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2D MOORE CA WITH NEW BOUNDARY CONDITIONS AND ITS REVERSIBILITY

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Abstract

In this paper, we examine two-dimensional cellular automata with the Moore neighborhood under specific conditions. Specifically, we delve into the characterization of 2D linear cellular automata defined by the Moore neighborhood, considering mixed boundary conditions over the field $\mathbb{Z}_p$. Lastly, we present the conditions that lead to the reversibility of the obtained rule matrices for 2D finite CAs.

Key words: cellular automata, boundary conditions, rule matrix, reverisibility.

Introduction

It is understood that a cellular automaton (CA) comprises a collection of cells organized in a grid with a specific shape. Each cell undergoes state changes over time according to predetermined rules influenced by the states of neighboring cells. Cellular automata (CAs) have been suggested for various applications including public-key cryptography, as well as in disciplines such as geography, anthropology, political science, sociology, physics, and others (see references [1], [2], [3]).

A system's configuration entails assigning states to all its cells. Each configuration dictates the subsequent one through a transition rule that operates locally, meaning a cell's state at time $(t+1)$ is solely influenced by the states of certain neighboring cells at time t. Under certain boundary conditions and with a linear transition rule, various outcomes emerge (refer to [13]). Typically, 2D cellular automata are explored within triangular, square, hexagonal, and pentagonal lattices (see [4], [8], [10], [11], [12]).

The mathematical representation of 2D finite cellular automata enables the description of these studied systems. Of crucial importance is determining the reversibility or irreversibility of these automata. Reversibility is a critical characteristic signifying the absence of Gardens of Eden in cellular automata. A reversible cellular automaton ensures that each configuration possesses a unique predecessor. It has been established that determining the reversibility of cellular automata for dimensions equal to or greater than two is undecidable (see [8], [9], [10], [11], [13]). This indicates that, in general, deriving the inverse of a given cellular automaton for higher dimensions via an algorithm is unattainable due to its intricate structure. Consequently, it is evident that determining inverses or instances of reversibility for 2D finite cellular automata presents a formidable challenge in the broader context ([5]).

In this paper, we continue to study 2D CA for Moore neighbors on the square lattice with mixing boundary conditions in [7]. These CA are investigated under new types of boundary conditions with the p-state spin value case, i.e., over the field $\mathbb{Z}_p$. We obtain the transition rule matrices of the Moore finite CA over some mixed boundary conditions. Then, we give the conditions under which the obtained rule matrices of 2D finite CAs are reversible.

Preliminary

The 2D finite CA consists of $m \times n$ cells arranged in m rows and n columns, where each cell takes one of the values of the field $\mathbb{Z}_p$. From now on, we will denote 2D finite CA order to $m \times n$ by 2D $\mathrm{CA_{m\times n}}$. A configuration of the system is an assignment of the states to all cells. Every configuration determines a next configuration via a linear transition rule that is local in the sense that the state of a cell at time $(t+1)$ depends only on the states of some of its neighbors at the time t using modulo p algebra.

The von Neumann and Moore neighborhood on CA lattice.

In 2D CA's theory, there are some classic types of neighborhoods, but in this paper we only restrict ourselves to the Moore neighborhood. This neighborhood was used in the well known Conway's Game of Life. It is similar to the notion of 8-connected pixels in computer graphics. In Figure 4, we illustrate the von Neumann and Moore neighborhoods. The von Neumann neighborhood the center cell is surrounded by four square cells (see Figure 4 (left)). The Moore neighborhood comprises eight square cells which surround the center cell $x_{(i;j)}$ (see Figure 4 (right)). From now on, we deal only with Moore neighborhood. Then the state $x_{i;j}^{(t+1)}$ of the cell $(i;j)$th at time $(t+1)$ is defined by the local rule function $\psi : \mathbb{Z}_p^8 \to \mathbb{Z}_p$ as follows:

$$
\begin{aligned}
x_{i,j}^{(t+1)} &= \psi(x_{i-1,j-1}, x_{i-1,j}, x_{i-1,j+1}, x_{i,j+1}, x_{i+1,j+1}, x_{i+1,j}, x_{i+1,j-1}, x_{i,j-1}) \\
&= ax_{i-1,j-1}^{(t)} + bx_{i-1,j}^{(t)} + cx_{i-1,j+1}^{(t)} + dx_{i,j+1}^{(t)} + ex_{i+1,j+1}^{(t)} + fx_{i+1,j}^{(t)} \\
&\qquad + gx_{i+1,j-1}^{(t)} + hx_{i,j-1}^{(t)} \pmod p
\end{aligned} \tag{1}
$$

where $a, b, c, d, e, f, g, h \in \mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$.

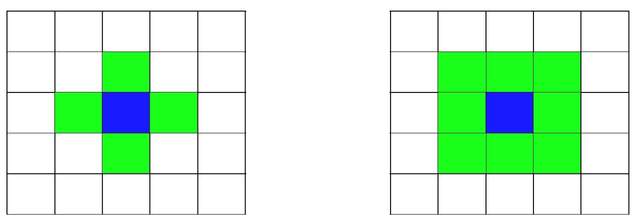

Рис. 4: von Neumann and Moore neighborhoods

The value of each cell for the next state may not depend upon all eight neighbors.

Remark 1. *If we assume $a = c = e = g = 0$, then all obtained above results hold for von Neumann neighborhood.*

Note that it is impossible to simulate a truly infinite lattice on a computer (unless the active region always remains finite). Therefore, we have to prescribe some boundary conditions (BC). Regarding the neighborhood of the boundary cells, four approaches exist:

- If the boundary cells are connected to 0-state, then CA is called *null boundary (NB) CA* (see Table 2A).
- If the boundary cells are adjacent to each other, then CA is called *periodic boundary (PB) CA* (see Table 2B).
- *An Adiabatic Boundary (AB) CA* is duplicating the value of the cell in an extra virtual neighbor (see Table 2C).
- *A Reflexive Boundary (RB) CA* is designed for the value of the left and right neighbors to be equal concerning the boundary cell (see Table 2D).

The rule matrix of Moore CA and mixed boundary condition

Now, we can characterize the rule matrix T_{R} under null boundary conditions. In order to characterize the corresponding rule, first we represent each state matrix of size $m \times n$ as a column vector of size $mn \times 1$. If the

0	0	0	0	0
0		$\mathcal{CA}_{3\times 3}$		0
0				0
0				0
0	0	0	0	0

(A)

$x_{(i+1,j+1)}$	$x_{(i+1,j-1)}$	$x_{(i+1,j)}$	$x_{(i+1,j+1)}$	$x_{(i+1,j-1)}$
$x_{(i-1,j+1)}$		$\mathcal{CA}_{3\times 3}$		$x_{(i-1,j-1)}$
$x_{(i,j+1)}$				$x_{(i.j-1)}$
$x_{(i+1,j+1)}$				$x_{(i+1,j-1)}$
$x_{(i-1,j+1)}$	$x_{(i-1,j-1)}$	$x_{(i-1,j)}$	$x_{(i-1,j+1)}$	$x_{(i-1,j-1)}$

(B)

$x_{(i-1,j-1)}$	$x_{(i-1,j-1)}$	$x_{(i-1,j)}$	$x_{(i-1,j+1)}$	$x_{(i-1,j+1)}$
$x_{(i-1,j-1)}$		$\mathcal{CA}_{3\times 3}$		$x_{(i-1,j+1)}$
$x_{(i,j-1)}$				$x_{(i,j+1)}$
$x_{(i+1,j-1)}$				$x_{(i+1,j+1)}$
$x_{(i+1,j-1)}$	$x_{(i+1,j-1)}$	$x_{(i+1,j)}$	$x_{(i+1,j+1)}$	$x_{(i+1,j+1)}$

(C)

$x_{(i,j)}$	$x_{(i,j-1)}$	$x_{(i,j)}$	$x_{(i,j+1)}$	$x_{(i,j)}$
$x_{(i-1,j)}$		$\mathcal{CA}_{3\times 3}$		$x_{(i-1,j)}$
$x_{(i,j)}$				$x_{(i,j)}$
$x_{(i+1,j)}$				$x_{(i+1,j)}$
$x_{(i,j)}$	$x_{(i,j-1)}$	$x_{(i,j)}$	$x_{(i,j+1)}$	$x_{(i,j)}$

(D)

Таблица 2: Boundary conditions on a 2D finite $\mathcal{CA}_{3\times 3}$: (A) – null; (B) – periodic; (C) – adiabatic; (D) – reflexive.

same rule is applied to all the cells in each evaluation, then those CA is called *uniform or regular*. Throughout the paper we deal with uniform CA.

Thus, the problem of finding a rule matrix of the corresponding rule is taken from the space of $m \times n$ matrices to the space of $\mathbb{Z}_p^{mn}$. In order to describe this problem more detailly we define the following map:

$$\varphi\colon \mathbf{M}_{m\times n}(\mathbb{Z}_p) \longrightarrow \mathbf{M}_{mn\times 1}(\mathbb{Z}_p)$$

which takes the t-th state $X^{(t)}$ given by

$$C(t) := \begin{pmatrix} x_{11}^{(t)} & x_{12}^{(t)} & \dots & x_{1n}^{(t)} \\ x_{21}^{(t)} & x_{22}^{(t)} & \dots & x_{2n}^{(t)} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1}^{(t)} & x_{m2}^{(t)} & \dots & x_{mn}^{(t)} \end{pmatrix} \longrightarrow X^{(t)} := (x_{11}^{(t)}, x_{12}^{(t)}, \dots, x_{1n}^{(t)}, \dots, x_{m1}^{(t)}, \dots, x_{mn}^{(t)})^T. \quad (2)$$

where the superscript T denotes the transpose and $\mathbf{M}_{m\times n}(\mathbb{Z}_p)$ is the set of matrices with entries $\{0, 1, 2, \dots, p-1\}$.

Thus, local rules will be assumed to act on $\mathbb{Z}_p^{mn}$ rather than $\mathbf{M}_{m\times n}(\mathbb{Z}_p)$. The matrix $C(t)$ is called *the configuration matrix (or information matrix)* of the 2-D finite CA at the time t and $C(0)$ is the initial information matrix of the 2-D finite CA. Therefore, one can conclude that $\varphi(C(t)) = X^{(t)}$.

Using the identification (2), we can define

$$T \cdot X^{(t)} = X^{(t+1)} \pmod p.$$

Let $e_{i,j} \in \mathbf{M}_{m\times n}(\mathbb{Z}_p^*)$ be the matrix units. Consider the following two sets:

$$X = \{e_{i,j},\ 1 \le i \le n,\ 1 \le j \le m\}, \quad Y = \{e_{0,i}, e_{m+1,i}, e_{j,0}, e_{j,n+1},\ 0 \le i \le n,\ 0 \le j \le m\}.$$

We define $\alpha, \eta, \pi, \rho\colon Y \to X$ mappings by the boundary conditions in the Table 2. Namely, α is adiabatic BC, η is null BC, π is periodic BC and ρ is reflexive BC.

Now consider a mapping $\varphi\colon \Gamma \to \Gamma$ where $\Gamma = \{\alpha, \eta, \pi, \rho\}$.Then we study the CA under boundary conditions that depends on the mapping φ. In other words, the boundary cells are evaluated depending upon $\varphi(x),\ x \in \Gamma$ (see Figure 5). A mixed boundary condition is defined when the boundary sides of the lattice are not in the same boundary condition, that is, as in Figure 6. I, II, III, and IV type boundary conditions in Figure 6 are described one of the boundary conditions from the set Γ.

In the paper [6] for the bijective function φ (i.e. the boundary condition as the form (A) in Figure 6) the characterization problem of 2D finite von Neumann CA is completely solved. We divide the other forms of mixed boundary conditions into two groups. The first group of mixed boundary condition consists two elements of the set Γ. The second one contains three elements of Γ. As showed in Figure 6, the first group of mixed boundary condition is illustrated in three appearances, i.e. (D), (E) and (F). In this paper we shall plan to research CA with boundary conditions of forms (E) and (F).

The characterization of CA with boundary condition of the form (E)

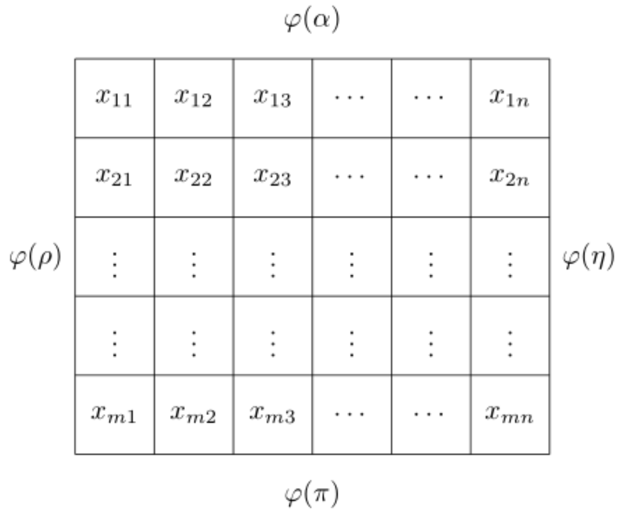

Рис. 5: Mixed boundary conditions

Considering that Γ has four elements and the form (E), then 12 types of mixing boundary conditions can be present. First, we give the rule matrices of 2D CA under the boundary conditions of all possible cases in the form (E).

Let us define ψ_E is non-bijective maps on Γ as

$$\psi_E(\alpha) = \psi_E(\pi) = \eta, \quad \psi_E(\rho) = \psi_E(\eta) = \rho, \tag{3}$$

If we study Moore neighborhood under the condition (3) there is ambiguity with setting boundary condition in the cells $x_{0,0}$, $x_{0,n+1}$, $x_{m+1,0}$, $x_{m+1,n+1}$. In order to distinguish this unclearness on the condition (3) we fix null boundary conditions for those cells.

First, we use some denotations. By T we refer to the rule matrix 2D $\mathcal{CA}_{m\times n}$ under the null boundary condition and this matrix T with the order $mn \times mn$ is

$$T = \begin{pmatrix} A & B & O & O & \dots & O & O & O \\ C & A & B & O & \dots & O & O & O \\ O & C & A & B & \dots & O & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & O & \dots & C & A & B \\ O & O & O & O & \dots & O & C & A \end{pmatrix}, \tag{4}$$

where

$$A = d\sum_{i=1}^{n-1}\epsilon_{i,i+1} + h\sum_{i=1}^{n-1}\epsilon_{i+1,i}, \quad B = f\sum_{i=1}^{n}\epsilon_{i,i} + e\sum_{i=1}^{n-1}\epsilon_{i,i+1} + g\sum_{i=1}^{n-1}\epsilon_{i+1,i},$$

$$C = b\sum_{i=1}^{n}\epsilon_{i,i} + c\sum_{i=1}^{n-1}\epsilon_{i,i+1} + a\sum_{i=1}^{n-1}\epsilon_{i+1,i},$$

$\epsilon_{i,j} \in \mathbf{M}_{n\times n}(\mathbb{Z}_p)$ are the unit matrices and $a, b, c, d, e, f, g, h \in \mathbb{Z}_p$.

To establish the transition rule matrix T_ψ^E structure under the boundary conditions defined by (3), it is needed to specify the action of T_ψ^E on the basis matrices $e_{i,j}$, respectively. Firstly, let us take the linear transition T_ψ^E from $m \times n$ matrix space structure to itself. The images $T_\psi^E(e_{i,j})$ of $e_{i,j}$ are connected to the four nearest neighbor elements considering the Moore neighborhood. Note that the boundary condition ψ does not play role for non-border cells. Hence, $T_\psi^E(e_{i,j})$ elements are equal to a linear sum of its eight neighbor elements. Thus, for non-border elements we have

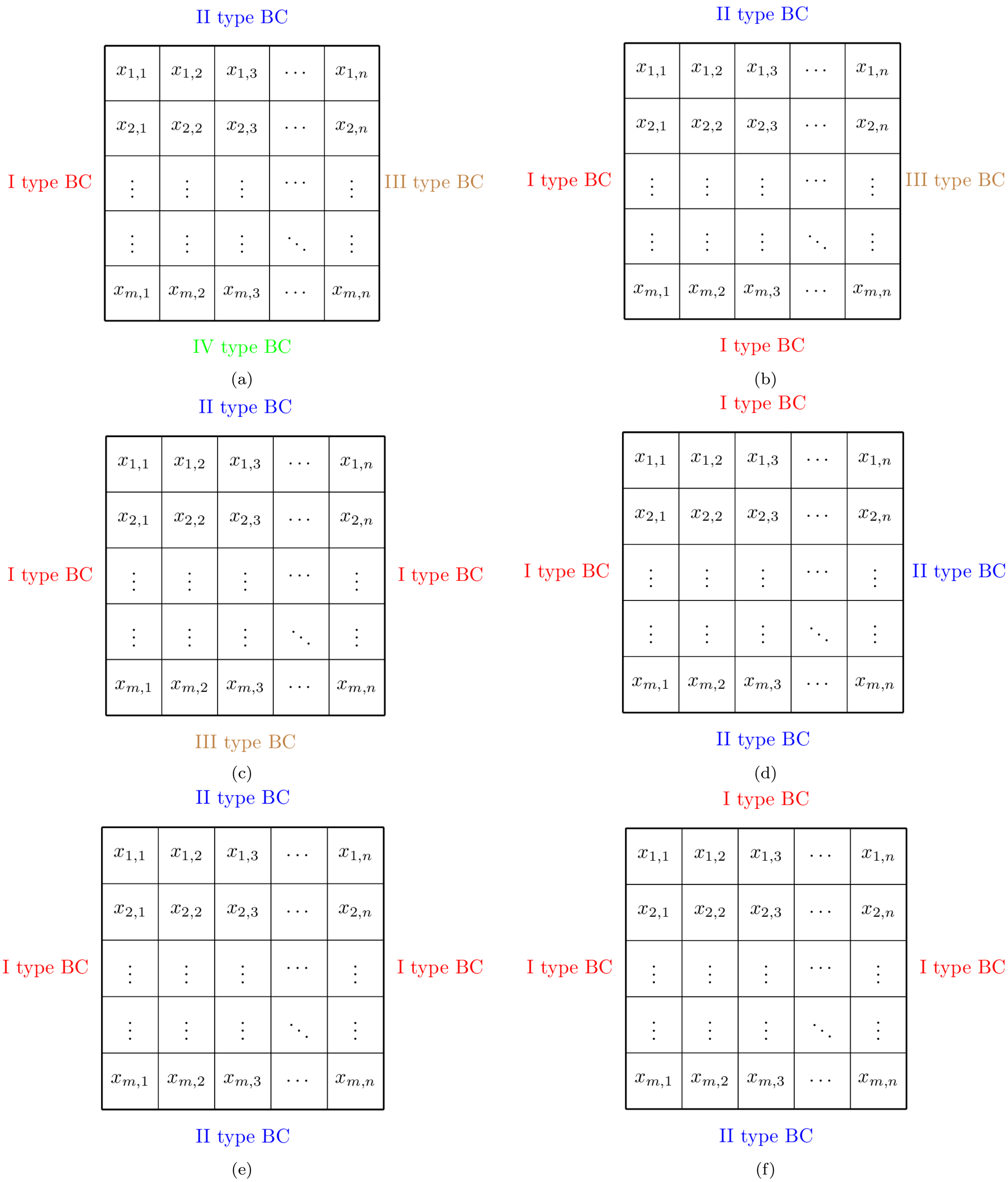

Рис. 6: Mixed boundary conditions: (A) all sides of the lattice have different types of boundary conditions; (B)-(C) three types of boundary conditions on the sides of the lattice; (D)-(E)-(F) two types of boundary conditions on the sides of the lattice.

$$T^E_\psi(e_{i,j}) = T(e_{i,j}) = ae_{i-1,j-1} + be_{i-1,j} + ce_{i-1,j+1} + de_{i,j+1} + ee_{i+1,j+1} + fe_{i+1,j} + ge_{i+1,j-1} + he_{i,j-1}. \quad (5)$$

Now, we define the action of T_ψ^E on the border elements. Border cells $e_{1,1}, e_{1,n}, e_{m,1}, e_{m,n}$ have five neighbors out of the configuration and cells $x_{0,0}$, $x_{0,n+1}$, $x_{m+1,0}$, $x_{m+1,n+1}$ are under null boundary conditions in mentioned above. Then we obtain

$$\begin{array}{rcl} T_\psi^E(e_{1,1}) & = & T(e_{1,1}) + b\psi(e_{0,1}) + c\psi(e_{0,2}) + h\psi(e_{1,0}) + g\psi(e_{0,2}), \\ T_\psi^E(e_{1,n}) & = & T(e_{1,n}) + a\psi(e_{0,n-1}) + b\psi(e_{0,n}) + d\psi(e_{1,n+1}) + e\psi(e_{2,n+1}), \\ T_\psi^E(e_{m,1}) & = & T(e_{m,1}) + a\psi(e_{m-1,0}) + d\psi(e_{m,0}) + f\psi(e_{m+1,1}) + e\psi(e_{m+1,2}), \\ T_\psi^E(e_{m,n}) & = & T(e_{m,n}) + c\psi(e_{m-1,n+1}) + d\psi(e_{m,n+1}) + f\psi(e_{m+1,n}) + g\psi(e_{m+1,n-1}), \end{array}$$

where $a,b,c,d,e,f,g,h \in \mathbb{Z}_p$.

Moreover, the border elements excepting $e_{1,1}, e_{1,n}, e_{m,1}, e_{m,n}$ have three neighbors out of the configuration. Thus, we get the following:

$$\begin{array}{rcl} T_\psi^E(e_{1,i}) & = & T(e_{1,i}) + a\psi(e_{0,i-1}) + b\psi(e_{0,i}) + c\psi(e_{0,i+1}), \\ T_\psi^E(e_{m,i}) & = & T(e_{m,i}) + g\psi(e_{m+1,i-1}) + f\psi(e_{m+1,i}) + e\psi(e_{m+1,i+1}), 2 \le i \le n-1 \\ T_\psi^E(e_{j,1}) & = & T(e_{j,1}) + a\psi(e_{j-1,0}) + h\psi(e_{j,0}) + g\psi(e_{j+1,0}), \\ T_\psi^E(e_{j,n}) & = & T(e_{j,n}) + c\psi(e_{j-1,n+1}) + d\psi(e_{j,n+1}) + e\psi(e_{j+1,n+1}), 2 \le j \le m-1, \end{array}$$

where $2 \le i \le n-1,\ 2 \le j \le m-1$ and $a,b,c,d,e,f,g,h \in \mathbb{Z}_p$.

If we consider the function ψ in (3) and the boundary conditions in Table 2, then the following result is true.

Theorem 1. *Let $T_\psi^E \colon \mathbb{Z}_p^{mn} \to \mathbb{Z}_p^{mn}$ be the rule matrix which takes the finite Moore CA over the configuration $C(t)$ of order $m \times n$ to the configuration $C(t+1)$ under the boundary condition of ψ_E. Then T_ψ^E has the following matrix form:*

$$T_\psi^E = T + \begin{pmatrix} A_1 & B_1 & O & O & \dots & O & O & O \\ C_1 & A_1 & B_1 & O & \dots & O & O & O \\ O & C_1 & A_1 & B_1 & \dots & O & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & O & \dots & C_1 & A_1 & B_1 \\ O & O & O & O & \dots & O & C_1 & A_1 \end{pmatrix}, \tag{6}$$

where $A_1 = h\epsilon_{1,2} + d\epsilon_{n,n-1}$, $B_1 = g\epsilon_{1,2} + e\epsilon_{n,n-1}$, $C_1 = a\epsilon_{1,2} + c\epsilon_{n,n-1}$, $O, \epsilon_{1,2}, \epsilon_{n,n-1} \in \mathbf{M}_{n\times n}(\mathbb{Z}_p)$, O is the zero matrix and $\epsilon_{1,2}$, $\epsilon_{n,n-1}$ are unit matrices.

Доказательство. Firstly, let us take the linear transition $T_\psi^E \colon \mathbf{M}_{m\times n}(\mathbb{Z}_p) \to \mathbf{M}_{m\times n}(\mathbb{Z}_p)$. The image $T_\psi^E(e_{i,j})$ of $e_{i,j}$ is connected to the four nearest neighbor elements considering the von Neumann neighborhood. Hence $T_\psi^E(e_{i,j})$ elements are equal to a linear sum of its five neighbor elements.

Let us denote by $E_{(i-1)n+j} = e_{i,j}$, $1 \le i \le m$, $1 \le j \le n$, the row vector $1 \times mn$ whose has the $((i-1)n+j)$-th (or (i,j)-th in matrix form) entry equals to 1 and the others are equal to zero. Then we have

$$T_\psi^E \cdot \begin{pmatrix} E_1 \\ \vdots \\ E_n \\ \vdots \\ \vdots \\ \vdots \\ E_{mn} \end{pmatrix} = T_\psi^E \cdot \begin{pmatrix} e_{1,1} \\ \vdots \\ e_{1,n} \\ \vdots \\ e_{i,j} \\ \vdots \\ e_{m,1} \\ \vdots \\ e_{m,n} \end{pmatrix} = \begin{pmatrix} T_\psi^E(e_{1,1}) \\ \vdots \\ T_\psi^E(e_{1,n}) \\ \vdots \\ T_\psi^E(e_{i,j}) \\ \vdots \\ T_\psi^E(e_{m,1}) \\ \vdots \\ T_\psi^E(e_{m,n}) \end{pmatrix} = \begin{pmatrix} (d+h)e_{1,2} + (e+g)e_{2,2} + fe_{2,1} \\ \vdots \\ (d+h)e_{1,n-1} + (e+g)e_{2,n-1} + fe_{2,n} \\ \vdots \\ ae_{i-1,j-1} + be_{i-1,j} + ce_{i-1,j+1} + de_{i,j+1} \\ +ee_{i+1,j+1} + fe_{i+1,j} + ge_{i+1,j-1} + he_{i,j-1} \\ \vdots \\ (a+c)e_{m-1,2} + be_{m-1,1} + (d+h)e_{m,2} \\ \vdots \\ (a+c)e_{m-1,n-1} + be_{m-1,n} + (d+h)e_{m,n-1} \end{pmatrix} =$$

$$\begin{pmatrix} A_1 & B_1 & O & O & \dots & O & O & O \\ C_1 & A_1 & B_1 & O & \dots & O & O & O \\ O & C_1 & A_1 & B_1 & \dots & O & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & O & \dots & C_1 & A_1 & B_1 \\ O & O & O & O & \dots & O & C_1 & A_1 \end{pmatrix} \cdot \begin{pmatrix} E_1 \\ \vdots \\ \vdots \\ \vdots \\ E_{mn} \end{pmatrix}.$$

Hence, the transition of the representation of matrix related to the equations above presented in () is obtained. So, the proof is complete. □

The matrix in Theorem 1 is described null-reflexive the boundary condition as the type (E). Now we give the rule matrices for null-periodic, null-adiabatic, periodic-reflexive, adiabatic-reflexive, adiabatic-periodic cases as the type (E). Let us formulate these boundary conditions accordingly using the following functions:

$$\begin{aligned} \mu_E(\alpha) = \mu_E(\pi) = \eta, \quad \mu_E(\rho) = \mu_E(\eta) = \pi, \\ \nu_E(\alpha) = \nu_E(\pi) = \eta, \quad \nu_E(\rho) = \nu_E(\eta) = \alpha, \\ \omega_E(\alpha) = \omega_E(\pi) = \pi, \quad \omega_E(\rho) = \omega_E(\eta) = \rho, \\ \tau_E(\alpha) = \tau_E(\pi) = \alpha, \quad \tau_E(\rho) = \tau_E(\eta) = \rho, \\ \gamma_E(\alpha) = \gamma_E(\pi) = \alpha, \quad \gamma_E(\rho) = \gamma_E(\eta) = \pi. \end{aligned}$$

Then the following theorem gives the rule matrices of 2D CA under the boundary conditions $\mu_E, \nu_E, \omega_E, \tau_E, \gamma_E$.

Theorem 2. *Let $T\colon \mathbb{Z}_p^{mn} \to \mathbb{Z}_p^{mn}$ be the rule matrices which takes the finite Moore CA over the configuration $C(t)$ of order $m \times n$ to the configuration $C(t+1)$ under the boundary conditions of $\mu_E, \nu_E, \omega_E, \tau_E, \gamma_E$. Then T_{R} have the following matrix forms:*

$$T_\mu^E = \begin{pmatrix} A_2 & B_2 & O & O & \dots & O & O & O \\ C_2 & A_2 & B_2 & O & \dots & O & O & O \\ O & C_2 & A_2 & B_2 & \dots & O & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & O & \dots & C_2 & A_2 & B_2 \\ O & O & O & O & \dots & O & C_2 & A_2 \end{pmatrix}, \qquad T_\nu^E = \begin{pmatrix} A_3 & B_3 & O & O & \dots & O & O & O \\ C_3 & A_3 & B_3 & O & \dots & O & O & O \\ O & C_3 & A_3 & B_3 & \dots & O & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & O & \dots & C_3 & A_3 & B_3 \\ O & O & O & O & \dots & O & C_3 & A_3 \end{pmatrix}$$

$$T_\omega^E = \begin{pmatrix} A_1 & B_1 & O & O & \dots & O & O & C_2 \\ C_1 & A_1 & B_1 & O & \dots & O & O & O \\ O & C_1 & A_1 & B_1 & \dots & O & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & O & \dots & C_1 & A_1 & B_1 \\ B_2 & O & O & O & \dots & O & C_1 & A_1 \end{pmatrix}, \qquad T_\tau^E = \begin{pmatrix} A_1 + C_3 & B_1 & O & O & \dots & O & O & O \\ C_1 & A_1 & B_1 & O & \dots & O & O & O \\ O & C_1 & A_1 & B_1 & \dots & O & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & O & \dots & C_1 & A_1 & B_1 \\ O & O & O & O & \dots & O & C_1 & A_1 + B_3 \end{pmatrix}$$

$$T_\gamma^E = \begin{pmatrix} A_2 + C_3 & B_2 & O & O & \dots & O & O & O \\ C_2 & A_2 & B_2 & O & \dots & O & O & O \\ O & C_2 & A_2 & B_2 & \dots & O & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & O & \dots & C_2 & A_2 & B_2 \\ O & O & O & O & \dots & O & C_2 & A_2 + B_3 \end{pmatrix},$$

where $\begin{array}{lll} A_2 = A + h\epsilon_{1,n} + d\epsilon_{n,1}, & B_2 = B + g\epsilon_{1,n} + e\epsilon_{n,1}, & C_2 = C + a\epsilon_{1,n} + c\epsilon_{n,1}, \\ A_3 = A + h\epsilon_{1,1} + d\epsilon_{n,n}, & B_3 = B + g\epsilon_{1,1} + e\epsilon_{n,n}, & C_3 = C + a\epsilon_{1,1} + c\epsilon_{n,n}, \end{array}$

$O, \epsilon_{1,1}, \epsilon_{n,n} \in \mathbf{M}_{n\times n}(\mathbb{Z}_p)$, *$O$ is the zero matrix and $\epsilon_{1,2}$, $\epsilon_{n,n-1}$ are unit matrices.*

In the beginning of the section, we refer that the number of all possible cases as the form (E) is 12. Above we gave rule matrices for 6 cases. The remaining 6 cases are generated by permutating the previous pairing boundary conditions with each other. Next theorem gives the rule matrices 2D CA under the boundary conditions $\psi_E, \mu_E, \nu_E, \omega_E, \tau_E, \gamma_E$ by rotating 180° degree in the lattice. We denote this boundary conditions by $\psi'_E, \mu'_E, \nu'_E, \omega'_E, \tau'_E, \gamma'_E$.

Theorem 3. Let $T\colon \mathbb{Z}_p^{mn} \to \mathbb{Z}_p^{mn}$ be the rule matrices which takes the finite Moore CA over the configuration $C(t)$ of order $m \times n$ to the configuration $C(t+1)$ under the boundary conditions of $\psi'_E, \mu'_E, \nu'_E, \omega'_E, \tau'_E, \gamma'_E$. Then T have the following matrix forms:

$$T^E_{\psi'} = \begin{pmatrix} A & B+C & O & O & \dots & O & O & O \\ C & A & B & O & \dots & O & O & O \\ O & C & A & B & \dots & O & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & O & \dots & C & A & B \\ O & O & O & O & \dots & O & B+C & A \end{pmatrix}, \quad T^E_{\mu'} = \begin{pmatrix} A & B & O & O & \dots & O & O & C \\ C & A & B & O & \dots & O & O & O \\ O & C & A & B & \dots & O & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & O & \dots & C & A & B \\ B & O & O & O & \dots & O & C & A \end{pmatrix},$$

$$T^E_{\nu'} = \begin{pmatrix} A+C & B & O & O & \dots & O & O & O \\ C & A & B & O & \dots & O & O & O \\ O & C & A & B & \dots & O & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & O & \dots & C & A & B \\ O & O & O & O & \dots & O & C & A+B \end{pmatrix} \quad T^E_{\omega'} = \begin{pmatrix} A_2 & B_2+C & O & O & \dots & O & O & B_4 \\ C_2 & A_2 & B_2 & O & \dots & O & O & O \\ O & C_2 & A_2 & B_2 & \dots & O & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & O & \dots & C_2 & A_2 & B_2 \\ C_4 & O & O & O & \dots & O & C_2+B & A_2 \end{pmatrix},$$

$$T^E_{\tau'} = \begin{pmatrix} A_4 & B_3+C & O & O & \dots & O & O & O \\ C_3 & A_3 & B_3 & O & \dots & O & O & O \\ O & C_3 & A_3 & B_3 & \dots & O & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & O & \dots & C_3 & A_3 & B_3 \\ O & O & O & O & \dots & O & C_3+B & A_5 \end{pmatrix} \quad T^E_{\gamma'} = \begin{pmatrix} A_4 & B_3 & O & O & \dots & O & O & C \\ C_3 & A_3 & B_3 & O & \dots & O & O & O \\ O & C_3 & A_3 & B_3 & \dots & O & O & O \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ O & O & O & O & \dots & C_3 & A_3 & B_3 \\ B & O & O & O & \dots & O & C_3 & A_5 \end{pmatrix},$$

where
$$\begin{array}{lll} A_2 = A + h\epsilon_{1,n} + d\epsilon_{n,1}, & B_2 = B + g\epsilon_{1,n} + e\epsilon_{n,1}, & C_2 = C + a\epsilon_{1,n} + c\epsilon_{n,1}, \\ A_3 = A + h\epsilon_{1,1} + d\epsilon_{n,n}, & B_3 = B + g\epsilon_{1,1} + e\epsilon_{n,n}, & C_3 = C + a\epsilon_{1,1} + c\epsilon_{n,n}, \\ A_4 = A_3 + a\epsilon_{1,1} + c\epsilon_{n,n}, & B_4 = a\epsilon_{1,n} + c\epsilon_{n,1}, & C_4 = g\epsilon_{1,n} + e\epsilon_{n,1}, \\ A_5 = A_3 + g\epsilon_{1,1} + e\epsilon_{n,n}, & & \end{array}$$

$O, \epsilon_{1,1}, \epsilon_{n,n} \in \mathbf{M}_{n\times n}(\mathbb{Z}_p)$, O is the zero matrix and $\epsilon_{1,2}$, $\epsilon_{n,n-1}$ are unit matrices.

Reversibility of the rule matrices

In this section, we will outline an algorithm to determine the reversibility of the 2D (linear) cellular automaton defined by the Moore rule under non-bijective boundary conditions represented by the function ϕ. Given that we have already derived the rule matrix T^{ϕ}_{R} corresponding to the 2D finite cellular automaton with the function ϕ, we can express the relationship between the column vectors $X(t)$ and the rule matrix T^{ϕ}_{R} as follows:

$$X^{(t+1)} = T^{\phi}_{\mathrm{R}} \cdot X^{(t)} \pmod{p}.$$

If the rule matrix T_{R} is non-singular, then we can express:

$$X^{(t)} = (T^{\phi}_{\mathrm{R}})^{-1} \cdot X^{(t+1)} \pmod{p}.$$

Hence, our primary objective is to investigate whether the rule matrices T^{ϕ}_{R} in Theorem 1, Theorem 2 and Theorem 3 are invertible or not. It is widely recognized that the 2D finite cellular automaton is reversible if and only if its rule matrix T^{ϕ}_{R} is non-singular. If the rule matrix T^{ϕ}_{R} achieves full rank, indicating it is invertible, then the 2D finite cellular automaton is reversible; otherwise, it is irreversible.

For solve the problem of reversibility we give the following lemma in [7].

Lemma 1. *Let $T \in \mathbf{M}_{mn\times mn}(\mathbb{Z}_p)$ be a matrix of the following form:*

$$\begin{pmatrix} A_m & X & O & \dots & O & O \\ B_{m-1} & A_{m-1} & X & \dots & O & O \\ O & B_{m-2} & A_{m-2} & \dots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \dots & A_2 & X \\ O & O & O & \dots & B_1 & A_1 \end{pmatrix}, \tag{7}$$

where all submatrices are $n \times n$, O is the zero matrix. If the submatrix X has full rank, then

$$rank(T) = (m-1)n + rank(P_m),$$

with $P_1 = A_1, \quad P_2 = -B_1 - A_1X^{-1}A_2, \quad P_k = -P_{k-2}X^{-1}B_{k-1} - P_{k-1}X^{-1}A_k, \quad k \in \{3, \dots, m\}.$

Remark 1.*Since the rank of the transpose of the matrix is equal to rank of itself, then we can prove similar result to Lemma 1 for matrices $T \in \mathbf{M}_{mn\times mn}(\mathbb{Z}_p)$ of the following form:*

$$\begin{pmatrix} A_1 & B_1 & O & \dots & O & O \\ X & A_2 & B_2 & \dots & O & O \\ O & X & A_3 & \dots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \dots & A_{m-1} & B_{m-1} \\ O & O & O & \dots & X & A_m \end{pmatrix}, \tag{1}$$

where all submatrices are $n \times n$, O is the zero matrix, I is the identity matrix. If the submatrix X has full rank, then

$$rank(T) = (m-1)n + rank(P_m),$$

where $P_1 = A_1, \quad P_2 = -A_1X^{-1}B_2, \quad P_k = -P_{k-2}X^{-1}B_{k-1} - P_{k-1}X^{-1}A_k, \quad k \in \{3, \dots, m\}.$

By Lemma 1 we can give the following result.

Proposition 1.*Consider the rule matrices T^E_ψ, T^E_μ, T^E_ν, T^E_τ, T^E_γ and $T^E_{\nu'}$. These matrices is being as the form in Lemma 1 or Remark 1, then it can be calculated their ranks by the formula in Lemma 1 or Remark 1.*

Remark 2. *According to the condition in Lemma 1, it was required that X has full rank. It should be considered B, C, B_1, C_1, B_2, C_2, B_3, C_3 as X.*

Note that the determinant of all matrices are mentioned in the last remark comes to calculate the following matrix

$$X = \begin{pmatrix} x & y & 0 & \dots & 0 & 0 & 0 \\ z & x & y & \dots & 0 & 0 & 0 \\ 0 & z & x & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & z & x & y \\ 0 & 0 & 0 & \dots & 0 & z & x \end{pmatrix}.$$

By using methods of evaluating higher-order determinants, for the determinant of X we obtain the following result.

Proposition 2.*Let X be a matrix $n \times n$. Then*

$$\det(X) = \Delta_n = x\Delta_{n-1} - yz\Delta_{n-2}$$

where $\Delta_n = \frac{\sum_{k=0}^{[\frac{n}{2}]} C_{n+1}^{2k+1}(x)^{n-2k}(x^2-4yz)^k}{2^n}$ *and $[\frac{n}{2}]$ is the integer part of $\frac{n}{2}$. Moreover, we can simplify this determinant under some conditions:*

- *if* $yz = 0$, *then* $\det(X) = x^n$;
- *if* $x = 0$, *then* $\det(X) = \begin{cases} 0, & n - odd; \\ (-yz)^{\frac{n}{2}}; & n - even, \end{cases}$
- *if* $yz \neq 0$, $y + z = x$, *and* $z \neq y$ *then* $\det(X) = \frac{y^{n+1} - z^{n+1}}{y - z}$;
- *if* $yz \neq 0$, $y + z = x$ *and* $z = y$, *then* $\det(X) = (n+1)y^n$.

Theorem 4. *Let T be the rule matrix T^E_ψ in Theorems 1. If the matrix B_1 and C_1 have the full rank, then*

$$rank(T) = (m-1)n + rank(P_m),$$

where the submatrix P_m is computed as in Remark 1.

Remark 3. *It can be calculated the rank of the rule matrices T_{μ}^{E}, T_{ν}^{E}, T_{τ}^{E}, T_{γ}^{E} and $T_{\nu'}^{E}$ in Theorems 2, 3 in the method of Theorem 4.*

Example. In order to illustrate the previous theorem, we take $m = 4$ and $n = 3$ and consider the rule matrix T_{ψ}^{E} with over the ternary field $\mathbb{Z}_3$. Thanks to Theorem 1 we have

$$T_{\psi}^{E} = \begin{pmatrix} A_1 & B_1 & O & O \\ C_1 & A_1 & B_1 & O \\ O & C_1 & A_1 & B_1 \\ O & O & C_1 & A_1 \end{pmatrix},$$

where O is the zero matrix and

$$A_1 = \begin{pmatrix} 0 & h+d & 0 \\ h & 0 & d \\ 0 & h+d & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} f & g+e & 0 \\ g & f & e \\ 0 & g+e & f \end{pmatrix}, \quad C_1 = \begin{pmatrix} b & a+c & 0 \\ a & b & c \\ 0 & a+c & b \end{pmatrix},$$

with $a, b, c, d, e, f, g, h \in \mathbb{Z}_3$.

Now, let $a = c = 0$, $b = d = e = f = g = h = 1$. Then,

$$A_1 = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}, C_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since C_1 has the full rank (i.e. $\det C_1 = 1$) we find the matrix P_4 in Remark 1. According to the recurrence relation in Lemma 1, we get the following matrices:

$$P_1 = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

Therefore, by applying the algorithm for computing the rank of the rule matrix T_{ψ}^{E} (see Theorem 4) we have

$$rank(T_{\mathrm{R}}^{\phi}) = (4-1) \cdot 3 + rank(P_4) = 9 + 1 = 10.$$

Hence, the rule matrix T_{ψ}^{E} does not have full rank. This implies that it is not reversible. In other words the mapping T_{ψ}^{E} is not surjective.

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Annotatsiya

Ushbu maqolada ba'zi shartlar ostida ikki o'lchamli Mur kletkali avtomatlari o'rganilgan. Xususan, $\mathbb{Z}_p$ maydonida aralash chegaraviy shartli Mur qo'shnilarili 2D chiziqli kletkali avtomatlarning tavsifini ko'rib chiqamiz. So'nggida, biz 2D chekli KA uchun olingan qoida matritsalarining teskarilanuvchanlik shartlari topilgan.

Kalit so'zlar: kletkali avtomatlar, chegaraviy shartlar, qoida matritsasi, teskarilanuvchanlik.

Аннотация

В этой статье мы исследуем двумерные клеточные автоматы с окрестностью Мура при определенных условиях. В частности, мы углубляемся в характеристику двумерных линейных клеточных автоматов, определяемых окрестностью Мура, рассматривая смешанные граничные условия над полем $\mathbb{Z}_p$. Наконец, мы представляем условия, которые приводят к обратимости полученных матриц правил для двумерных конечных КА.

Ключевые слова: клеточные автоматы, граничные условия, матрица правил, обратимость.