UDC 517.956.6

ON SMOOTHNESS OF THE SOLUTION TO A NONLOCAL BOUNDARY VALUE PROBLEM OF PERIODIC TYPE FOR A FOURTH-ORDER MIXED-TYPE EQUATION OF THE SECOND KIND

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RESUME

In this paper, using the modified Galerkin method and the methods of a priori estimates, "ε-regularization", we study the unique solvability and smoothness of a regular generalized solution of a nonlocal boundary value problem of periodic type for a mixed type equation of the second kind of the fourth order in Sobolev spaces.

Key words: equations of mixed type, nonlocal boundary value problem, ε-regularization, Faedo-Galerkin, Sobolev spaces.

As is known, A.V. Bitsadze in his research has shown that the Dirichlet problem for a second-order mixed-type equation is ill-posed [2]. The question naturally arises: is it possible to replace the conditions of the Dirichlet problem with other conditions that cover the entire boundary and ensure the well-posedness of the problem ? For the first time, such boundary value problems (nonlocal boundary value problems) for a second-order mixed-type equation were proposed and studied by F.I. Frankl [13]. Problems for second-order mixed-type equations of the second kind, close in formulation to the ones under consideration, were studied in bounded domains in [6]-[9], [14], [16], [20]. Nonlocal boundary value problems for a high-order partial differential equation without degeneracy were studied by many researchers; a complete bibliography of these studies is given in books [11], [19] for a high-order mixed-type equation with local boundary conditions in various spaces, they were discussed in [4], [5], [18] and with nonlocal boundary conditions such problems were studied insufficiently [10].

In this paper, applying the results obtained in [4], [5], [8], [9] and the modified Galerkin method, methods of a priori estimates and "ε-regularization", we study the unique solvability of a regular generalized solution to one nonlocal boundary value problem of periodic type for the fourth-order mixed-type equation of the second kind in Sobolev space.

In domain $Q = (0,1) \times (0,T) = \{(x,t); 0 < x < 1; 0 < t < T < +\infty\}$ we consider the fourth-order mixed-type equation of the second kind:

$$L_2 u = Pu + Mu = f(x,t), \tag{1}$$

where $P_u = \sum_{i=0}^{4} K_i(x,t) D_t^i u$, $Mu = au_{xxxx} - bu_{xxtt} - cu_{xx}$, $K_4(x,t) = K_4(t)$, $D_t^i u = \frac{\partial^i u}{\partial t^i}$ $(i = 0,1,2,3,4)$, $D_t^0 u = u$.

Let the following conditions be satisfied for the coefficients of equation (1):

$$K_4(t) \in C^3(0,T) \cap C[0,T];\ K_i(x,t) \in C^2(Q) \cap C(\overline{Q});\ a,b,c - \text{const s} > 0,$$

$$K_4(0) = K_4(T) = 0;\ K_{4t}(0) = K_{4t}(T);\ K_i(x,0) = K_i(x,T);\ i = 0,1,2,3, \text{ for all } x \in [0,1].$$

Equation (1) refers to equations of mixed type of the second kind since no restrictions are imposed on the sign of function $K_4(t)$ with respect to variable t inside segment $[0,T]$ [3], [8], [9].

1. Nonlocal boundary value problem of periodic type: Find solution $u(x,t)$ to equation (1) from Sobolev space $W_2^4(Q)$, that satisfies the following boundary conditions:

$$\gamma D_t^q \left. u\right|_{t=0} = D_t^q \left. u\right|_{t=T};\ q = 0,1,2, \tag{2}$$

$$D_x^p \left. u\right|_{x=0} = D_x^p \left. u\right|_{x=1};\ p = 0,1,2,3, \tag{3}$$

where γ is a non-zero value, which will be specified below.

In what follows, we need to determine the following definitions and auxiliary propositions.

Let $\overrightarrow{e}(e_t, e_x); (e_t = \cos(\overrightarrow{e}, t), e_x = \cos(\overrightarrow{e}, x))$ be the unit vector of the internal normal to boundary ∂Q. When obtaining various a priori estimates, we often use Cauchy's inequality with σ [12], that is

$$\forall u, \vartheta \geq 0;\ \forall \sigma > 0;\ 2u \cdot \vartheta \leq \sigma u^2 + \sigma^{-1} \vartheta^2.$$

The class of smooth functions from space $W_2^4(Q)$, satisfying conditions (2)–(3) we denote by C_L.

Definition 1. *We call function $u(x,t)$ a regular solution to problem (1)–(3) if $u \in C_L$ and it satisfies equation (1) almost everywhere in domain Q.*

Theorem 1. *Let the above conditions be satisfied for the coefficients of equation (1); $K_1(x,t) > 0$ is a sufficiently large function and, in addition, let the following inequalities be satisfied for the coefficients of equation (1): $-(2K_3 - 3K_{4t} + 3\lambda K_4) \geq \delta_3 > 0$, $2K_1 - K_{2t} + \lambda K_2 \geq \delta_2 > 0$, $\lambda K_0 - K_{0t} \geq \delta_1 > 0$, for any $(x,t) \in \overline{Q}$, where $\lambda = \frac{2}{T} \ln|\gamma| > 0$, $|\gamma| > 1$. Then if for any $f(x,t) \in L_2(Q)$, there is a regular solution $u(x,t)$ to problem (1)–(3) from Sobolev space $W_2^4(Q)$, then it is unique and the following estimate holds for it:*

$$\|u\|^2_{W_2^2(Q)} \leq c\,\|f\|_0^2. \tag{4}$$

Proof. We will prove the uniqueness of the solution to problem (1)–(3) using the method of energy integrals. Let there exist a regular generalized solution to problem (1)–(3) $u(x,t)$ from Sobolev space $W_2^4(Q)$. Consider the following identity:

$$2\int_Q Lu \cdot e^{-\lambda t} \cdot u_t \, dxdt = 2\int_Q f \cdot e^{-\lambda t} \cdot u_t \, dxdt. \tag{5}$$

Due to the conditions of Theorem 1 and Cauchy's inequality with σ [12], and based on the boundary conditions (2), (3), by integrating identity (5), we easily obtain the following inequality:

$$\begin{aligned}\left|2\int_Q e^{-\lambda t} Lu \cdot u_t \, dxdt\right| \geq \int_Q e^{-\lambda t}\Big\{&-(2K_3 - 3K_{4t} + 3\lambda K_4)u_{tt}^2 + \lambda a u_{xx}^2 + \lambda b u_{xt}^2 + \lambda c u_x^2 \\ &+(2K_1 - K_{2t} + \lambda K_2)u_t^2 + (\lambda K_0 - K_{0t})u^2\Big\} dxdt - 2\sigma\|u_{tt}\|_0^2 \\ &-5\lambda^4 K\sigma^{-1}\|u_t\|_0^2 + \int_{\partial Q} e^{-\lambda t}\Big(-2K_4 u_{ttt}u_t + 2(K_{4t} - \lambda K_4)u_{tt}u_t \\ &+K_4 u_{tt}^2 - 2K_3 u_{tt}u_t - 2K_2 u_t^2 - K_0 u^2 + a u_{xx}^2 + b u_{xt}^2 + c u_x^2\Big) e_t \, ds \\ &+\int_{\partial Q} e^{-\lambda t}\Big\{2a u_{xxx}u_t\big(a u_{xx}u_{tx} - 2b u_{xxt}u_t - 2c u_x u_t\big)\Big\} e_x \, ds.\end{aligned} \tag{6}$$

where $K = \max\left\{\|K_4\|^2_{C^2[0,T]}, \|K_3\|^2_{C^1(Q)}\right\}$. The conditions of Theorem 1 ensure that the integral over domain Q is non-negative and that the boundary integrals vanish. Then from inequality (6), we obtain:

$$\begin{aligned}&\left|-2\int_Q Lu \cdot e^{-\lambda t} \cdot u_t \, dxdt\right| \geq \int_Q e^{-\lambda t}\, \{\delta_3 \cdot u_{tt}^2 + \lambda a u_{xx}^2 + \lambda b u_{xt}^2 + \delta_2 u_t^2 + \\ &+\lambda c u_x^2 + \delta_1 \cdot u^2\}\, dxdt - 2\sigma \cdot \|u_{tt}\|_0^2 - 5\lambda^4\sigma^{-1}K \cdot \|u_t\|_0^2.\end{aligned} \tag{7}$$

Now, applying the Cauchy's inequality with σ at the left side of (7) above and choosing in inequality (7) constant numbers δ_3 and δ_2 such that $\delta_3 - 3\sigma \geq \delta_{03} > 0$, $\delta_2 - 5\lambda^4\sigma^{-1}K \geq \delta_{02} > 0$, then denoting $\delta = \min\{\delta_{03}, \lambda a, \lambda b, \lambda c, \delta_{02}, \delta_1\}$, from inequality (7), we obtain the first a priori estimate for solving problem (1)–(3):

$$\|u\|^2_{W_2^2(Q)} \leq c_1 \|f\|^2_{L_2(Q)}.$$

In what follows, we denote various positive constants by A_i.

Now we will prove the uniqueness of a regular solution to problem (1)–(3).

We prove the theorem by contradiction. Let problem (1)–(3) have two solutions $u_1(x,t)$, $u_2(x,t)$. Then new function $\vartheta(x,t) = u_1(x,t) - u_2(x,t)$ satisfies the homogeneous equation (1) with conditions (2)–(3) and the first inequality $\|\vartheta\|_2^2 \leq 0$ holds for it. This implies the uniqueness of a regular solution to problem $u_1(x,t) = u_2(x,t)$.

Now we will prove the solvability of a regular solution to problem (1)–(3).

2. Fifth-order equation with a small parameter (auxiliary problem).

We prove the solvability of problem (1)–(3) using the "ε-regularization" method combined with the modified Galerkin method and a priori estimates. Namely, in the domain $Q = (0, 1) \times (0, T)$, we consider a family of fifth-order equations with a small parameter.

$$L_\varepsilon u_\varepsilon = -\varepsilon \frac{\partial \Delta^2 u_\varepsilon}{\partial t} + L u_\varepsilon = f(x, t), \tag{8}$$

with nonlocal boundary conditions of periodic type:

$$\gamma D_t^q \left. u_\varepsilon \right|_{t=0} = D_t^q \left. u_\varepsilon \right|_{t=T}; \; q = 0, 1, 2, 3, 4, \tag{9}$$

$$D_x^p \left. u_\varepsilon \right|_{x=0} = D_x^p \left. u_\varepsilon \right|_{x=1}; \; p = 0, 1, 2, 3, \tag{10}$$

where ε is a small positive number, $D_z^q w = \frac{\partial^q w}{\partial z^q}$, $q = 0, 1, 2, 3, 4$; $D_z^0 w = w$;

$\Delta^2 u = \left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} \right)^2 u = \left(\frac{\partial^4 u}{\partial t^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial t^2} + \frac{\partial^4 u}{\partial x^4} \right)$ is the biharmonic operator.

Below we use a fifth-order equation with a small parameter (8) as a "ε-regulating" equation for a fourth-order mixed-type equation of the second kind (1) [3]-[9].

We will denote the class of functions such that $u_\varepsilon(x, t) \in W_2^4(Q)$, $\frac{\partial \Delta^2 u_\varepsilon}{\partial t} \in L_2(Q)$, satisfying corresponding conditions (9)–(10) by $V(Q)$.

Definition 2. *We call function $u_\varepsilon(x, t)$ a regular solution to problem (8), (9)–(10), if $u_\varepsilon \in V(Q)$ and it satisfies equation (8) almost everywhere in domain Q.*

Theorem 2. *Let all the conditions of Theorem 1 be satisfied and, in addition, let the following conditions be satisfied for the coefficients of equation (8):*

$$-(2K_3 + (2j - 3)K_{4t} + 3\lambda K_4) \geq \delta > 0, \; j = 0, 1, 2.$$

Then for any function $f(x, t) \in W_2^1(Q)$, such that $\gamma f(x, 0) = f(x, T)$, there is a unique regular solution $u_\varepsilon(x, t)$ to problem (8), (9)–(10), from space $V(Q)$ and the following estimates are valid for it:

I) $\varepsilon \cdot (\|u_{\varepsilon ttt}\|_0^2 + \|u_{\varepsilon ttx}\|_0^2 + \|u_{\varepsilon txx}\|_0^2) + \|u_\varepsilon\|_2^2 \leq c_1 \|f\|_0^2$,

II) $\varepsilon \left\| \frac{\partial}{\partial t} \Delta^2 u_\varepsilon \right\| + \|u_\varepsilon\|_4^2 \leq c_2 \|f\|_1^2$.

Proof. The proof of inequality I) is done in the same way as the proof of the first estimate of Theorem 1, from which it follows that there is a unique regular solution to problem (8), (9)–(10) [3]-[9].

Let us give the proof of the first a priori estimate I).

Let $\phi_j(x, t) \in W_2^4(Q)$ be the eigenfunctions of the following problem:

$$-\Delta^2 \phi_j = -\left(\frac{\partial^4 \phi_j}{\partial t^4} + \frac{\partial^4 \phi_j}{\partial x^4} \right) = \mu_j^2 \phi_j, \tag{11}$$

$$D_t^q \left. \phi_j \right|_{t=0} = D_t^q \left. \phi_j \right|_{t=T}, \quad q = 0, 1, 2, 3, \tag{12}$$

$$D_x^q \left. \phi_j \right|_{x=0} = D_x^q \left. \phi_j \right|_{x=1} = 0, \quad q = 0, 1, 2, 3. \tag{13}$$

From the general theory [1], [12], [17] of linear self-adjoint elliptic operators, it is known that all eigenfunctions of problem (11)–(13) belong to $W_2^4(Q)$ and form a complete orthonormal system in $L_2(Q)$. Using these sequences of functions, we construct a solution to the auxiliary problem:

$$P\omega_j \equiv \exp\left(-\frac{\lambda t}{2}\right)\frac{\partial \omega_j}{\partial t} = \phi_j, \tag{14}$$

$$\gamma \cdot \omega_j(x,0) = \omega_j(x,T), \tag{15}$$

where γ is a constant $\neq 0$, and $|\gamma| > 1$. Obviously, problem (14), (15) is uniquely solvable and its solution has the following form:

$$P^{-1}\phi_j = \omega_j = \int_0^t \exp\left(\frac{\lambda\tau}{2}\right)\phi_j d\tau + \frac{1}{\gamma-1}\cdot\int_0^T \exp\left(\frac{\lambda t}{2}\right)\phi_j dt.$$

It is clear that functions $\omega_j(x,t) \in W_2^5(Q)$ are linearly independent. Indeed, if $\omega_j(x,t) \in W_2^5(Q)$ for some set of sequences of functions $\omega_1, \omega_2, \ldots, \omega_N$, then acting on this sum with operator P, we obtain $\sum_{j=1}^N c_j P\omega_j = \sum_{j=1}^N c_j\phi_j = 0$, and it follows that for all $j = \overline{1,N}$ coefficients are $c_j = 0$. Note that the following conditions for function $\omega_j(x,t) \in W_2^5(Q)$ follow from the construction of function $\phi_j(x,t)$:

$$\gamma \cdot D_t^q\, \omega_j\big|_{t=0} = D_t^q\, \omega_j\big|_{t=T}, \quad q = 0,1,2,3,4, \tag{16}$$

$$D_x^p\omega_j\big|_{x=0} = D_x^p\omega_j\big|_{x=1}, \quad p = 0,1,2,3. \tag{17}$$

Now, an approximate solution to problem (8)–(10) is sought in the form $u_\varepsilon^N(x,t) = \sum_{j=1}^N c_j\omega_j(x,t)$, where coefficients c_j for any j from 1 to N are defined as a solution to the linear algebraic system:

$$2\int_Q L_\varepsilon u_\varepsilon^N \cdot \exp\left(-\frac{\lambda t}{2}\right)\phi_j dxdt = 2\int_Q f\cdot \exp\left(-\frac{\lambda t}{2}\right)\phi_j dxdt. \tag{18}$$

Let us prove the unique solvability of algebraic system (18). Multiplying each equation from (18) by c_j and summing over j from 1 to N, considering boundary conditions (16)–(17) and algebraic system (18), we obtain the following identity:

$$2\int_Q L_\varepsilon u_\varepsilon^N \cdot \exp(-\lambda t) u_{\varepsilon t}^N dxdt = 2\int_Q f\cdot\exp(-\lambda t) u_{\varepsilon t}^N dxdt, \tag{19}$$

from which, due to the conditions of Theorem 2, by integrating identity (19) we obtain estimate I) for the approximate solution of problem (8)–(10), i.e.

$$\varepsilon\cdot(\left\|u_{\varepsilon ttt}^N\right\|_0^2 + \left\|u_{\varepsilon ttx}^N\right\|_0^2 + \left\|u_{\varepsilon txx}^N\right\|_0^2) + \left\|u_\varepsilon^N\right\|_2^2 \le c_1\left\|f\right\|_0^2. \tag{20}$$

This implies the solvability of algebraic system (18) [9], [12]. By the weak compactness theorem [12], [17], estimate (20) allows us to pass to the limit at $N\to\infty$ and conclude that a certain subsequence $\{u_\varepsilon^{N_k}(x,t)\}$ weakly converges, due to the uniqueness of the solution (Theorem 1), in space $V(Q)$ to the sought-for solution to problem (8)–(10), which has the properties specified in Theorem 2 [12], [17]. For $u_\varepsilon(x,t)$, under (20), the following inequality holds:

$$\varepsilon\cdot(\left\|u_{\varepsilon ttt}\right\|_0^2 + \left\|u_{\varepsilon ttx}\right\|_0^2 + \left\|u_{\varepsilon txx}\right\|_0^2) + \left\|u_\varepsilon\right\|_2^2 \le c_1\left\|f\right\|_0^2. \tag{21}$$

Now, passing to the limit at $N \to \infty$ in (18), we obtain the only weak generalized solution to problem (8)–(10).

Let us prove the second a priori estimate II).

Using problem (11)–(15), from identity (18), we obtain:

$$-\frac{1}{\mu_j^2}\int_Q L_\varepsilon u_\varepsilon^N \cdot \exp\left(-\frac{\lambda t}{2}\right)\Delta^2 P\omega_j\,dxdt = -\frac{1}{\mu_j^2}\int_Q f \cdot \exp\left(-\frac{\lambda t}{2}\right)\Delta^2 P\omega_j\,dxdt. \qquad (22)$$

Multiplying each equation (22) by $2 \cdot \mu_j^2 c_j$, summing over j from 1 to N and considering conditions (16)–(17), from (22), we obtain the following identity:

$$-2\int_Q (L_\varepsilon u_\varepsilon^N - f)\cdot e^{-\lambda t} P u_\varepsilon^N\,dxdt = 0, \qquad (23)$$

where $Pu_\varepsilon^N \equiv \frac{\partial \Delta^2 u_\varepsilon^N}{\partial t} - 2\lambda\frac{\partial^2}{\partial t^2}\Delta u_\varepsilon^N + 3\lambda^2\frac{\partial}{\partial t}\Delta u_\varepsilon^N - \frac{\lambda}{2}u_{\varepsilon tt}^N + \frac{\lambda^2}{16}u_{\varepsilon t}^N$.

Integrating (23), under conditions of Theorem 2 and boundary conditions (16), (17), we obtain the following inequality:

$$\begin{aligned}
&c_2\|f\|_1^2 \ge \varepsilon\left\|\tfrac{\partial\Delta^2 u_\varepsilon^N}{\partial t}\right\|_0^2 + \textstyle\int_Q e^{-\lambda t}\Big\{-(2K_3 + K_{4t} + 3\lambda K_4)u_{\varepsilon tttt}^{2N}\\
&-(2K_3 - K_{4t} + 3\lambda K_4)u_{\varepsilon xxtt}^{2N} - (2K_3 + K_{4t} + 3\lambda K_4)u_{\varepsilon tttx}^{2N}\\
&+\lambda a u_{\varepsilon xxxx}^{2N} + \lambda b u_{\varepsilon xxtt}^{2N} + \lambda a u_{\varepsilon xxxt}^{2N}\Big\}dxdt\\
&+\rho\left\|u_\varepsilon^N\right\|_3^2 - N_1\sigma\left(\left\|u_{\varepsilon tttt}^{2N}\right\|_0^2 + \left\|u_{\varepsilon ttxx}^N\right\|_0^2 + \left\|u_{\varepsilon tttx}^N\right\|_0^2\right)\\
&-N_2\sigma\left(\left\|u_{\varepsilon xxxx}^N\right\|_0^2 + \left\|u_{\varepsilon ttxx}^N\right\|_0^2 + \left\|u_{\varepsilon txxx}^N\right\|_0^2\right) - c(\sigma^{-1},\lambda,K)\left\|u_\varepsilon^N\right\|_3^2\\
&+\textstyle\int_{\partial Q} e^{-\lambda t}B(u_\varepsilon^N(s),K_i(s))\,ds, \quad i=\overline{0,4}
\end{aligned} \qquad (24)$$

where ρ, N_i $(i = 1, 2)$ are the positive numbers, depending on the norm of function $K_i(x,t)$, $i = \overline{0,3}$, in space $C^3(\overline{Q})$, $K = \max\{\|K_4(t)\|_{C^3[0,T]}, \|K_i(x,t)\|_{C^2(\overline{Q})}\}$, σ, $c(\sigma^{-1})$ are the coefficients of Cauchy's inequality σ [12], $B(u_\varepsilon^N(s), K_i(s))$ are functions, depending on traces of function $u_\varepsilon^N(x,t)$, $K_i(x,t)$ on the boundary of domain Q. Let $\delta_0 = \min\{\delta_3, \lambda a, \lambda b, \lambda c, \delta_2, \delta_1\}$. We introduce the notation by $N = \max\{N_1, N_2\}$. Under conditions of Theorem 2 and boundary conditions (16), (17) and $\gamma^2 = e^{\lambda T}$, we obtain that in (24) the boundary integrals vanish. Now, choosing σ in such a way that $\delta_0 - N\sigma \ge \sigma_0 > 0$, $\rho - c(\sigma^{-1},\lambda,K) \ge \rho_0 > 0$, from inequality (24) we obtain the necessary second estimate:

$$\varepsilon\left\|\frac{\partial\Delta^2 u_\varepsilon^N}{\partial t}\right\|_0^2 + \left\|u_\varepsilon^N\right\|_4^2 \le c_2\cdot(\|f\|_0^2 + \|f_t\|_0^2) \le c_2\|f\|_1^2. \qquad (25)$$

The constant on the right side of (25) does not depend on N, therefore, from (25) the second estimate for the approximate solution of problem (8)–(10) follows. Estimate (21) together with estimate (25) allow us to pass to the limit at $N \to \infty$ and conclude that a certain subsequence $\{u_\varepsilon^{N_k}(x,t)\}$ converges weakly, due to the uniqueness of the solution to problem (8)–(10) in $V(Q)$, together with derivatives of the fourth and fifth orders, to the sought-for solution $u_\varepsilon(x,t)$ to problem (8)–(10), which has the properties specified in Theorem 2 [12], [17]. Therefore, under (25), the following inequality is true:

$$\varepsilon \left\| \frac{\partial}{\partial t}\Delta u_\varepsilon \right\|_0^2 + \|u_\varepsilon\|_2^2 \le c_2 \cdot (\|f\|_0^2 + \|f_t\|_0^2) \le c_2 \|f\|_1^2 . \tag{26}$$

This implies the existence of a regular generalized solution $u_\varepsilon(x,t)$ to problem (8)–(10) from space $V(Q)$. This proves Theorem 2.

3. Existence of a solution to problem (1)–(3).

Let us proceed to proving the solvability of problem (1)–(3).

Theorem 3. *Let all the conditions of Theorem 2 be satisfied. Then the solution to problem (1)–(3) from $W_2^4(Q)$ exists and it is unique.*

Proof. The uniqueness of the solution to problem (1)–(3) in the space $W_2^4(Q)$ is proven in Theorem 1. Now, we will prove the existence of a solution to problem (1)–(3) in the space $W_2^4(Q)$. To do so, we consider equation (8) and boundary conditions (9) and (10) for $\varepsilon > 0$ in the domain Q. Since all conditions of Theorem 2 are satisfied, there exists a unique regular solution to problem (8)–(10) for $\varepsilon > 0$ in $V(Q)$, and the first and second estimates hold for it. It follows that, from the set of functions $\{u_\varepsilon(x,t)\}$, $\varepsilon > 0$, a weakly convergent subsequence can be extracted in $V(Q)$, such that $\{u_{\varepsilon_i}(x,t)\} \to u(x,t)$ as $\varepsilon_i \to 0$. We show that the limit function $u(x,t)$ satisfies the equation $Lu = f$ (equation (1)) almost everywhere in the domain Q. Indeed, since the subsequence $\{u_{\varepsilon_i}(x,t)\}$ converges weakly in $W_2^4(Q)$, the subsequence $\{\sqrt{\varepsilon_i}\frac{\partial \Delta^2 u_{\varepsilon_i}(x,t)}{\partial t}\}$ is uniformly bounded in $L_2(Q)$, and the operator L is linear, we have

$$Lu - f = Lu - Lu_{\varepsilon_i} + \varepsilon_i \frac{\partial \Delta^2 u_{\varepsilon_i}}{\partial t} = L(u - u_{\varepsilon_i}) + \varepsilon_i \frac{\partial \Delta^2 u_{\varepsilon_i}}{\partial t}. \tag{27}$$

From equality (27), passing to the limit at $\varepsilon_i \to 0$, we obtain a unique solution to problem (1)–(3) [3]-[5], [9]. Thus, Theorem 3 is proven.

4. Smoothness of the solution to problem (1)–(3).

Now let us study the smoothness of the solution to problem (1)–(3) in Sobolev spaces $W_2^{m+4}(Q)$, when $0 \le m$ is the finite integer. Below, for simplicity, we assume that the coefficients of equation (1) are sufficiently differentiable functions in closed domain $\overline{Q}$.

Theorem 4. *Let the conditions of Theorem 3 be satisfied. In addition, let $p = 0,1,2,3,\dots,m$, $q = 0,1,2,3,\dots,m$, with $-2(K_3 + mK_{4t}) - (2j-3)K_{4t} + \lambda K_4 \ge \delta > 0$, $j = 0,1,2$, for all $(x,t) \in \overline{Q}$, and let $D_t^{p+1} K_4|_{t=0} = D_t^{p+1} K_4|_{t=T}$, $D_t^q K_i|_{t=0} = D_t^q K_i|_{t=T}$ for $i = 0,1,2,3$. Then, for any function $f(x,t) \in W_2^{m+1}(Q)$ such that $\gamma D_t^q f|_{t=0} = D_t^q f|_{t=T}$ for all $x \in [0,1]$, there exists a unique solution to problem (1)–(3) in the Sobolev space $W_2^{m+4}(Q)$, where $m = 0,1,2,3,\dots$ is a finite integer.*

Proof. Taking into account the conditions of Theorems 2 and 3 for $\varepsilon > 0$ and nonlocal conditions at $t = 0$, $t = T$, from the following equality

$$(e^{-\frac{\lambda t}{2}} \cdot L_\varepsilon u_\varepsilon)\big|_{t=0}^{t=T} = \left(-\varepsilon \cdot e^{-\frac{\lambda t}{2}} \cdot \frac{\partial}{\partial t}\Delta^2 u_\varepsilon + e^{-\frac{\lambda t}{2}} \cdot Lu_\varepsilon\right)\big|_{t=0}^{t=T} = (e^{-\frac{\lambda t}{2}} \cdot f(x,t))\big|_{t=0}^{t=T}$$

we obtain $\gamma \cdot D_t^5 u_\varepsilon|_{t=0} = D_t^5 u_\varepsilon|_{t=T}$.

It follows that function $\vartheta_\varepsilon(x,t) = u_{\varepsilon t}(x,t)$ belongs to class $V(Q)$ and satisfies the following equation:

$$T_\varepsilon \vartheta_\varepsilon \equiv L_\varepsilon \vartheta_\varepsilon + K_{4t}\vartheta_{\varepsilon ttt} = f_t - \sum_{i=0}^{3} K_{it} D_t^i u_\varepsilon \equiv F_\varepsilon. \tag{28}$$

From Theorem 3, it follows that the family of functions $\{F_\varepsilon\}$ is uniformly bounded in space $L_2(Q)$,

$$\|F_\varepsilon\|_0^2 \le c_1 \cdot \|f\|_1^2 .$$

Then under conditions of Theorem 3, it is easy to obtain that the coefficients of operator $T_\varepsilon(\varepsilon > 0)$ satisfy the conditions of Theorem 4; hence, based on estimates I), II) and Theorem 3 for function $\{\vartheta_\varepsilon(x,t)\}$, we obtain similar estimates:

$$\varepsilon \cdot (\|\vartheta_{\varepsilon ttt}\|_0^2 + \|\vartheta_{\varepsilon ttx}\|_0^2 + \|\vartheta_{\varepsilon txx}\|_0^2) + \|\vartheta_\varepsilon\|_2^2 \le c_1 \|f\|_0^2 , \tag{29}$$

$$\varepsilon \left\| \frac{\partial}{\partial t}\Delta^2 \vartheta_\varepsilon \right\|_0^2 + \|\vartheta_\varepsilon\|_4^2 \le c_2 \|f\|_1^2 . \tag{30}$$

Then, functions $\{u_\varepsilon\}$ satisfy the following parabolic equation:

$$\Pi u_\varepsilon \equiv u_{\varepsilon t} - u_{\varepsilon xx} = f + \varepsilon \frac{\partial}{\partial t}\Delta^2 u_\varepsilon - \sum_{i=0}^{4} K_i D_t^i u_\varepsilon - M u_\varepsilon + u_{\varepsilon t} - u_{\varepsilon xx} \equiv \Phi_\varepsilon, \tag{31}$$

with conditions:

$$\gamma \, u_\varepsilon|_{t=0} = u_\varepsilon|_{t=T} , \tag{32}$$

$$D_x^p u_\varepsilon|_{x=0} = D_x^p u_\varepsilon|_{x=1} , \quad p = 0, 1, \tag{33}$$

and $\Phi_\varepsilon \in W_2^1(Q)$; by virtue of what was proven above, the family of functions $\{\Phi_\varepsilon\}$ is uniformly bounded in space $W_2^1(Q)$, i.e.

$$\|\Phi_\varepsilon\|_1^2 \le c_2 \cdot (\|f\|_1^2 + \|f_{tt}\|_0^2) \le c_2 \|f\|_2^2 < c_4 \|f\|_4^2 . \tag{34}$$

Hence, based on a priori estimates for parabolic equations [3],[9],[12] and inequality (34), we obtain:

$$\|u_\varepsilon\|_5^2 \le c_4 \|f\|_4^2 . \tag{35}$$

Repeating similar reasoning, we prove the following inequalities:

$$\|u_\varepsilon\|_{m+1}^2 \le c_{m+2} \|f\|_m^2 , \quad m = 1, 2, 3, \ldots \tag{36}$$

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REZYUME

Ushbu maqolada biz modifitsirlangan Galerkin usuli, aprior baholash va "ε-regulyarizatsiya" usullari yordamida Sobolev fazolarida to'rtinchi tartibli ikkinchi tur aralash tipdagi tenglama uchun davriy nolokal chegaraviy masala yechimining mavjudligi va yagonaligini hamda silliqligini o'rganamiz.

Kalit so'zlar: aralash tipdagi tenglama, nolokal chegaraviy masala, "ε-regulyarizatsiya" usuli, Faedo-Galerkin usuli, Sobolev fazolari.

РЕЗЮМЕ

В данной работе с использованием модифицированного метода Галеркина, методов априорных оценок и "ε-регуляризации" исследуется однозначная разрешимость и гладкость регулярного обобщенного решения нелокальной краевой задачи периодического типа для уравнения смешанного типа второго рода четвертого порядка в пространствах Соболева.

Ключевые слова: уравнения смешанного типа, нелокальная краевая задача, метод "ε-регуляризации", метод Фаэдо-Галеркина, пространства Соболева.