UDC 512.554.1

ABELIAN EXTENSIONS OF SOLVABLE LEIBNIZ ALGEBRA WITH NATURALLY GRADED FILIFORM NILRADICAL OF MAXIMAL CODIMENSION

Sheraliyeva S. A.
V.I. Romanovskiy Institute of Mathematics Uzbekistan Academy of Sciences.Tashkent, Uzbekistan
abdiqodirovna@mail.ru

RESUME

Using the method of central extensions, we can construct only those Leibniz algebras with nontrivial centers. Therefore, to identify Leibniz algebras with trivial centers, we apply the method of abelian extensions. In this paper, we provide a classification of one-dimensional abelian extensions of solvable Leibniz algebras whose nilradical is a naturally graded filiform Leibniz algebra of maximal codimension. We give explicit descriptions of these extensions and determine their structures up to isomorphism.

Key words: Leibniz algebra, solvability, nilpotency, filiform Leibniz algebras, abelian extension.

1. Introduction

Leibniz algebras were introduced and systematically studied by J.-L. Loday in the early 1990s [8]. However, similar algebraic structures had previously been considered by Bloh under the name D-algebras [2]. While investigating the homology theory of Lie algebras, Loday observed that the antisymmetry condition of the Lie bracket was not essential for establishing the derivation property on chain complexes. This insight led him to define what is now known as a Leibniz algebra.

A right (or, equivalently, left) Leibniz algebra is a nonassociative algebra in which the right (respectively, left) multiplication operator satisfies the derivation rule. In this way, Leibniz algebras naturally generalize Lie algebras by relaxing the antisymmetry condition, while preserving key structural properties such as the Leibniz identity.

The extension method is widely regarded as one of the most effective approaches for classifying algebras. The study of abelian extensions in the context of Leibniz algebras was initially introduced in [3]. Central extensions of Leibniz algebras were examined in [10], where a complete classification of central extensions for null-filiform Leibniz algebras was provided. Additionally, the notion of non-abelian extensions was introduced in [9]. Furthermore, based on the techniques developed in [9], a method for determining abelian extensions of solvable Leibniz algebras has been presented in [6]. It is important to note that the extensions of solvable Leibniz algebras and superalgebras have been examined in [4] and [11], with a focus on the central extensions of solvable algebras in these works.

In this work, we focus on solvable Leibniz algebras whose nilradical is a naturally graded filiform Leibniz algebra of maximal codimension. Specifically, we investigate their one-dimensional abelian extensions and describe their algebraic structures. Our aim is to provide a complete characterization of these extensions.

2. Preliminaries

In this section, we recall several fundamental definitions and preliminary results that will be used in the subsequent sections of the paper.

Definition 1. *A vector space with a bilinear multiplication $(L, [\cdot,\cdot])$ is called a Leibniz algebra if for any $x, y, z \in L$ the so-called Leibniz identity*

$$[[x, y], z] = [[x, z], y] + [x, [y, z]],$$

holds.

For a given Leibniz algebra $(L,[\cdot,\cdot])$, the sequences of two-sided ideals are defined recursively as follows:

$$L^1=L,\ L^{k+1}=[L^k,L],\ k\geq 1, \qquad L^{[1]}=L,\ L^{[s+1]}=[L^{[s]},L^{[s]}],\ s\geq 1.$$

Definition 2. *A Leibniz algebra L is said to be nilpotent (respectively, solvable), if there exists $n\in\mathbb{N}$ ($m\in\mathbb{N}$) such that $L^n=\{0\}$ ($L^{[m]}=\{0\}$).*

The maximal nilpotent ideal of a Leibniz algebra is said to be the nilradical of the algebra.

Definition 3. *A Leibniz algebra L is called filiform if $\dim L^i=n-i$, for $2\leq i\leq n$, where $n=\dim L$.*

Definition 4. *Given a nilpotent Leibniz algebra L with nilindex s, put $L_i=L^i/L^{i+1},\ 1\leq i\leq s-1$, and $Gr(L)=L_1\oplus L_2\oplus\cdots\oplus L_{s-1}$. Define the product in the vector space $Gr(L)$ as follows:*

$$[x+L^{i+1},y+L^{j+1}]=[x,y]+L^{i+j+1},$$

where $x\in L^i/L^{i+1}, y\in L^j/L^{j+1}$. Then $[L_i,L_j]\subseteq L_{i+j}$ and we obtain the graded algebra $Gr(L)$. If $Gr(L)$ and L are isomorphic, then we say that the algebra L is naturally graded.

Now, we present a method for constructing abelian extensions of solvable Leibniz algebras.

Definition 5. *A representation of a Leibniz algebra $(L,[\cdot,\cdot])$ is a triple (V,l,r), where V is a vector space equipped with two linear maps $l:L\to gl(V)$ and $r:L\to gl(V)$, such that the following equalities hold:*

$$r_{[x,y]}=r_y\circ r_x-r_x\circ r_y,\quad l_{[x,y]}=r_y\circ l_x-l_x\circ r_y,\quad l_x\circ l_y=-l_x\circ r_y,\quad \forall x,y\in L. \tag{1}$$

Here $[\cdot,\cdot]$ is the commutator Lie bracket on $gl(V)$, the vector space of linear transformations on V.

Let L be a Leibniz algebra and let V be an abelian (i.e., trivial bracket) Leibniz algebra. Suppose, $l:L\to gl(V)$, $r:L\to gl(V)$ are linear maps defining the left and right actions of L on V, respectively. Let $\omega:L\otimes L\to V$ be a bilinear map satisfying

$$\omega([x,y],z)-\omega(x,[y,z])-\omega([x,z],y)-l_x\omega(y,z)-r_y\omega(x,z)+r_z\omega(x,y)=0. \tag{2}$$

We refer to bilinear maps ω that satisfy the compatibility condition described above as 2-cocycles on L with respect to the pair (l,r). The set of all such 2-cocycles is denoted by $Z^2(L,l,r)$.

The 2-coboundary on L with respect to the pair (l,r) is defined as

$$df(x,y)=f([x,y])-l_{\varphi(x)}f(y)-r_{\varphi(y)}f(x),\quad x,y\in L,$$

where $\varphi\in\mathrm{Aut}(L)$ and $f\in\mathrm{Hom}(L,V)$. The set of such bilinear maps is denoted by $B^2(L,l,r)$.

For $\omega\in Z^2(L,l,r)$, we define on the vector space $\widehat{L}=L\oplus V$ the bilinear product $[-,-]_{(l,r,\omega)}$ by

$$[x+a,y+b]_{(l,r,\omega)}=[x,y]_L+\omega(x,y)+l_xb+r_ya,\quad x,y\in L,\quad a,b\in V. \tag{3}$$

The algebra $L_\omega=(\widehat{L},[-,-]_{(l,r,\omega)})$ is called an abelian extension of L by V. One can easily check that L_ω is a Leibniz algebra if and only if $\omega\in Z^2(L,l,r)$.

We now establish the conditions under which two abelian extensions of a given algebra are isomorphic. Given two extensions $\widehat{L}_1=L(\omega^1,l^1,r^1)$ and $\widehat{L}_2=L(\omega^2,l^2,r^2)$.

Proposition 1.[6] *Two Leibniz algebras $\widehat{L}_1=L(\omega^1,l^1,r^1)$ and $\widehat{L}_2=L(\omega^2,l^2,r^2)$ are isomorphic if and only if there exists $\varphi\in\mathrm{Aut}(L)$, $\psi\in\mathrm{Aut}(V)$ and $f\in\mathrm{Hom}(L,V)$, such that*

$$\omega^2(\varphi(x),\varphi(y))+l^2_{\varphi(x)}f(y)+r^2_{\varphi(y)}f(x)-f([x,y])=\psi\big(\omega^1(x,y)\big),$$

$$l^2_{\varphi(x)}\psi(a)=\psi\big(l^1_xa\big),\quad r^2_{\varphi(y)}\psi(a)=\psi\big(r^1_ya\big),$$

for any $x,y\in L$, $a\in V$.

Given a Leibniz algebra L, an abelian module V and associated left and right actions (l,r) the group $\mathrm{Aut}\,L\times\mathrm{Aut}\,V$ acts on the space of 2-cocycles (L,l,r) by:

$$\omega'(x,y)=\psi\big(\omega(\varphi(x),\varphi(y))\big),\quad l'_x(a)=\psi\big(l_{\varphi(x)}\psi^{-1}(a)\big),\quad r'_x(a)=\psi\big(r_{\varphi(x)}\psi^{-1}(a)\big).$$

In this action, we say that ψ is an intertwining operator for l' and $l\circ\varphi$ (respectively, r' and $r\circ\varphi$). This action preserves cohomology classes up to coboundaries: Two extensions $\widehat{L}_1 = L(\omega^1, l^1, r^1)$ and $\widehat{L}_2 = L(\omega^2, l^2, r^2)$ are isomorphic if and only if:

$$\omega^2 - \psi\circ\omega^1\circ\varphi \in B^2(L, l^2, r^2),$$

and ψ intertwines l^2 with $l^1\circ\varphi$, and r^2 with $r^1\circ\varphi$. Thus, we obtain the following proposition.

Proposition 2.[6] *Let $\widehat{L}_1 = L(\omega^1, l^1, r^1)$ and $\widehat{L}_2 = L(\omega^2, l^2, r^2)$ be extensions of the solvable Leibniz algebra L by the abelian algebra V. Then the Leibniz algebras $\widehat{L}_1$ and $\widehat{L}_2$ are isomorphic if and only if ω^1 and ω^2 are in the same $\operatorname{Aut} L\times\operatorname{Aut} V$ orbit in $\bigcup\limits_{l,r} H^2(L,l,r)$.*

3. Main result

In this section, we construct an abelian extension of a solvable Leibniz algebra whose nilradical is naturally graded filiform algebra and possesses the maximal possible codimension.

It is known that, any complex n-dimensional naturally graded filiform non-Lie Leibniz algebra is isomorphic to one of the following non-isomorphic algebras [1]:

$$F_n^1 : [e_i, e_1] = e_{i+1}, \quad 2\le i\le n-1, \quad F_n^2 : \begin{cases} [e_1, e_1] = e_3, \\ [e_i, e_1] = e_{i+1}, \quad 3\le i\le n-1. \end{cases}$$

The complete classification of solvable Leibniz algebras whose nilradical is isomorphic to F_n^1 or F_n^2 has been given in [5]. We consider the solvable Leibniz algebra whose nilradical is F_n^1 and whose codimension is maximal. Up to isomorphism, there exists a unique such solvable Leibniz algebra, with the multiplication defined as follows:

$$L(F_n^1) : \begin{cases} [e_1, x] = e_1, & [x, e_1] = -e_1, \quad [e_i, e_1] = e_{i+1}, \quad 2\le i\le n-1, \\ [e_i, x] = (i-1)e_i, & [e_i, y] = e_i, \quad 2\le i\le n. \end{cases}$$

Note that any automorphism of the algebra $L(F_n^1)$ has the following form:

$$\begin{cases} \varphi(e_1) = ae_1, \quad \varphi(x) = be_1 + x, & \varphi(y) = y, \\ \varphi(e_j) = a^{j-2}ce_j + a^{j-2}\sum\limits_{i=j+1}^{n} \frac{(-1)^{i+j-1}cb^{i-j}}{(i-j)!}e_i, & 2\le j\le n, \end{cases}$$

where $a, c\in\mathbb{C}^*$, $b\in\mathbb{C}$.

Now we describe all 2-coboundaries on $L(F_n^1)$ with respect to the pair (l, r), by the one-dimensional abelian algebra $V = \langle e_{n+1}\rangle$.

For the basis elements $\{e_1, e_2, \dots, e_n, x, y\}$ and $f\in\operatorname{Hom}(L(F_n^1), V)$, we put

$$f(e_i) = c_ie_{n+1}, \quad 1\le i\le n, \quad f(x) = c_{n+1}e_{n+1}, \quad f(y) = c_{n+2}e_{n+1}.$$

We consider linear maps $l, r : g\to\operatorname{End}(V)$, such that

$$\begin{aligned} l_x(e_{n+1}) = \alpha_1 e_{n+1}, \quad r_x(e_{n+1}) = \alpha_2 e_{n+1}, \\ l_y(e_{n+1}) = \beta_1 e_{n+1}, \quad r_y(e_{n+1}) = \beta_2 e_{n+1}, \end{aligned} \tag{4}$$

for some scalars $\alpha_1, \alpha_2, \beta_1, \beta_2\in\mathbb{C}$.

For any automorphism $\varphi\in\operatorname{Aut}(L(F_n^1))$, consider the map

$$df(x, y) = f\big([x, y]\big) - l_{\varphi(x)}f(y) - r_{\varphi(y)}f(x),$$

where f is a linear map. We obtain the following result.

Proposition 3. *Any 2-coboundary with respect to the pair (l, r) for the algebras $L(F_n^1)$ has the following forms:*

$$B^2:\begin{cases} df(e_i,e_1)=c_{i+1}, & 2\le i\le n-1, & df(e_1,x)=(1-\alpha_2)c_1,\\ df(e_1,y)=-\beta_2c_1, & df(x,e_1)=-(1+\alpha_1)c_1, & df(y,e_1)=-\beta_1c_1,\\ df(e_i,x)=((i-1)-\alpha_2)c_i, & df(e_i,y)=(1-\beta_2)c_i, & 2\le i\le n,\\ df(x,e_i)=-\alpha_1c_i, & df(y,e_i)=-\beta_1c_i, & 2\le i\le n,\\ df(x,x)=-(\alpha_1+\alpha_2)c_{n+1}, & df(x,y)=-\alpha_1c_{n+2}-\beta_2c_{n+1}, & \\ df(y,x)=-\beta_1c_{n+1}-\alpha_2c_{n+2}, & df(y,y)=-(\beta_1+\beta_2)c_{n+2}. & \end{cases}$$

Now, using the algorithm for constructing solvable Leibniz algebras, we derive all possible extensions of the solvable Leibniz algebra $L(F_n^1)$ by the one-dimensional abelian algebra $V=\langle e_{n+1}\rangle$. The condition (1), to the linear maps $l, r: L(F_n^1)\to \mathrm{End}(V)$, which defined as (4), leads to the following system of equations:

$$\begin{aligned} &\alpha_1(\alpha_1+\alpha_2)=0, \quad \alpha_1(\beta_1+\beta_2)=0,\\ &\beta_1(\alpha_1+\alpha_2)=0, \quad \beta_1(\beta_1+\beta_2)=0. \end{aligned} \tag{5}$$

In the following proposition, we describe all 2-cocycles on $L(F_n^1)$ with respect to the pair (l, r) taking values in the one-dimensional abelian algebra $V=\langle e_{n+1}\rangle$.

Proposition 4. *A basis of $Z^2(L(F_n^1), l, r)$ consists of the following cocycles:*

1. $\alpha_1=\beta_1=\beta_2=0,\ \alpha_2=1:$

$$\begin{array}{lll} \omega(e_i,e_1)=b_{i,1}, & 2\le i\le n-1, & \omega(x,e_1)=b_{n+1,1},\\ \omega(e_i,x)=(i-2)b_{i-1,1}, & 3\le i\le n, & \omega(e_2,y)=b_{2,n+2},\\ \omega(e_i,y)=b_{i-1,1}, & 3\le i\le n, & \omega(y,e_1)=b_{n+2,1},\\ \omega(x,x)=b_{n+1,n+1}, & \omega(y,x)=b_{n+2,n+1}. & \end{array} \tag{a}$$

2. $\alpha_1=\beta_1=\beta_2=0,\ \alpha_2=2:$

$$\begin{array}{lll} \omega(e_1,e_1)=b_{1,1}, & \omega(e_1,x)=b_{1,n+1}, & \omega(x,e_1)=b_{1,n+1},\\ \omega(e_i,e_1)=b_{i,1}, & 2\le i\le n-1, & \omega(e_2,x)=b_{2,n+1},\\ \omega(e_i,x)=(i-3)b_{i-1,1}, & 3\le i\le n, & \omega(e_2,y)=-b_{2,n+1},\\ \omega(e_3,y)=b_{3,n+2}, & \omega(e_i,y)=b_{i-1,1}, & 4\le i\le n,\\ \omega(x,x)=b_{n+1,n+1}, & \omega(y,x)=b_{n+2,n+1}. & \end{array} \tag{b}$$

3. $\alpha_1=\beta_1=0,\ \alpha_2=1,\ \beta_2=1:$

$$\begin{array}{lll} \omega(e_1,y)=b_{1,n+2}, & \omega(e_i,e_1)=b_{i,1}, & 2\le i\le n-1,\\ \omega(e_2,x)=b_{2,n+1}, & \omega(e_i,x)=(i-2)b_{i-1,1}, & 3\le i\le n,\\ \omega(e_2,y)=b_{2,n+2}, & \omega(x,e_1)=b_{1,n+2}, & \omega(x,y)=b_{n+1,n+1},\\ \omega(x,x)=b_{n+1,n+1}, & \omega(y,x)=b_{n+2,n+1}, & \omega(y,y)=b_{n+2,n+1}. \end{array} \tag{c}$$

4. $\alpha_1=\beta_1=0,\ \alpha_2=n,\ \beta_2=1:$

$$\begin{array}{lll} \omega(e_1,x)=(n-1)b_{1,n+1}, & \omega(e_1,y)=b_{1,n+1}, & \omega(x,e_1)=b_{1,n+1},\\ \omega(e_i,e_1)=b_{i,1}, & 2\le i\le n, & \omega(e_2,x)=b_{2,n+1},\\ \omega(e_i,x)=(i-n-1)b_{i-1,1}, & 3\le i\le n, & \omega(x,y)=b_{n+1,n+2},\\ \omega(x,x)=nb_{n+1,n+2}, & \omega(y,y)=b_{n+2,n+2}, & \omega(y,x)=nb_{n+2,n+2}, \end{array} \tag{d}$$

5. $\alpha_1 = \beta_1 = 0,\ (\alpha_2, \beta_2) \notin \{(1,0),(2,0),(1,1),(n,1)\}:$

$$\begin{array}{lll}
\omega(e_1,x) = (\alpha_2-1)b_{1,n+1}, & \omega(e_1,y) = \beta_2 b_{1,n+1}, & \omega(e_2,x) = (\alpha_2-1)b_{2,n+1},\\
\omega(e_i,e_1) = b_{i,1}, & 2 \le i \le n-1, & \omega(e_2,y) = (\beta_2-1)b_{2,n+1},\\
\omega(e_i,x) = (i-1-\alpha_2)b_{i-1,1}, & \omega(e_i,y) = (1-\beta_2)b_{i-1,1}, & 3 \le i \le n,\\
\omega(x,e_1) = b_{1,n+1}, & \omega(x,x) = \alpha_2 b_{n+1,n+1}, & \omega(x,y) = \beta_2 b_{n+1,n+1},\\
\omega(y,x) = \alpha_2 b_{n+2,n+1}, & \omega(y,y) = \beta_2 b_{n+2,n+1}. &
\end{array} \quad (e)$$

6. $\alpha_1 = -\alpha_2 = -1,\ \beta_1 = -\beta_2 = 0:$

$$\begin{array}{lll}
\omega(e_1,x) = b_{1,n+1}, & \omega(e_1,y) = b_{1,n+2}, & \omega(e_2,y) = b_{2,n+2},\\
\omega(e_i,e_1) = b_{i,1}, & 2 \le i \le n-1, & \omega(x,e_1) = -b_{1,n+1},\\
\omega(e_i,x) = (i-2)b_{i-1,1}, & \omega(x,e_i) = b_{i-1,1}, & 3 \le i \le n,\\
\omega(e_i,y) = b_{i-1,1}, & 3 \le i \le n, & \omega(x,e_2) = b_{2,n+2},\\
\omega(y,e_1) = -b_{1,n+2}, & \omega(x,y) = b_{n+1,n+2}, & \omega(y,x) = -b_{n+1,n+2}.
\end{array} \quad (f)$$

7. $\alpha_1 = -\alpha_2,\ \beta_1 = -\beta_2,\ (\alpha_1, \beta_1) \neq (-1,0):$

$$\begin{array}{ll}
\omega(e_1,x) = (\alpha_1+1)b_{1,n+1}, & \omega(e_1,y) = \beta_1 b_{1,n+1},\\
\omega(e_2,x) = (\alpha_1+1)b_{2,n+1}, & \omega(e_2,y) = (\beta_1+1)b_{2,n+1},\\
\omega(e_i,e_1) = b_{i,1}, & 2 \le i \le n-1,\\
\omega(e_i,x) = (\alpha_1+i-1)b_{i-1,1}, & 3 \le i \le n,\\
\omega(e_i,y) = (\beta_1+1)b_{i-1,1}, & 3 \le i \le n,\\
\omega(x,e_1) = -(\alpha_1+1)b_{1,n+1}, & \omega(x,e_2) = -\alpha_1 b_{2,n+1},\\
\omega(x,e_i) = -\alpha_1 b_{i-1,1}, & 3 \le i \le n,\\
\omega(y,e_1) = -\beta_1 b_{1,n+1}, & \omega(y,e_2) = -\beta_1 b_{2,n+1},\\
\omega(y,e_i) = -\beta_1 b_{i-1,1}, & 3 \le i \le n,\\
\omega(x,y) = b_{n+1,n+2}, & \omega(y,x) = -b_{n+1,n+2}.
\end{array} \quad (h)$$

Proof. For any $\omega \in Z^2(L(F_n^1), l, r)$, let us denote $\omega(a,c) = b_{i,j}e_{n+1}$, $1 \le i,j \le n+2$, where $a,c \in \{e_1, e_2, \dots, e_n, x, y\}$. Using equality (2), we compute the cocycles and obtain the following relations:

$$\begin{array}{lll}
\beta_1 b_{1,1} = 0, & \beta_2 b_{1,1} = 0, & \alpha_1 b_{1,1} = 0,\\
(2-\alpha_2)b_{1,1} = 0, & b_{1,i} = 0, & 2 \le i \le n,\\
b_{i,j} = 0, & 2 \le j \le i \le n, & \beta_1 b_{n,1} = 0,\\
(1-\beta_2)b_{n,1} = 0, & \alpha_1 b_{n,1} = 0, & (n-\alpha_2)b_{n,1} = 0,\\
(\alpha_1+1)b_{1,n+1} = (\alpha_2-1)b_{n+1,1}, & b_{1,n+1} = (\alpha_1+\alpha_2-1)b_{n+1,1}, &\\
\beta_2 b_{1,n+1} = (\alpha_2-1)b_{1,n+2}, & \beta_1 b_{1,n+1} = (\alpha_2-1)b_{n+2,1}, &\\
\beta_1 b_{1,n+2} = \beta_2 b_{n+2,1}, & \beta_2 b_{n+1,1} = (1+\alpha_1)b_{1,n+2}, &\\
b_{1,n+2} = \alpha_1 b_{n+2,1} + \beta_2 b_{n+1,1}, & (1-\alpha_2)b_{n+2,1} = \beta_1 b_{n+1,1}, & (\beta_1+\beta_2)b_{n+2,1} = 0,\\
b_{i+1,n+1} = (i-\alpha_2)b_{i,1}, & b_{n+1,i+1} = -\alpha_1 b_{i,1}, & 2 \le i \le n-1,\\
\alpha_1 b_{i,n+1} = (\alpha_2+1-i)b_{n+1,i}, & (\alpha_1+\alpha_2)b_{n+1,i} = 0, & 2 \le i \le n,\\
b_{n+2,i+1} = -\beta_1 b_{i,1}, & 2 \le i \le n-1, &\\
(\beta_2-1)b_{i,n+1} = (\alpha_2+1-i)b_{i,n+2}, & \alpha_1 b_{i,n+2} = (\beta_2-1)b_{n+1,i}, & 2 \le i \le n,\\
\beta_1 b_{n+1,n+1} = 0, & \alpha_1 b_{n+1,n+1} = 0, & \beta_1 b_{n+2,n+2} = 0,\\
\alpha_1 b_{n+2,n+2} = 0, & \alpha_2 b_{n+2,i} = -\beta_1 b_{n+1,i}, & 2 \le i \le n,\\
\beta_1 b_{i,n+1} = (\alpha_2-i+1)b_{n+2,i}, & \alpha_1 b_{n+2,i} = -\beta_2 b_{n+1,i}, & 2 \le i \le n,\\
(\beta_2-1)b_{n+2,i} = \beta_1 b_{i,n+2}, & (\beta_1+\beta_2)b_{n+2,i} = 0, & 2 \le i \le n,
\end{array}$$

$$(\alpha_1+\alpha_2)b_{n+1,n+2}=\beta_2 b_{n+1,n+1}, \qquad (\beta_1+\beta_2)b_{n+2,n+1}=\alpha_2 b_{n+2,n+2},$$
$$\alpha_1 b_{n+2,n+1}-\alpha_2 b_{n+1,n+2}+\beta_2 b_{n+1,n+1}=0, \quad \alpha_2 b_{n+2,n+2}+\beta_1 b_{n+1,n+2}-\beta_2 b_{n+2,n+1}=0.$$

By equation (5), we deduce that either $\alpha_1=\beta_1=0$, or $\alpha_1=-\alpha_2$ and $\beta_1=-\beta_2$. Accordingly, to determine the values of the coefficients $b_{i,j}$, we consider the following two cases:

Case 1. Let $\alpha_1=\beta_1=0$. Then we have

$$\begin{array}{lll} b_{1,i}=0, & 2\le i\le n, & \\ b_{i,j}=0, & 2\le j\le i\le n, & b_{1,n+1}=(\alpha_2-1)b_{n+1,1}, \\ b_{1,n+2}=\beta_2 b_{n+1,1}, & b_{i+1,n+1}=(i-\alpha_2)b_{i,1}, & 2\le i\le n-1, \\ b_{n+1,i}=0, & b_{n+2,i}=0, & 2\le i\le n, \end{array}$$

and the following restrictions

$$\begin{array}{ll} \beta_2 b_{1,1}=0, & (2-\alpha_2)b_{1,1}=0, \\ (1-\beta_2)b_{n,1}=0, & (n-\alpha_2)b_{n,1}=0, \\ \beta_2 b_{n+2,1}=0, & (\alpha_2-1)b_{n+2,1}=0, \\ (\beta_2-1)b_{i,n+1}=(\alpha_2+1-i)b_{i,n+2}, & 2\le i\le n, \\ \alpha_2 b_{n+1,n+2}=\beta_2 b_{n+1,n+1}, & \beta_2 b_{n+2,n+1}=\alpha_2 b_{n+2,n+2}, \end{array}$$

Now, consider the following subcases:

1.1. If $\alpha_2=1$, $\beta_2=0$, then we obtain $b_{i,n+1}=(i-2)b_{i,n+2}$, $2\le i\le n$, and $b_{1,1}=b_{n,1}=b_{n+1,n+2}=b_{n+2,n+2}=0$. Hence, we conclude that any 2-cocycles on $L(F_n^1)$ with respect to the pair (l,r) is given by the expression in (a).

1.2. If $\alpha_2=2$, $\beta_2=0$, then we obtain $b_{i,n+1}=(i-3)b_{i,n+2}$, $2\le i\le n$, and $b_{1,n+2}=b_{n,1}=b_{n+2,1}=b_{n+1,n+2}=b_{n+2,n+2}=0$. Hence, we get that any 2-cocycles on $L(F_n^1)$ with respect to the pair (l,r) has the form (b).

1.3. If $\alpha_2=1$, $\beta_2=1$, then we obtain $b_{n+1,n+2}=b_{n+1,n+1}$, $b_{n+2,n+1}=b_{n+2,n+2}$, and $b_{1,1}=b_{1,n+1}=b_{n,1}=b_{n+2,1}=b_{i,n+2}=0$, $3\le i\le n$. Hence, we get that any 2-cocycles on $L(F_n^1)$ with respect to the pair (l,r) has the form (c).

1.4. If $\alpha_2=n$, $\beta_2=1$, then we obtain $b_{1,n+1}=(n-1)b_{1,n+2}$, $nb_{n+1,n+2}=b_{n+1,n+1}$, $b_{n+2,n+1}=nb_{n+2,n+2}$, and $b_{1,1}=b_{n,1}=b_{n+2,1}=b_{i,n+2}=0$, $2\le i\le n$. Hence, we get that any 2-cocycles on $L(F_n^1)$ with respect to the pair (l,r) has the form (d).

1.5. If $(\alpha_2,\beta_2)\notin\{(1,0),(2,0),(1,1),(n,1)\}$ then we obtain

$$(\beta_2-1)b_{i,n+1}=(\alpha_2+1-i)b_{i,n+2}, \quad 2\le i\le n,$$

$$\alpha_2 b_{n+1,n+2}=\beta_2 b_{n+1,n+1}, \quad \beta_2 b_{n+2,n+1}=\alpha_2 b_{n+2,n+2},$$

and $b_{1,1}=b_{n,1}=b_{n+2,1}=0$. Hence, we get that any 2-cocycles on $L(F_n^1)$ with respect to the pair (l,r) has the form (e).

Case 2. Let $\alpha_1=-\alpha_2$ and $\beta_1=-\beta_2$. Then we have

$$\begin{array}{lll} b_{1,i}=0, & 1\le i\le n, & b_{n,1}=0, \\ b_{i,j}=0, & 2\le j\le i\le n, & b_{n+1,n+1}=0, \\ b_{n+2,n+2}=0, & b_{1,n+1}=-b_{n+1,1}, & b_{1,n+2}=\alpha_1 b_{n+2,1}-\beta_1 b_{n+1,1}, \\ b_{n+2,n+1}=-b_{n+1,n+2}, & b_{i,n+1}=(i-1+\alpha_1)b_{i-1,1}, & b_{n+1,i}=-\alpha_1 b_{i-1,1}, \\ b_{n+2,i}=-\beta_1 b_{i-1,1}, & 3\le i\le n, & \end{array}$$

and the following restrictions

$$\begin{array}{lll} \beta_1 b_{1,n+2} = -\beta_1 b_{n+2,1}, & \beta_1 b_{1,n+1} = (\alpha_1+1) b_{1,n+2}, & \beta_1 b_{1,n+1} = -(\alpha_1+1) b_{n+2,1}, \\ \alpha_1 b_{2,n+1} = -(\alpha_1+1) b_{n+1,2}, & \beta_1 b_{2,n+1} = -(\alpha_1+1) b_{n+2,2}, & \\ \alpha_1 b_{i,n+2} = -(\beta_1+1) b_{n+1,i}, & \beta_1 b_{i,n+2} = -(\beta_1+1) b_{n+2,i} & 2 \le i \le n, \end{array}$$

Now, consider the following subcases:

2.1. If $\alpha_1 = -1$, $\beta_1 = 0$, then we obtain $b_{i,n+2} = b_{n+1,i}$, $2 \le i \le n$, and $b_{2,n+1} = b_{n+2,i} = 0$, $2 \le i \le n$. Hence, we get that any 2-cocycles on $L(F_n^1)$ with respect to the pair (l, r) has the form (f).

2.2. If $(\alpha_1, \beta_1) \neq (-1, 0)$, then we obtain $b_{n+2,1} = -b_{1,n+2}$, and

$$\beta_1 b_{1,n+1} = (\alpha_1+1) b_{1,n+2}, \quad \alpha_1 b_{2,n+1} = -(\alpha_1+1) b_{n+1,2}, \quad \beta_1 b_{2,n+1} = -(\alpha_1+1) b_{n+2,2},$$

$$\alpha_1 b_{i,n+2} = -(\beta_1+1) b_{n+1,i}, \quad \beta_1 b_{i,n+2} = -(\beta_1+1) b_{n+2,i}, \quad 2 \le i \le n.$$

In this case, we conclude that any 2-cocycles on $L(F_n^1)$ for the pair (l, r) have the form (h). This concludes the proof of Proposition 4.

By Propositions 3 and 4, we obtain the following corollary:

Corollary. *We have the following:*

I. *If* $\alpha_1 = \beta_1 = \beta_2 = 0$, $\alpha_2 = 1$, *then* $\dim H^2(L(F_n^1), l, r) = 1$, $H^2(L(F_n^1), l, r) = \langle \omega(y, e_1) \rangle$;

II. *If* $\alpha_1 = \beta_1 = \beta_2 = 0$, $\alpha_2 = 2$, *then* $\dim H^2(L(F_n^1), l, r) = 2$, $H^2(L(F_n^1), l, r) = \langle \omega(e_1, e_1), \omega(e_3, y) \rangle$;

III. *If* $\alpha_1 = \beta_1 = 0$, $\alpha_2 = 1$, $\beta_2 = 1$, *then* $\dim H^2(L(F_n^1), l, r) = 2$, $H^2(L(F_n^1), l, r) = \langle \omega(e_2, x), \omega(e_2, y) \rangle$;

IV. *If* $\alpha_1 = \beta_1 = 0$, $\alpha_2 = n$, $\beta_2 = 1$, *then* $\dim H^2(L(F_n^1), l, r) = 1$, $H^2(L(F_n^1), l, r) = \langle \omega(e_n, e_1) \rangle$;

V. *If* $\alpha_1 = \beta_1 = 0$, $(\alpha_2, \beta_2) \notin \{(1,0), (2,0), (1,1), (n,1)\}$, *then* $\dim H^2(L(F_n^1), l, r) = 0$;

VI. *If* $\alpha_1 = -\alpha_2 = -1$, $\beta_1 = \beta_2 = 0$, *then* $\dim H^2(L(F_n^1), l, r) = 2$, $H^2(L(F_n^1), l, r) = \langle \omega(e_1, x) - \omega(x, e_1), \omega(e_1, y) - \omega(y, e_1) \rangle$;

VII. *If* $\alpha_1 = -\alpha_2$, $\beta_1 = -\beta_2$, $(\alpha_1, \beta_1) \neq (-1, 0)$, *then* $\dim H^2(L(F_n^1), l, r) = 0$.

Now, we can formulate the following result.

Theorem. *Let* $\widehat{L}$ *be an extension of the solvable Leibniz algebra* $L(F_n^1)$ *by the abelian algebra* $V = \langle e_{n+1} \rangle$. *Then* $\widehat{L}$ *is isomorphic to one of the following non-isomorphic algebras:*

$$\widehat{L}_1 : \begin{array}{ll} [e_i, e_1] = e_{i+1}, & 2 \le i \le n-1, \\ [e_i, y] = e_i, & 2 \le i \le n, \\ [e_i, x] = (i-1) e_i, & 2 \le i \le n, \\ [e_1, x] = e_1, & [x, e_1] = -e_1, \\ [y, e_1] = e_{n+1}, & [e_{n+1}, x] = e_{n+1}, \end{array} \qquad \widehat{L}_2 : \begin{array}{ll} [e_1, e_1] = e_{n+1}, & [e_1, x] = e_1, \\ [e_i, e_1] = e_{i+1}, & 2 \le i \le n-1, \\ [e_i, y] = e_i, & 2 \le i \le n, \\ [e_i, x] = (i-1) e_i, & 2 \le i \le n, \\ [x, e_1] = -e_1, & [e_{n+1}, x] = 2e_{n+1}, \end{array}$$

$$\widehat{L}_3 : \begin{array}{ll} [e_1, e_1] = e_{n+1}, & [e_1, x] = e_1, \\ [e_i, e_1] = e_{i+1}, & 2 \le i \le n-1, \\ [e_2, y] = e_2, & [e_3, y] = e_3 + e_{n+1}, \\ [e_i, y] = e_i, & 4 \le i \le n, \\ [e_i, x] = (i-1) e_i, & 2 \le i \le n, \\ [x, e_1] = -e_1, & [e_{n+1}, x] = 2e_{n+1}, \end{array} \qquad \widehat{L}_4 : \begin{array}{ll} [e_i, e_1] = e_{i+1}, & 2 \le i \le n-1, \\ [e_2, y] = e_2, & [e_2, x] = e_2 + e_{n+1}, \\ [e_i, y] = e_i, & 3 \le i \le n, \\ [e_i, x] = (i-1) e_i, & 3 \le i \le n, \\ [e_1, x] = e_1, & [x, e_1] = -e_1, \\ [e_{n+1}, x] = e_{n+1}, & [e_{n+1}, y] = e_{n+1}, \end{array}$$

$$\widehat{L}_5(\delta): \begin{array}{ll} [e_i,e_1]=e_{i+1}, & 2\le i\le n-1,\\ [e_2,y]=e_2+e_{n+1}, & [e_2,x]=e_2+\delta e_{n+1},\\ [e_i,y]=e_i, & 3\le i\le n,\\ [e_i,x]=(i-1)e_i, & 3\le i\le n,\\ [e_1,x]=e_1, & [x,e_1]=-e_1,\\ [e_{n+1},x]=e_{n+1}, & [e_{n+1},y]=e_{n+1}, \end{array} \qquad \widehat{L}_6: \begin{array}{ll} [e_i,e_1]=e_{i+1}, & 2\le i\le n,\\ [e_i,y]=e_i, & 2\le i\le n,\\ [e_i,x]=(i-1)e_i, & 2\le i\le n,\\ [e_1,x]=e_1, & [x,e_1]=-e_1,\\ [e_{n+1},x]=ne_{n+1}, & [e_{n+1},y]=e_{n+1}, \end{array}$$

$$\widehat{L}_7: \begin{array}{lll} [e_1,x]=e_1+e_{n+1}, & [e_i,e_1]=e_{i+1}, & 2\le i\le n-1,\\ [e_i,x]=(i-1)e_i, & [e_i,y]=e_i, & 2\le i\le n,\\ [e_{n+1},x]=e_{n+1}, & [x,e_1]=-e_1-e_{n+1}, & [x,e_{n+1}]=-e_{n+1}, \end{array}$$

$$\widehat{L}_8(\delta): \begin{array}{llll} [e_1,x]=e_1+\delta e_{n+1}, & [e_1,y]=e_{n+1}, & [e_i,e_1]=e_{i+1}, & 2\le i\le n-1,\\ [e_{n+1},x]=e_{n+1}, & [e_i,x]=(i-1)e_i, & [e_i,y]=e_i, & 2\le i\le n,\\ [x,e_1]=-e_1-\delta e_{n+1}, & [x,e_{n+1}]=-e_{n+1}, & [y,e_1]=-e_{n+1}. & \end{array}$$

Proof. By Corollary, we have that $H^2(L(F_n^1),l,r)=0$ in cases V and VII. Thus, it is enough to consider the remaining cases.

I. $\alpha_1=\beta_1=\beta_2=0$, $\alpha_2=1$. In this case, the second cohomology group satisfies $\dim H^2(L(F_n^1),l,r)=1$ and 2-cocycle defined by $\omega(y,e_1)=e_{n+1}$ form a basis on this space. That is, $H^2(L(F_n^1),l,r)=\langle\overline{\omega}\rangle$.

Consider an automorphism $\varphi\in \mathrm{Aut}(L(F_n^1))$, and an automorphism $\psi\in\mathrm{Aut}(V)$ satisfying $\psi(e_{n+1})=\lambda e_{n+1}$. Then the action of these automorphisms on the cohomology class $\delta\overline{\omega}\in H^2(L(F_n^1),l,r)$ results in a new class $\delta'\overline{\omega}$, where

$$\delta'=\delta\lambda a,\quad l'=l,\quad r'=r.$$

By choosing $\lambda=1$ and $a=\frac{1}{\delta}$, we can normalize the representative so that $\delta'=1$.

Thus, the corresponding extended algebra $\widehat{L}_1=L(F_n^1)\oplus\{e_{n+1}\}$ has nontrivial products given by

$$[y,e_1]=\omega(y,e_1)=e_{n+1},\quad [e_{n+1},x]=r_xe_{n+1}=e_{n+1}.$$

Hence, we obtain the algebra $\widehat{L}_1$.

II. $\alpha_1=\beta_1=\beta_2=0$, $\alpha_2=2$. In this case, $\dim H^2(L(F_n^1),l,r)=2$, and the basis of this space is given by the 2-cocycles

$$\omega_1(e_1,e_1)=e_{n+1},\qquad \omega_2(e_3,y)=e_{n+1}.$$

Automorphisms $\varphi\in\mathrm{Aut}(L(F_n^1))$ and $\psi\in\mathrm{Aut}(V)$ act the element $\delta_1\overline{\omega_1}+\delta_2\overline{\omega_2}$ to $\delta_1'\overline{\omega_1}+\delta_2'\overline{\omega_2}$ as

$$\delta_1'=\delta_1\lambda a^2,\quad \delta_2'=\delta_2\lambda ac.$$

- If $\delta_2=0$, then $\delta_2'=0$ and $\delta_1\neq0$. Taking $\lambda=1$, $a=\sqrt{\frac{1}{\delta_1}}$, we can suppose $\delta_1'=1$ and obtain the algebra $\widehat{L}_2$.
- If $\delta_2\neq0$, then choosing $\lambda=1$, $c=\frac{\delta_1}{\delta_2}a$, we get $\delta_1'=\delta_2'$ and obtain the algebra $\widehat{L}_3$.

III. $\alpha_1=\beta_1=0$, $\alpha_2=1$, $\beta_2=1$. In this case, $\dim H^2(L(F_n^1),l,r)=2$, and the basis of this space is given by the 2-cocycles

$$\omega_1(e_2,x)=e_{n+1},\qquad \omega_2(e_2,y)=e_{n+1}.$$

Automorphisms $\varphi\in\mathrm{Aut}(L(F_n^1))$ and $\psi\in\mathrm{Aut}(V)$ act the element $\delta_1\overline{\omega_1}+\delta_2\overline{\omega_2}$ to $\delta_1'\overline{\omega_1}+\delta_2'\overline{\omega_2}$ as

$$\delta_1'=\delta_1\lambda c,\quad \delta_2'=\delta_2\lambda c.$$

- If $\delta_2=0$, then $\delta_2'=0$ and $\delta_1\neq0$. Taking $\lambda=1$, $c=\frac{1}{\delta_1}$, we can suppose $\delta_1'=1$ and obtain the algebra $\widehat{L}_4$.

- If $\delta_2 \neq 0$, then choosing $\lambda = 1$, $c = \frac{1}{\delta_2}$, we get $\delta_2' = 1$ and obtain the algebra $\widehat{L}_5(\delta)$.

IV. $\alpha_1 = \beta_1 = 0$, $\alpha_2 = n$, $\beta_2 = 1$. In this case $\dim H^2(L(F_n^1), l, r) = 1$ and $H^2(L(F_n^1), l, r) = \langle \overline{\omega} \rangle$, where $\omega(e_n, e_1) = e_{n+1}$. Automorphisms $\varphi \in \mathrm{Aut}(L(F_n^1))$ and $\psi \in \mathrm{Aut}(V)$ act the element $\delta\overline{\omega}$ to $\delta'\overline{\omega}$, as follows:

$$\delta' = \delta\lambda a^{n-1}c.$$

Taking $\lambda = \frac{1}{\delta}$ and $a = c = 1$, we can suppose $\delta' = 1$. Thus, the new products of the algebra $\widehat{L} = L(F_n^1) \oplus \{e_{n+1}\}$ are given by:

$$[e_n, e_1] = \omega(e_n, e_1) = e_{n+1}, \quad [e_{n+1}, x] = r_x e_{n+1} = n e_{n+1}, \quad [e_{n+1}, y] = r_y e_{n+1} = e_{n+1}.$$

Therefore, we obtain the algebra $\widehat{L}_6$.

V. $\alpha_1 = -\alpha_2 = -1$, $\beta_1 = \beta_2 = 0$. In this case $\dim H^2(L(F_n^1), l, r) = 2$, and the basis of this space is given by the 2-cocycles

$$\omega_1(e_1, x) = e_{n+1}, \quad \omega_1(x, e_1) = -e_{n+1}, \qquad \omega_2(e_1, y) = e_{n+1}, \quad \omega_2(y, e_1) = -e_{n+1}.$$

Automorphisms $\varphi \in \mathrm{Aut}(L(F_n^1))$ and $\psi \in \mathrm{Aut}(V)$ act the element $\delta_1\overline{\omega_1} + \delta_2\overline{\omega_2}$ to $\delta_1'\overline{\omega_1} + \delta_2'\overline{\omega_2}$ as

$$\delta_1' = \delta_1\lambda a, \quad \delta_2' = \delta_2\lambda a.$$

- If $\delta_2 = 0$, then $\delta_2' = 0$ and $\delta_1 \neq 0$. Taking $\lambda = 1$, $a = \frac{1}{\delta_1}$, we can suppose $\delta_1' = 1$ and obtain the algebra $\widehat{L}_7$.
- If $\delta_2 \neq 0$, then choosing $\lambda = 1$, $a = \frac{1}{\delta_2}$, we get $\delta_2' = 1$ and obtain the algebra $\widehat{L}_8(\delta)$.

This completes the proof.

REFERENCES

1. Ayupov Sh. A., Omirov B. A., Rakhimov I. S. Leibniz Algebras: Structure and Classification. Taylor and Francis Group Publisher, 2019, ISBN 0367354810, P. 333.
2. Bloh A. A generalization of the concept of a Lie algebra. Sov. Math. Dok., 1965, V. 165(3), P. 1450–1452.
3. Casas J. M., Faro E., Vieites A. M. Abelian extensions of Leibniz algebras. Communications in Algebra, 1999, V. 27(6), P. 2833–2846.
4. Camacho L. M., Navarro R. M., Omirov B. A. Central extensions of some solvable Leibniz superalgebras. Linear Algebra and its Applications, 2023, V. 656(1), P. 63–91.
5. Karimjonov I. A. Classification of Leibniz algebras with a given nilradical and with some corresponding Lie algebras. PhD thesis, Santiago de Compostella, 2017.
6. Khudoyberdiyev A. Kh., Sheraliyeva S. A. Abelian extensions of five-dimensional solvable Leibniz algebras. arXiv:2506.23528.
7. Ladra M., Masutova K. K., Omirov B. A. Solvable Leibniz algebra with non-Lie and non-split naturally graded filiform nilradical and its rigidity. Linear Algebra Appl, 2016, V. 507, P. 513–517.
8. Loday J.-L. Une version non commutative des algèbres de Lie: les algèbres de Leibniz. Enseign. Math., 1993, V. 39(3-4), P. 269–293.
9. Liu J., Sheng Y., Wang Q. On non-abelian extension of Leibniz algebras. Communications in Algebra, 2018, V. 46(2), P. 574–587.
10. Rakhimov I. S., Langari S. L., Langari M. B. On central extensions of null-filiform Leibniz algebras. J. Algebra, 2009, V. 3(6), P. 271–280.

11. Sheraliyeva S. Extension of some solvable Leibniz algebra. Bulletin of the Institute of Mathematics, 2023, V. 6(4), P. 78–89.

REZYUME

Markaziy kengaytma metodidan foydalanib, markazi noldan farqli bo'lgan algebralarni qurishimiz mumkin. Shuning uchun, markazi trivial bo'lgan Leybnits algebralarini aniqlash uchun Abel kengaytmalar metodidan foydalanamiz. Ushbu ishda nilradikali maksimal ko-o'lchamli tabiiy usulda gradiurlangan filiform Leybnits algebrasi bo'lgan yechiluvchi Leybnits algebralarining bir o'lchovli Abel kengaytmalarining klassifikatsiyasi keltirilgan. Bu kengaytmalar aniq ifodalangan va ularning strukturalari izomorfizm aniqligini tasniflangan.

Kalit so'zlar: Leybnits algebrasi, yechiluvchanlik, nilpotentlik, filiform Leybnits algebralari, Abel kengatma.

РЕЗЮМЕ

Методом центральных расширений можно построить только такие алгебры Лейбница, центр которых нетривиален. Поэтому для определения алгебр Лейбница с тривиальным центром мы используем метод абелевых расширений. В данной работе представлена классификация одномерных абелевых расширений разрешимых алгебр Лейбница, нильрадикал которых является естественно градуированной филлиформной алгеброй Лейбница максимальной коразмерности. Даются явные описания таких расширений и определяются их структуры с точностью до изоморфизма.

Ключевые слова: Алгебра Лейбница, разрешимость, нильпотентность, филиформные алгебры Лейбница, абелевое расширение.