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SOME HOMOTOPY PROPERTIES OF n-FOLD SYMMETRIC PRODUCT OF THE SPACE X

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RESUME

In this paper, we study some homotopy properties of the space n-fold symmetric product of the space X. We prove that if mappings $f, g : X \to Y$ are homotopic, then the mappings $\mathcal{F}_n f, \mathcal{F}_n g : \mathcal{F}_n X \to \mathcal{F}_n Y$ are also homotopic. Also shown that the functor of n-fold symmetric product $\mathcal{F}_n$ is a covariant homotopy functor. Besides proved that if spaces X and Y are homotopically equivalent, then the spaces $\mathcal{F}_n X$ and $\mathcal{F}_n Y$ are also homotopically equivalent.

Key words: Functor, n-fold symmetric product, homotopy, homotopically equivalent.

Recently, the topological properties on hyperspaces with the Vietoris topology and the homotopy properties of the topological spaces have been studied by many authors ([1], [2], [3], [4], [5], [6]).

In [1] the connection between a finally compact, pceudocompact, extremely disconnected, $\aleph$-space and its hyperspace is studied. And in the work [2] have been studied the connection between a uniformly connected, uniformly pseudocompact, $P-$precompact and its hyperspace. In the works [3] and [4] have been studied some cardinal and homotopy properties of the superextension λX of a topological space X. And in [4] proved that the superextension functor λ preserves homotopy, i.e. that it is a homotopy functor. In [5] showed that the functor of Permutation Degree SP^n_G preserves the homotopy and the retraction of topological spaces. And in [6] have been studied some homotopy properties of the space of complete linked systems.

Recall that a covariant functor is a mapping $\mathcal{F}$ which assigns to a topological space X the space $\mathcal{F}(X)$, and to a continuous mapping $f : X \to Y$, the mapping $\mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{F}(Y)$ satisfying the following conditions:

1) $\mathcal{F}$ preserves identity, that is, if id_X is the identity mapping of X, then $\mathcal{F}(id_X) = id_{\mathcal{F}(X)}$;

2) $\mathcal{F}$ preserves composition, that is, if $f : X \to Y$ and $g : Y \to Z$ are continuous mappings, then we have

$$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f).$$

We refer the reader to the book [7] and the article [8] for more information about functors. Some metric properties of n-fold symmetric product of the space X is studied in the work [9]. In this paper we study some homotopy properties and retractions of n-fold symmetric product of the space X.

All of our space are Hausdorff unless otherwise indicated. The symbol N stands for the set of positive integers and R stands for the set of real numbers. Given a space X, we define its hyperspaces as the following sets:

1) $CL(X) = \{A \subset X \mid A \text{ is closed and nonempty }\}$;

2) $2^X = \{A \in CL(X) \mid A \text{ is compact }\}$;

3) $\mathcal{F}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ points }\}$, $n \in N$ (see [9, 10]).

$CL(X)$ is topologized by the Vietoris topology defined as the topology generated by

$$\beta = \{\langle U_1, ..., U_k\rangle \mid U_1, ..., U_k \text{ are open subsets of } X,\ k \in N\},$$

where $\langle U_1, ..., U_k\rangle = \{A \in CL(X) \mid A \subset \bigcup U_j \text{ and } A \cap U_j \neq \emptyset \text{ for each } j \in \{1, ..., k\}\}$.

Note that, by definition, 2^X, $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ are subsets of $CL(X)$. Hence, they are topologized with the appropriate restriction of the Vietoris topology. Moreover,

1) $CL(X)$ is called the *hyperspace of nonempty closed subsets of* X;

2) 2^X is called the *hyperspace of nonempty compact subsets of* X;

3) $\mathcal{F}_n(X)$ is called the $n-$*fold symmetric product of* X;

4) $\mathcal{F}(X)$ is called the *hyperspace of finite subsets of* X.

On the other hand, it is obvious that $\mathcal{F}(X) = \bigcup_{n=1}^{\infty} \mathcal{F}_n(X)$ and $\mathcal{F}_n(X) \subset \mathcal{F}_{n+1}(X)$ for each $n \in N$ (see [9, 10]).

Remark 1. Let X be a space and let $n \in N$.
1) $\mathcal{F}_n(X)$ is closed in $\mathcal{F}(X)$;
2) $f_1 : X \twoheadrightarrow \mathcal{F}_1(X)$, $(x \mapsto x)$, is a homeomoerphism;
3) Every $\mathcal{F}_m(X)$ is a closed subset of $\mathcal{F}_n(X)$ for each $m, n \in N$, $m < n$ (see [11]).

Notation 1. If $U_1, U_2, ..., U_n$ are open subsets of a space X, then $\langle U_1, U_2, ..., U_n\rangle_{\mathcal{F}(X)}$ denotes the intersection of the open set $\langle U_1, U_2, ..., U_n\rangle$ of the Vietotis topology, with $\mathcal{F}(X)$ (see [12]).

Notation 2. Let X be a space. If $\{x_1, x_2, ..., x_r\}$ is a point of $\mathcal{F}(X)$ and $\{x_1, x_2, ..., x_r \in \langle U_1, U_2, ..., U_n\rangle_{\mathcal{F}(X)}\}$, then for each $j \leq r$, we let $U_{x_j} = \cap\{U \in \{U_1, U_2, ..., U_s\} : x_j \in U\}$ Observe that $\langle U_{x_1}, U_{x_2}, ..., U_{x_r}\rangle_{\mathcal{F}(X)} \subset \langle U_1, U_2, ..., U_s\rangle_{\mathcal{F}(X)}$ (see [13]).

For some undefined or related concepts, we refer the reader to [14], [15] and [16].

Now we will consider some homotopic properties of n-fold symmetric product of the space X. We begin with definitions of notions that will be used in this section. We mainly follow terminology from [15] and [16].

Continuous mappings $f, g : X \to Y$ are said to be *homotopic* if there is a continuous mapping $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. The mapping H is called a *homotopy* between f and g and we write $f \simeq g$ [15].

Let $f : X \to Y$ is a continuous mapping and $A \in \mathcal{F}_n X$. Let's assume that $(\mathcal{F}_n f)(A) = f(A)$. Then the mapping $\mathcal{F}_n f : \mathcal{F}_n X \to \mathcal{F}_n Y$ also is a continuous mapping.

For the functor of n-fold symmetric product $\mathcal{F}_n$ the following theorem holds.

Theorem 1. If mappings $f, g : X \to Y$ are homotopic, then the mappings $\mathcal{F}_n f, \mathcal{F}_n g : \mathcal{F}_n X \to \mathcal{F}_n$ are also homotopic.

Proof. Assume that the mappings $f, g : X \to Y$ are homotopic. Then there exists a continuous mapping $H : X \times I \to Y$ such that $H(x, 0) = f(x)$, and $H(x, 1) = g(x)$. On the other hand, we have that $(\mathcal{F}_n f)(A) = f(A)$ and $(\mathcal{F}_n g)(A) = g(A)$ for all $A \in \mathcal{F}_n X$. Now we can define the mapping $(\mathcal{F}_n H)(A, t) = H(a, t)$, where $A \in \mathcal{F}_n X$ and $t \in [0, 1]$. It is clear that since the mapping H is continuous, the mapping $\mathcal{F}_n H$ is also continuous. Now we will show that the mapping $\mathcal{F}_n H$ is a homotopy between the mappings $\mathcal{F}_n f$ and $\mathcal{F}_n g$. Indeed $(\mathcal{F}_n H)(A, 0) = H(A, 0) = f(A) = (\mathcal{F}_n f)(A)$, and $(\mathcal{F}_n H)(A, 1) = H(A, 1) = g(A) = (\mathcal{F}_n g)(A)$. This means that $\mathcal{F}_n f \simeq \mathcal{F}_n g$. Theorem 1 is proved.

From the Theorem 1 we get the following corollary.

Corollary 1. If the mapping H is homotopy between f and g, then the mapping also $\mathcal{F}_n H$ is a homotopy between $\mathcal{F}_n f$ and $\mathcal{F}_n g$.

A continuous mapping $f : X \to Y$ is said to be a *homotopy equivalence* if there exists a continuous mapping $g : Y \to X$ such that the compositions $g \circ f$ and $f \circ g$ are homotopic to the identity mappings on X and Y respectively. Two topological spaces X and Y is said to be *homotopically equivalent* (notation, $X \simeq Y$) if there exists a homotopy equivalence $f : X \to Y$ (see [15]).

Proposition 1. If the mapping $f : X \to Y$ is homotopy equivalence, then the mapping $\mathcal{F}_n f : \mathcal{F}_n X \to \mathcal{F}_n Y$ is also homotopy equivalence.

Proof. Suppose that the spaces X and Y are homotopically equivalent. Then there exists two continuous mappings $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. It means that there are two homotopy $H_1(y,t)$ and $H_2(x,t)$ such that $H_1(y,0) = (f \circ g)(y)$, $H_1(y,1) = y$, and $H_2(x,0) = (g \circ f)(x)$, $H_2(x,1) = x$. We can define the compositions $\mathcal{F}_n f \circ \mathcal{F}_n g : \mathcal{F}_n Y \to \mathcal{F}_n Y$, and $\mathcal{F}_n g \circ \mathcal{F}_n f : \mathcal{F}_n X \to \mathcal{F}_n X$ of the mappings $\mathcal{F}_n f : \mathcal{F}_n X \to \mathcal{F}_n Y$ and $\mathcal{F}_n g : \mathcal{F}_n Y \to \mathcal{F}_n X$ as follows: $(\mathcal{F}_n f \circ \mathcal{F}_n g)(B) = (f \circ g)(B)$ for all $B \in \mathcal{F}_n Y$, and $(\mathcal{F}_n g \circ \mathcal{F}_n f)(A) = (g \circ f)(A)$ for all $A \in \mathcal{F}_n X$. One can easily check that the mapping $(\mathcal{F}_n H_1)(B,t) = H_1(B,t)$ is a homotopy between $\mathcal{F}_n f \circ \mathcal{F}_n g$ and $id_{\mathcal{F}_n Y}$, where $B \in \mathcal{F}_n Y$ and $t \in [0,1]$. Indeed, $(\mathcal{F}_n H_1)(B,0) = H_1(B,0) = (f \circ g)(B) = (\mathcal{F}_n f \circ \mathcal{F}_n g)(B)$ for all $B \in \mathcal{F}_n Y$, and $(\mathcal{F}_n H_1)(B,1) = H_1(B,1) = B = id_{\mathcal{F}_n Y}$. Similarly, the mapping $(\mathcal{F}_n H_2)(A,t) = H_2(A,t)$ is a homotopy between $\mathcal{F}_n g \circ \mathcal{F}_n f$ and $id_{\mathcal{F}_n X}$, where $A \in \mathcal{F}_n X$ and $t \in [0,1]$. Indeed, $(\mathcal{F}_n H_2)(A,0) = H_2(A,0) = (g \circ f)(A) = (\mathcal{F}_n f \circ \mathcal{F}_n g)(A)$ for all $A \in \mathcal{F}_n X$, and $(\mathcal{F}_n H_2)(A,1) = H_2(A,1) = A = id_{\mathcal{F}_n X}$. It means that $\mathcal{F}_n X$ and $\mathcal{F}_n Y$ are homotopically equivalent. Proposition 1 is proved.

Corollary 2. If spaces X and Y are homotopically equivalent, then the spaces $\mathcal{F}_n X$ and $\mathcal{F}_n Y$ are also homotopically equivalent.

By a *covariant homotopy functor* we mean a functor $\mathcal{F}$ in the category **Top** of topological spaces and their continuous mappings satisfying

(*) $\mathcal{F}$ preserves homotopy, that is, if a mapping $H(x,t)$ is a homotopy between continuous mappings $f, g : X \to Y$, then $F(H(x,t))$ is also a homotopy between mappings $F(f), F(g) : F(X) \to F(Y)$.

Theorem 2. The functor of n-fold symmetric product $\mathcal{F}_n$ is a covariant homotopy functor.

Proof. Now we will show that the functor $\mathcal{F}_n$ satisfies the above three conditions.

(i) Let id_X be identity mapping in the topological space X. Then we have that $(\mathcal{F}_n id_X)(A) = id_X(A) = A$. It means that the mapping $\mathcal{F}_n id_X$ is an identity mapping in the topological space $\mathcal{F}_n X$.

(ii) Let $f : X \to Y$ and $g : Y \to Z$ be the continuous mappings. Then it follows that $\mathcal{F}_n(g \circ f)(A) = (g \circ f)(A) = g(f(A)) = (\mathcal{F}_n g)(f(A)) = \mathcal{F}_n g(A) \circ \mathcal{F}_n f(A)$.

(*) It follows easily from Theorem 1. Theorem 2 is proved.

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REZYUME

Ushbu maqolada biz X fazoning n-darajali simmetrik ko'paytma fazosining ba'zi gomotopik xossalarini o'rganamiz. Agar $f, g : X \to Y$ akslantirishlar o'zaro gomotop bo'lsa, u holda $\mathcal{F}_n f, \mathcal{F}_n g : \mathcal{F}_n X \to \mathcal{F}_n Y$ akslantirishlar ham o'zaro gomotop ekanligi isbotlangan. Bundan tashqari, n-darajali simmetrik ko'paytma funktori $\mathcal{F}_n$ kovariant gomotopik funktor ekanligi ko'rsatilgan. Shu bilan birga, agar X va Y fazolar gomotopik ekvivalent bo'lsa, u holda $\mathcal{F}_n X$ va $\mathcal{F}_n Y$ fazolar ham gomotopik ekvivalent ekanligi isbotlangan.

Kalit so'zlar: Funktor, n-darajali simmetrik ko'paytma, gomotopiya, gomotopik ekvivalentlik.

РЕЗЮМЕ

В данной работе изучаются некоторые гомотопические свойства пространства n-кратного симметрического произведения пространства X. Доказывается, что если отображения $f, g : X \to Y$ гомотопны, то отображения $\mathcal{F}_n f, \mathcal{F}_n g : \mathcal{F}_n X \to \mathcal{F}_n Y$ также гомотопны. Также показано, что функтор n-кратного симметрического произведения $\mathcal{F}_n$ является ковариантным гомотопическим функтором. Кроме того, доказано, что если пространства X и Y гомотопически эквивалентны, то пространства $\mathcal{F}_n X$ и $\mathcal{F}_n Y$ также гомотопически эквивалентны.

Ключевые слова: Функтор, n-кратное симметричное произведение, гомотопия, гомотопическая эквивалентность.