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An abstract characterization of Schatten's ideal $\mathcal{C}_2$

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RESUME

In this paper we establish abstract characterizations of the sequence space l^2 and the Schatten ideal $\mathcal{C}^2$, and present our main results in this direction.

Key words: symmetric sequence spaces, Banach ideals of compact operators, Schatten ideals, duality between symmetric sequence spaces and Banach ideals of compact operators.

Introduction

The Hilbert space l^2 and the Hilbert–Schmidt class $\mathcal{C}_2$ of compact operators occupy a central position in functional analysis. Their distinguished role arises from the fact that they are the unique spaces among the classical sequence and operator ideals where the geometry is governed by the parallelogram law, and the norm admits a Hilbert space structure. It is therefore natural to ask whether one can recover l^2 and $\mathcal{C}_2$ from purely abstract axioms, without appealing directly to their standard constructions.

In this paper we address this question within the framework of symmetric sequence spaces and fully symmetric ideals of compact operators. Recall that a symmetric sequence space is a Banach space of real sequences closed under rearrangements, while a fully symmetric space additionally respects submajorization. The operator-theoretic counterpart is provided by fully symmetric ideals in $\mathcal{K}(\mathcal{H})$, which are in one-to-one correspondence with fully symmetric sequence spaces via the Calkin correspondence.

Our starting point is a simple but fundamental observation in l^2: if $x, y \in l^2$ have disjoint supports, then they are orthogonal with respect to the scalar product, and consequently

$$\|x+y\|_2^2 = \|x\|_2^2 + \|y\|_2^2.$$

This property suggests a natural characterization principle: one may single out l^2 among all symmetric sequence spaces precisely as the unique space where the "Pythagorean identity" holds for disjointly supported vectors.

The first main result of the paper establishes this principle rigorously. We prove that if $(E, \|\cdot\|_E) \subset c_0$ is a fully symmetric sequence space satisfying

$$\|x+y\|_E^2 = \|x\|_E^2 + \|y\|_E^2 \quad \text{whenever } x \cdot y = 0,$$

then necessarily $E = l^2$ and $\|\cdot\|_E = \|\cdot\|_2$. In other words, l^2 is the only fully symmetric sequence space with the Pythagorean property.

The second main result translates this characterization to the setting of operator ideals. Making use of the Calkin correspondence between fully symmetric sequence spaces and ideals in $\mathcal{K}(\mathcal{H})$, we show that if $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a fully symmetric ideal satisfying

$$\|A+B\|_{\mathcal{C}_E}^2 = \|A\|_{\mathcal{C}_E}^2 + \|B\|_{\mathcal{C}_E}^2,$$

for all self-adjoint operators A, B with $AB = 0$, then necessarily

$$(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E}) = (\mathcal{C}_2, \|\cdot\|_2).$$

Thus the Hilbert–Schmidt ideal $\mathcal{C}_2$ is uniquely characterized among fully symmetric operator ideals by the same Pythagorean identity.

Symmetric sequence spaces

Let l^∞ (respectively, c_0) be the Banach space of bounded (respectively, converging to zero) sequences $\{\xi_n\}_{n=1}^\infty$ of complex numbers equipped with the uniform norm $\|\{\xi_n\}\|_\infty = \sup\limits_{n\in\mathbb{N}} |\xi_n|$, where $\mathbb{N}$ is the set of natural numbers. If $2^{\mathbb{N}}$ is the σ-algebra of all subsets of $\mathbb{N}$ and $\mu(\{n\}) = 1$ for each $n \in \mathbb{N}$, then $(\mathbb{N}, 2^{\mathbb{N}}, \mu)$ is a σ-finite measure space such that $L^\infty(\mathbb{N}, 2^{\mathbb{N}}, \mu) = l^\infty$ and

$$L^1(\mathbb{N}, 2^{\mathbb{N}}, \mu) = l^1 = \left\{ \{\xi_n\}_{n=1}^\infty \subset \mathbb{C} : \ \|\{\xi_n\}\|_1 = \sum_{n=1}^{\infty} |\xi_n| < \infty \right\} \subset l^\infty,$$

where $\mathbb{C}$ is the field of complex numbers.

For any subset $E \subset l^\infty$ we denote $E_h = \{\{\xi_n\}_{n=1}^\infty \in E : \ \xi_n \in \mathbb{R} \ \text{ for each } \ n\}$, where $\mathbb{R}$ is the field of real numbers. It is know that $(l_h^\infty, \|\cdot\|_\infty)$ and $((c_0)_h, \|\cdot\|_\infty)$ are Banach lattices with respect to the natural partial order

$$\{\xi_n\} \le \{\eta_n\} \iff \xi_n \le \eta_n \ \text{ for all } \ n \in \mathbb{N}.$$

If $\xi = \{\xi_n\}_{n=1}^\infty \in l^\infty$, then the *non-increasing rearrangement* $\xi^* : (0, \infty) \to (0, \infty)$ of ξ is defined by

$$\xi^*(t) = \inf\{\lambda : \mu\{|\xi| > \lambda\} \le t\}, \quad t > 0,$$

(see, for example, [1, Ch. 2, Definition 1.5]). As such, the non-increasing rearrangement of a sequence $\{\xi_n\}_{n=1}^\infty \in l^\infty$ can be identified with the sequence $\xi^* = \{\xi_n^*\}_{n=1}^\infty$, where

$$\xi_n^* = \inf\left\{ \sup_{n \notin F} |\xi_n| : F \subset \mathbb{N}, \ |F| < n \right\}.$$

If $\{\xi_n\} \in c_0$, then $\xi_n^* \downarrow 0$; in this case there exists a bijection $\pi : \mathbb{N} \to \mathbb{N}$ such that $|\xi_{\pi(n)}| = \xi_n^*$, $n \in \mathbb{N}$.

Hardy-Littlewood-Polya partial order in the space l^∞ is defined as follows:

$$\xi = \{\xi_n\} \prec\prec \eta = \{\eta_n\} \iff \sum_{n=1}^{m} \xi_n^* \le \sum_{n=1}^{m} \eta_n^* \ \text{ for all } \ m \in \mathbb{N}.$$

A non-zero linear subspace $E \subset l^\infty$ with a Banach norm $\|\cdot\|_E$ is called a *symmetric* (*fully symmetric*) sequence space if

$$\eta \in E, \ \xi \in l^\infty, \ \xi^* \le \eta^* \ (\text{resp., } \ \xi^* \prec\prec \eta^*) \implies \xi \in E \ \text{ and } \ \|\xi\|_E \le \|\eta\|_E.$$

Every fully symmetric sequence space is a symmetric sequence space. The converse is not true in general. At the same time, any separable symmetric sequence space is a fully symmetric space.

If $(E, \|\cdot\|_E)$ is a symmetric sequence space, then

$$\|\xi\|_E = \| \, |\xi| \, \|_E = \|\xi^*\|_E \quad \text{for all } \ \xi \in E.$$

Besides, $(E_h, \|\cdot\|_E)$ is a Banach lattice with respect to the partial order induced from l^∞.

We say that the norm in a symmetric sequence space $(E, \|\cdot\|_E)$ is said to have the *Fatou property* if from the conditions

$$0 \le x^{(n)} \le x^{(n+1)} \in E, \quad n \in \mathbb{N}, \quad \sup_n \|x^{(n)}\|_E < \infty,$$

it follows that

$$\sup_{n\ge1} x^{(n)} \in E \quad \text{and} \quad \|x\|_E = \sup_{n\ge1} \|x^{(n)}\|_E.$$

It is known [1, Chapter II, §2.4, Theorem 2.4.2, pp. 44-46] that the norm of every fully symmetric sequence space $((E, \|\cdot\|_E)$ has the Fatou property. But general symmetric sequence spaces (not fully symmetric) do not always have the Fatou property.

Immediate examples of fully symmetric sequence spaces are $(l^\infty, \|\cdot\|_\infty)$, $(c_0, \|\cdot\|_\infty)$ and the Banach spaces

$$l^p = \left\{ \xi = \{\xi_n\}_{n=1}^{\infty} \in l^\infty : \ \|\xi\|_p = \left(\sum_{n=1}^{\infty} |\xi_n|^p \right)^{1/p} < \infty \right\}, \ 1 \leq p < \infty.$$

For any symmetric sequence space $(E, \|\cdot\|_E)$ the following continuous embeddings hold [1, Ch. 2, § 6, Theorem 6.6]: $(l^1, \|\cdot\|_1) \subset (E, \|\cdot\|_E) \subset (l^\infty, \|\cdot\|_\infty)$. Besides, $\|\xi\|_E \leq \|\xi\|_1$ for all $\xi \in l^1$ and $\|\xi\|_\infty \leq \|\xi\|_E$ for all $\xi \in E$.

If there is $\xi \in E \setminus c_0$, then $\xi^* \geq \alpha \mathbf{1}$ for some $\alpha > 0$, where $\mathbf{1} = \{1, 1, ...\}$. Consequently, $\mathbf{1} \in E$ and $E = l^\infty$. Therefore, either $E \subset c_0$ or $E = l^\infty$.

Symmetric operator spaces

Now, let $(\mathcal{H}, (\cdot, \cdot))$ be an infinite-dimensional Hilbert space over $\mathbb{C}$, and let $(\mathcal{B}(\mathcal{H}), \|\cdot\|_\infty)$ be the $C^\star$–algebra of all bounded linear operators in $\mathcal{H}$. Denote by $\mathcal{K}(\mathcal{H})$ ($\mathcal{F}(\mathcal{H})$) the two-sided ideal of compact (respectively, finite rank) linear operators in $\mathcal{B}(\mathcal{H})$. It is well known that, for any proper two-sided ideal $\mathcal{I} \subset \mathcal{B}(\mathcal{H})$, we have $\mathcal{F}(\mathcal{H}) \subset \mathcal{I}$, and if $\mathcal{H}$ is separable, then $\mathcal{I} \subset \mathcal{K}(\mathcal{H})$ (see, for example, [5, Proposition 2.1]). At the same time, if $\mathcal{H}$ is a non-separable Hilbert space, then there exists a proper two-sided ideal $\mathcal{I} \subset \mathcal{B}(\mathcal{H})$ such that $\mathcal{K}(\mathcal{H}) \subsetneqq \mathcal{I}$.

Denote $\mathcal{B}_h(\mathcal{H}) = \{x \in \mathcal{B}(\mathcal{H}) : x = x^*\}$, $\mathcal{B}_+(\mathcal{H}) = \{x \in \mathcal{B}_h(\mathcal{H}) : x \geq 0\}$, and let $\tau : \mathcal{B}_+(\mathcal{H}) \to [0, \infty]$ be the *canonical trace* on $\mathcal{B}(\mathcal{H})$, that is,

$$\tau(x) = \sum_{j \in J} (x\varphi_j, \varphi_j), \quad x \in \mathcal{B}_+(\mathcal{H}),$$

where $\{\varphi_j\}_{j \in J}$ is an orthonormal basis in $\mathcal{H}$ (see, for example, [5, Ch. 7, E. 7.5]).

Let $\mathcal{P}(\mathcal{H}) = \{e \in \mathcal{B}(\mathcal{H}) : e = e^2 = e^*\}$ be the lattice of projectors in $\mathcal{B}(\mathcal{H})$. If $\mathbf{1}$ is the identity of $\mathcal{B}(\mathcal{H})$ and $e \in \mathcal{P}(\mathcal{H})$, we will write $e^\perp = \mathbf{1} - e$.

Let $x \in \mathcal{B}(\mathcal{H})$, and let $\{e_\lambda(|x|)\}_{\lambda \geq 0}$ be the spectral family of projections for the absolute value $|x| = (x^*x)^{1/2}$ of x, that is, $e_\lambda(|x|) = \{|x| \leq \lambda\}$. If $t > 0$, then the *t-th generalized singular number* of x, or the *non-increasing rearrangement* of x, is defined as

$$\mu_t(x) = \inf\{\lambda > 0 : \ \tau(e_\lambda(|x|)^\perp) \leq t\}$$

(see [2]).

A non-zero linear subspace $X \subset \mathcal{B}(\mathcal{H})$ with a Banach norm $\|\cdot\|_X$ is called *symmetric* (*fully symmetric*) if the conditions

$$x \in X, \ y \in \mathcal{B}(\mathcal{H}), \ \mu_t(y) \leq \mu_t(x) \quad \text{for all} \quad t > 0$$

(respectively,

$$x \in X, \ y \in \mathcal{B}(\mathcal{H}), \ \int_0^s \mu_t(y)dt \leq \int_0^s \mu_t(x)dt \quad \text{for all} \quad s > 0 \ \ (\text{writing } y \prec\prec x))$$

imply that $y \in X$ and $\|y\|_X \leq \|x\|_X$.

The spaces $(\mathcal{B}(\mathcal{H}), \|\cdot\|_\infty)$ and $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty)$ as well as the classical Banach two-sided ideals

$$\mathcal{C}^p = \{x \in \mathcal{K}(\mathcal{H}) : \ \|x\|_p = \tau(|x|^p)^{1/p} < \infty\}, \ 1 \leq p < \infty,$$

are examples of fully symmetric spaces.

It should be noted that for every symmetric space $(X, \|\cdot\|_X) \subset \mathcal{B}(\mathcal{H})$ and all $x \in X$, $a, b \in \mathcal{B}(\mathcal{H})$,

$$\|x\|_X = \| \, |x| \, \|_X = \|x^*\|_X, \quad axb \in X, \quad \text{and} \quad \|axb\|_X \leq \|a\|_\infty \|b\|_\infty \|x\|_X.$$

Remark 1. *If $X \subset \mathcal{B}(\mathcal{H})$ is a symmetric space and there exists a projection $e \in \mathcal{P}(\mathcal{H}) \cap X$ such that $\tau(e) = \infty$, that is, $\dim e(\mathcal{H}) = \infty$, then $\mu_t(e) = \mu_t(\mathbf{1}) = 1$ for every $t \in (0, \infty)$. Consequently, $\mathbf{1} \in X$ and $X = \mathcal{B}(\mathcal{H})$. If $X \neq \mathcal{B}(\mathcal{H})$ and $x \in X$, then $e_\lambda(|x|)^\perp = \{|x| > \lambda\}$ is a finite-dimensional projection, that is, $\dim e_\lambda(|x|)^\perp(\mathcal{H}) < \infty$ for all $\lambda > 0$. This means that $x \in \mathcal{K}(\mathcal{H})$, hence $X \subset \mathcal{K}(\mathcal{H})$. Therefore, either $X = \mathcal{B}(\mathcal{H})$*

or $X \subset \mathcal{K}(\mathcal{H})$. Thus, if $\mathcal{H}$ is non-separable, then there exists a proper two-sided ideal $\mathcal{I} \subset \mathcal{B}(\mathcal{H})$ such that $\mathcal{K}(\mathcal{H}) \subsetneqq \mathcal{I}$ and $(\mathcal{I}, \|\cdot\|_\infty)$ is a Banach space which is not a symmetric subspace of $\mathcal{B}(\mathcal{H})$.

Throughout this paper we assume that the Hilbert space $\mathcal{H}$ is separable.

If $x \in \mathcal{K}(\mathcal{H})$, then $|x| = \sum\limits_{n=1}^{m(x)} s_n(x)p_n$ (if $m(x) = \infty$, the series converges uniformly), where $\{s_n(x)\}_{n=1}^{m(x)}$ is the set of singular values of x, that is, the set of eigenvalues of the compact operator $|x|$ in the decreasing order, and p_n is the projection onto the eigenspace corresponding to $s_n(x)$. Consequently, the non-increasing rearrangement $\mu_t(x)$ of $x \in \mathcal{K}(\mathcal{H})$ can be identified with the sequence $\{s_n(x)\}_{n=1}^\infty$, $s_n(x) \downarrow 0$ (if $m(x) < \infty$, we set $s_n(x) = 0$ for all $n > m(x)$).

Duality between symmetric sequence and operator spaces

Let $(X, \|\cdot\|_X) \subset \mathcal{K}(\mathcal{H})$ be a symmetric space. Fix an orthonormal basis $\{\varphi_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$. Let p_n be the one-dimensional projection on the subspace $\mathbb{C} \cdot \varphi_n \subset \mathcal{H}$. It is clear that the set

$$E(X) = \left\{ \xi = \{\xi_n\}_{n=1}^\infty \in c_0 : \ x_\xi = \sum_{n=1}^\infty \xi_n p_n \in X \right\}$$

(the series converges uniformly), is a symmetric sequence space with respect to the norm $\|\xi\|_{E(X)} = \|x_\xi\|_X$. Consequently, each symmetric subspace $(X, \|\cdot\|_X) \subset \mathcal{K}(\mathcal{H})$ uniquely generates a symmetric sequence space $(E(X), \|\cdot\|_{E(X)}) \subset c_0$. The converse is also true: every symmetric sequence space $(E, \|\cdot\|_E) \subset c_0$ uniquely generates a symmetric space $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E}) \subset \mathcal{K}(\mathcal{H})$ by the following rule (see, for example, [4, Ch. 3, Section 3.5]):

$$\mathcal{C}_E = \{x \in \mathcal{K}(\mathcal{H}) : \{s_n(x)\} \in E\}, \quad \|x\|_{\mathcal{C}_E} = \|\{s_n(x)\}\|_E.$$

In addition,

$$E(\mathcal{C}_E) = E, \ \|\cdot\|_{E(\mathcal{C}_E)} = \|\cdot\|_E, \ \mathcal{C}_{E(\mathcal{C}_E)} = \mathcal{C}_E, \ \|\cdot\|_{\mathcal{C}_{E(\mathcal{C}_E)}} = \|\cdot\|_{\mathcal{C}_E}.$$

We will call the pair $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ a *Banach ideal of compact operators* (cf. [3, Ch. III]). It is known that $(\mathcal{C}^p, \|\cdot\|_p) = (\mathcal{C}_{l^p}, \|\cdot\|_{\mathcal{C}_{l^p}})$ for all $1 \le p < \infty$ and $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty) = (\mathcal{C}_{c_0}, \|\cdot\|_{\mathcal{C}_{c_0}})$.

Hardy-Littlewood-Polya partial order in the Banach ideal $\mathcal{K}(\mathcal{H})$ is defined by

$$x \prec\prec y, \ x, y \in \mathcal{K}(\mathcal{H}) \iff \{s_n(x)\} \prec\prec \{s_n(y)\}.$$

We say that a Banach ideal $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is *fully symmetric* if conditions $y \in \mathcal{C}_E$, $x \in \mathcal{K}(\mathcal{H})$, $x \prec\prec y$ entail that $x \in \mathcal{C}_E$ and $\|x\|_{\mathcal{C}_E} \le \|y\|_{\mathcal{C}_E}$. It is clear that $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a fully symmetric ideal if and only if $(E, \|\cdot\|_E)$ is a fully symmetric sequence space.

Examples of fully symmetric ideals include $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty)$ as well as the Banach ideals $(\mathcal{C}^p, \|\cdot\|_p)$ for all $1 \le p < \infty$. It is clear that $\mathcal{C}^1 \subset \mathcal{C}_E \subset \mathcal{K}(\mathcal{H})$ for every symmetric sequence space $E \subset c_0$ with $\|x\|_{\mathcal{C}_E} \le \|x\|_1$ and $\|y\|_\infty \le \|y\|_{\mathcal{C}_E}$ for all $x \in \mathcal{C}^1$ and $y \in \mathcal{C}_E$.

Abstract characterizations of spaces l^2 and $\mathcal{C}^2$

Let now consider $E = l^2 = \{x = \{x_n\}_{n=1}^\infty \subset \mathbb{C} : \sum_{n=1}^\infty |x_n|^2 < \infty\}$. It is known that l^2 is a Hilbert space with respect to the scalar product

$$(x, y) = \sum_{n=1}^\infty x_n \cdot \overline{y}_n, \quad x = \{x_n\}_{n=1}^\infty, \ y = \{y_n\}_{n=1}^\infty \in l^2.$$

In the linear space l^2 we define multiplication of elements as follows:

$$x \cdot y = \{x_n \cdot y_n\}_{n=1}^\infty, \quad x, y \in l^2.$$

And let $|x| = \{|x_n|\}_{n=1}^\infty$ for all $x = \{x_n\}_{n=1}^\infty \in l^\infty$.

Proposition 2. *If $x, y \in l^2$ and $x \cdot y = \theta = \{0, 0, 0, \dots\}$, then $(x, y) = 0$.*

Proof. From $x, y \in l^2$ and $x \cdot y = \theta$ it follows that $x_n \cdot y_n = 0$ for all $n \in \mathbb{N}$. Hence,

$$(x, y) = \sum_{n=1}^{\infty} x_n \cdot y_n = 0.$$

The Proposition 2 is proved.

Remark 3. *The converse of Proposition 2 is not true in general.*

In every Hilbert space, in particular in l^2, the following important result is known.

Proposition 4. *If $x, y \in l^2$ and $(x, y) = 0$, then $\|x + y\|_2^2 = \|x\|_2^2 + \|y\|_2^2$.*

Proof. Let $x = \{x_n\}_{n=1}^{\infty}$, $y = \{y_n\}_{n=1}^{\infty} \in l^2$ with $(x, y) = 0$. Then

$$\|x + y\|_2^2 = (x + y, x + y) = (x, x) + (x, y) + (y, x) + (y, y) = \|x\|_2^2 + \|y\|_2^2.$$

The Proposition 4 is proved.

Corollary 5. *If $x, y \in l_2$ and $x \cdot y = 0$, then $\|x + y\|_2^2 = \|x\|_2^2 + \|y\|_2^2$.*

Motivated by Corollary 5, we would like to characterize other symmetric sequence spaces $(E, \|\cdot\|_E)$ for which the property of Corollary 5 holds. We prove the following theorem on the abstract characterization of l^2.

Theorem 6. *Let $(E, \|\cdot\|_E) \subset c_0$ be a fully symmetric space such that*

$$\|x + y\|_E^2 = \|x\|_E^2 + \|y\|_E^2, \quad \forall x, y \in E,\ x \cdot y = \theta,$$

where multiplication in E is defined as in l^2. Then $E = l^2$ and $\|x\|_E = \|x\|_2$ for all $x \in E$.

Proof. Let $(E, \|\cdot\|_E)$ be a fully symmetric space with $E \subset c_0$ and let $e_n = \{0, 0, \ldots, 1, 0, \ldots\}$ with 1 at the n-th position, $n \in \mathbb{N}$. Then $e_n \in E$ for any n. Furthermore, it can be assumed without loss of generality that $\|e_n\|_E = 1$. Also it is clear that $e_n^* = \{1, 0, 0, \ldots\} \in E$ and $\|e_n^*\|_E = \|e_n\|_E = 1$, for any $n \in \mathbb{N}$.

We fix $x = \{x_n\}_{n=1}^{\infty} \in E$. Let $x^* = \{x_n^*\}_{n=1}^{\infty} \in E$. Consider

$$x_k = \sum_{n=1}^{k} x_n^* e_n.$$

Then from the Proposition 4 we have

$$\|x_k\|_E^2 = \sum_{n=1}^{k} |x_n^*|^2.$$

Since $(E, \|\cdot\|_E)$ is fully symmetric, the norm has the Fatou property (see section "Symmetric sequence spaces"). Hence,

$$\|x^*\|_E = \sup_{k \geq 1} \|x_k\|_E = \sup_{k \geq 1} \sqrt{\sum_{n=1}^{k} |x_n^*|^2} < \infty,$$

which implies $x^* \in l^2$.

Thus, $\sum\limits_{n=1}^{\infty} (x_n^*)^2 < \infty$, i.e. $x^* \in l^2$. Since $(l^2, \|\cdot\|_2)$ is a symmetric space, it follows that $x^* \in l^2$, and consequently $x \in l^2$.

If $x = \{x_n\}_{n=1}^{\infty} \in l^2$, then $x^* = \sum\limits_{n=1}^{\infty} x_n^* e_n \in l^2$, and hence

$$\sum_{n=1}^{\infty} (x_n^*)^2 < \infty.$$

Again, considering the sequence $x_k = \sum\limits_{n=1}^{k} x_n^* e_n \in E$, we have $0 \leq x_k \uparrow x^*$. By the Fatou property, $\|x_k\|_E \to \|x^*\|_E$ and $x^* \in E$, so $x \in E$.

Moreover,

$$\|x_k\|_E = \sqrt{\sum_{n=1}^{k}(x_n^*)^2} \to \|x^*\|_E,$$

which implies

$$\|x\|_E = \|x^*\|_E = \sqrt{\sum_{n=1}^{\infty}(x_n^*)^2} = \|x^*\|_2 = \|x\|_2.$$

Thus $E = l^2$ and for all $x \in E$ we have $\|x\|_E = \|x\|_2$. The Theorem 6 is proved.

Using Calkin's correspondence (see section "Duality between symmetric sequence and operator spaces") we obtain the following variant of Theorem 6 for the Schatten's ideal $\mathcal{C}^2$, which is called the abstract characterization of $\mathcal{C}^2$.

Theorem 7. *Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a fully symmetric ideal $\mathcal{C}_E \subset \mathcal{K}(\mathcal{H})$ such that*

$$\|A+B\|^2_{\mathcal{C}_E} = \|A\|^2_{\mathcal{C}_E} + \|B\|^2_{\mathcal{C}_E}$$

for all $A, B \in \mathcal{C}_E$ with $A = A^*$, $B = B^*$, and $A \cdot B = 0$. Then $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E}) = (\mathcal{C}^2, \|\cdot\|_2)$.

Proof. Let $E(\mathcal{C}_E) = E$ be a fully symmetric sequence space with $E \subset c_0$, since $\mathcal{C}_E \subset \mathcal{K}(\mathcal{H})$. Let $x = \{x_n\}_{n=1}^{\infty}, y = \{y_n\}_{n=1}^{\infty} \in E$, with $x \cdot y = \{x_n \cdot y_n\}_{n=1}^{\infty} = \theta$.

Fix an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty} \in \mathcal{H}$. Let consider two diagonal operators $A : \mathcal{H} \to \mathcal{H}$, $A(\varphi_n) = |x_n| \cdot \varphi_n$, and $B : \mathcal{H} \to \mathcal{H}$, $B(\varphi_n) = |y_n| \cdot \varphi_n$ for all $n \in \mathbb{N}$. Therefore, for all $a = \sum\limits_{n=1}^{\infty} a_n\varphi_n \in \mathcal{H}$ it follows that $A(a) = \sum\limits_{n=1}^{\infty}(a_n|x_n|) \cdot \varphi_n$, and $B(a) = \sum\limits_{n=1}^{\infty}(a_n|y_n|) \cdot \varphi_n$.

It is clear that $A = A^*$ and $B = B^*$, $AB = 0$. Moreover, $\{s_n(A)\}_{n=1}^{\infty} = |x|^* \in E_h$ and $s_n(B) = |y|^* \in E_h$. Therefore $A, B \in \mathcal{C}_E$. From the condition of theorem we have that

$$\|A+B\|^2_{\mathcal{C}_E} = \|A\|^2_{\mathcal{C}_E} + \|B\|^2_{\mathcal{C}_E}.$$

On the other hand, it is easy to see that the operator $A+B$ is again diagonal with respect to the chosen basis, and its singular-value sequence is given by $\{s_n(A+B)\}_{n=1}^{\infty} = (|x|+|y|)^* = |x+y|^*$ (the last equation is true since $x_n y_n = 0$ for all $n \in \mathbb{N}$).

Thus, we obtain the equality

$$\|x+y\|^2_E = \||x+y|^*\|^2_E = \|\{s_n(A+B)\}_{n=1}^{\infty}\|^2_E = \|A+B\|^2_{\mathcal{C}_E} =$$

$$= \|A\|^2_{\mathcal{C}_E} + \|B\|^2_{\mathcal{C}_E} = \||x|^*\|^2_E + \||y|^*\|^2_E = \|x\|^2_E + \|y\|^2_E.$$

By the Theorem 6 this implies $E = l^2$ and $\|\cdot\|_E = \|\cdot\|_2$ on E. Finally, returning to the ideal via the Calkin correspondence we conclude

$$(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E}) = (\mathcal{C}_2, \|\cdot\|_2),$$

which completes the proof of Theorem 7.

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REZYUME

Ushbu maqolada biz l^2 ketma-ketlik fazosi va Schattenning $\mathcal{C}^2$ idealining abstrakt xarakterizatsiyalarini keltiramiz hamda shu yo'nalishda asosiy natijalarimizni taqdim etamiz.

Kalit so'zlar: simmetrik ketma-ketlik fazolari, kompakt operatorlarning Banax ideallari, Shatten ideallari, simmetrik ketma-ketlik fazolari va kompakt operatorlarning Banax ideallari orasidagi bog'lanish.

РЕЗЮМЕ

В данной статье мы устанавливаем абстрактные характеристики пространства последовательностей l^2 и идеала Шаттена $\mathcal{C}^2$ и представляем наши основные результаты в этом направлении.

Ключевые слова: симметричные пространства последовательностей, банаховы идеалы компактных операторов, идеалы Шаттена, двойственность между симметричными пространствами последовательностей и банаховыми идеалами компактных операторов.