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STRONG LAW OF LARGE NUMBERS FOR BANACH SPACE-VALUED RANDOM FIELDS WITH WEIGHTS

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RESUME

We consider random fields with values in infinite dimensional spaces. We assume that random fields can be represented as a functional of random fields consisting of independent identically distributed random variables. For such random fields we prove strong law of large numbers.

Key words: random field, strong law of large numbers, Banach space.

Let $\left\{Y_i,\ i\in Z^d\right\}$ be a real-valued random field with $d\geq 2$. We assume that $\left\{Y_i,\ i\in Z^d\right\}$ can be represented as

$$Y_i=g(\varepsilon_{i-s},\,s\in Z^d)\quad i\in Z^d \tag{1}$$

where $(\varepsilon_j)_{j\in Z^d}$ are iid random variables and g is a measurable function.

We define the dependence measure as follows: let $(\varepsilon_j^{|})_{j\in Z^d}$ be an iid copy of $(\varepsilon_j)_{j\in Z^d}$ and consider for any positive integer n the coupled version Y_i^* of Y_i defined by

$$Y_i^*=g(\varepsilon_{i-s}^*,\,s\in Z^d),\ \ i\in Z^d,$$

where for any j in Z^d,

$$\varepsilon_j^*=\begin{cases}\varepsilon_j, & j\neq 0\\ \varepsilon_0^{|}, & j=0\end{cases}$$

denote

$$\delta_{i,p}=\|Y_i-Y_i^*\|_p=\left(E\left|Y_i-Y_i^*\right|^p\right)^{\frac{1}{p}}.$$

In [1] a central limit theorem for $\left\{Y_i,\ i\in Z^d\right\}$ was proved under some additional conditions. Generalizations of such random fields in infinite dimensional Banach spaces were considered in [4]. In [4] the central limit theorem was proved for random fields with values in some Banach spaces. Strong law of large numbers for random fields with values in infinite dimensional spaces was proved in [5].

Let $\left\{\xi(t)\,,\ t\in Z^2\right\}$ be a random field of independent identically distributed random variables with values in a separable Hilbert space H (with inner product $(\cdot,\cdot)$ and a norm $\|\cdot\|=\sqrt{(\cdot,\cdot)}$) and let $\left\{X(t),\ t\in Z^2\right\}$ be a random field with values in H. By $\{e_j,\ j\geq 1\}$ we denote an orthonormal basis of H. Thus, we can write

$$X(t)=\sum_{j=1}^{\infty}X^{(j)}(t)\ e_j,\quad t\in Z^2.$$

Assume that the following representations hold

$$X^{(j)}(t)=g_j(\xi(t-s),\ \ s\in Z^2),\qquad j=1,2,... \tag{2}$$

where $g_j, j=1,2,...$ are measurable functions.

Let $\{\xi^{|}(t),\ t\in Z^2\}$ be an independent copy of $\left\{\xi(t)\,,\ t\in Z^2\right\}$ and

$$\overline{X}^{(j)}(t)=g_j(\xi^*(t-s),\quad s\in Z^2)$$

where

$$\overline{X}^{(j)}(t) = g_j(\xi^*(t-s), \quad s \in Z^2),$$

$$\xi^*(j) = \begin{cases} \xi(j), & if \ j \neq 0 \\ \xi^|(0), & if \ j = 0 \end{cases}.$$

Denote

$$\delta_j(t,p) = \left(E \left| X^{(j)}(t) - \overline{X}^{(j)}(t) \right|^p \right)^{1/p},$$

$$\Delta_p(j) = \sum_{t \in Z^2} \delta_j(t,p).$$

$\{G_n, \, n = 1, 2, ...\}$ – is a sequence of finite sets from Z^2 such that $G_k \subset G_{k+1}, \ k = 1, 2, ...,$

$$|G_k| = card\, G_k, \ f(|G_n|) = \sum_{i \in G_n} a_i^2, \ S_{G_n} = \sum_{i \in G_n} a_i X_i.$$

where a_i-real numbers.

Random fields with values in c_0 (a space of all sequences $\{x_1, \, x_2, ... x_n\}$ such that $\lim\limits_{n \to \infty} x_n = 0$ with a norm $\|x\| = \sup\limits_n |x_n|$) and $l_p (1 \leq p \leq 2)$ (a space of the sequences $x = (x^{(1)}, x^{(2)}, ...)$ such that $\sum\limits_{i=1}^{\infty} \left| x^{(i)} \right|^p < \infty$ with a norm $\|x\| = \left(\sum\limits_{i=1}^{\infty} \left| x^{(i)} \right|^p \right)^{1/p}$) can be defined analogously. In these spaces as a basis we take the following standard basis $\{e_i, \ i \geq 1\}$, $e = (0, ..., 0, 1, 0, ...)$.

We establish the strong law of large numbers for $\left\{ X(t), \ t \in Z^2 \right\}$. In [2] the strong laws of large numbers were proved for mixing random variables with values in Banach spaces. In [3] the strong law of large numbers were proved for mixing random fields with values in some Banach spaces.

Main results

Now we formulate our results.

Theorem 1. *Let $\left\{ X_i, \ i \in Z^2 \right\}$ be a random field with values in H defined by equation (2). Assume that the following conditions hold:*

1. $EX_i = 0, \quad \sum\limits_{j=1}^{\infty} (\Delta_2(j))^2 < \infty.$

2. *There exists an integer valued increasing function $\varphi(m)$ such that*

$$\sum_{m=1}^{\infty} \frac{f\left(\left| G_{\varphi(m)} \right| \right)}{\left| G_{\varphi(m)} \right|^2} < \infty, \tag{3}$$

$$\sum_{m=1}^{\infty} \frac{\left| G_{\varphi(m+1)} \backslash G_{\varphi(m)} \right| \left(f\left(\left| G_{\varphi(m+1)} \right| \right) - f\left(\left| G_{\varphi(m)} \right| \right) \right)}{\left| G_{\varphi(m)} \right|^2} < \infty. \tag{4}$$

Then as $n \to \infty$

$$\frac{S_{G_n}}{|G_n|} \ \to \ 0 \ \ a.s..$$

Theorem 2. *Let $\left\{ X_i, \ i \in Z^2 \right\}$ be a c_0 -space-valued random field defined by equation (2). Assume that the following conditions hold:*

1. $EX(t) = 0, \quad \sum\limits_{j=1}^{\infty} \Delta_2^2(j) < \infty.$

2. There exists an integer valued increasing function $\varphi(m)$ such that

$$\sum_{m=1}^{\infty} \frac{f\left(\left|G_{\varphi(m)}\right|\right)}{\left|G_{\varphi(m)}\right|^{2}} < \infty,$$

$$\sum_{m=1}^{\infty} \frac{\left|G_{\varphi(m+1)} \backslash G_{\varphi(m)}\right| \left(f\left(\left|G_{\varphi(m+1)}\right|\right) - f\left(\left|G_{\varphi(m)}\right|\right)\right)}{\left|G_{\varphi(m)}\right|^{2}} < \infty.$$

Then as $n \to \infty$

$$\frac{S_{G_n}}{|G_n|} \to 0 \quad a.s..$$

Theorem 3. *Let $\left\{X_i,\ i \in Z^2\right\}$ be a $l_p (1 \le p \le 2)$ space-valued random field defined by equation (2). Assume that the following conditions hold:*

1. $EX(t) = 0, \quad \sum\limits_{j=1}^{\infty} (\Delta_2(j))^{p/2} < \infty, \quad 1 \le p \le 2.$

2. *There exists an integer valued increasing function $\varphi(m)$ such that*

$$\sum_{m=1}^{\infty} \frac{f\left(\left|G_{\varphi(m)}\right|\right)}{\left|G_{\varphi(m)}\right|^{2}} < \infty,$$

$$\sum_{m=1}^{\infty} \frac{\left|G_{\varphi(m+1)} \backslash G_{\varphi(m)}\right| \left(f\left(\left|G_{\varphi(m+1)}\right|\right) - f\left(\left|G_{\varphi(m)}\right|\right)\right)}{\left|G_{\varphi(m)}\right|^{2}} < \infty.$$

Then as $n \to \infty$

$$\frac{S_{G_n}}{|G_n|} \to 0 \quad a.s..$$

Proof of theorems.

We now present the proof of the above theorems.

In the proof of the above theorems, we use the following theorems from [5].

Theorem 4. *Let $\left\{X(t),\, t \in Z^2\right\}$ be a random field with values in H defined by equation (2). Assume that the following conditions hold:*

$$EX(t) = 0, \quad \sum_{j=1}^{\infty} \left(\Delta_2(j)\right)^2 < \infty.$$

Then there is a constant $C > 0$ such that the following inequality holds:

$$E\left\|S_G\right\|^2 \le \ C \ |G| \ \sum_{j=1}^{\infty} \left(\Delta_2(j)\right)^2.$$

Theorem 5. *Let $\left\{X(t),\, t \in Z^2\right\}$ be a c_0 - space-valued random field defined by (2). Assume that the following conditions hold:*

$$EX(t) = 0, \quad \sum_{j=1}^{\infty} \Delta_2^2(j) < \infty.$$

Then there is a constant $C > 0$ such that the following inequality holds:

$$E\left\|S_G\right\|^p \le C \ |G| \ ^{p/2} \sum_{j=1}^{\infty} \Delta_2^2\left(j\right).$$

Theorem 6. *Let $\{X(t),\, t \in Z^2\}$ be a $l_p(1 \le p \le 2)$ space - valued random field defined by (2). Assume that the following conditions hold:*

$$EX(t) = 0, \qquad \sum_{j=1}^{\infty} (\Delta_2(j))^{p/2} < \infty, \quad 1 \le p \le 2.$$

Then there is a constant $C > 0$ for which the following inequality holds:

$$E\left\|S_G\right\|^p \le C \cdot |G|^{p/2} \cdot \sum_{j=1}^{\infty} (\Delta_2(j))^{p/2}, \quad \textit{for} \;\; 1 \le p \le 2.$$

Proof of Theorem 1.

Using the inequality from Theorem 4 we obtain:

$$E\left\|\sum_{i\in G} a_i X_i\right\|^2 \le C \sum_{i\in G} a_i^2 \sum_{j=1}^{\infty} (\Delta_2(j))^2 = Cf\left(|G|\right) \sum_{j=1}^{\infty} (\Delta_2(j))^2. \tag{5}$$

Using Chebyshev inequality and (5) we have for any $\varepsilon > 0$

$$P\left(\frac{\left\|S_{G_{\varphi(m)}}\right\|}{\left|G_{\varphi(m)}\right|} > \varepsilon\right) \le \frac{E\left\|S_{G_{\varphi(m)}}\right\|^2}{\varepsilon^2\left|G_{\varphi(m)}\right|^2} \le \frac{C\left(\sum\limits_{i\in G_{\varphi(m)}} a_i^2\right)\sum\limits_{i\in G_{\varphi(m)}} (\Delta_2(i))^2}{\varepsilon^2\left|G_{\varphi(m)}\right|^2} = \frac{Cf\left(\left|G_{\varphi(m)}\right|\right)}{\varepsilon^2\left|G_{\varphi(m)}\right|^2}$$

It follows from Borel-Cantelli lemma that as $m \to \infty$

$$\frac{1}{\left|G_{\varphi(m)}\right|}\left\|S_{G_{\varphi(m)}}\right\| \to 0 \;\; \text{a.s.} \tag{6}$$

Again using Chebyshev inequality we have

$$P\left(\frac{\max\limits_{\varphi(m)<n\le\varphi(m+1)}\left\|S_{G_n} - S_{G_{\varphi(m)}}\right\|}{\left|G_{\varphi(m)}\right|} > \varepsilon\right) \le \frac{E\max\limits_{\varphi(m)<n\le\varphi(m+1)}\left\|S_{G_n} - S_{G_{\varphi(m)}}\right\|^2}{\varepsilon^2\left|G_{\varphi(m)}\right|^2} \le$$

$$\le \frac{\left|G_{\varphi(m+1)} - G_{\varphi(m)}\right| \sum\limits_{i\in G_{\varphi(m+1)}\setminus G_{\varphi(m)}} E\left\|a_i X_i\right\|^2}{\varepsilon^2\left|G_{\varphi(m)}\right|^2} \le \frac{\left|G_{\varphi(m+1)} - G_{\varphi(m)}\right| \sum\limits_{i\in G_{\varphi(m+1)}\setminus G_{\varphi(m)}} a_i^2 E\left\|X_i\right\|^2}{\varepsilon^2\left|G_{\varphi(m)}\right|^2} =$$

$$= \frac{\left|G_{\varphi(m+1)} - G_{\varphi(m)}\right| E\left\|X_1\right\|^2 \left(f\left(\left|G_{\varphi(m+1)}\right|\right) - f\left(\left|G_{\varphi(m)}\right|\right)\right)}{\varepsilon^2\left|G_{\varphi(m)}\right|^2}.$$

From (4) and Borel-Cantelli lemma we have as $m \to \infty$

$$\frac{\max\limits_{\varphi(m)<n\le\varphi(m+1)}\left\|S_{G_n} - S_{G_{\varphi(m)}}\right\|}{\left|G_{\varphi(m)}\right|} \to 0 \;\; \text{a.s.} \tag{7}$$

From (6) and (7) follows the statement of the theorem.

Theorems 2-3 can be proved using the same method and therefore we omit their proofs.

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REZYUME

Cheksiz o'lchamli fazolarda qiymat qabul qiluvchi tasodifiy maydonlarni ko'rib chiqamiz. Tasodifiy maydonlarni bog'liqsiz bir xil taqsimlangan tasodifiy miqdorlardan tuzilgan tasodifiy maydonlarning funksionali sifatida yozish mumkin deb faraz qilamiz. Bunday tasodifiy maydonlar uchun biz kuchaytirilgan katta sonlar qonunini isbotlaymiz.

Kalit so'zlar: tasodifiy maydon, kuchaytirilgan katta sonlar qonuni, Banax fazosi.

РЕЗЮМЕ

Рассматриваются случайные поля со значениями в бесконечномерных пространствах. Мы предполагаем, что случайные поля могут быть представлены как функционал от случайных полей состоящих из независимых одинаково распределенных случайных величин. Для таких случайных полей мы доказываем усиленный закон больших чисел.

Ключевые слова: случайное поле, усиленный закон больших чисел, банаховое пространсво.