quantities, such as airborne and surface dust concentrations, or two interdependent species in a heterogeneous medium. The spatial variable is $x \in \mathbb{R}^N$ and time $t > 0$. The weights $\rho_j(x) = |x|^{n_j}$ $(j = 1,2,3)$ encode the inhomogeneity of the medium. The diffusion operator incorporates nonlinear exponents $m_i \geq 1$, $p \geq 2$, $k > 0$, $l_i > 0$, giving rise to a doubly nonlinear and possibly singular structure.

The source terms involve time-dependent growth factors $\gamma(t)e^{\alpha_i t}$ and nonlinear coupling between u_1 and u_2 via the exponents k_i, with ε_i and α_i as positive parameters governing the transfer rate.

The mathematical formulation of the problem is therefore to determine nonnegative functions

$$u_i(x,t) \geq 0, \quad (x,t) \in \mathbb{R}^N \times (0,T),\ i = 1,2,$$

satisfying equations (14)–(15) in the weak sense, that is, belonging to the appropriate weighted Lebesgue and Sobolev spaces and fulfilling the corresponding integral identities for all compactly supported test functions $\eta \in C_0^1(\mathbb{R}^N \times (0,T))$.

The central analytical questions addressed in this study are as follows:

- To establish sufficient conditions on the parameters and initial data under which global-in-time weak solutions exist.
- To identify the role of the weighted coefficients $\rho_j(x)$, nonlinear diffusion exponents, and source terms in the qualitative behavior of solutions.
- To validate the theoretical findings by constructing and analyzing numerical solutions using the Peaceman–Rachford splitting method in combination with the Thomas algorithm.

This problem formulation connects the nonlinear analysis of degenerate and singular parabolic equations with computational approaches, providing a unified framework for theoretical study and numerical simulation.

Global-in-time existence of solutions. In this section we state and prove a global existence result for system (1)–(2). We begin by listing the standing assumptions and the definition of weak solution. Throughout this section we assume that the coefficient functions $\rho_j : \mathbb{R}^N \to (0,\infty)$, $j = 1,2,3$, are measurable and satisfy the uniform bounds

$$0 < \rho_j^- \leq \rho_j(x) \leq \rho_j^+ < \infty \qquad \text{for a.e. } x \in \mathbb{R}^N,$$

and that the time-factor $\gamma \in L^\infty_{\text{loc}}([0,\infty))$ and the constants $\varepsilon_i, \alpha_i > 0$ are fixed. The exponents satisfy

$$m_i \geq 1, \quad l_i > 0, \quad k > 0, \quad k_i > 0, \quad p \geq 2, \qquad i = 1,2.$$

Finally we assume the initial data

$$u_{i,0} \in L^2_{\rho_1}(\mathbb{R}^N) := L^2(\mathbb{R}^N; \rho_1(x)\,dx), \qquad u_{i,0} \geq 0, \quad i = 1,2.$$

Definition 1: We say that the pair (u_1,u_2) is a *weak solution* of system (1)–(2) on $\mathbb{R}^N \times (0,T)$ if for $i = 1,2$:

- $u_i \geq 0$ a.e.,
- $u_i \in L^\infty(0,T; L^2_{\rho_1}(\mathbb{R}^N))$ and $u_i^{m_i-1}|\nabla u_i^k|^{p-2}\nabla u_i^{l_i} \in L^1_{\text{loc}}(\mathbb{R}^N \times (0,T))$,
- for every test function $\varphi \in C_c^\infty(\mathbb{R}^N \times [0,T))$ and a.e. $t \in (0,T)$ the integral identity holds:

$$\begin{aligned}\int_{\mathbb{R}^N} \rho_1(x)u_i(x,t)\varphi(x,t)\,dx - \int_{\mathbb{R}^N} \rho_1(x)u_{i,0}(x)\varphi(x,0)\,dx \\ = \int_0^t \int_{\mathbb{R}^N} \Big(\rho_1(x)u_i\partial_\tau\varphi - \rho_2(x)u_i^{m_i-1}|\nabla u_i^k|^{p-2}\nabla u_i^{l_i}\cdot\nabla\varphi \\ + \gamma(\tau)e^{\alpha_i\tau}\varepsilon_i\rho_3(x)u_{3-i}^{k_i}\varphi\Big)\,dx\,d\tau.\end{aligned}$$

Theorem 1: Under the standing assumptions above, for any nonnegative initial data $u_{i,0} \in L^2_{\rho_1}(\mathbb{R}^N)$ there exists a global-in-time weak solution (u_1, u_2) of system (1)–(2) satisfying

$$u_i \in L^\infty_{\text{loc}}([0,\infty); L^2_{\rho_1}(\mathbb{R}^N)) \cap L^p_{\text{loc}}([0,\infty); W^{1,p}_{\rho_2,\text{loc}}(\mathbb{R}^N)),$$

and the energy estimate

$$\sup_{0\le t\le T} \sum_{i=1}^{2} \int_{\mathbb{R}^N} \rho_1(x) u_i^2(x,t)\,dx + \int_0^T \sum_{i=1}^{2} \int_{\mathbb{R}^N} \rho_2(x) u_i^{m_i-1} |\nabla u_i^k|^p \,dx\,dt \le C(T), \tag{16}$$

for every $T > 0$, where $C(T) > 0$ depends on T, the norms of the initial data and the structural constants.

Proof: We give the main steps; full rigorous details follow standard lines (Galerkin approximation, a priori bounds, compactness, passage to the limit).

1. Regularized problem and approximation. Construct smooth, compactly supported approximations $u^n_{i,0} \to u_{i,0}$ in $L^2_{\rho_1}$ and, for each n, consider a regularized problem (e.g. add a small viscosity and truncate nonlinearities) for which classical solutions exist on $[0,T]$.

2. Energy identity. Multiply the i-th equation by u_i and integrate over $\mathbb{R}^N$. Using integration by parts (justified for the regularized problem) and the positivity of the weights, we obtain for each i

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} \rho_1 u_i^2\,dx &+ \int_{\mathbb{R}^N} \rho_2 u_i^{m_i-1}|\nabla u_i^k|^p\,dx \\ &\le \int_{\mathbb{R}^N} \gamma(t) e^{\alpha_i t} \varepsilon_i \rho_3 u_{3-i}^{k_i} u_i\,dx + (\text{transport / lower order terms if present}).\end{aligned}$$

The right-hand coupling term is controlled by HГ¶lder and Young inequalities: for any $\delta > 0$,

$$\int \rho_3 u_{3-i}^{k_i} u_i \le \delta \int \rho_2 u_i^{m_i-1}|\nabla u_i^k|^p\,dx + C(\delta)\,\Phi(u_{3-i}),$$

where $\Phi(u_{3-i})$ denotes quantities depending on norms of u_{3-i} (these are controlled recursively). Choosing δ small allows absorption of diffusion terms to the left. Summing the two energy inequalities for $i = 1, 2$ yields (16).

3. Uniform bounds and extension. The energy estimate (16) provides uniform bounds for the approximations in the spaces appearing in Theorem **Theorem 1**. In particular, we get uniform $L^\infty(0,T; L^2_{\rho_1})$ bounds and $L^p(0,T; W^{1,p}_{\rho_2,\text{loc}})$ bounds for u_i^k.

4. Compactness and passage to the limit. By standard compactness results (Aubin–Lions lemma adapted to weighted spaces, see e.g. [14] for the abstract framework), a subsequence of approximations converges strongly in $L^2_{\text{loc}}(\mathbb{R}^N \times (0,T))$ to functions u_i. Weak/measurewise convergence of the fluxes combined with monotonicity properties of the nonlinear diffusion operator allows passage to the limit in the nonlinear terms. The limit pair (u_1, u_2) satisfies the integral identity of the weak solution.

5. Global-in-time extension. The a priori energy bound depends only on T and the initial data norms. Therefore the local solution can be continued stepwise for all $t > 0$, yielding a global-in-time weak solution.

Theorem 2. Let $a \in (a_c, N + n_1)$, and suppose that the initial data is given by

$$u_{i,0}(x) = \lambda \varphi_i(x), \quad \lambda > 0, \quad i = 1, 2,$$

where each $\varphi_i(x) \in F^a$ is a nonnegative function satisfying the asymptotic condition

$$\lim_{|x|\to\infty} |x|^a \varphi_i(x) = M > 0.$$

Then there exists a threshold $\lambda_0 = \lambda_0(\varphi_1, \varphi_2) > 0$ such that for any $\lambda < \lambda_0$, the corresponding solution $u_i(x,t)$ to system (1)вЂ“(2) admits the following large-time asymptotic behavior:

$$t^{\theta_i}\left|u_i(x,t) - U^{(i)}_{\lambda M,a}(x,t)\right| \to 0, \quad \text{as } t \to \infty,$$

uniformly on compact subsets of $\mathbb{R}^N$, where each $U^{(i)}_{\lambda M,a}(x,t)$ is a self-similar solution constructed from the initial profile $\lambda M|x|^{-a}$, and

$$\theta_i = \frac{a}{a(m_i + k_i(p-2) + l_i - 1) + p + n_1 - n_2}.$$

Proof. We introduce the radially symmetric self-similar solution $U^{(i)}_{\lambda M,a}(x,t)$ to describe the large-time asymptotic behavior of solutions to the degenerate coupled parabolic system (1).

Let $U^{(i)}_{\lambda M,a}(x,t)$ denote the self-similar solution of the corresponding *uncoupled* limit problem (obtained by neglecting the coupling term) with algebraically decaying initial data:

$$U^{(i)}_{\lambda M,a}(x,t) = t^{-a\alpha_i} f^{(i)}_M(r), \quad r = |x|t^{-\alpha_i},$$

where

$$\alpha_i = \frac{1}{a(m_i + k_i(p-2) + l_i - 1) + p + n_1 - n_2}.$$

The profile $f^{(i)}_M(r)$ satisfies a second-order nonlinear ODE obtained via radial reduction of the diffusion operator, together with the asymptotic condition

$$\lim_{r\to\infty} r^a f^{(i)}_M(r) = M > 0, \quad f^{(i)}_M(r) > 0, \quad f^{(i)\prime}_M(0) = 0.$$

To establish the existence of such a profile $f^{(i)}_M(r)$, we consider the following Cauchy problem for $g(r)$ with initial value $\eta > 0$:

$$\begin{cases} \left(g^{m_i-1}\left|(g^{k_i})'\right|^{p-2}g'\right) + \dfrac{N+n_2-1}{r} g^{m_i-1}\left|(g^{k_i})'\right|^{p-2}g' \\ \qquad +\alpha_i r^{n_1-n_2+1}g' + a\alpha_i r^{n_1-n_2} g = 0, \quad r > 0, \\ g(0) = \eta, \quad g'(0) = 0. \end{cases}$$

By standard ODE theory for degenerate nonlinear equations, one shows that $g(r)$ is positive, smooth for $r > 0$, and decays algebraically as $r \to \infty$. In fact,

$$\lim_{r\to\infty} r^a g(r) = M(\eta),$$

where $M(\eta)$ is a continuous and strictly monotone function of $\eta > 0$.

Moreover, the scaling invariance of the ODE yields

$$g_\eta(r) = \eta\, g_1(\eta^\sigma r)\,, \quad \sigma = -\frac{m_i + k_i(p-2) + l_i - 1}{p + n_1 - n_2},$$

which implies

$$M(\eta) = \eta^{\,1-a\sigma} M(1).$$

Hence, for each $M > 0$, there exists a unique $\eta > 0$ such that $M(\eta) = M$, and the corresponding self-similar profile $f^{(i)}_M(r)$ satisfies the desired asymptotic behavior. Furthermore, qualitative analysis of the ODE shows that $f^{(i)}_M(r)$ is monotone non-increasing.

Therefore, the radially symmetric self-similar solution $U^{(i)}_{\lambda M,a}(x,t)$ accurately describes the asymptotic behavior of the solutions to the full PDE system (1)вЂ“(2) in the limit $t \to \infty$, provided λ is sufficiently small so that the coupling term acts only as a perturbation.

Numerical methods. In order to approximate the continuous problem on a computationally feasible framework, the infinite spatial domain $\mathcal{R}^N$ is truncated to a sufficiently large bounded region $\Omega = [0, L]$ (in one spatial dimension) or its higher-dimensional analog. The truncation length L is chosen such that the influence of the boundary on the solution dynamics remains negligible over the considered time interval.

The domain Ω is then partitioned into a uniform mesh with spatial step size

$$h = \frac{L}{N}, \quad N \in \mathcal{N},$$

where N denotes the number of subdivisions. The grid points are defined as

$$x_j = jh, \quad j = 0, 1, \ldots, N.$$

The approximate numerical solution at location x_j and time $t^n = n\tau$ is denoted by $u_i^n(j)$, with $\tau > 0$ being the temporal discretization parameter.

For the spatial derivatives, finite difference operators are introduced. The first-order forward and backward difference quotients are given by

$$\nabla_h^+ u(j) = \frac{u(j+1) - u(j)}{h}, \qquad \nabla_h^- u(j) = \frac{u(j) - u(j-1)}{h},$$

while the centered approximation of the gradient reads

$$\nabla_h u(j) = \frac{u(j+1) - u(j-1)}{2h}.$$

These discrete operators are employed to approximate nonlinear fluxes of the type

$$\nabla(\rho_2(x)\, u^{m_i-1}\, |\nabla(u^k)|^{p-2}\, \nabla(u^{l_i})),$$

appearing in the governing equations.

The weighted coefficients $\rho_j(x)$ are discretized directly at the nodal points, i.e.,

$$\rho_j(x_n) = |x_n|^{n_j}, \qquad j = 1, 2, 3,$$

so that the heterogeneity of the medium is incorporated into the discrete scheme without additional approximation.

The numerical solution of nonlinear coupled parabolic systems with heterogeneous coefficients presents considerable computational challenges, particularly when both diffusion and reaction-coupling terms exhibit strong nonlinearities. In order to ensure stability and efficiency, we adopt the Peaceman-Rachford splitting method, which belongs to the class of alternating direction implicit schemes. The fundamental idea is to decompose the time integration into two fractional steps, thereby treating the diffusion and reaction contributions separately.

Diffusion step (first half-step). At the first stage, only the nonlinear diffusion operator is advanced over a half-time step, while the reaction-coupling terms are neglected. The semi-discrete formulation reads

$$\frac{u_i^{n+1/2}(j) - u_i^n(j)}{\tau/2} = \nabla_h(\rho_2(x_j)\, (u_i^n(j))^{m_i-1}\, |\nabla_h(u_i^n(j))^k|^{p-2}\, \nabla_h(u_i^{n+1/2}(j))^{l_i}),$$

where ∇_h denotes the discrete gradient operator defined on the uniform mesh. This formulation ensures that the diffusion terms are treated implicitly, which provides enhanced stability properties even in the presence of nonlinear degenerate diffusion.

Reaction-coupling step (second half-step). In the second stage, the updated values $u_i^{n+1/2}$ serve as input to evolve the system under the effect of the nonlinear reaction and inter-component coupling, again over a half-time step:

$$\frac{u_i^{n+1}(j) - u_i^{n+1/2}(j)}{\tau/2} = \gamma(t^n)\, e^{\alpha_i t^n}\, \varepsilon_i\, \rho_3(x_j)\, (u_{3-i}^{n+1/2}(j))^{k_i}.$$

Since this step involves only local algebraic updates, it is computationally inexpensive. Moreover, the splitting ensures that nonlinear couplings are treated explicitly but in a stable manner due to the staggered update.

The implicit discretization of the diffusion step leads to a nonlinear system of algebraic equations at each half-time step. After linearization, the resulting system can be written in the block tridiagonal form

$$A_j\, u_i^{n+1/2}(j-1) + B_j\, u_i^{n+1/2}(j) + C_j\, u_i^{n+1/2}(j+1) = F_j, \qquad j = 1, \ldots, N-1,$$

where the matrices A_j, B_j, C_j encode the contributions of the discretized nonlinear fluxes, and F_j denotes the right-hand side incorporating previous time-level information.

In the scalar case, this system reduces to the standard tridiagonal structure, which can be solved efficiently using the Thomas algorithm. The algorithm performs a forward elimination followed by a backward substitution, yielding a solution in $\mathbb{O}(N)$ operations.

For the coupled two-component system considered here, the discretization naturally leads to a block tridiagonal system, where each coefficient A_j, B_j, C_j is itself a small matrix corresponding to the interaction between the two components. Nevertheless, the Thomas algorithm generalizes straightforwardly to this block setting. The block tridiagonal solver retains linear computational complexity with respect to the number of spatial nodes, while the cost of inverting small blocks remains negligible.

The proposed Peaceman-Rachford splitting scheme is inherently stable for linear parabolic equations. For the nonlinear coupled system under consideration, stability is investigated through discrete energy estimates. Specifically, we multiply the discrete equations by appropriate test functions and apply discrete integration by parts (summation by parts). This procedure yields a discrete energy inequality of the form

$$E^{n+1} + \tau \mathbb{D}^{n+1} \leq E^n + C\tau,$$

where E^n denotes the discrete energy of the numerical solution at the n-th time step, $\mathbb{D}^n$ represents the dissipation due to nonlinear diffusion, and C is a constant depending only on the data of the problem. Such an inequality guarantees that the discrete energy remains uniformly bounded with respect to n, thereby preventing unphysical growth of the numerical solution.

A key role in establishing stability is played by the monotonicity of the nonlinear diffusion operator and the positivity of the weight functions $\rho_j(x)$. These structural properties ensure that the flux terms do not destabilize the discrete dynamics, even in the presence of nonlinear interactions between the two components u_1 and u_2.

Convergence of the numerical solution towards a weak solution of the continuous problem is obtained via a compactness argument. First, the stability estimates provide uniform bounds in discrete Sobolev norms, independent of the discretization parameters h and τ. These bounds allow us to extract subsequences converging weakly in the corresponding functional spaces.

To strengthen the convergence and identify the weak limit with the exact solution, we employ discrete compactness results analogous to the Aubin-Lions lemma. In particular, the boundedness of temporal increments in L^2 and the spatial regularity inherited from the diffusion operator imply compactness in L^2_{loc}. Passing to the limit in the discrete equations, we conclude that the limit function is indeed a weak solution of the original system.

To complement the theoretical results, we perform numerical experiments based on the Peaceman-Rachford scheme combined with the block Thomas algorithm. The computations are carried out in one spatial dimension, with spatial domain $x \in [0, 1]$ and time horizon $t \in [0, 2]$. Unless otherwise stated, the discretization parameters are chosen as $h = 0.05$ and $\tau = 0.01$, ensuring sufficient resolution in both space and time.

The nonlinear exponents are taken as $m_1 = 1.2$, $m_2 = 1.4$, and $p = 2.5$, while the coupling exponents are $q_1 = 4.5$ and $q_2 = 5.0$. The initial conditions are selected in a compactly supported form, reflecting localized dust concentration in air and on surfaces. At the boundary $x = 0$, nonlinear Robin-type conditions are imposed to model particle exchange, while at $x = 1$ homogeneous Neumann conditions are prescribed.

Figure 1 displays the evolution of the solutions $u(x, t)$ and $\nu(x, t)$ at successive time levels. The numerical solutions exhibit rapid diffusion and nonlinear damping near the origin, while coupling effects enhance interaction between the two components.

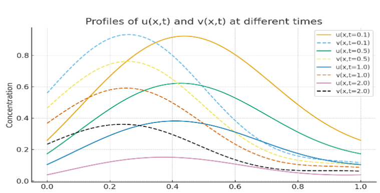

Figure 1: Two-dimensional spatial profiles of $u(x, t)$ and $\nu(x, t)$ at different times ($t = 0.1, 0.5, 1.0, 2.0$).

To better capture the spatio-temporal dynamics, we plot three-dimensional surfaces of the solutions (Figures 2 and 3). These surfaces clearly demonstrate the dissipative nature of the solutions and highlight the asymmetry introduced by nonlinear diffusion and weighted coefficients.

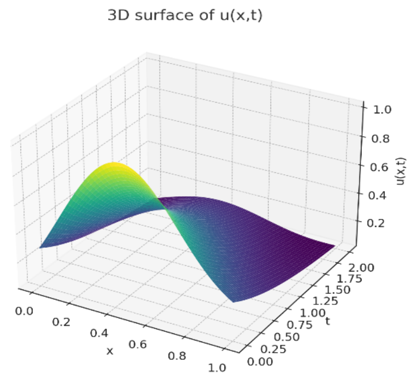

Figure 2: Three-dimensional surface plot of $u(x,t)$ over space and time.

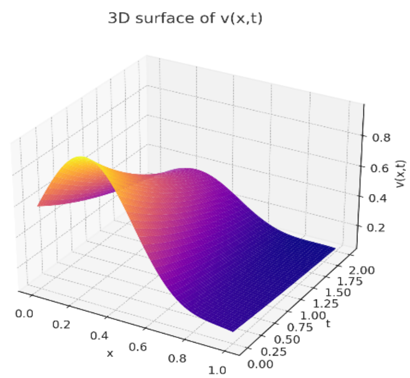

Figure 3: Three-dimensional surface plot of $\nu(x,t)$ over space and time.

The supplementary material contains the full 3D animation, while Figure 4 shows selected frames.

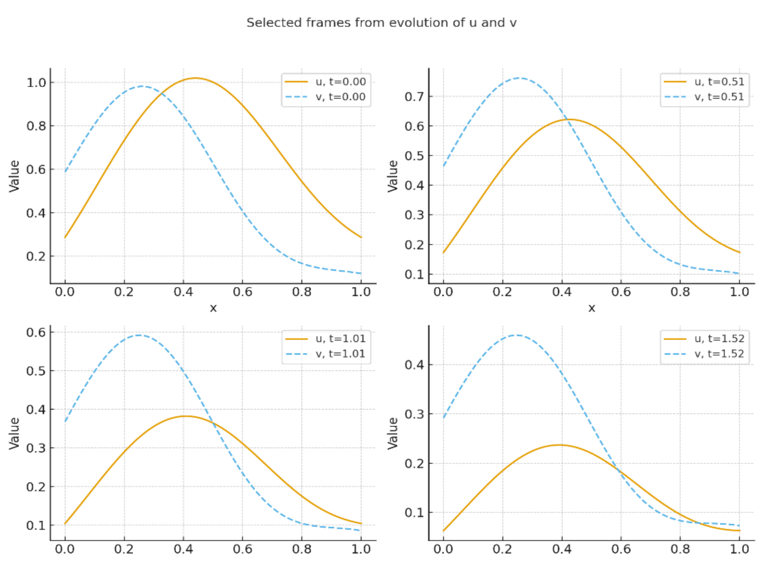

Figure 4: Selected frames from the 3D animation of the coupled system, showing nonlinear diffusion and coupling dynamics.

Conclusion. The analysis carried out for problem (1)-(2) has established sufficient conditions ensuring the global-in-time existence and large-time asymptotics of weak solutions. It has been shown that, under smallness assumptions on the initial data, the long-time dynamics of the coupled degenerate parabolic system are governed by radially symmetric self-similar solutions of the corresponding uncoupled limit problem.

The obtained results demonstrate that diffusion effects dominate the asymptotic regime, while the nonlinear coupling terms act only as perturbations. The scaling exponents characterizing the decay and spreading rates were identified explicitly, and the associated self-similar profiles were proven to exist, to be positive and monotone, and to decay algebraically at infinity.

These findings provide a rigorous theoretical framework for understanding the asymptotic structure of nonlinear weighted parabolic systems of type (1)-(2). Furthermore, the results offer a foundation for subsequent numerical simulations and potential applications to models of heterogeneous media and anomalous diffusion phenomena.

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REZYUME

Ushbu maqolada biz ikkita o‘zaro kuchli bog‘langan ta‘sirlashuvchi maydonlarning bog‘langan dinamikasini tasvirlovchi og‘irlikli chiziqsiz parabolik tenglamalar sistemalar sinfini ko‘rib chiqamiz. Energiya baholari va integral tengsizliklardan foydalanib, tegishli og‘irlikli Sobolev fazolarida kuchsiz yechimlarning global mavjudligi uchun yetarli shartlar aniqlanadi. Shuningdek, yechimlarni samarali yaqinlashtirish maqsadida Peaceman-Rachford sonli hisoblash usuli va Tomas algoritmi(haydash usuli) asosida sonli sxema ishlab chiqiladi. Taklif etilgan hisoblash doirasi ikki va uch o‘chovli sonli simulyatsiyalar yordamida, dinamik sirt grafigi va animatsiyalangan profillar bilan amalga oshiriladi va namoyish etiladi. Natijalar global yechimning sifat jihatlarini ko‘rsatib beradi, analitik natijalarni tasdiqlaydi hamda chiziqsiz diffuziya, og‘irlikli bir jinsli bo‘lmagan va komponentlararo bog‘lanish o‘rtasidagi o‘zaro ta‘sir haqida qo‘shimcha tushuncha beradi.

Kalit so‘zlar: Nochiziqli parabolik tenglamalar sistemasi, o‘zgaruvchan koeffitsiyentlar, global mavjudlik, chekli ayirmalar usuli, bir jinsli bo‘lmagan muhit, sonli modellashtirish.

РЕЗЮМЕ

В данной работе рассматривается класс нелинейных взвешенных параболических систем, описывающих связанную динамику двух взаимодействующих скалярных полей. Установлены достаточные условия глобального существования слабых решений в соответствующих взвешенных пространствах Соболева с использованием энергетических оценок и интегральных неравенств. Кроме того, разработана численная схема, основанная на методе расщепления Писмана-Рэчфорда в сочетании с алгоритмом Томаса, позволяющая эффективно приближать решения. Предложенная вычислительная методика реализована и проиллюстрирована на двух- и трёхмерных численных экспериментах, включая динамические поверхностные графики и анимированные профили. Полученные результаты демонстрируют качественные свойства глобального решения, подтверждают аналитические выводы и дают дополнительное представление о взаимодействии нелинейной диффузии, весовой неоднородности и межкомпонентной связи.

Ключевые слова: нелинейная параболическая система, переменные коэффициенты, глобальное существование, метод конечных разностей, неоднородная среда, численное моделирование.