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LIE ALGEBRA OF DIVERGENCE-FREE VECTOR FIELDS AND SOLENOIDALITY OF KILLING VECTOR FIELDS

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RESUME

The article investigates solenoidal vector fields and demonstrates that they form a Lie algebra with respect to their Lie bracket, while also proving that Killing vector fields are solenoidal. These fields are pivotal in the Helmholtz decomposition theorem and have diverse applications, such as in the design of solenoid valves and electromagnets. Furthermore, the proven theorems confirm that the space of solenoidal vector fields constitutes a Lie algebra, with the Lie bracket acting as the multiplication operation.

Key words: Vector field, Killing vector fields, Solenoidality, Lie bracket, Lie Algebra.

Vector fields whose divergence is calculated in different physical applications have very different physical meanings. However, everywhere the divergence of the vector field is directly related to flows through closed surfaces S.

The very concept of flow originally arose in hydrodynamics, when describing the motion of an incompressible fluid, the bulk density of which is the same at all points in space: $\rho = const$. Since the concept of hydrodynamic flow is closest to life, therefore its discussion makes sense.

In addition to the gradient and divergence discussed above, in applications one more 1st order differential operation, called a rotor, is often encountered, which maps a vector field into a vector field.

Divergence and curl are defined invariantly. If the system of Cartesian coordinates $Oxyz$ is introduced in the three-dimensional Euclidean space R^3, and the vector field $\vec{X}(x,y,z) = P(x,y,z)\vec{j} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$ is given, then they can be calculated by the following formulas

$$div\vec{X}(M_0) = \frac{\partial P(M_0)}{\partial x} + \frac{\partial Q(M_0)}{\partial y} + \frac{\partial R(M_0)}{\partial z},$$
$$rot\vec{X}(M_0) = (R_y - Q_z)\vec{i} + (P_z - R_x)\vec{j} + (Q_x - P_y)\vec{k}.$$

Now, using the properties of differential operations, we will prove several propositions that can be used in problems in the theory of vector analysis.

Consider the scalar field $u = xyz$. Let us prove that any integral curve of the potential vector field $X = gradu$ is the intersection of two surfaces of the second order. That is, we prove the following

Proposition 1. Any integral curve of a potential vector field $X = gradu$ is an intersection of two second-order surfaces.

Proof. For a vector field $X(x, y, z) = gradu(x, y, z) = yz\partial_1 + xz\partial_2 + xy\partial_3$,

where $\partial_1, \partial_2, \partial_3$ are basic vector fields, the system defining integral curves has the form

$$\begin{cases} \dot{x} = yz, \\ \dot{y} = xz, \\ \dot{z} = xy. \end{cases}$$

Where

$$\frac{x^2}{2} - \frac{y^2}{2} = C_1, \tag{1}$$

$$\frac{y^2}{2} - \frac{z^2}{2} = C_2. \tag{2}$$

Equations (1) and (2) define two families of hyperbolic cylinders whose generators are parallel to the axes Oz and Ox, respectively, as well as two pairs of planes, $x = \pm y$ and $y = \pm z$, at $C_1 = C_2 = 0$.

Any integral curve of a vector field X, is a line of intersection of two surfaces, which are obtained at fixed values of constants C_1 and C_2, from families (1) and (2).

If $C_1 = C_2 = 0$, then the line of intersection of the planes $x = y$ and $y = z$ is a straight line passing through the origin. Its canonical equation has the form

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} \tag{3}$$

And the vector field $X(x, y, z)$ at the points of the straight line (3) has the form

$$X(x, y, z) = x^2\partial_1 + x^2\partial_2 + x^2\partial_3.$$

Definition 1. A scalar field that depends only on the distance of a point to the origin is called spherical.

Proposition 2. The gradient vector field of a spherical scalar field is a potential vector field.

Proof. Let a scalar field $u = f(r)$ be given, where $r = \sqrt{x^2 + y^2 + z^2}$. The scalar field $u = f(r)$ depends only on the distance of the point (x, y, z) to the origin, therefore it is spherical. Find the gradient field of $u = f(r)$:

$$gradu = gradf(r) = \left\{ \frac{\partial}{\partial x} f(r), \frac{\partial}{\partial y} f(r), \frac{\partial}{\partial z} f(r) \right\} =$$

$$= \left\{ f'(r)\frac{x}{r}, f'(r)\frac{y}{r}, f'(r)\frac{z}{r} \right\} = f'(r)\frac{\vec{r}}{r}.$$

It follows from the relation $X = f'(r)\frac{\vec{r}}{r} = gradf(r)$ that the vector field $X = f'(r)\frac{\vec{r}}{r}$ is potential, and the function $f(r)$ is its potential.

Corollary 1. The Coulomb field $X = \frac{C}{r^2} \cdot \frac{\vec{r}}{r} (C = const)$ is a potential field.

Indeed, the Coulomb field $X = \frac{C}{r^2} \cdot \frac{\vec{r}}{r}$ is a special case of the potential field $f'(r)\frac{\vec{r}}{r}$ considered in Proposition 2 and having the potential $f(r)$. Therefore, sloping $f'(r) = \frac{C}{|\vec{r}|^2}$, we find $f(r) = -\frac{C}{r} + C_1$, where C_1 is an arbitrary constant. This means that the Coulomb field is potentially and can be represented as $X = \frac{C}{r^2} \cdot \frac{\vec{r}}{r} = gradf(r)$, where $f(r) = C_1 - \frac{C}{r}$ is its potential.

It should be noted that the potential of any vector potential field is determined up to a constant term. This term does not affect the coordinates of the vector field, which is obtained by differentiating the potential.

Let the vector field $X = f(r)\vec{r}$.

Proposition 3. If X is a vector field of a spherical scalar field, then it is solenoidal in any region that does not contain the origin.

Proof. A vector field X is called solenoidal if it is a vortex of some field Y, i.e. $X = rotY$. In this case, the vector field Y is called the vector potential of the field X. A necessary condition for such a ratio is the equality $divX = divrotY = 0$. Therefore, the flow of the solenodial field through a closed surface is zero. If the vector field is the velocity field of a continuous medium or liquid, then the flow of this field through a closed surface characterizes the total power of sources or sinks.

Divergence is a point characteristic of the distribution of sources and sinks. In the case of a Solenodial field, there are no sources and sinks. An example of such a field is the field of magnetic intensity. This component of the electromagnetic field is different in that it is not generated by static elements such as static electric charge. The absence of magnetic charges in nature from a mathematical point of view is a property of the solenoidal nature of the magnetic field.

Consider the vector field X of a spherical scalar field $X = f(r)\vec{r}$, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}, r = |\vec{r}|$ and determine the form of the function $f(r)$ for which the field X is solenoidal. By definition of divergence, we find

$$divX = \frac{\partial}{\partial x}(f(r)x) + \frac{\partial}{\partial y}(f(r)y) + \frac{\partial}{\partial z}(f(r)z) = f'(r)\frac{x^2}{r} +$$
$$+f(r) + f'(r)\frac{y^2}{r} + f(r) + f'(r)\frac{z^2}{r} + f(r) = f'(r)r + 3f(r).$$

It follows from the solenoidality condition $divX = 0$ that $f'(r)r + 3f(r) = 0$. Next, we find a solution to the equation $f'(r)r + 3f(r) = 0$:

$$\frac{df(r)}{f(r)} = -3\frac{dr}{r}, \ \ln|f(r)| = -3\ln r + \ln C,$$

where $f(r) = \frac{C}{r^3}$, where C is an arbitrary constant.

Hence, the divergence of the spherical vector field $X = f(r)\vec{r}$ is equal to zero only when $f(r) = \frac{C}{r^3}$, i.e. only in the case of a Coulomb field $X = \frac{C}{r^3}\vec{r}$. This field is solenoidal in any region that does not contain the origin.

Proposition 4. If X is a vector field of a spherical scalar field, then it is irrotational.

Proof. The vector field X, whose rotor is equal to zero, is called irrotational. In this case, if $X = X^1\partial_1 + X^2\partial_2 + X^3\partial_3$ (here $\partial_1 = \{1,0,0\}, \partial_2 = \{0,1,0\}, \partial_3 = \{0,0,1\}$ are basic vector fields) then

$$\begin{cases} rot_x X = \frac{\partial X^3}{\partial y} - \frac{\partial X^2}{\partial z} = 0, \\ rot_y X = \frac{\partial X^1}{\partial z} - \frac{\partial X^3}{\partial x} = 0, \\ rot_z X = \frac{\partial X^2}{\partial x} - \frac{\partial X^1}{\partial y} = 0. \end{cases} \tag{4}$$

Note that conditions (4) coincide with the conditions for the potentiality of the field X. This means that there is a scalar field $u(x,y,z)$ whose gradient is $X : X = gradu$.

Thus, the equation $rotX = 0$ expresses the condition of the potentiality of the field X.

Consider a spherical vector field $X = f(r)\vec{r}, \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}, r = |\vec{r}|$. By definition of a curl, we find

$$rotX = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(r)x & f(r)y & f(r)z \end{vmatrix} =$$

$$(\frac{\partial}{\partial y} f(r)z - \frac{\partial}{\partial z} f(r)y)\vec{i} + \left(\frac{\partial}{\partial z} f(r)x - \frac{\partial}{\partial x} f(r)z\right)\vec{j} + + \left(\frac{\partial}{\partial x} f(r)y - \frac{\partial}{\partial y} f(r)x\right)\vec{k} =$$

$$f'(r)\left(\frac{yz}{r} - \frac{zy}{r}\right)\vec{i} + f'(r)\left(\frac{xz}{r} - \frac{zx}{r}\right)\vec{j} + f'(r)\left(\frac{yx}{r} - \frac{xy}{r}\right)\vec{k} = 0.$$

It follows from this that the curl of any spherical vector field is equal to zero, i.e. the spherical vector field is vortex-free field.

Let M be a smooth connected Riemannian manifold of dimension n, and let a smooth vector field X be given on the manifold M.

Definition 2. A vector field X on M is called a Killing vector field if the one-parameter group of local transformations generated by the field X consists of isometries [6].

Example 2. In three-dimensional Euclidean space $M = R^3(x,y,z)$ there are six linearly independent Killing fields over the field of real numbers: $X_1 = \partial_1, X_2 = \partial_2, X_3 = \partial_3$, $X_4 = z\partial_2 - y\partial_3, X_4 = -z\partial_1 + x\partial_3$, $X_6 = y\partial_1 - x\partial_2$.

The transformation groups generated by the vector fields X_1, X_2, X_3 are the groups of parallel translations in the direction of the axes Ox, Oy and Oz respectively, and the last three are the groups of rotations around the axes Ox, Oy and Oz respectively.

The last three fields are also Killing fields on the two-dimensional sphere S^2.

The Killing vector field has the following properties [12]:

1. The Lie bracket of two Killing fields again gives a Killing field.

2. A linear combination of Killing fields over the field of real numbers is also a Killing field. Therefore, the set of all Killing vector fields on the manifold M, denoted by $K(M)$, forms a Lie algebra over the field of real numbers.

Theorem 1. [12] The Lie algebra $K(M)$ of the Killing vector fields of a connected Riemannian manifold M has dimension at most $\frac{1}{2}n(n+1)$, where $n = dimM$. If $dimK(M) = \frac{1}{2}n(n+1)$, then M is a manifold of constant curvature.

Proposition 5. The Killing vector field in three-dimensional Euclidean space is a solenoidal vector field.

Proof. As shown in the proof of Proposition 3, in order for a vector field to be potential or irrotational, it is sufficient that condition $divX = 0$ be satisfied. It is known that in three-dimensional Euclidean space any Killing field can be represented as a linear combination of the following basis Killing vector fields:

$$X_1 = \partial_1,\ X_2 = \partial_2,\ X_3 = \partial_3,\ X_4 = z\partial_2 - y\partial_3,\ X_4 = -z\partial_1 + x\partial_3,\ X_6 = y\partial_1 - x\partial_2.$$

It is easy to see that the divergence $divX = \frac{\partial X^1}{\partial x} + \frac{\partial X^2}{\partial y} + \frac{\partial X^3}{\partial z}$ of the basis vector fields is equal to zero, i.e. they are solenoidal.

The Killing vector field in three-dimensional Euclidean space can be written as

$$X = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4 + \lambda_5 X_5 + \lambda_6 X_6,$$

where λ_1, λ_2, λ_3, λ_4, λ_5, λ_6 are real numbers. Using the divergence property

$$div(\lambda X + \mu Y) = \lambda divX + \mu divY,$$

we can state that $divX = 0$.

Theorem 2. The Lie bracket of two solenoidal vector fields in three-dimensional Euclidean space is a solenoidal vector field.

Proof. Let $X = X^i\partial_i$ and $Y = Y^i\partial_i$ be given in R^3 - solenoidal vector fields. Consider the Lie bracket of these vector fields, which is defined in local coordinates as follows $[X, Y] = (X^i\frac{\partial Y^j}{\partial x^i} - Y^i\frac{\partial X^j}{\partial x^i})\partial_j$, where ∂_j are basic vector fields. Since the considered vector fields X and Y are solenoidal, it can be argued that $X = rot\bar{X}, Y = rot\bar{Y}$. Then, we prove that if $Z = [X, Y]$ then $Z = rot\bar{Z}$. It is known that if $Z = rot\bar{Z}$, then $divZ = 0$.

Let's check the divergence of the vector field, which was obtained using the Lie bracket of two solenoidal vector fields:

$$\sum_{j=1}^{3}\frac{\partial[X,Y]^j}{\partial x^j} = \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\sum_{k=1}^{3}\left(X^k\frac{\partial Y^j}{\partial x_k} - Y^k\frac{\partial X^j}{\partial x_k}\right)\right) = \frac{\partial X^1}{\partial x_1}\cdot\frac{\partial Y^1}{\partial x_1} + X^1\frac{\partial^2 Y^1}{\partial x_1^2} -$$

$$-\frac{\partial Y^1}{\partial x_1}\cdot\frac{\partial X^1}{\partial x_1} - Y^1\frac{\partial^2 X^1}{\partial x_1^2} + \frac{\partial X^2}{\partial x_1}\cdot\frac{\partial Y^1}{\partial x_2} + X^2\frac{\partial^2 Y^1}{\partial x_1\partial x_2} - \frac{\partial Y^2}{\partial x_1}\cdot\frac{\partial X^1}{\partial x_2} - Y^2\frac{\partial^2 X^1}{\partial x_1\partial x_2} +$$

$$+\frac{\partial X^3}{\partial x_1}\cdot\frac{\partial Y^1}{\partial x_3} + X^3\frac{\partial^2 Y^1}{\partial x_1\partial x_3} - \frac{\partial Y^3}{\partial x_1}\cdot\frac{\partial X^1}{\partial x_3} - Y^3\frac{\partial^2 X^1}{\partial x_1\partial x_3} + \frac{\partial X^1}{\partial x_2}\cdot\frac{\partial Y^2}{\partial x_1} + X^1\frac{\partial^2 Y^1}{\partial x_1\partial x_2} -$$

$$-\frac{\partial Y^1}{\partial x_2}\cdot\frac{\partial x^2}{\partial x_1} - Y^1\frac{\partial^2 x^2}{\partial x_1\partial x_2} + \frac{\partial x^2}{\partial x_2}\cdot\frac{\partial Y^2}{\partial x_2} + x^2\frac{\partial^2 Y^2}{\partial x_2^2} - \frac{\partial Y^2}{\partial x_2}\cdot\frac{\partial x^2}{\partial x_2} - Y^2\frac{\partial^2 x^2}{\partial x_2^2} +$$

$$+\frac{\partial x^3}{\partial x_2}\cdot\frac{\partial Y^2}{\partial x_3} + x^3\frac{\partial^2 Y^2}{\partial x_2\partial x_3} - \frac{\partial Y^3}{\partial x_2}\cdot\frac{\partial x^2}{\partial x_3} - Y^3\frac{\partial^2 x^2}{\partial x_2\partial x_3} + \frac{\partial x^1}{\partial x_3}\cdot\frac{\partial Y^3}{\partial x_1} + x^1\frac{\partial^2 Y^1}{\partial x_1\partial x_2} -$$

$$-\frac{\partial Y^1}{\partial x_3}\cdot\frac{\partial x^3}{\partial x_1} - Y^1\frac{\partial^2 x^3}{\partial x_2\partial x_3} + \frac{\partial x^2}{\partial x_3}\cdot\frac{\partial Y^3}{\partial x_2} + x^2\frac{\partial^2 Y^3}{\partial x_2\partial x_3} - \frac{\partial Y^2}{\partial x_3}\cdot\frac{\partial x^3}{\partial x_2} - Y^2\frac{\partial^2 x^3}{\partial x_2\partial x_3} +$$

$$+\frac{\partial x^3}{\partial x_3}\cdot\frac{\partial Y^3}{\partial x_3}+x^3\frac{\partial^2 Y^3}{\partial x_3^2}-\frac{\partial Y^3}{\partial x_3}\cdot\frac{\partial x^3}{\partial x_3}-Y^3\frac{\partial^2 x^3}{\partial x_3^2}.$$

If we take into account that the considered vector fields X and Y are solenoidal, then from the relations

$$\sum_{j=1}^{3}\frac{\partial x^j}{\partial x_j}=0, \sum_{j=1}^{3}\frac{\partial Y^j}{\partial x_j}=0,$$

it follows that

$$\sum_{j=1}^{3}\frac{\partial [X,Y]^j}{\partial x_j}=0,$$

which was required to prove.

Solenoidal vector fields, with zero divergence, are crucial in fluid dynamics for describing incompressible flows and in electromagnetism for adhering to Gauss's law, which indicates the absence of "magnetic charges."These fields are essential in the Helmholtz decomposition theorem and are used in various applications, including the design of solenoid valves and electromagnets. Additionally, Theorem 2 establishes that the space of solenoidal vector fields forms a Lie algebra, with their Lie bracket functioning as the multiplication operation.

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REZYUME

Maqolada solenoidal vektor maydonlarini o'rganilgan va ularning Lie algebrasi tashkil etishi ko'rsatilgan, shuningdek, Killing vektor maydonlari solenoidal ekanligini isbotlangan.

Kalit so'zlar: Vektor maydon, Killing vektor maydonlari, Solenoidallik, Lie qavsi.

РЕЗЮМЕ

В статье изучены соленоидальные векторные поля и показано, что они образуют алгебру Ли относительно их скобок Ли, а также доказано, что векторные поля Киллинга являются соленоидальными.

Ключевые слова: Векторные поля, Векторные поля Киллинга, Соленоидальность, Скобка Ли.