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ON THE POINT SPECTRUM OF THE SCHRODINGER OPERATOR FOR A SYSTEM CONSISTING OF TWO IDENTICAL INFINITELY HEAVY BOSONS AND ONE LIGHT FERMION ON THE THREE-DIMENSIONAL LATTICE

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RESUME

We consider the Hamiltonian of a system of three quantum mechanical particles (two identical bosons and a fermion) on the one-dimensional lattice interacting by means of zero-range attractive or repulsive potentials. We investigate the point spectrum of the three-particle discrete Schrodinger operator $H(K)$, $K \in \mathbb{T}$ which possesses infinitely many eigenvalues depending on repulsive or attractive interactions, under the assumption that the bosons in the system have infinite mass.

Key words: Discrete Schrödinger operator, Point spectrum, Threshold resonance, Zero-range pair potentials, Threshold eigenvalues, Fredholm' determinant

INTRODUCTION

In physics, stable composite objects often form due to attractive forces, allowing constituents to lower their energy by binding together. While repulsive forces typically cause particles to scatter in free space, within structured environments such as periodic potentials, and in the absence of dissipation, stable composite objects can exist even with repulsive interactions due to the lattice band structure [1]. The Bose-Hubbard model, which describes repulsive pairs, serves as the theoretical foundation for various applications.

The work [1] highlights the critical connection between the Bose-Hubbard model [2],[3] and atoms in optical lattices, paving the way for numerous interesting developments and applications, as discussed in [4]. In cold atom lattice physics, stable repulsively bound objects are expected to be common, potentially leading to composites with fermions [5] or Bose-Fermi mixtures [6], and can even form with more than two particles in a similar manner.

The Efimov effect, first discovered by V. Efimov in 1970 [7], is one of the most fascinating phenomena in physics. It occurs in three-body systems in three-dimensional space that interact through short-range pairwise potentials. It is possible to adjust the couplings of the interactions so that none of the particle pairs have a negative energy bound state, but at least two pairs exhibit a resonance at zero energy. The existence of this effect for a three-particle Schrodinger operator was discovered by Yafaev [8] and in the discrete case by others [9,10,11,12,13,14]. This asymptotics problem was reduced to studying the asymptotics of the number of eigenvalues of Faddeev-type compact integral operators.

Recently, the authors [15,16,17] considered perturbations of the system of three arbitrary quantum particles on lattices $\mathbb{Z}^d$, $d = 1, 2$, interacting through attractive zero-range pairwise potentials. They established that the three-particle Schrödinger operators posses infinitely many negative eigenvalues for all positive non-zero point interactions, under the assumption that two particles in the system have infinite mass. Also, note that the author of [17] obtained asymptotics for these eigenvalues.

The main goal of the paper is to investigate the existence of infinite number of bound states of the three-particle discrete Schrödinger operator associated to a system of two identical bosons and a fermion, where the bosons have infinite mass and the fermion has a finite mass. This investigation is conducted on the three-dimensional lattice $\mathbb{Z}^3$ and involves repulsive or attractive zero-range pairwise interactions. We show that infinitely many eigenvalues may arise from threshold eigenvalues and threshold resonances or only from threshold eigenvalues depending on the interaction energies.

It should be noted that, unlike the last three articles [15,16,17], we study eigenvalues below and above the essential spectrum of the unperturbed operator for all repulsive or attractive zero-range pairwise interactions.

The paper is organized as follows. In Section 1, we introduce the three-particle discrete Schrödinger operator $H(K)$ and the two-particle discrete Schrödinger operators associated with subsystems of the system of two identical bosons and a fermion. In Section 2, we study the essential spectrum of the three-particle disceret Schrödinger operator $H(K)$. The eigenvalues of $H(K)$ below and above the spectrum of the non-perturbed operator $H_0(K)$ are investigated in Section 3. Section 4 is devoted to showing main result, Theorem 1. Section 5 is devoted to the description of the threshold resonance. The last section is conclusion.

Throughout the paper we adopt the following notations: $\mathbb{Z}^3$ is the three-dimensional lattice, $\mathbb{T}^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3 = (-\pi,\pi]^3$ is the three-dimensional torus (the first Brillouin zone, i.e., the dual group of $\mathbb{Z}^3$) equipped with the Haar measure, the subscripts $\alpha,\beta,\gamma \in \{1,2,3\}$ are pairwise different numbers.

THREE-PARTICLE DISCRETE SCHRÖDINGER OPERATOR ON THE LATTICE $\mathbb{Z}^3$

Let us consider the discrete Schrödinger operator $H(K)$, where $K \in \mathbb{T}^3$, associated with a system consisting of two identical bosons and a fermion moving on the one-dimensional lattice $\mathbb{Z}$ (see [15,17]).

$$H(K) = H_0(K) - V$$

with zero-range attractive potentials

$$V = V_1 + V_2 + V_3,$$

where

$$(V_1 f)(p,q) = \frac{\lambda}{(2\pi)^3}\int\limits_{\mathbb{T}^3} f(p,t)dt, \quad (V_2 f)(p,q) = \frac{\lambda}{(2\pi)^3}\int\limits_{\mathbb{T}^3} f(t,q)dt$$

$$(V_3 f)(p,q) = \frac{\mu}{(2\pi)^3}\int\limits_{\mathbb{T}^3} f(t,p+q-t)dt, \quad f \in L_s^2((\mathbb{T}^3)^2),\ p,q \in \mathbb{T}^3, \tag{1}$$

and numbers λ and μ serve as the parameters of boson-fermion interaction and boson-boson interaction, respectively.

Here the numbers λ and μ indicate repulsive pair-wise interaction when $\lambda < 0$ and $\mu < 0$, and attractive pair-wise interaction when $\lambda > 0$ and $\mu > 0$. The operator $H_0(K)$ is defined on the Hilbert space $L_s^2((\mathbb{T}^3)^2)$ by

$$\big(H_0(K)f\big)(p,q) = E(K;p,q)f(p,q), \quad f \in L_s^2((\mathbb{T}^3)^2),$$

and

$$E(K;p,q) = \varepsilon_{\mathfrak{b}}(p) + \varepsilon_{\mathfrak{b}}(q) + \varepsilon_{\mathfrak{f}}(K-p-q), \quad p,q \in \mathbb{T}^3.$$

Here, the real-valued continuous function $\varepsilon_{\mathfrak{b}}(\cdot)$ and $\varepsilon_{\mathfrak{f}}(\cdot)$, referred to as *the dispersion relation* associated with the free boson and fermion is defined as

$$\varepsilon_{\mathfrak{b}}(p) = \frac{1}{m}\varepsilon(p), \quad \varepsilon_{\mathfrak{f}}(p) = \frac{1}{\mathfrak{m}}\varepsilon(p), \quad \varepsilon(p) = \sum_{j=1}^{3}\big(1-\cos p_j\big), \quad p \in \mathbb{T}^3, \tag{2}$$

respectively, and m and $\mathfrak{m}$ represents the mass of the boson and fermion, respectively.

Let $k \in \mathbb{T}^3$ and $L_k^2(\mathbb{T}^3)$ be a linear subspace of the Hilbert space $L^2(\mathbb{T}^3)$ defined by

$$L_k^2(\mathbb{T}^3) = \{f \in L^2(\mathbb{T}^3) | f(p) = f(k-p)\}.$$

A two-particle discrete Schrödinger operator corresponding to the subsystem {bozon,fermion} and {bozon,bozon}, of the three-particle system acts on the Hilbert space $L^2(\mathbb{T}^3)$ and $L_k^2(\mathbb{T}^3)$ as

$$h_1(k) = h_1^0(k) - v_1, \quad \text{and} \quad h_2(k) = h_2^0(k) - v_2, \quad k \in \mathbb{T}^3, \tag{3}$$

respectively.

Here, the operators $h_\alpha^0(k)$

$$\big(h_1^0(k)f\big)(p) = E_k^{(1)}(p)f(p), \quad f \in L^2(\mathbb{T}^3),$$

and

$$\big(h_2^0(k)f\big)(p) = E_k^{(2)}(p)f(p), \quad f \in L_k^2(\mathbb{T}^3),$$

where

$$E_k^{(1)}(p) = \varepsilon_{\mathfrak{b}}(p) + \varepsilon_{\mathfrak{f}}(k-p), \quad E_k^{(2)}(p) = \varepsilon_{\mathfrak{b}}(p) + \varepsilon_{\mathfrak{b}}(k-p), \quad p \in \mathbb{T}^3. \tag{4}$$

The operators v_1 and v_2 are defined as

$$(v_1 f)(p) = \frac{\lambda}{(2\pi)^3}\int\limits_{\mathbb{T}^3} f(q)dq, \quad f \in L^2(\mathbb{T}^3),\, p \in \mathbb{T}^3.$$

and

$$(v_2 f)(p) = \frac{\mu}{(2\pi)^3}\int\limits_{\mathbb{T}^3} f(q)dq, \quad f \in L_k^2(\mathbb{T}^3),\, p \in \mathbb{T}^3,$$

respectively.

SPECTRAL PROPERTIES OF THE TWO-PARTICLE DISCRETE SCHRÖDINGER OPERATORS WHEN $m = \infty$ AND $0 < \mathfrak{m} < \infty$

With $m = \infty$ and $0 < \mathfrak{m} < \infty$ and the equality (2), the functions (4) can be written as

$$E_k^{(1)}(p) = \varepsilon_{\mathfrak{f}}(k-p) = \varepsilon(k-p)/\mathfrak{m},\ E_k^{(2)}(p) = 0,\ p \in \mathbb{T}^3.$$

Consequently, since the potentials v_α, $\alpha = 1, 2$ have a convolution-type property, all three two-particle Schrödinger operators do not depend on the quasi-momentum $k \in \mathbb{T}^3$,

$$h_1 := h_1(k), \quad \text{and} \quad h_2 = h_2(k).$$

Then, the operators $h_1(k)$ and $h_2(k)$ act as

$$h_1(k)f(p) = \varepsilon_{\mathfrak{f}}(p)f(p) - (v_1 f)(p), \quad f \in L^2(\mathbb{T}^3) \quad \text{and} \qquad h_2(k)f(p) = -(v_2 f)(p), \quad f \in L_k^2(\mathbb{T}^3).$$

As v_1 is a finite rank operator, according to the Weyl theorem, the essential spectrum $\sigma_{ess}\big(h_1(k)\big)$ of the operator $h_1(k)$ in (3) coincides with the spectrum $\sigma\big(h_1^0(k)\big)$ of the non-perturbed operator $h_1^0(k)$. More specifically,

$$\sigma_{ess}\big(h_1(k)\big) = \big[E_{\min}^{(1)}(k),\, E_{\max}^{(1)}(k)\big],$$

where

$$E_{\min}^{(1)}(k) \equiv \min_{p\in\mathbb{T}^3} E_k^{(1)}(p), \quad E_{\max}^{(1)}(k) \equiv \max_{p\in\mathbb{T}^3} E_k^{(1)}(p).$$

Therefore, in our case we have

$$\sigma_{ess}\big(h_1(k)\big) = \big[0, 6/\mathfrak{m}\big] \quad \text{and} \quad \sigma_{ess}\big(h_2(k)\big) = \{0\}.$$

The Fredholm determinants associated with the operators $h_1(k)$ are defined as

$$\Delta(\lambda; z) = 1 - \lambda d_0(z), \quad d_0(z) = \frac{1}{(2\pi)^3}\int\limits_{\mathbb{T}^3} \frac{ds}{\varepsilon_{\mathfrak{f}}(s) - z}, \quad z \in \mathbb{C}\setminus\big[0, 6/\mathfrak{m}\big] \tag{5}$$

Lemma 1. *The number $z \in \mathbb{C}\setminus\big[0, 6/\mathfrak{m}\big]$ is an eigenvalue of $h_1(k)$ if and only if $\Delta(\lambda; z) = 0$.*

Proof. *The equation*

$$h_1(k)f = zf \quad i.e., \quad f = \lambda\big(h_1^0(k) - z\big)^{-1} vf$$

has a non-trivial solution if and only if

$$\Delta(\lambda; z)C = 0, \quad C \in \mathbb{C},$$

has a non-trivial solution.

Therein, the solutions $C \in \mathbb{C}$ and $f \in L^2(\mathbb{T}^3)$ are connected by the following relations

$$C = vf \quad and \quad f = \lambda\big(h_1^0(k) - z\big)^{-1} C.$$

Set

$$\lambda_0 = \frac{1}{d_0(0)} \quad \text{and} \quad \lambda_0^* = \frac{1}{d_0(6/\mathfrak{m})},$$

where $d_0(0)$ and $d_0(6/\mathfrak{m})$ are continuation of $d_0(z)$ at the edges of the essential spectrum of $h_1(k)$. One may see that $\lambda_0^* = -\lambda_0$.

Lemma 2. *(a) If $\lambda < -\lambda_0$ (resp. $\lambda > \lambda_0$, then there exists a unique simple eigenvalue $z = z_1^0$ of $h_1(k)$ in the interval $\big(6/\mathfrak{m}, \infty\big)$ (resp. $\big(-\infty, 0\big)$). Moreover, z_1^0 does not depend on $k \in \mathbb{T}^3$.*

(b) If $\lambda = -\lambda_0$ (resp. $\lambda = \lambda_0$, then the operator $h_\alpha(k)$ has a threshold resonance at $6/\mathfrak{m}$ (resp. at 0).

(c) If $-\lambda_0 \le \lambda < 0$ (resp. $0 < \lambda \le \lambda_0$), then $h_\alpha(k)$ has no eigenvalues in the interval $\big(6/\mathfrak{m}, \infty\big)$ (resp. $\big(-\infty, 0\big)$).

Proof. *We prove (a) and (c) when $\lambda < 0$. These assertions can be proven in a similar way when $\lambda > 0$.* *The function $\Delta(\lambda; z)$ is monotonic increasing in $\big(6/\mathfrak{m}, \infty\big)$ and $\Delta(\lambda; z) > 1$ in $(-\infty, 0)$. Since*

$$\lim_{z \to +\infty} \Delta(\lambda; z) = 1 \quad and \quad \lim_{z \to 6/\mathfrak{m}+} \Delta(\lambda; z) = 1 + \frac{\lambda}{\lambda_0}$$

the intermediate-value theorem implies the existence of a unique simple zero $z = z_1^0$, $z_1^0 \in \big(6/\mathfrak{m}, \infty\big)$ of the function $\Delta(\lambda; \cdot)$, for $\lambda < -\lambda_0$, and $\Delta(\lambda; z)$ has no zeros in $(-\infty, 0) \cup \big(6/\mathfrak{m}, \infty\big)$ for $-\lambda_0 \le \lambda < 0$.

(b) The equation

$$h_1(k)f = 6/\mathfrak{m}f$$

has a non-trivial solution if

$$\Delta(\lambda; 6/\mathfrak{m})C = 0, \quad C \in \mathbb{C}, \quad that\ is \quad \lambda = -\lambda_0,$$

and the solution

$$f = \lambda\big(h_1^0(k) - 6/\mathfrak{m}\big)^{-1}\varphi, \quad that\ is \quad f(p) = \frac{\varphi(p)}{\varepsilon_{\mathfrak{f}}(p) - 6/\mathfrak{m}},$$

where

$$\varphi(p) = vf = \frac{1}{(2\pi)^3}\int_{\mathbb{T}^3} f(t)dt \equiv \text{constant}$$

As $\varepsilon_{\mathfrak{f}}(p)$ has a unique nondegenarate maximum at the point (π, π, π), the function f belongs in the Banach space $L^1(\mathbb{T}^3)$, but not in $L^2(\mathbb{T}^3)$, and it means $6/\mathfrak{m}$ is a threshold resonance.

Now we can summarize the results of this section in the following lemma.

Lemma 3.

$$\sigma_{\text{disc}}(h_1(k)) = \begin{cases} \varnothing, & if \quad 0 < \lambda \le \lambda_0, \\ \{z_1^0\}, & if \quad \lambda > \lambda_0, \end{cases}$$

$$\sigma(h_1(k)) = \begin{cases} [0, 6/\mathfrak{m}], & if \quad -\lambda_0 \le \lambda \le \lambda_0, \\ \{z_1^0\} \cup [0, 6/\mathfrak{m}], & if \quad |\lambda| > \lambda_0 \end{cases}$$

and

$$\sigma_{\text{disc}}(h_2(k)) = \{-\mu\}, \quad \sigma(h_2(k)) = \{-\mu\} \cup \{0\}$$

hold.

ESSENTIAL SPECTRUM OF $H(K)$

One of the notable outcomes in the spectral theory of multi-particle continuous Schrödinger operators involves characterizing the essential spectrum of the Schrödinger operators in terms of cluster operators (the HVZ-theorem. See Refs. [18,19,20,21,22] for the discrete case and [23] for a pseudo-relativistic operator).

Lemma 4. *The essential spectrum of $H(K)$ satisfies the relation*

$$\sigma_{ess}\big(H(K)\big)=\bigcup_{k\in\mathbb{T}^3}\Big\{\sigma\big(h_1(K-k)\big)+\varepsilon_{\mathfrak{b}}(k)\Big\}\cup\bigcup_{k\in\mathbb{T}^3}\Big\{\sigma\big(h_2(K-k)\big)+\varepsilon_{\mathfrak{f}}(k)\Big\}.$$

Proof. *The proof can be found in [17,20].*

THE ESSENTIAL SPECTRUM OF $H(K)$ with $m=\infty$ AND $\mathfrak{m}<\infty$

Due to Lemma 3 and the relations $\varepsilon_{\mathfrak{b}}(p)=0$ and $\varepsilon_{\mathfrak{f}}(p)=\varepsilon(p)/\mathfrak{m}$, we obtain

$$\bigcup_{k\in\mathbb{T}^3}\Big\{\sigma\big(h_1(K-k)\big)+\varepsilon_{\mathfrak{b}}(k)\Big\}=\sigma\big(h_1(k)\big),$$

$$\bigcup_{k\in\mathbb{T}^3}\Big\{\sigma\big(h_2(K-k)\big)+\varepsilon_{\mathfrak{f}}(k)\Big\}=\bigcup_{k\in\mathbb{T}^3}\Big\{\{-\mu\}\cup\{0\}+\varepsilon_{\mathfrak{f}}(k)\Big\}=\big[-\mu,6/\mathfrak{m}-\mu\big]\cup\big[0,6/\mathfrak{m}\big].$$

According the last two relations and Lemmas 3, 4 we have

Lemma 5. *For the essential spectrum of the main operator $H(K)$, we have*

$$\sigma_{ess}(H(K))=[0,6/\mathfrak{m}]\cup\left(\Lambda\cup[-\mu,6/\mathfrak{m}-\mu]\right),$$

where

$$\Lambda=\begin{cases}\varnothing & \textit{if}\quad \lambda\in[-\lambda_0,\lambda_0],\\ \{z_1^0\} & \textit{if}\quad \lambda\in\mathbb{R}\setminus[-\lambda_0,\lambda_0].\end{cases}$$

THE POINT SPECTRUM OF $H(K)$ FOR $m=\infty$ AND $\mathfrak{m}<\infty$

One can show easily that the subspace

$$\mathcal{A}_0=\{f\in L_s^2(\mathbb{T}^3\times\mathbb{T}^3)|f(p,q)=g(p+q),\ g\in L^2(\mathbb{T}^3)\}$$

is invariant under the operator $H(K)$, and so is $\mathcal{A}_0^{\perp}=L_s^2(\mathbb{T}^3\times\mathbb{T}^3)\ominus\mathcal{A}_0$.

Therefore, we have

$$\sigma_{pp}\big(H(K)\big)=\sigma_{pp}\big(A_0(K)\big)\cup\sigma_{pp}\big(A_1(K)\big),$$

where $A_0(K)$*and* $A_1(K)$ are restrictions of $H(K)$ on the linear subspaces $\mathcal{A}_0$ and $\mathcal{A}_0^{\perp}$, respectively.

Since $\mathcal{A}_0$ and $L^2(\mathbb{T}^3)$ are isomorphic, the operator $A_0(K)$ is unitarily equivalent to the operator B_0 on $L^2(\mathbb{T}^3)$, where

$$B_0=E_0(K)-\mu I-2\lambda v, \tag{6}$$

$E_0(K)$ denotes the multiplication by the function $\varepsilon_{\mathfrak{f}}(K-p)$, I is the identity operator, and v is an integral operator defined by

$$(vf)(p)=\frac{1}{(2\pi)^3}\int\limits_{\mathbb{T}^3}f(q)dq,\quad f\in L^2(\mathbb{T}^3),\,p\in\mathbb{T}^3.$$

The operator $A_1(K)$ takes the form

$$A_1(K)=H_0(K)-\lambda V_1-\lambda V_2.$$

Let $U_K:L_s^2(\mathbb{T}^3\times\mathbb{T}^3)\to L_s^2(\mathbb{T}^3\times\mathbb{T}^3)$ be a unitary operator defined as

$$(U_Kf)(p,q)=f\big(-K/2+p,-K/2+q\big). \tag{7}$$

It establishes a unitary equivalence between $H(K)$ and $H(0)$, and so we can proof the coming statements for $H(0)$.

SPECTRUM OF $A_0(K)$. The following lemma describes the behaviour of the eigenvector of $A_0(K)$ in the linear space $\mathcal{H}_0$.

Lemma 6. *(a) If $\lambda < -\lambda_0/2$, then $A_0(K)$ has a unique eigenfunction in the space $\mathcal{H}_0$ with an eigenvalue $\eta \in (6/\mathfrak{m} - \mu, \infty)$.*

(b) If $\lambda > \lambda_0/2$, then $A_0(K)$ has a unique eigenfunction in the space $\mathcal{H}_0$ with an eigenvalue $\eta \in (-\infty, -\mu)$.

(c) If $-\lambda_0/2 \le \lambda \le \lambda_0/2$, then $A_0(K)$ has no eigenfunctions in the space $\mathcal{H}_0$.

(d) If $2\lambda = -\lambda_0/2$ (resp. $\lambda = \lambda_0/2$), then $A_0(K)$ has a resonance energy at $-\mu$ (resp. $6/\mathfrak{m} - \mu$) (see. Definition 1).

Proof. *Recall that*

$$\mathcal{H}_0 = \{f(p,q) = g(p+q) | g \in L^2(\mathbb{T}^3)\}.$$

Again, we prove the theorem for $A_0(0)$, using unitary equivalence of $A_0(0)$ and $A_0(K)$.

The equation

$$A_0(0)f = zf, \quad f \in \mathcal{H}_0,$$

has a solution if and only if the equation

$$c\,(1 - 2\lambda d_0(\mu + z)) = 0, \quad c \in \mathbb{C}, \tag{8}$$

has one, and their solutions are related by

$$f(p,q) := g(p+q) = \frac{2\lambda}{\varepsilon_{\mathfrak{f}}(-p-q) - z} c, \tag{9}$$

where

$$c = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} g(t)dt.$$

Equation (8) has a nontrivial solution if and only if

$$\Delta(2\lambda; \mu + z) = 1 - 2\lambda d_0(\mu + z) = 0.$$

According to Lemma 1, if $2\lambda < -\lambda_0$ (resp. $2\lambda > -\lambda_0$), then $\Delta(2\lambda; \mu + z)$ has a unique eigenvalue η in $(-6/\mathfrak{m}, \infty)$ (resp. in $(-\infty, -\mu)$).

(b) Lemma 1 gives that $\Delta(2\lambda; \mu + z)$ has no zeros in $(-\infty, \infty)$.

(c) If $2\lambda = \pm\lambda_0$, then the solution given in (9) lies in $L^1((\mathbb{T}^3)^2) \setminus L^2((\mathbb{T}^3)^2)$, that is, $A_0(0)$ has a resonance energy at $-\mu$ or $6/\mathfrak{m} - \mu$.

SPECTRUM OF $A_1(K)$. As the operator $A_1(K)$ does not contain the parameter μ, Lemma 5 implies that

$$\sigma_{ess}\big(A_1(K)\big) = \{z_1^0\} \cup [0, 6/\mathfrak{m}].$$

In order to study we define the following integral depending on $n \in \mathbb{Z}$:

$$d_n(z) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{T}^3} \frac{e^{i(n,t)}}{\varepsilon_{\mathfrak{f}}(t) - z} dt, \quad z \in \mathbb{C} \setminus [0, 6/\mathfrak{m}]. \tag{10}$$

Lemma 7. *(a) For any fixed $n \in \mathbb{Z}^3$, the function $d_n(z)$ is positive and monotonically increasing in the interval $(-\infty, 0)$ as a function of z.*

(b) The following limit exists

$$d_n(0) = \lim_{z \to 0-} d_n(z) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{T}^3} \frac{e^{i(n,t)}}{\varepsilon_{\mathfrak{f}}(t) - 0} dt. \tag{11}$$

Additionally, for any $z \in \mathbb{R} \setminus [0, 6/\mathfrak{m}]$, the relation

$$\lim_{n\to\infty} d_n(z) = 0 \tag{12}$$

holds.

Proof. *The first part of the lemma easily follows from the conditionally negative definiteness of the function $\varepsilon_{\mathfrak{f}}(\cdot)$ (see [24, Lemma 4]).*

(b) The asymptotic relation $\varepsilon_{\mathfrak{f}}(q) - 0 \approx \frac{1}{2}q^2$ near the origin implies the existence of the relation 11.

We represent the operators V_1 and V_2 in (1) as $V_1 = \lambda\Phi_1^*\Phi_1$ and $V_2 = \lambda\Phi_2^*\Phi_2$, respectively, where the operators $\Phi_1, \Phi_2 : L^2((\mathbb{T}^3)^2) \to L^2(\mathbb{T}^3)$ are defined as

$$(\Phi_1 f)(p) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int\limits_{\mathbb{T}^3} f(p,t)dt \quad \text{and} \quad (\Phi_2 f)(q) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int\limits_{\mathbb{T}^3} f(t,q)dt. \tag{13}$$

Lemma 8. *The number $z \in \mathbb{C} \setminus \sigma_{ess}(A_1(K))$ is an eigenvalue of the operator $A_1(K)$ if and only if $D(z) = 0$, where*

$$D(z) = \frac{1}{\Delta^2(\lambda; z)}\prod_{n\in\mathbb{Z}^3}\delta_n^+(\lambda; z), \tag{14}$$

with

$$\delta_n^+(\lambda; z) = 1 - \lambda\big(d_0(z) + d_n(z)\big). \tag{15}$$

If $z_n \in \mathbb{C} \setminus \sigma_{ess}(A_1(K))$ is an eigenvalue of $A_1(K)$ and $\delta_n^+(\lambda; z_n) = 0$, $n \in \mathbb{Z}^3$, then the corresponding eigenfunction is of the form

$$f_n(p,q) = \frac{\lambda}{\varepsilon_{\mathfrak{f}}(K-p-q) - z_n}\Big[\cos\left(n, (-K/2+p)\right) + \cos\left(n, (-K/2+q)\right)\Big]. \tag{16}$$

Proof. *Given the unitary equivalence of $H(K)$ and $A_1(K)$ to $H(0)$ and $A_1(0)$, respectively, we first establish the claim for the latter operators.*

Let $z \in \mathbb{C} \setminus \sigma_{ess}\big(A_1(0)\big)$ be an eigenvalue of $A_1(0)$, and let f be the corresponding eigenfunction, then

$$f = R_0(z)\big[\lambda V_1 + \lambda V_2\big]f, \tag{17}$$

where $R_0(z) = (H_0(0) - zI)^{-1}$ is a resolvent of $H_0(0)$.

The last equation has a non-trivial solution if and only if the system of two linear equations

$$\tilde{\varphi}_\alpha = \Phi_\alpha\left(R_0(z)[\lambda\Phi_1^*\tilde{\varphi}_1 + \lambda\Phi_2^*\tilde{\varphi}_2]\right), \ \alpha = 1,2, \ \tilde{\varphi}_1, \tilde{\varphi}_2 \in L^2(\mathbb{T}^3) \tag{18}$$

on the space $L^2(\mathbb{T}^3) \oplus L^2(\mathbb{T}^3)$ has a non-zero solution.

Solutions of the equations (17) and (18) are related by

$$f(p,q) = R_0(z)[\lambda\Phi_1^*\tilde{\varphi}_1 + \lambda\Phi_2^*\tilde{\varphi}_2], \tag{19}$$

and

$$\tilde{\varphi}_\alpha = \Phi_\alpha f, \quad \alpha = 1,2. \tag{20}$$

Since f is a symmetric function, $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are same function, and only there arguments are different, that is, the argument of $\tilde{\varphi}_1$ is the variable p, while q is an independent variable of $\tilde{\varphi}_1$:

$$\tilde{\varphi}_2(p) = \tilde{\varphi}_1(p). \tag{21}$$

We note that the functions

$$\boldsymbol{\Delta}_\alpha := I - \lambda\Phi_\alpha R_0(z)\Phi_\alpha^*, \quad z \in \mathbb{C} \setminus \sigma_{ess}(H_0), \quad \alpha = 1,2,$$

are nonzero for any $z \in \mathbb{C} \setminus \sigma_{ess}(A_1(0))$, and they are multiplication operator by a number or by the Fredholm determinant $\Delta(\lambda; z)$ in(5), and due to Lemma 5 the inequlity $\Delta(\lambda; z) \neq 0$ holds, and hence $\mathbf{\Delta}^{-1}$ exist and it is a multiplication operator by a number

$$1/\Delta(\lambda; z).$$

Then, the solutions $\tilde{\varphi}_\alpha, \alpha = 1, 2$, of the equation (18) satisfy the following system of integral equations

$$\begin{cases} \tilde{\varphi}_1 = \lambda \mathbf{\Delta}^{-1} \mathcal{Q} \tilde{\varphi}_2, \\ \tilde{\varphi}_2 = \lambda \mathbf{\Delta}^{-1} \mathcal{Q}^* \tilde{\varphi}_1, \end{cases} \tag{22}$$

where

$$\mathcal{Q} = \Phi_1 R_0(z) \Phi_2^* \tag{23}$$

is the integral operator on $L^2(\mathbb{T}^3)$ defined as

$$(\mathcal{Q}g)(p) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{T}^3} \frac{g(t)dt}{\varepsilon_{\mathfrak{f}}(-p-t) - z}, \quad g \in L^2(\mathbb{T}^3) \tag{24}$$

and $\mathcal{Q}^*$ is the adjoint of $\mathcal{Q}$.

Using the substitution method, the system (22) can be reduced into the form

$$\tilde{\varphi}_1 = Q(z)\tilde{\varphi}_1, \quad i.e., \quad Q(z) = \lambda^2 \mathbf{\Delta}^{-1} \mathbf{\Delta}^{-1} \mathcal{Q}\mathcal{Q}^*. \tag{25}$$

Moreover, if $\varphi = (\tilde{\varphi}_1, \tilde{\varphi}_2)$ is a solution to (22), then $\tilde{\varphi}_1$ is an eigenfunction of $Q(z)$ corresponding to the eigenvalue 1. Conversely, suppose that $\tilde{\varphi}_1$ is an eigenfunction corresponding to the eigenvalue 1 of the operator $Q(z)$. Then $\varphi = (\tilde{\varphi}_1, \tilde{\varphi}_2)$, with $\tilde{\varphi}_2 = \frac{\lambda}{\Delta(\lambda;z)} \mathcal{Q}\tilde{\varphi}_1$ is a solution to (22) (i.e. (18)). Notice that the multiplicities of the linearly independent eigenvectors $\tilde{\varphi}_1$ and φ coincide.

We also note that the function f defined in (19) is an eigenfunction of $A_1(0)$ corresponding to an eigenvalue $z \in \mathbb{C} \setminus \sigma_{ess}(A_1(0))$. Moreover, the multiplicity of the eigenvalues z of $A_1(0)$ is the same as the multiplicity of the eigenvalue $\kappa = 1$ of $Q(z)$.

The operator $Q(z)$ is a convolution-type trace-class integral operator. The standard Fourier transform $\mathcal{F}_1 : L^2(\mathbb{T}^3) \to \ell^2(\mathbb{Z}^3)$,

$$(\mathcal{F}_1 g)(n) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{T}^3} e^{i(n,p)} g(p)dp, \quad g \in L^2(\mathbb{T}^3),\, n \in \mathbb{Z}^3,$$

establishes that $\widehat{Q}(z) := \mathcal{F}_1 Q(z) \mathcal{F}_1^*$ acts as a multiplication operator on the space $\ell^2(\mathbb{Z}^3)$ by the function

$$\kappa_n(z) = \frac{\lambda^2}{\Delta^2(\lambda; z)} d_n(z) d_{-n}(z), \; n \in \mathbb{Z}^3. \tag{26}$$

Thus, the spectrum of $\widehat{Q}(z)$ consists of the following union

$$\sigma(\widehat{Q}(z)) = \{0\} \cup \bigcup_{n \in \mathbb{Z}^3} \{\kappa_n(z)\},$$

with the space of eigenfunctions

$$\hat{\varphi}_n(m) = \delta_{n,m}, \; n, m \in \mathbb{Z}^3,$$

where $\delta_{\cdot,\cdot}$ is the Kroneker delta function on $\mathbb{Z}^3$.

Note that the compact operator $Q(z)$ has eigenvalues $\kappa_n(z)$, $n \in \mathbb{Z}^3$ with the corresponding eigenfunctions

$$\psi_n(p) = e^{i(n,p)}, \; n \in \mathbb{Z}^3, p \in \mathbb{T}^3. \tag{27}$$

Therefore, the determinant of the operator $I - Q(z)$ can be written as the following product

$$\det(I - Q(z)) = \prod_{n \in \mathbb{Z}^3} (1 - \kappa_n(z)), \tag{28}$$

which takes the form (14), since $d_{-n}(z) = d_n(z)$ and (26).

Let $\kappa_n(z_n) = 1$ be an eigenvalue of $Q(z_n)$, then ψ_n is the first component of the solution $\varphi_n = (\psi_n, \tilde{\psi}_n)$ of equation (18), and the second component is defined as

$$\tilde{\psi}_n(q) := \tilde{\varphi}_2(q) = \lambda \Delta_2^{-1} Q^* \psi_n(q) = \frac{\lambda d_n(z_n)}{\Delta(\lambda; z_n)} \tilde{\psi}_{-n}(q). \tag{29}$$

Using the equality (21) in the last relation, we get

$$e^{i(n,q)} = \lambda \Delta_2^{-1} Q^* \psi_n(q) = \frac{\lambda d_n(z_n)}{\Delta(\lambda; z_n)} e^{-i(n,q)},$$

which is contradiction.

However, if $\vartheta_n(p) = (e^{i(n,p)} + e^{-i(n,p)})/2 = \cos((n,p))$, it satisfies (21) and (29) if and only if

$$\frac{\lambda d_n(z_n)}{\Delta(\lambda; z_n)} = 1, \quad i.e. \quad \delta_n^+(\lambda; z) = 1 - \lambda\big(d_0(z_n) + d_n(z_n)\big) = 0.$$

However, for the functions $\theta_n(p) = (e^{i(n,p)} - e^{-i(n,p)})/2i = \sin((n,p))$ the relation

$$\theta_n = \lambda \Delta_2^{-1} Q^* \theta_n = -\frac{\lambda d_n(z_n)}{\Delta(\lambda; z_n)} \theta_n,$$

holds, which contradicts with (21). It implies

$$-\frac{\lambda d_n(z_n)}{\Delta(\lambda; z_n)} = 1, \quad i.e. \quad \delta_n^-(\lambda; z) = 1 - \lambda\big(d_0(z_n) - d_n(z_n)\big) = 0.$$

Since the system $\{\vartheta_n, \theta_n\}$, $n \in \mathbb{Z}^3$ is complete in $\ell^2(\mathbb{Z}^3)$, the last three relations and (26) allow us to use the Fredholm determinant (14) instead of (28).

Accordingly, the number z_n is an eigenvalue of $A_1(0)$ and the corresponding eigenfunction can be found by (19) as

$$f_n^0(p,q) = \frac{\lambda}{\varepsilon_{\mathfrak{f}}(-p-q) - z} \Big[\vartheta_n(p) + \vartheta_n(q)\Big].$$

According to the relation $H(0) = U_K^* H(K) U_K$, the number z_n is also an eigenvalue of $H(K)$ with the eigenfunction $f_n = U_K f_n^0$, i.e., (16), where the unitary operator U_K is defined in (7).

EIGENVALUES OF $H(K)$ BELOW AND ABOVE THE SPECTRUM OF $H_0(K)$

First, we study the function $\delta_n^+(\lambda; z)$ in () for any $n \in \mathbb{Z}^3$.

Lemma 9. *For any fixed $n \in \mathbb{Z}^3$, the function $d_0(z) + d_n(z)$ is positive and monotonically increasing in the interval $(-\infty, 0)$ as a function of z. Additionally,*

$$\lim_{z \to 0-} \big(d_0(z) + d_n(z)\big) = d_0(0) + d_n(0), \tag{30}$$

$$\lim_{z \to -\infty} \big(d_0(z) + d_n(z)\big) = 0. \tag{31}$$

and for any $z \in (-\infty, 0)$

$$\lim_{n \to \infty} \big(d_0(z) + d_n(z)\big) = d_0(z) \tag{32}$$

hold.

Proof. *The equality*

$$d_0(z) + d_n(z) = \frac{1}{(2\pi)^3} \int\limits_{\mathbb{T}^3} \frac{1 + \cos\big(n,t\big)}{\varepsilon_{\mathfrak{f}}(t) - z} dt$$

and Lemma 7 imply the positivity and monotonicity of $d_0(z)+d_n(z)$ in $(-\infty,0)$. The limits (30), (31) and (12) follow from Lemma 7.

Let z_1^0 be a zero of $\Delta_\alpha(\cdot)=0$, i.e. an eigenvalues of $h_1(k)$ in the interval $\mathbb{R}\setminus[0,6/\mathfrak{m}]$ (see 2). Note that z_1^0 exists iff $\lambda\in\mathbb{R}\setminus[-\lambda_0,\lambda_0]$.

The number defined by

$$\lambda_n^+=\frac{1}{d_0(0)+d_n(0)},$$

satisfies

$$0<\lambda_0^+<\lambda_n^+<\lambda_0,\ n\in\mathbf{Z}^3,\quad \text{and}\quad \lambda_0^+=\frac{\lambda_0}{2}. \tag{33}$$

Lemma 10. *Let $n\in\mathbb{Z}^3$. (a) If $\lambda\le\lambda_n^+$, then $\delta_n^+(\lambda,z)$ has no zeros in the interval $\big(-\infty,0\big)$.*

(b) If $\lambda_n^+<\lambda$, then $\delta_n^+(\lambda,z)$ has a unique zero $z_n\in\big(-\infty,0\big)$, and it satisfies $z_n<z_1^0$.

Proof. *Due to Lemma 9 the functions $d_0(z)+d_n(z)$ are positive and monotonically increasing in the interval $(-\infty,0)$, and so the proof of the existence of z_n is quite similar to the proof of the existence of z_1^0 in Lemma 2.*

Let $\lambda>0$. The inequalities $z_n<z_1^0$ follow the inequalities

$$\delta_n^+(\lambda;z)<\Delta(\lambda;z)$$

and the monotonicity of the last three functions.

EIGENVALUES OF $H(K)$ ABOVE $\big[0,6/\mathfrak{m}\big]$

The unitary operator $U_{\pi/2}$ in (7) is used to establish the equalities

$$U_{\pi/2}H_0(K)U_{\pi/2}=\frac{6}{\mathfrak{m}}-H_0(K)\quad \text{and}\quad U_{\pi/2}VU_{\pi/2}=V$$

which implies the relation

$$U_{\pi/2}\big(H_0(K)-V\big)U_{\pi/2}=\frac{6}{\mathfrak{m}}-\big(H_0(K)+V\big). \tag{34}$$

The final relationship enables us to shift the investigation of the eigenvalues of $H(K)$ from above the interval $\big[0,6/\mathfrak{m}\big]$ to below it.

Lemma 11. *Let $n\in\mathbb{Z}^3$. (a) If $-\lambda_n^+\ge\lambda$, then $\delta_n^+(\lambda,z)$ has no zeros in the interval $\big(6/\mathfrak{m},\infty\big)$.*

(b) If $\lambda<-\lambda_n^+$, then $\delta_n^+(\lambda,z)$ has a unique zero $z_n\in\big(6/\mathfrak{m},\infty\big)$, and it satisfies $6/\mathfrak{m}<z_1^0<z_n$.

Proof. *The proof is a consequence of Lemma 10 and the identity 34.*

MAIN THEOREM

Recall that z_1^0 be a zero of $\Delta(\lambda;\cdot)=0$, i.e. an eigenvalues of $h_1(k)$, and $z_1^0<0$ if $\lambda>0$ and $6/\mathfrak{m}<z_1^0$ if $\lambda<0$. Let η be an eigenvalue of $H(K)$ mentioned in Lemma 6.

Now, we are ready to formulate the main result of the paper.

Theorem 1. *Assume $\mu\in\mathbb{R}$ and $\lambda\in\mathbb{R}$.*

(a) Let $|\lambda|\le|\lambda_0/2|$. Then

$$\sigma_{pp}\big(H(K)\big)=\varnothing.$$

(b) Let $|\lambda_0/2|<|\lambda|<|\lambda_0|$. Then

$$\sigma_{pp}\big(H(K)\big)=\bigcup_{|n|\le n_0}\{z_n\}\cup\{\eta\},$$

and $6/\mathfrak{m}<z_1^0<z_n$ if $\lambda<0$; $z_n<z_1^0<0$ if $\lambda>0$. Here n_0 is a positive integer number depending on λ.

(c) Let $|\lambda_0| \le |\lambda|$. Then

$$\sigma_{pp}\big(H(K)\big) = \bigcup_{n\in\mathbb{Z}^3} \{z_n\} \cup \{\eta\},$$

and $6/\mathfrak{m} < z_1^0 < z_n$ if $\lambda < 0$; $z_n < z_1^0 < 0$ if $\lambda > 0$.

Proof. *Lemmas 6, 10 and 11 provide the proof. For example, by combining the assertions (a) in Lemma 6, (b) in Lemma 10, we get the proof of the part (a) of the theorem.*

Remark 1. According to Lemma 5

$$\sigma_{ess}\big(H(K)\big) = \{z_1^0\} \cup \Big(\big[-\mu, 6/\mathfrak{m}-\mu\big] \cup \big[0, 6/\mathfrak{m}\big]\Big).$$

Theorem 1 demonstrates that z_n, ζ_n, or η could be within $\sigma_{ess}\big(H(K)\big)$ or within a gap of the essential spectrum.

Theorem 2. *Let $|\lambda| \ge |\lambda_0|$. Then*

$$\lim_{n\to\infty} z_n = z_1^0.$$

Proof. *Due to Theorem 1, there exists a sequence $\{z_n\}$ of eigenvalues of $H(K)$ below z_1^0. As $\delta_n^+(\lambda; z_n) = 0$ and $\Delta_1(\lambda; z_1^0) = 0$, using the intermediate theorem we get*

$$\delta_n^+(\lambda; z_1^0) = \frac{d}{dz}\delta_n^+(\lambda; \xi_n)(z_1 - z_n),\ \ z_n < \xi_n < z_1^0,$$
$$\delta_n^+(\lambda; z_1^0) = \delta_n^+(\lambda; z_1^0) - \Delta_1(\lambda; z_1^0) = -\lambda d_n(z_1^0)$$

and hence

$$|z_n - z_1^0| = d_n(z_1^0)\left|\frac{\lambda}{\frac{d}{dz}\big(\delta_n^+(\lambda;\xi_n)\big)}\right|$$

Since $0 < C < \left|\frac{d}{dz}\big(\delta_n^+(\lambda;\xi_n)\big)\right|$ for some $C > 0$ and $n \in \mathbb{Z}^d$ we get

$$|z_n - z_1^0| < C|d_n(z_1^0)|$$

which together the limit (12) we get the first limit of the theorem.

THRESHOLD RESONANCES

In this section, we investigate how η or z_n arises from or are absorbed into the essential spectrum $\sigma_{ess}\big(H(K)\big)$

Let $\nu = -\mu$ or $\nu = 0$ or $\nu = 6/\mathfrak{m} - \mu$ or $\nu = 6/\mathfrak{m}$.

Definition 1.(Threshold eigenvalue and threshold resonance). *Let f be a solution of $H(K)f = \nu f$.*

(1) If $f \in L_s^2((\mathbb{T}^3)^2)$, ν is called a lower threshold eigenvalue of $H(K)$.

(2) If $f \in L_s^1((\mathbb{T}^3)^2) \setminus L_s^2((\mathbb{T}^3)^2)$, ν is called a lower threshold resonance of $H(K)$.

From the continuity of $d_n(z)$ at 0 and $6/\mathfrak{m}$, the function $D(z)$ can be defined at 0 and $6/\mathfrak{m}$. Let us denote it by $D(\nu)$.

Lemma 12. *(a) The number ν is a threshold eigenvalue (threshold resonance) of the operator $A_1(K)$ if and only if $D(\nu) = 0$, where $D(z)$ is defined by (14).*

If $D_n(\nu) = 0$ for some $n \in \mathbb{Z}^3$, the solution of the equation $A_1(K)f = \nu f$ can be written as

$$f_n(p,q) = \frac{\lambda}{\varepsilon_{\mathfrak{f}}(K-p-q) - \nu}\Big(\cos\big(n, (-K/2+p)\big) + \cos\big(n, (-K/2+q)\big)\Big)$$

and $f \in L_s^1((\mathbb{T}^3)^2) \setminus L_s^2((\mathbb{T}^3)^2)$.

Proof. *We prove the lemma for $K = 0$.*

(a) The proof of the part (a) is similar to the proof of Lemma 8.

The continuation of the operators $Q(z)$ in (23) and $\mathcal{Q}(z)$ in (18) at $z=\nu$ are defined by

$$Q(\nu)=\lambda\mathbf{\Delta}^{-1}(\nu)\Phi_1(H_0-\nu)^{-1}\Phi_2^*$$

and

$$\mathcal{Q}(\nu)=\lambda^2\mathbf{\Delta}^{-2}(\nu)\Phi_1(H_0-\nu)^{-1}\Phi_2^*\Phi_2(H_0-\nu)^{-1}\Phi_1^*,$$

respectively.

The equation $A_1(K)f=\nu f$, $f\in L_s^1((\mathbb{T}^3)^2)$, has a solution if and only if the equation

$$Q(\nu)\varphi=\varphi,\quad \varphi\in L^2(\mathbb{T}^3), \tag{35}$$

has a solution and

$$\mathcal{Q}(\nu)\varphi=\varphi. \tag{36}$$

Here the solutions are connected by the relations

$$f(p,q)=\frac{\lambda}{\varepsilon_{\mathfrak{f}}(-p-q)-\nu}\big(\varphi(p)+\varphi(q)\big) \tag{37}$$

and

$$\varphi=\Phi_1 f. \tag{38}$$

Since $\mathcal{Q}(\nu)$ is a Hilbert-Schmidt operator, and it has eigenvalues $\kappa_n(\nu)=\lambda^2 d_n^2(\nu)/\Delta^2(\lambda;\nu)$, $n\in\mathbb{Z}^3$ with the corresponding eigenfunctions

$$\psi_n(p)=e^{i(n,p)},\ n\in\mathbb{Z}^3,\ p\in\mathbb{T}^3. \tag{39}$$

Since $\mathcal{Q}(\nu)\psi_n=\psi_{-n}$, which is contradiction with (36), the function $\vartheta_n=(\psi_n+\psi_{-n})/2=\cos((n,\cdot))$ satisfies (36), i.e.

$$\vartheta_n=\frac{\lambda d_n(\nu)}{\Delta(\lambda;\nu)}\vartheta_n.$$

and hence

$$\delta_n^+(\lambda;\nu)=1-\lambda\big(d_0(\nu)+d_n(\nu)\big)=0.$$

Since the system $\{\vartheta_n,\theta_n\}$, $n\in\mathbb{Z}^3$ is complete in $\ell^2(\mathbb{Z}^3)$, the last three relations and (14) allow us to use the Fredholm determinant (14) instead of (28).

Accordingly, the number z_n is an eigenvalue of $A_1(0)$ and the corresponding eigenfunction can be found by (19) as

$$f_n^0(p,q)=\frac{\lambda}{\varepsilon_{\mathfrak{f}}(-p-q)-\nu}\Big[\vartheta_n(p)+\vartheta_n(q)\Big].$$

According to the relation $H(0)=U_K^*H(K)U_K$, the equation $H(K)f=\nu f$ has a solution of the form

$$f_n(p,q)=\frac{\lambda}{\varepsilon_{\mathfrak{f}}(K-p-q)-\nu}\Big[\vartheta_n(-K/2+p)+\vartheta_n(-K/2+q)\Big].$$

(b) As the function $1/\varepsilon_{\mathfrak{f}}(p)$ is integrable on $\mathbb{T}^3$, the inclusion $f^0\in L_s^1((\mathbb{T}^3)^2)$ is obvious.

Consider the relation

$$\int_{\mathbb{T}^3}\int_{\mathbb{T}^3}|f_n^0(p,q)|^2dpdq=4\int_{\mathbb{T}^3}\sin^2(n,q)dq\int_{\mathbb{T}^3}\frac{\cos^2(n,t/2)}{\varepsilon_{\mathfrak{f}}^2(t)}dt. \tag{40}$$

As $\varepsilon_{\mathfrak{f}}(p)\approx p^2$ near the origin and $\varepsilon_{\mathfrak{f}}(p)>c$ for some $c>0$ when p is not close to 0, the last integral satisfies the conditions

$$\int_{\mathbb{T}^3}\int_{\mathbb{T}^3}|f_n^0(p,q)|^2dpdq=\infty.$$

Thus $f \in L^1((\mathbb{T}^3)^2) \setminus L^2((\mathbb{T}^3)^2)$.

For any $n \in \mathbb{Z}^3$, define

$$T_n = \{m \in \mathbb{Z}^3 | d_m(0) = d_n(0)\} \quad \text{and} \quad \tau_n = |T_n|.$$

As $d_n(0) < d_0(0)$ and the function $\varepsilon_{\mathfrak{f}}(p)$ is invariant with respect to the permutations of its arguments $p^{(j)}$ and $p^{(k)}$, $j, k = 1, 2, 3$, the set T_n is singleton only for $n = 0$.

Notice that $\lambda_0/2 \le \lambda_n^+ < \lambda_0$ and $\lambda_n^+ \to \lambda_0$ as $n \to \infty$.

Theorem 3. *Let* $n \in \mathbb{Z}^3 \setminus \{0\}$.

(a) Let $|\lambda| = \lambda_0/2$*, then the operator* $H(K)$ *has a threshold resonance of multiplicity one at* 0 *if* $\lambda = \lambda_0/2$*; at* $6\mathfrak{m}$ *if* $\lambda = -\lambda_0/2$*. Moreover* $\sigma_p(H(K)) = \varnothing$.

Moreover, if $\mu > 0$ *(resp.* $\mu < 0$*), then* $H(K)$ *has a threshold resonance at the left edge (resp. the right edge) of the two-particle branch* $[-\mu, 6/\mathfrak{m} - \mu]$ *of the essential spectrum* $\sigma_{ess}(H(K))$.

(b) Let $|\lambda| = \lambda_n^+$, $n \neq 0$*. Then, the operator* $H(K)$ *has a threshold resonance of multiplicity one and threshold eigenvalue of multiplicity* $\tau_n - 1$ *at* 0 *if* $\lambda >$*; at* $6\mathfrak{m}$ *if* $\lambda < 0$.

(c) If $|\lambda| \neq \lambda_n^+$ *the operator* $H(K)$ *has a neither threshold resonance nor threshold eigenvalues at the thresholds of the essential spectrum.*

Proof. *Due to Lemma 5, the interval* $[-\mu, 6/\mathfrak{m} - \mu]$ *is a component of the essential spectrum* $\sigma_{ess}(H(K))$*, and hence Lemmas 6 and 12 imply the proof of the statement (a).*

The proof of the remaining statements follows from Lemma 12.

CONCLUSION

The discrete Schrödinger operator corresponding to the Hamiltonian of a system of three quantum mechanical particles (two identical bosons and a fermion) with masses $m = \infty$ and $\mathfrak{m} < \infty$, respectively, is considered on the three-dimensional lattice for all non-zero point interactions. The point spectrum of the three-particle discrete Schrödinger operator, which may posses infinitely many eigenvalues, has been studied for all non-zero point interactions.

Moreover, we showed the conditions for the appearance of threshold resonances and threshold eigenvalues. The dependence of their multiplicities on the parameters was explicitly derived.

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REZYUME

Biz uchta kvant mexanik zarrachalardan (ikki bir xil bozon va bitta fermiondan) iborat sistemaning Hamiltonianini koʻrib chiqamiz. Bu zarrachalar bir oʻlchamli panjarada nuqtada tortuvchi yoki itaruvchi potensiallar orqali oʻzaro ta’sirlashadi.

Bozonlarning massasi cheksiz deb faraz qilingan holda, biz uch zarrachali diskret Shredinger operatori $H(K), K \in \mathbb{T}$ ning nuqtali spektrini oʻrganamiz. Bu operator, oʻzaro tortuvchi yoki itaruvchi ta’sir kuchlariga bogʻliq holda, cheksiz koʻp xos qiymatlarga ega boʻladi.

Kalit so‘zlar: diskret Shro’dinger operatori, nuqtali spektr, bo’sag’a rezonasi, kontakt potensial, boʻsagʻa xos qiymat, Fredgolm determinanti

Аннотация

Мы рассматриваем гамильтониан системы трех квантово-механических частиц (два идентичных бозона и фермион) на одномерной решетке, взаимодействующих посредством притягивающих или отталкивающих потенциалов нулевого радиуса действия.

Мы исследуем точечный спектр трехчастичного дискретного оператора Шредингера $H(K)$, $K \in \mathbb{T}$, который обладает бесконечным числом собственных значений, зависящих от отталкивающих или притягивающих взаимодействий, в предположении, что бозоны в системе имеют бесконечную массу.

Ключевые слова: дискретный оператор Шредингера, точечный спектр, пороговый резонанс, контактный потенциал, пороговое собственное значение, определитель Фредгольма