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The extension of a *-isomorphism between two real W*-algebras to a *-isomorphism of the corresponding *-algebras of all m-measurable operators

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RESUME

The subadditive measures (denoted by m) on projectors of the real von Neumann algebra are considered. A theorem on the extension of a *-isomorphism between two real von Neumann algebras to a *-isomorphism of the corresponding *-algebras of all m-measurable operators has been proved.

Key words: Real von Neumann algebras, projection, subadditive measure.

INTRODUCTION

In the non-commutative theory of integration, started by I.E.Segal in (see [1]), the measure τ satisfies the subadditivity condition. Therefore, it seems interesting to study subadditive measures on projectors of operator algebras, thereby finding out to what extent Segal's theory is based on the "subadditivity"of the measure τ. In the article (see [2]), Leszek J.Ciach considered subadditivity measures on projectors of the von Neumann algebra (W*-algebra). The author of the work gives some partial answers to the question posed above. In this article, subadditive measures (denoted by m) on projectors of the real von Neumann algebra are considered. A theorem on the extension of a *-isomorphism between two real von Neumann algebras to a *-isomorphism of the corresponding *-algebras of all m-measurable operators has been proved.

PRELIMINARIES

Let A be an algebra and let $P(A)$ be the set of all projections of A. By *subadditive measure* on A we mean a mapping $m : P(A) \to [0, \infty]$ with the following properties:

i) $m(0) = 0$, and $m(p) = 0$ implies $p = 0$ (faithfulness);

ii) $p \leq q$ implies $m(p) \leq m(q)$ (monotonicity);

iii) $p \sim q$ implies $m(p) = m(q)$,

iv) $m(p \vee q) \leq m(p) + m(q)$ (subadditivity);

v) $p_n \uparrow p$ implies $m(p_n) \uparrow m(p)$.

The measure m is called *finite* if $m(\mathbf{1}) < \infty$, where $\mathbf{1}$ is the unit of algebra A. Making use of the relation $p \vee q - q \sim p - p \wedge q$, it is not hard to notice that instead of (iv) one may assume

iv*) $m(p \vee q) = m(p + q) \leq m(p) + m(q)$ for $p \perp q$.

Now, let $B(H)$ be the algebra of all bounded linear operators on a complex Hilbert space H. A weakly closed *-subalgebra M containing the identity operator $\mathbf{1}$ in $B(H)$ is called a *von Neumann algebra* (or W*-*algebra*). A real *-subalgebra $R \subset B(H)$ with the identity $\mathbf{1}$ is called a *real von Neumann algebra* (or *real W*-algebra*), if it is weakly closed and $R \cap iR = \{0\}$ (see [3], [4], [5]).

Theorem 1. *Let $R \subset B(H)$ be a real von Neumann algebra and let m be subadditive measure on R. We extend the subadditive measure m at $A = R + iR$ as*

$$\overline{m}(e) = m(s(f)),$$

where $e \in P(A)$ with $e = f + ig$ and $s(f)$ is the support of the element f. Then $\overline{m} : P(A) \to [0,\infty]$ is the subadditive measure on A.

The proof of the theorem is carried out with a direct verification of condition i)-v) for the map $\overline{m}$.

Definition. *Let $R \subset B(H)$ be a real von Neumann algebra.*

1) An operator a acting in a Hilbert space H $(a : D(a) \to H)$ is called affiliated with the algebra R (symbolically, $a\eta R$) if $au' = u'a$, for any unitary operator $u' \in R'$, where R' is commutant of R.

2) A linear subspace $D \subset H$ is called affiliated with the algebra R if $u'(D) \subset D$ for any unitary operator $u' \in R'$ and denoted by $D\eta R$.

3) A linear space $D\eta R$ is said to be m-dense in H if for any $\varepsilon > 0$ there exists a projection $p \in R$ such that $p(H) \subset D$ and $m(p^{\perp}) \leq \varepsilon$.

4) An operator a is called m-measurable, if (i) $a\eta R$; (ii) $D(a)$ is m-dense; (iii) a is closed.

It is easy to check that if $a\eta R$, then a is m-measurable if and only if a is $\overline{m}$-measurable. By $L_m = L_m(R)$ we denote the set of all m-measurable (relative to algebra R) operators. It is also easy to show that R is dense in L_m the topology of m-convergence i.e. $\overline{R}^m = L_m$. Similarly, by $L_{\overline{m}} = L_{\overline{m}(A)}$ we denote the set of all $\overline{m}$-measurable (relative to algebra $A = R + iR$) operators.

If a and b are m-measurable operators, then the closures of $a+b$ and ab are called, respectively, the *m-sum* and the *m-product* of a and b. The space L_m together with the operations: m-sum, m-product, the adjoint of the operator and the natural operation of multiplication by real numbers, is real *-algebra and $L_m + iL_m = L_{\overline{m}}$.

MAIN RESULT

Let $a \in L_m$ and $|a| = \int_0^{\infty} \lambda de_{\lambda}$ be the spectral decomposition of the element modulus $|a|$.
We define the mapping $\ell_m : L_m \to [0,\infty)$ as

$$\ell_m(a) = \inf\{\varepsilon > 0 : m(e_{\varepsilon}^{\perp}) \leq \varepsilon\}.$$

Then the functional $\rho_r : L \times L \to [0,\infty)$ defined as $\rho(a,b) = \ell_m(a-b)$ $(a, b \in L_m)$ is a metric in L_m invariant with respect to the translations, i.e. for any $a, b, c \in L_m$ the equalities are satisfied $d(a,b) = d(a+c, b+c)$ and $d(a,b) = d(a^*, b^*)$. Moreover, the convergence in this metric is equivalent to m-convergence.

The main result of the article is the following theorem.

Theorem 2. *Let R_1 and R_2 be *-isomorphic real von Neumann algebras through a $*$-isomorphism θ, whereas m' and m'' are subadditive measures on $P(R_1)$ and $P(R_2)$, respectively. If $m'(p) = m''(\theta(p))$ for all $p \in P(R_1)$, then θ extends uniquely to a $*$-isomorphism $\overline{\theta} : L_{m'} \to L_{m''}$ continuous in the topologies of m'-convergence and of m''-convergence, respectively.*

Proof. Let $a \in R_1$ and $|a| = \int_0^{\infty} \lambda de_{\lambda}$. It is obvious that $\theta(a) = \int_0^{\infty} \lambda d\theta(e_{\lambda})$. Let $a = u|a|$ be the polar decomposition of a (see Theorem 1.25 [5]). Then it is easy to see that $\theta(u)\theta(|a|)$ is the polar decomposition of $\theta(a)$. Since $m'(p) = m''(\theta(p))$ $(p \in P(R_1))$, then we have

$$\ell_{m'}(a) = \inf\{\varepsilon > 0 : m(e_{\varepsilon}^{\perp}) \leq \varepsilon\} = \inf\{\varepsilon > 0 : m(\theta(e_{\varepsilon}^{\perp})) \leq \varepsilon\} = \ell_{m''}(\theta(a))$$

i.e. $\ell_{m'}(a) = \ell_{m''}(\theta(a))$. Since R_1 is m'-dense in $L_{m'}$, then for any $a \in L_{m'}$ there exists a sequence $\{a_n\}_{n=1}^{\infty} \subset R_1$ such that $a_n \xrightarrow{m'} a$. Hence we have $\rho_1(a_n, a) \to 0$, therefore $\ell_{m'}(a_n - a) \to 0$. From the above equality we obtain $\ell_{m'}(a_n - a) = \ell_{m''}(\theta(a_n) - b_a) \to 0$, where b_a is some element from $L_{m''}$. Hence we have $\rho_2(\theta(a_n), b_a) \to 0$, therefore $\theta(a_n) \xrightarrow{m''} b_a$. We put $\overline{\theta}(a) := b_a$. Then it is directly verified that the map $\overline{\theta} : L_{m'} \to L_{m''}$ is a *-isomorphism with $\overline{\theta}|_{R_1} = \theta$ and $\theta(a_n) \xrightarrow{m''} \overline{\theta}(a)$. The theorem is proven.

Corollary 1. *The *-isomorphism $\theta : R_1 \to R_2$ we extend on $A_1 = R_1 + iR_1$ as*

$$\widetilde{\theta} : A_1 \to A_2, \qquad \widetilde{\theta}(x+iy) = \theta(x) + i\theta(y),$$

*and similarly a *-isomorphism* $\overline{\theta}: L_{m'} \to L_{m''}$ *we also extend on* $L_{\overline{m}'} = L_{m'} + iL_{m'}$ *as*

$$\widetilde{\overline{\theta}}: L_{\overline{m}'} \to L_{\overline{m}''}, \qquad \widetilde{\overline{\theta}}(a+ib) = \overline{\theta}(a) + i\overline{\theta}(b).$$

Then we directly obtain that $\overline{m}'(p) = \overline{m}''(\widetilde{\overline{\theta}}(p))$, $p \in P(A_1)$.

Corollary 2. *For any* $a \in L_{m'}(R_1)$ *and* $\sigma > 0$ *we have*

$$\overline{\theta}(|a|^\sigma) = |\overline{\theta}(a)|^\sigma.$$

Proof. Let $|a| = \int\limits_0^\infty \lambda de_\lambda$. Then it is clear that $|a|^\sigma = \int\limits_0^\infty \lambda^\sigma de_\lambda$. Hence we obtain

$$\overline{\theta}(|a|^\sigma) = \overline{\theta}(\int\limits_0^\infty \lambda^\sigma de_\lambda) = \int\limits_0^\infty \lambda^\sigma d\theta(e_\lambda) = |\overline{\theta}(a)|^\sigma.$$

The corollary is proven.

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REZYUME

Haqiqiy fon Neyman algebrasining proyektorlarida subadditiv o'lchovlar (m bilan belgilanadi) ko'rib chiqiladi. Ikki haqiqiy fon Neyman algebrasi orasidagi *-izomorfizmning barcha m-o'lchovli operatorlarning mos *-algebralarining *-izomorfizmiga kengayishi haqidagi teorema isbotlangan.

Kalit so'zlar: Haqiqiy fon Neyman algebralari, proyeksiya, subadditiv o'lchov.

РЕЗЮМЕ

Рассматриваются субаддитивные меры (обозначаемые m) на проекторах вещественной алгебры фон Неймана. Доказана теорема о продолжении *-изоморфизма между двумя вещественными алгебрами фон Неймана до *-изоморфизма соответствующих *-алгебр всех m-измеримых операторов.

Ключевые слова: вещественные алгебры фон Неймана, проекция, субаддитивная мера.