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# HARMONIC FUNDAMENTAL SOLUTIONS AND BOUNDARY INTEGRAL EQUATIONS FOR WAVE PROPAGATION IN ELASTIC, SCALAR AND POROELASTIC MEDIA

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**Abstract**

This study explores integral formulations and harmonic fundamental solutions for various media, including elastodynamic, scalar, and poroelastic media. The integral formulations are expressed in terms of boundary variables linking the variables in the internal domain with the boundary conditions. The work highlights the significance of fundamental solutions in solving boundary integral equations, enabling numerical solutions using the Boundary Element Method (BEM). Detailed derivations and applications for each medium are presented, emphasizing methods to address singularities and reduce computational complexity. The results offer foundational insights for engineers and researchers applying BEM to complex media.

***Key words:*** Integral equations, harmonic solutions, Boundary Element Method, wave equations, poroelastics, elastodynamics, scalar media, computational mechanics.

## Introduction

The boundary integral formulation for each of the three media is presented below. Using data derived from the governing equations, it links the fundamental variables at internal points to their values and derivatives at the boundary. These equations connect the actual field variables with those from a simplified, well-known “fundamental solution”. Together, the formulation and fundamental solutions form the basis for solving the problem numerically using the Boundary Element Method (BEM).

## Literature Review

Fundamental solutions to elastodynamic integral equations-developed by Stokes, Cruse-Rizzo, and Kupradze-enable modeling of elastic media, including half-spaces and point source problems, for practical engineering use. In scalar media, wave propagation governed by the Helmholtz equation facilitates pressure modeling, with modified fundamental solutions easing boundary computations and reducing computational demands, particularly in fluid environments. For poroelastic media, integral formulations describe solid matrix displacements and fluid stresses, with detailed coupling of phases as outlined by Dominguez and Aznarez. The Boundary Element Method (BEM) connects these theoretical models with numerical solutions, efficiently handling singularities and solving for internal variables using boundary data.

## Research Methodology

**Integral formulation in harmonic elastodynamics:** In the absence of body forces, the displacement field in a reduced elastodynamic state of a bounded domain $\Omega$ with boundary $\Gamma$ is given by:

$$u_j^k + \int\limits_{\Gamma} t_{ji}^* u_i d\Gamma = \int\limits_{\Gamma} t_{ji}^* t_i d\Gamma \quad (1)$$

Here, $u_j^k$ is the displacement in direction $j$ at point $k$, where the external force is applied. $u_i$ and $t_i$ are the displacement and stress components in direction $i$ of the actual problem. $u_{ji}^*$ and $t_{ji}^*$ represent the displacement and stress components of the fundamental solution due to a unit point load in direction $j$, satisfying the Navier equation in the frequency domain: $\mu\nabla^2 u^*_{ji} + (\lambda+\mu)\nabla e^*_{j,i} - \rho\omega^2 u^*_{j,i} + \delta(x-x_k)\delta_{ij}$. The Dirac delta function models the point nature of the excitation.

**Integral formulation in harmonic scalar problems:** The equation equivalent to the previous one in the case of scalar media that relates the pressure at a point $k$ belonging to the domain $\Omega$ with the variables pressure and its derivative in the contour $\Gamma$, is [1, 2]:

$$p^k + \int\limits_\Gamma \left(\frac{\partial p}{\partial n}\right)^n p d\Gamma = \int\limits_\Gamma p^* \frac{\partial p}{\partial n} d\Gamma \tag{2}$$

where $p^k$ is the value of the pressure at the internal point $k$, $n$ the normal to the contour and $p^*$ is the fundamental solution that satisfies $\Omega$ the Helmholtz equation for a point source $k$ pulsating in an infinite medium with frequency $\omega$:

$$\nabla^2 p^* + \left(\frac{\omega}{c}\right)^2 p^* + \delta(x-x_k) = 0 \tag{3}$$

## Analysis and results

**Comprehensive formulation in harmonic poroelasticity:** The integral formulation for the case of poroelastic media is given by the following four equations [2,3-4]:

$$u^k_j + \int\limits_\Gamma t^*_{ji} u_i d\Gamma - \int\limits_\Gamma U^*_{nj}\tau d\Gamma = \int\limits_\Gamma u^*_{ji} t_i d\Gamma - \int\limits_\Gamma \tau^*_j U_n d\Gamma \tag{4}$$

$$-J\tau^k + \int\limits_\Gamma t^*_{oi} u_i d\Gamma - \int\limits_\Gamma \left(U^*_{no} - JX_i^{'*} n_i\right)\tau d\Gamma = \int\limits_\Gamma u^*_{oi} t_i d\Gamma - \int\limits_\Gamma \tau^*_o U_n d\Gamma \tag{5}$$

where $J = \frac{1}{i\omega b - \omega^2 \rho_{22}}$

Equations (4) relate displacement in directions $j = 1, 2, 3$ at an internal point $k$ in the domain $\Omega$ to displacements $u_i, U_n$ and stresses $t_i, \tau$ in the porous medium along the boundary $\Gamma$, where $U_n$ is the normal displacement of the fluid phase. The terms $u^*_{ji}$ and $t^*_{ji}$ are the fundamental displacements and tractions of the solution in the solid matrix due to a point load in the direction $j$.

For the same load, $\tau^*_j$ and $U^*_{nj}$ represent the equivalent stress and normal fluid displacement at the boundary. These known values form the fundamental solution when the load acts in the solid phase. Equation (5) provides an integral representation of the equivalent stress at an internal fluid-phase point $k$ in $\Omega$. This stress is related to boundary displacements $u_i$, $U_n$ and stresses $t_i, \tau$. Here, $u^*_{oi}$ and $t^*_{oi}$ are the displacement and traction components of the solid skeleton caused by a point source applied in the fluid phase (subscript $o$, with $j = 4$). Consequently, $\tau^*_o$ and $U^*_{no}$ are the response of the fluid to stress and normal displacement. These four terms define the fundamental solution when the load is applied in the fluid phase [3,5].

**Harmonic fundamental solution:** In the previous section's integral formulation, several terms involve the fundamental solution a known solution to simplified problems with few restrictions. These allow the formulation of boundary integral equations, solvable approximately via the Boundary Element Method (BEM), as discussed next. The relevant problems and their solutions for each medium described in this chapter are presented below.

**Fundamental elastodynamic solution:** In this case the problem consists of a point load applied at a point in an infinite, homogeneous, elastic, linear and isotropic medium. The resulting stresses and displacements constitute a classical problem that was solved by Stockes in the time domain, by Cruse-Rizzo in the Laplace transformed domain and some years earlier by Kupradze for harmonic problems. For a point $x$ that is a distance $r$ from the point of application $\xi$, the displacement in direction $k$ for a load applied in direction $l$ is given by [3,6]:

$$u^*_{lk}(x,\xi,\omega) = \frac{1}{4\pi\mu}(\psi\delta_{lk} - \chi r_l r_k) \tag{6}$$

where:

$$\psi = \sum_{m=1}^{2}\left[1-\left(\frac{z_1}{z_2}\right)^2\delta_{m1}\right]\left(\frac{1}{z_m^2r^2}-\frac{1}{z_mr}+\delta_{m2}\right)E_m$$
$$\chi = \sum_{m=1}^{2}\left[1-\left(\frac{z_1}{z_2}\right)^2\delta_{m1}\right]\left(\frac{3}{z_m^2r^2}-\frac{3}{z_mr}+1\right)E_m \tag{7}$$

In these expressions $E_m = \frac{1}{r}e^{-ik_mr}$, $r = |x-\xi|$, $z_1 = -ik_1$, $z_2 = -ik_2$

Starting from the solution in displacements (7) and using the law of material behavior, the stresses for a surface of normal n are:

$$t_{lk}^{*}(x,\xi,\omega) = \frac{1}{4\pi}\left[\frac{\partial r}{\partial n}(A\delta_{lk}+Br_lr_k)+(Ar_ln_l+Cr_ln_k)\right] \tag{8}$$

where:

$$A = \frac{d\psi}{dr}-\frac{\chi}{r};\quad B = 2\left(2\frac{\chi}{r}-\frac{d\chi}{dr}\right);\quad C = \frac{\lambda}{\mu}\left(\frac{d\psi}{dr}-\frac{d\chi}{dr}-2\frac{\chi}{r}\right)-2\frac{\chi}{r} \tag{9}$$

Fundamental solution in scalar wave propagation problems: The pressure at any point $x$ in a scalar medium as a result of the application of a point source is $\xi$ given by the expression:

$$p^{*}(x,\xi,\omega) = \frac{1}{4\pi r}e^{-ikr} \tag{10}$$

Where $r = |x-\xi|$ and $k = \frac{\omega}{c}$, where $c$ is the speed of wave propagation in the medium. The derived variable, that is, the pressure flux on a surface with normal n is given by:

$$\frac{\partial p^{*}}{\partial n} = -\frac{1}{4\pi}\left(\frac{1}{r^2}+\frac{ik}{r}\right)e^{-ikr}\frac{\partial r}{\partial n} \tag{11}$$

In this type of medium, equation (2) enables the exclusion of certain boundaries if the fundamental solution satisfies the boundary conditions of the real problem. This applies to water in the presented models, where the free surface boundary can be omitted. Consequently, the Boundary Element Method (BEM) reduces the number of degrees of freedom, enhancing computational efficiency.

The modified fundamental solution is derived by introducing two point sources: a positive source at the domain point $\xi$, where the integral equation is evaluated, and a negative source at its mirror image with respect to the free surface. Figure 1 illustrates the placement of these sources.

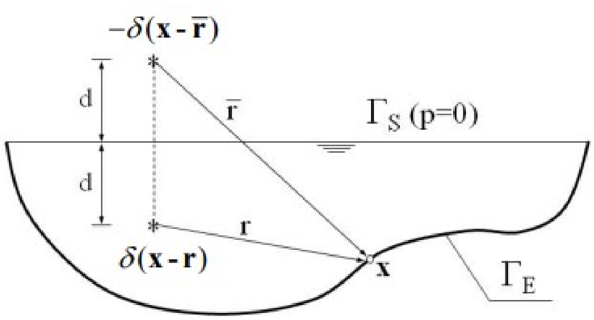


Рис. 14: Position of the charges for obtaining the fundamental source-image solution in scalar problems.

By making this double placement of the source we arrive at the following fundamental solution, which obviously leads to zero pressures at the points of the free surface of the medium.

$$\hat{p}^{*}(x,\xi,\omega) = \frac{1}{4\pi}\left(\frac{1}{r}e^{-ikr}-\frac{1}{\bar{r}}e^{-ik\bar{r}}\right) \tag{12}$$

Where $\bar{r} = |x - \bar{\xi}|$. As already indicated, using $\hat{p}^*$ only it is necessary to apply the integral equation to the part of the contour that in the figure appears with the designation $\Gamma_E$.

Fundamental poroelastic solution: The basic solution for poroelastic media is presented below. In this case, the load can be applied both to the solid matrix and to the fluid phase of the porous medium, which leads, depending on the response considered, to the values contained in tables 1 and 2.

**Load applied in $l$ direction on the solid matrix. Displacement response of the solid matrix in direction $k$.**

$$u^*_{lk}(x,\xi,\omega) = \frac{1}{4\pi\mu}\left(\psi\delta_{lk} - \tilde{\chi} r_l r_k\right) \tag{13}$$

**Load applied in $l$ direction on the solid matrix. Equivalent stress response of the fluid phase.**

$$\tau^*_l(x,\xi,\omega) = \frac{i\omega\eta}{4\pi\mu}\tilde{\phi}_l r_l \tag{14}$$

**Point source in the fluid phase. Response to solid displacements in the direction $k$.**

$$u^*_{0k}(x,\xi,\omega) = \frac{\gamma}{4\pi}\tilde{\phi}_k \tag{15}$$

**Point source in the fluid phase. Equivalent stress response of the fluid phase.**

$$\tau^*_o(x,\xi,\omega) = \frac{1}{4\pi}\tilde{\kappa} \tag{16}$$

**Auxiliary expressions:**

$$\begin{aligned}
\psi &= \sum_{m=1}^{2}(-1)^m\frac{\mu}{(\lambda+2\mu)z_{21}}\left(\frac{i\omega}{K} - z_m^2\right)(\delta_{m1}+\delta_{m2}) + \delta_{m3}\left[\left(\frac{1}{z_m^2 r^2} - \frac{1}{z_m r} + \delta_{m3}\right)E_m\right]\\
\chi &= \sum_{m=1}^{2}(-1)^m\frac{\mu}{(\lambda+2\mu)z_{21}}\left(\frac{i\omega}{K} - z_m^2\right)(\delta_{m1}+\delta_{m2}) + \delta_{m3}\left[\left(\frac{3}{z_m^2 r^2} - \frac{1}{z_m r} + 1\right)E_m\right]\\
\tilde{\phi} &= \sum_{m=1}^{2}\frac{(-1)^{m+1}}{(\lambda+2\mu)z_{21}}z_m\left(\frac{1}{z_m r} - 1\right)E_m\\
\tilde{\kappa} &= \sum_{m=1}^{2}\frac{(-1)^{m+1}}{z_{21}}\left(\frac{\mu}{\lambda+2\mu}z_3^2 - z_m^2\right)E_m
\end{aligned} \tag{17}$$

Table 1. Fundamental poroelastic solution in terms of the fundamental variables: Solid matrix displacements and equivalent stress in the fluid phase.

**Load applied in $l$ direction on the solid matrix. Stress vector in the solid matrix in direction $k$.**

$$t^*_{lk}(x,\xi,\omega) = \frac{1}{4\pi}\left[\frac{\partial r}{\partial n}\left(\tilde{A}\delta_{lk} + \tilde{B}_l r_k\right) + \left(\tilde{A} r_k n_l + \tilde{C} r_l n_k\right)\right] \tag{18}$$

**Point source in the fluid phase. Stress vector in the solid matrix in direction $k$.**

$$\tau^*_{ok}(x,\xi,\omega) = \frac{\gamma}{4\pi}\left(\frac{\partial r}{\partial n}\tilde{F}_k + \tilde{G}_k\right) \tag{19}$$

**Load applied in $l$ direction on the solid matrix. Displacement response in the fluid phase.**

$$U^*_{nl}(x,\xi,\omega) = \frac{1}{4\pi}\left(\frac{\partial r}{\partial n}\tilde{D} r_l + \tilde{E}\eta_l\right) \tag{20}$$

**Point source in the fluid phase. Displacement response in the fluid phase.**

$$U^*_{no} - JX'_j\eta_l = \frac{1}{4\pi}\frac{\partial r}{\partial n}\tilde{H} \tag{21}$$

**Auxiliary expressions:**

$$\tilde{A} = \frac{d\tilde{\psi}}{dr} - \frac{\tilde{\chi}}{r}, \qquad \tilde{B} = 2\left(\frac{2\tilde{\chi}}{r} - \frac{d\tilde{\chi}}{dr}\right)$$
$$\tilde{C} = \frac{\lambda}{\mu}\left(\frac{d\tilde{\psi}}{dr} + \frac{d\tilde{\chi}}{dr} - 2\frac{\tilde{\chi}}{r}\right) - 2\frac{\tilde{\chi}}{r} + \frac{Q}{R}i\omega\tilde{\phi}$$
$$\tilde{D} = i\omega\eta J\left(\frac{d\tilde{\phi}}{dr} - \frac{\tilde{\phi}}{r}\right) - \frac{Z}{\mu}\tilde{\chi}$$
$$\tilde{E} = i\omega\eta J\frac{\tilde{\phi}}{r} + \frac{Z}{\mu}\tilde{\psi}, \qquad \tilde{F} = 2\mu\left(\frac{d\tilde{\phi}}{dr} - \frac{\tilde{\phi}}{r}\right)$$
$$\tilde{G} = \lambda\left(\frac{d\tilde{\phi}}{dr} + 2\frac{\tilde{\phi}}{r}\right) - 2\mu\frac{\tilde{\phi}}{r} + \frac{Q}{R\gamma}\tilde{\kappa}, \qquad \tilde{H} = J\frac{d\tilde{\kappa}}{dr} + Z\tilde{\phi} \tag{22}$$

Table 2. Fundamental poroelastic solution in terms of the derived variables: Stress vector in the solid matrix and normal displacement in the fluid phase associated with a surface with exterior normal $n$.

In all the above equations $\bar{r} = |x - \bar{\xi}|$, $E_m = \frac{1}{r}e^{z_m r}$, $z_m = -ik_m\,(m = 1, 2, 3)$ and $z_{21} = z_2^2 - z_1^2$.

Comprehensive Contour Formulation: The application of the BEM to problems in viscoelastic, scalar, and poroelastic media requires that the integral formulations in equations [(1), (2), (4), (5)] involve only boundary variables. These equations link internal fundamental variables to their values and derivatives on the boundary. To ensure consistency, placement points must lie on the boundary. However, this introduces challenges, as the integrands become singular at these points [8,3].

The usual way to overcome this fact is by means of a process of passing to the limit, replacing the real contour $\Gamma$ by an approximate one that avoids the singularity, composed of two contours, $(\Gamma - \Gamma_e)$ and $\Gamma_e$, where $\Gamma_e$ is an infinitesimal spherical volume of radius $\varepsilon \to 0$ with center at the placement point (figure 2). With this technique each of the contour integrals can be decomposed into two others extended to the contours $\Gamma - \Gamma_\varepsilon$ and $\Gamma_\varepsilon$.

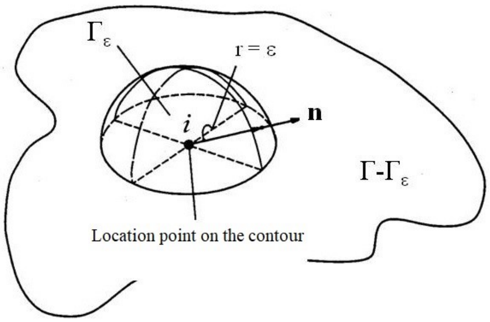


Рис. 15: Contour decomposition $\Gamma - \Gamma_e$ and $\Gamma_e$ “avoiding” the singularity.

To describe the process, this procedure will be applied to the case of viscoelastic regions. Starting from the integral equation (1) we have:

$$u^i_l + \int\limits_{\Gamma-\Gamma_\varepsilon} t^*_{lk}u_k d\Gamma + \int\limits_{\Gamma_\varepsilon} t^*_{lk}u_k d\Gamma = \int\limits_{\Gamma-\Gamma_\varepsilon} u^*_{lk}t_k d\Gamma + \int\limits_{\Gamma_\varepsilon} u^*_{lk}t_k d\Gamma \tag{23}$$

In order to achieve our goal of having variables appear only on the contour, it is necessary to study the behavior of these integrals when $\varepsilon \to 0$. Thus, the integrals on $\Gamma - \Gamma_\varepsilon$ do not present problems since the contour on which they extend does not include the singularity and in the limit they must be understood in the sense of the Cauchy Principal Value (CPV) [4, 5p].

$$\lim_{\varepsilon \to 0} \int_{\Gamma-\Gamma_\varepsilon} t^*_{lk} u_k d\Gamma = \text{CPV} \int_{\Gamma} t^*_{lk} u_k d\Gamma, \quad \lim_{\varepsilon \to 0} \int_{\Gamma-\Gamma_\varepsilon} u^*_{lk} t_k d\Gamma = \text{CPV} \int_{\Gamma} u^*_{lk} t_k d\Gamma \tag{24}$$

If limits are taken on the integrals along $\Gamma_e$:

$$\lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} u^*_{lk} t_k d\Gamma = 0, \quad u^i_l + \lim_{\varepsilon \to 0} \int_{\Gamma_\varepsilon} t^*_{lk} u_k d\Gamma = c^i_{lk} u^i_k. \tag{25}$$

Where $c^i_{lk}$, called the free term, with a value equal to that appearing in elastostatics, is a constant that depends on the geometry of the contour at the point of application of the load $\xi$ and $\nu$. Taking into account (24) and (25) (23) can be written, omitting for convenience the acronym “CPV” from the expressions (24), as follows:

$$c^i_{lk} u^i_k + \int_{\Gamma_\varepsilon} t^*_{lk} u_k d\Gamma = \int_{\Gamma_\varepsilon} u^*_{lk} t_k d\Gamma \tag{26}$$

Or in a more compact matrix notation in which the placement in the three directions is recorded together [6, 4p]:

$$c^i u^i + \int_{\Gamma} p^* u d\Gamma = \int_{\Gamma} u^* p d\Gamma \tag{27}$$

where $u$ and $p$ will be the vectors of the field variables, $u^*$ and $p^*$ the tensors of the fundamental solution and $c^i$ the tensor of the elastostatic free term at the collocation point (as is obvious $i$ $c^i = I$ (These are internal points) [5, 7p]:

$$c^i = \begin{pmatrix} c^i_{11} & c^i_{12} & c^i_{13} \\ c^i_{21} & c^i_{22} & c^i_{23} \\ c^i_{31} & c^i_{32} & c^i_{33} \end{pmatrix} \tag{28}$$

Following an analogous procedure for scalar wave propagation problems starting from equation (2), the integral formulation on the contour in this case becomes [9,3]:

$$c^i p^i + \int_{\Gamma} \left(\frac{\partial p}{\partial n}\right)^* p d\Gamma = \int_{\Gamma} p^* \frac{\partial p}{\partial n} d\Gamma \tag{29}$$

where the free term takes the value $c^i = \frac{\theta}{4\pi}$($\theta$ is the solid angle of the contour at the point).

In the case of poroelastic media, carrying out the limit step of equations (4) and (5), a matrix equation of the type (27) is obtained where the tensor corresponding to the free term has an expression of the type [7, 3p]:

$$c^i = \begin{pmatrix} c^i_{11} & c^i_{12} & c^i_{13} & 0 \\ c^i_{21} & c^i_{22} & c^i_{23} & 0 \\ c^i_{31} & c^i_{32} & c^i_{33} & 0 \\ 0 & 0 & 0 & -Jc^i \end{pmatrix} \tag{30}$$

In this case$i$ $c$ depends on the geometry of the contour in $i$ $x$, the Poisson ratio of the drained material and the value of $J = \frac{1}{i\omega b - \omega^2 \rho_{22}}$ [10, 5p].

**Conclusion/Recommendations**

This article has detailed the integral formulations and fundamental solutions for elastodynamic, scalar, and poroelastic media, emphasizing their application in Boundary Element Methods. By addressing challenges such as singularities and computational complexity, these methodologies offer powerful tools for solving complex mechanical and physical problems. The work highlights the importance of fundamental solutions in enabling numerical solutions, laying the groundwork for further research and application in engineering and computational sciences.

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### Annotatsiya

Ushbu tadqiqot ishida turli xil muhitlar, jumladan, elastik, skalyar va g‘ovak elastik muhitlar uchun integral tenglamalar va asosiy garmonik yechimlar o‘rganilgan. Integral tenglamalar chegaraviy o‘zgaruvchilar nuqtai nazaridan ifodalanadi va ichki soha o‘zgaruvchilarini chegaraviy shartlarga bog‘laydi. Ushbu ish asosiy yechimlarning chegaraviy integral tenglamalarni yechishda muhim hisoblanadi va Chegaraviy elementlar usuli (CHEU) yordamida sonli yechimlarni ta’minlaydi. Har bir muhit uchun tenglamalar va yechimlar keltirilgan bo‘lib, ularni hal qilish usullari va hisoblash murakkabligini kamaytirish yo‘llari yoritilgan. Natijalar CHEU ni murakkab muhitlarga qo‘llashda muhandislar va tadqiqotchilar uchun asosiy tushunchalarni taqdim etadi.



***Kalit so‘zlar:*** Integral tenglamalar, garmonik yechimlar, Chegaraviy elementlar usuli, to‘lqin tenglamalari, g‘ovak elastik, elastik, skalyar muhitlar, hisoblash mexanikasi.



### Аннотация

В данном исследовании рассматриваются интегральные формулировки и гармонические фундаментальные решения для различных сред, включая системы в области эластодинамики, скалярных полей и пороупругости. Интегральные формулировки выражены через переменные на границе, связывая внутренние переменные области с граничными условиями. Работа подчеркивает важность фундаментальных решений в решении граничных интегральных уравнений, что позволяет численно решать их с использованием метода граничных элементов (МГЭ). Приведены подробные выводы и примеры применения для каждой среды, с акцентом на методы устранения сингулярностей и уменьшения вычислительной сложности. Полученные результаты предоставляют базовые знания для инженеров и исследователей, применяющих метод граничных элементов к сложным средам.

***Ключевые слова:*** интегральные уравнения, гармонические решения, метод граничных элементов, волновые уравнения, пороэластичность, эластодинамика, скалярные среды, вычислительная механика.