UDC 519.21

ON THE ASYMPTOTIC OF BRANCHING REDUCED PROCESSES

Khusanbaev Ya. M.
V.I.Romanovskiy Institute of Mathematics of Uzbekistan Academy of Sciences, Tashkent
yakubjank@mail.ru
Toshkulov X. A.
Samarkand State University named after Sharof Rashidov, Samarkand
kh.toshkulov@mathinst.uz

RESUME

In this paper we investigate the subcritical and critical reduced processes generated by the Gal'ton-Watson branching processes. The limit theorems for such processes in the case when in the population at the initial moment there are a large number of particles are obtained.

Key words: branching process, reduced process, limit theorem.

Introduction

Let $\{Z(k), k \geq 0\}$- random Gal'ton-Watson branching process starting with one particle, in which the number of immediate descendants of one particle has a generating function

$$f(s) = Es^{\xi} = \sum_{k=0}^{\infty} p_k s^k, \ \ 0 \leq s \leq 1.$$

Let $A = f'(1) < \infty$. Branching process $\{Z(k), k \geq 0\}$ called subcritical, critical or supercritical if $A < 1, \ A = 1$ or $A > 1$ respectively.

Let us denote by $Z(m,n)$ number of particles at moment m $(m \leq n)$ in process $\{Z(k), k \geq 0\}$, whose descendants exist at the moment n. Random process $\{Z(m,n), 0 \leq m \leq n\}$ called the reduced process generated by the Gal'ton-Watson branching process $\{Z(k), k \geq 0\}$. Reduced process $\{Z(m,n), 0 \leq m \leq n\}$ called subcritical, critical or supercritical if the corresponding Gal'ton-Watson branching process $\{Z(k), k \geq 0\}$ is accordingly. Reduced processes for Gal'ton-Watson processes were introduced by Fleischmann and Prehn [1]. Fleischmann and Sigmund-Schulze [2] proved a functional limit theorem (under the assumption $Z(0) = 1, Z(n) > 0$) in which the convergence in Skorokhod space of critical reduced processes to the Yule process (with a modified time scale) has been established. Liu and Vatutin [3] proved conditional limit theorems (under conditions $0 < Z(n) \leq \tau(n)$) for critical reduced processes starting with one particle $(Z(0) = 1)$ and with a finite variance in the number of immediate descendants of one particle, where $\tau(n) = O(n)$, or $\tau(n) = o(n)$ as $n \to \infty$.

In this paper the conditional limit theorems for subcritical and critical reduced processes are proved.

Let us denote by $f_n(s)$ n- th iteration$f(s)$:

$$f_0(s) = s, \ f_1(s) = f(s), ..., f_n(s) = f_{n-1}(f(s)) .$$

If

$$A = f'(1) = E\xi = 1, \ \sigma^2 = f''(1) = var\xi \in (0, \infty) , \tag{1}$$

then it is known (see, for example, [4]Chapter I, paragraph 9) that

$$P(Z(n) > 0 / Z(0) = 1) = 1 - f_n(0) \sim \frac{2}{\sigma^2 n} \ \ \text{as} \ \ n \to \infty, \tag{2}$$

and Yaglom's conditional limit theorem also holds:

$$\lim_{n\to\infty} P\left(\frac{2Z(n)}{\sigma^2 n} < y \Big/ Z(0)=1, Z(n)>0\right) = 1-e^{-y},\ y\geq 0. \tag{3}$$

Fleischmann and Sigmund-Schulze [2]proved for critical reduced processes that for $s\in[0,1]$ and for anyone $t\in[0,1)$

$$\lim_{n\to\infty} E\left[s^{Z([nt],n)} \Big/ Z(0)=1, Z(n)>0\right] = \frac{(1-t)\,s}{1-ts}, \tag{4}$$

where is the sign $[a]$, hereinafter, denotes the integer part of the number a.

If

$$A = f'(1) = E\xi < 1,\ \sum_{k=2}^{\infty} p_k k\log k < \infty, \tag{5}$$

that is known (see for example, [4], Ch. I, paragraph 9), that there is a finite number $K>0$ such that

$$P\left(Z(n)>0 / Z(0)=1\right) = 1-f_n(0) \sim KA^n + o(A^n) \quad \text{as } n\to\infty, \tag{6}$$

and Yaglom's conditional limit theorem also holds: there are limits

$$\lim_{n\to\infty} P\left(Z(n)=k/Z(0)=1, Z(n)>0\right) = b_k, \tag{7}$$

and $\sum_{k=1}^{\infty} b_k = 1$, and the generating function

$$g(s) = \sum_{k=1}^{\infty} b_k s^k,\ \ 0\leq s\leq 1$$

satisfies the functional equation

$$1-g(f(s)) = A(1-g(s)),\ 0\leq s\leq 1. \qquad 8$$

Joffe and Spitzer [5] proved that if $A<1$ and $\sum_{k=1}^{\infty} p_k k\log k<\infty$, then for anyone $k\in N_0 = N\cup\{0\}$ there are limits

$$\lim_{n\to\infty} P\left(Z(n)=k/Z(0)=cA^{-n}+o\left(A^{-n}\right)\right) = d_k(c),$$

where $c>0$ is constant,

$$d_k(c)\geq 0,\ \sum_{k=0}^{\infty} d_k(c) = 1$$

and

$$\sum_{k=0}^{\infty} d_k(c)\,s^k = e^{-Kc(1-g(s))},$$

where $K>0$ from the relation (6), and the function $g(s)$ the same as in (8).

In what follows we need the following lemmas.

Lemma 1. The following formula hold

$$E\left[Z(m,n)/Z(0)=k, Z(n)>0\right] = \frac{k\left(1-f_{n-m}(0)\right)}{1-f_n^k(0)}A^m. \tag{9}$$

Lemma 2. The following formula hold

$$var\left[Z(m,n)/Z(0)=k, Z(n)>0\right] = \frac{k\left(1-f_{n-m}(0)\right)^2 varZ(m)}{1-f_n^k(0)} +$$

$$+\frac{kf_{n-m}(0)\left(1-f_{n-m}(0)\right)A^m}{1-f_n^k(0)} - \frac{k^2 f_n^k(0)\left(1-f_{n-m}(0)\right)^2 A^{2m}}{\left(1-f_n^k(0)\right)^2}, \tag{10}$$

Proof of Lemma 1. It is easy to verify that

$$E\left[s^{Z(m,n)}\Big/Z(0)=1\right]=E\left[E\left[s^{Z(m,n)}\Big/Z(0)=1,Z(m)\right]\right]=$$

$$=E\left[\left[f_{n-m}(0)+(1-f_{n-m}(0))\,s\right]^{Z(m)}\Big/Z(0)=1\right]=$$

$$=f_m\left(f_{n-m}(0)+(1-f_{n-m}(0))\,s\right)=f_m\left(\varphi_{n-m}(s)\right),$$

where $\varphi_k(s)=f_k(0)+(1-f_k(0))\,s$.
Now, according to the additively property of branching processes, we have

$$E\left[s^{Z(m,n)}\Big/Z(0)=k\right]=\left\{E\left[s^{Z(m,n)}\Big/Z(0)=1\right]\right\}^k=f_m^k\left(\varphi_{n-m}(s)\right). \tag{11}$$

In the last relation we take the left derivative at the point $s=1$:

$$E\left[Z(m,n)\,/Z(0)=k\right]=\left[E\left[s^{Z(m,n)}\Big/Z(0)=k\right]\right]'_{s=1}=\left[f_m^k\left(\varphi_{n-m}(s)\right)\right]'_{s=1}=$$

$$=kf_m^{k-1}\left(\varphi_{n-m}(s)\right)f'_m\left(\varphi_{n-m}(s)\right)|_{s=1}\left(1-f_{n-m}(0)\right)=$$

$$=kf'_m(1)\left(1-f_{n-m}(0)\right)=kA^m\left(1-f_{n-m}(0)\right).$$

Now, taking into account (11), we have

$$E\left[s^{Z(m,n)}\Big/Z(0)=k,Z(n)>0\right]=\frac{\left[f_m\left(\varphi_{n-m}(s)\right)\right]^k-f_n^k(0)}{1-f_n^k(0)}. \tag{12}$$

Here $I(A)$ denotes indicator of the event A.Taking left derivatives from both sides of the last relation at the point $s=1$, we have

$$E\left[Z(m,n)\,/Z(0)=k,Z(n)>0\right]=\left[E\left[s^{Z(m,n)}\Big/Z(0)=k,Z(n)>0\right]\right]'_{s=1}=$$

$$=\frac{k\left[f_m(1)\right]^{k-1}f'_m(1)\left(1-f_{n-m}(0)\right)}{1-f_n^k(0)}=\frac{k\left(1-f_{n-m}(0)\right)}{1-f_n^k(0)}A^m,$$

that proves the validity of the relationship (9).

Proof of Lemma 1 is complete.

Proof of Lemma 2. By virtue of (12) we have

$$\left\{E\left[s^{Z(m,n)}\Big/Z(0)=k,Z(n)>0\right]\right\}''=\frac{k(k-1)\left[f_m\left(\varphi_{n-m}(s)\right)\right]^{k-2}\left[f'_m\left(\varphi_{n-m}(s)\right)\right]^2\left(1-f_{n-m}(0)\right)^2}{1-f_n^k(0)}+$$

$$+\frac{k\left[f_m\left(\varphi_{n-m}(s)\right)\right]^{k-1}f''_m\left(\varphi_{n-m}(s)\right)\left(1-f_{n-m}(0)\right)^2}{1-f_n^k(0)}.$$

Consequently

$$E\left[Z(m,n)\left(Z(m,n)-1\right)/Z(0)=k,Z(n)>0\right]=\frac{k\left(1-f_{n-m}(0)\right)^2}{1-f_n^k(0)}f''_m(1)+$$

$$+\frac{k(k-1)\left(1-f_{n-m}(0)\right)^2}{1-f_n^k(0)}A^{2m}.$$

Now, substituting the last expression and (9) into the formula

$$var\left[Z(m,n)\,/Z(0)=k,Z(n)>0\right]=$$

$$=E\left[Z(m,n)\left(Z(m,n)-1\right)/Z(0)=k,Z(n)>0\right]+E\left[Z(m,n)\,/Z(0)=k,Z(n)>0\right]-$$

$$-\left[E\left[Z(m,n)\,/Z(0)=k,Z(n)>0\right]\right]^2$$

we arrive at relation (10).
The proof of Lemma 2 is complete.

Formulation of the results.

Theorem 1. Let the condition (5) be satisfied. Then for $s \in [0,1]$ and for anyone $t \in [0,1)$

$$E\left[s^{Z([nt],n)} \big/ Z(0)=1, Z(n)>0\right] \to s \text{ as } n \to \infty.$$

Corollary 1. Let condition (5) is satisfied. Then for anyone $t \in [0,1)$

$$P\left(Z([nt],n)=1/Z(0)=1, Z(n)>0\right) \to 1 \text{ as } n \to \infty.$$

Theorem 2. Let condition (1) is satisfied and $Z(0)=\psi(n)$ with probability 1, where $\psi(n)=o(n)$ as $n \to \infty$. Then for $s \in [0,1]$ and for anyone $t \in [0,1)$

$$E\left[s^{Z([nt],n)} \big/ Z(0)=\psi(n), Z(n)>0\right] \to \frac{(1-t)s}{1-ts} \text{ as } n \to \infty. \tag{13}$$

Corollary 2. Under the conditions of the theorem 2

$$P\left(Z([nt],n)=k/Z(0)=\psi(n), Z(n)>0\right) \to (1-t)t^k, \ k \in N. \tag{14}$$

Theorem 3. Let $A<1$, $\sum p_k k \log k < \infty$ and $Z(0)=\psi(n)$ with probability 1, where $\psi(n)$ such that $A^n\psi(n) \to 0$ as $n \to \infty$. Then for $s \in [0,1]$ and for anyone $t \in [0,1)$

$$E\left[s^{Z([nt],n)} \big/ Z(0)=\psi(n), Z(n)>0\right] \to s \text{ as } n \to \infty.$$

Corollary 3. Under the conditions of the theorem 3

$$P\left(Z([nt],n)=1/Z(0)=\psi(n), Z(n)>0\right) \to 1 \text{ as } n \to \infty.$$

Theorem 4. Let $A<1$, $\sum p_k k \log k < \infty$ and $Z(0)=\psi(n)$ with probability 1, where $A^n\psi(n) \to 0$ as $n \to \infty$. Then for $m \in N$, $s \in [0,1]$ and for anyone $t \in [0,1)$

$$E\left[s^{Z(n-m,n)} \big/ Z(0)=\psi(n), Z(n)>0\right] \to 1-A^{-m}\left(1-g\left(\varphi_m(s)\right)\right) \text{ as } n \to \infty,$$

where $g(s)$ satisfies equation (8), $\varphi_m(s)=f_m(0)+(1-f_m(0))s$.

Remark 1. From Corollary 1 it follows that if it is known that the population has not degenerated by the moment n, then at the moment nt there is almost certainly only one particle whose descendants will survive until the moment n. It should be noted that the number of particles at the moment nt in the branching process $\{Z(k), k \geq 0\}$ itself differs significantly from the degenerate distribution law in unity.

Remark 2. Comparison of statement (4) with the result of Theorem 2 leads to the conclusion that if at the beginning of the development of the population there are $\varphi(n)=o(n)$ particles, then the number of particles at the moment nt that have descendants at the moment n, have in the asymptotic such a distribution as a process starting with one particle.

Proof of the results.

Proof of Theorem 1. According to Taylor's formula

$$f_{[nt]}\left(\varphi_{n-[nt]}(s)\right)=f_{[nt]}(1)-f'_{[nt]}(\theta(n,s))\left(K(1-s)A^{n-[nt]}+o\left(A^{n-[nt]}\right)\right), \tag{15}$$

where $\theta(n,s)$ such that $1-KA^{n-[nt]} \leq \theta(n,s) \leq 1$.
From (15), according the fact that $f'_{[nt]}(1)=A^{[nt]}$, we get

$$f_{[nt]}\left(\varphi_{n-[nt]}(s)\right)=1-K(1-s)A^n+o(A^n). \tag{16}$$

By virtue of (12) and (16), we have

$$E\left[s^{Z([nt],n)}\Big/Z(0)=1,Z(n)>0\right]=\frac{f_{[nt]}\left(f_{n-[nt]}(0)+\left(1-f_{n-[nt]}(0)\right)s\right)-f_n(0)}{1-f_n(0)}\sim$$

$$\sim\frac{1-K(1-s)A^n+o(A^n)-1+KA^n+o(A^n)}{KA^n+o(A^n)}\to s$$

as $n\to\infty$, which completes the proof of the theorem 1.

Proof of Theorem 2. By virtue of (11), we have

$$E\left[s^{Z(m,n)}\Big/Z(0)=\psi(n)\right]=\left[f_m\left(\varphi_{n-m}(s)\right)\right]^{\psi(n)}. \tag{17}$$

Since the critical Galton–Watson process degenerates with probability 1, then

$$f_n(0)\to 1 \quad \text{as } n\to\infty. \tag{18}$$

Therefore uniformly converges for$s\in[0,1]$ and for anyone $t\in[0,1)$

$$\varphi_{n-[nt]}(s)\to 1 \quad \text{as } n\to\infty. \tag{19}$$

By virtue of (18) and (19), we can choose a natural number $r=r(t,s)$ so that the chain of inequalities holds

$$f_r(0)\le\varphi_{n-[nt]}(s)\le f_{r+1}(0). \tag{20}$$

taking into account relation (2) we have

$$\varphi_{n-[nt]}(s)\sim 1-\frac{2(1-s)}{\sigma^2(1-t)n} \quad \text{as } n\to\infty. \tag{21}$$

Now relations (2), (20) and (21) allow us to conclude that

$$\frac{1}{r+1}\le\frac{1-s}{n(1-t)}\le\frac{1}{r}.$$

Hence,

$$r\sim\frac{n(1-t)}{1-s} \quad \text{as } n\to\infty. \tag{22}$$

From here and from (20) we obtain

$$f_{[nt]+r}(0)=f_{[nt]}\left(f_r(0)\right)\le f_{[nt]}\left(\varphi_{n-[nt]}(s)\right)\le f_{[nt]}\left(f_{r+1}(0)\right)=f_{[nt]+r+1}(0)$$

which in turn allows us to conclude

$$f_{[nt]}\left(\varphi_{n-[nt]}(s)\right)\sim f_{[nt]+r}(0) \tag{23}$$

Therefore, from (12), (22)-(23) and (2), using the asymptotic relation

$$e^{-x}\sim 1-x,\ \ x\to 0, \tag{24}$$

we get

$$E\left[s^{Z([nt],n)}/Z(0)=\psi(n),Z(n)>0\right]=\frac{e^{-\frac{\psi(n)}{n}\cdot\frac{2}{\sigma^2}\cdot\frac{1-s}{1-st}}-e^{-\frac{2}{\sigma^2}\cdot\frac{\psi(n)}{n}}}{1-e^{-\frac{2}{\sigma^2}\cdot\frac{\psi(n)}{n}}}+o(1)\sim$$

$$\sim\frac{1-\frac{2}{\sigma^2}\cdot\frac{\psi(n)}{n}\cdot\frac{1-s}{1-st}-1+\frac{2}{\sigma^2}\cdot\frac{\psi(n)}{n}}{\frac{2}{\sigma^2}\cdot\frac{\psi(n)}{n}}1-\frac{1-s}{1-st}=\frac{s(1-t)}{1-st} \text{ as } n\to\infty.$$

The proof of Theorem 2 is complete.

Proof of Theorem 3. Taking (6) into account and applying relation (24) we have

$$E\left[Z\left([nt],n\right)/Z\left(0\right)=\psi\left(n\right),Z\left(n\right)>0\right]=\frac{\psi\left(n\right)\left(1-f_{n-[nt]}\left(0\right)\right)}{1-f_{n}^{\psi(n)}\left(0\right)}A^{[nt]}=$$

$$=\frac{\psi\left(n\right)\left(KA^{n-[nt]}+o\left(A^{n-[nt]}\right)\right)}{1-\left(1-KA^{n}+o\left(A^{n}\right)\right)^{\psi(n)}}A^{[nt]}\sim\frac{\psi\left(n\right)\left(KA^{n}+o\left(A^{n}\right)\right)}{\psi\left(n\right)KA^{n}+o\left(\psi\left(n\right)A^{n}\right)}\to 1 \quad (25)$$

as $n\to\infty$. Further, according to (10), (6), (24) we have

$$var\left[Z\left([nt],n\right)/Z\left(0\right)=\psi\left(n\right),Z\left(n\right)>0\right]=\frac{\psi\left(n\right)\left(1-f_{n-[nt]}\left(0\right)\right)^{2}varZ\left([nt]\right)}{1-f_{n}^{\psi(n)}\left(0\right)}+$$

$$+\frac{\psi\left(n\right)f_{n-[nt]}\left(0\right)\left(1-f_{n-[nt]}\left(0\right)\right)A^{[nt]}}{1-f_{n}^{\psi(n)}\left(0\right)}-\frac{\psi^{2}\left(n\right)f_{n}^{\psi(n)}\left(0\right)\left(1-f_{n-[nt]}\left(0\right)\right)^{2}A^{2[nt]}}{\left(1-f_{n}^{\psi(n)}\left(0\right)\right)^{2}}\sim$$

$$\sim\frac{\sigma^{2}KA^{n-[nt]}}{A\left(1-A\right)}\to 0 \quad (26)$$

as $n\to\infty$. Now from (25) and (26) as well as from Chebyshev's inequality the statement of Theorem 3 follows.

Proof of Theorem 4. According (6), (12) and (16) we have for any $m\in N$

$$E\left[s^{Z(n-m,n)}\Big/Z\left(0\right)=\psi\left(n\right),Z\left(n\right)>0\right]=\frac{f_{n-m}^{\psi(n)}\left(\varphi_{m}\left(s\right)\right)-f_{n}^{\psi(n)}\left(0\right)}{1-f_{n}^{\psi(n)}\left(0\right)}\sim$$

$$\sim\frac{\left(1-K\left(1-g\left(\varphi_{m}\left(s\right)\right)\right)A^{n-m}+o\left(A^{n}\right)\right)^{\psi(n)}-\left(1-KA^{n}+o\left(A^{n}\right)\right)^{\psi(n)}}{1-\left(1-KA^{n}+o\left(A^{n}\right)\right)^{\psi(n)}}\sim$$

$$\sim\frac{1-K\left(1-g\left(\varphi_{m}\left(s\right)\right)\right)\psi\left(n\right)A^{n-m}+o\left(\psi\left(n\right)A^{n}\right)-1+K\psi\left(n\right)A^{n}+o\left(\psi\left(n\right)A^{n}\right)}{K\psi\left(n\right)A^{n}+o\left(\psi\left(n\right)A^{n}\right)}\sim$$

$$\sim 1-A^{-m}\left(1-g\left(\varphi_{m}\left(s\right)\right)\right)$$

as $n\to\infty$, which completes the proof of the theorem 4.

The proofs of the corollary follows immediately from the continuity theorem for generating functions.

REFERENCES

1. Fleischmann K., Prehn U. Ein Grenzfersatz für subkritische Verzweigungsprozesse mit eindlich vielen Typen von Teilchen // Math. Nachr., 64 (1974), 233-241.

2. Fleischmann K., Siegmund-Schultze R. The structure of reduced critical Galton-Watson processes// Math. Nachr., 79 (1977), 357-362.

3. Fleischmann K., Siegmund-Schultze R. The structure of reduced critical Galton-Watson processes// Math. Nachr., 79 (1977), 357-362.

4. Liu M., Vatutin V. Reduced processes for small populations// Theory Probab. Appl., 63:4 (2018).

5. Athreya K. B., Ney P. E. Branching Processes. Berlin, Germany: Springer-Verlag. 1972.287p.

6. Joffe A., Spitzer F. On multitype branching processes with // Journal of Math. Analysis and Applications.v.19(1967), 409-430.

7. Harris T. Theory of branching random processes. M.Mir.1966. 355c. (in Russian).

8. Gikhman I.I., Skorokhod A.V. Theory of random processes. volume 1. From “Science”, 1971, 664 pp. (in Russian).

REZYUME

Ushbu maqolada biz Gal’ton-Votson tarmoqlanuvchi jarayonlarining subkritik va kritik bo’lgan hollarini o’rganamiz. Bunday jarayonlar uchun populyatsiyada boshlang’ich momentda juda ko’p zarrachalar mavjud bo’lganda limit teoremalar olingan.

Kalit so‘zlar: tarmoqlanish jarayon, qisqartirilgan jarayon, limit teorema.

РЕЗЮМЕ

В данной работе исследуются докритические и критические редуцированные процессы, порождаемые ветвящимися процессами Гальтона-Уотсона. Получены предельные теоремы для таких процессов в случае, когда в популяции в начальный момент имеется большое число частиц.

Ключевые слова: ветвящийся процесс, предельная теорема.