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SOME REMARK ON THE INVARIANT MEASURE OF CIRCLE MAPS WITH BREAK POINTS

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RESUME

In this paper we study invariant measure of circle homeomorphisms with break type of singularities. It is proven that invariant measure of circle homeomorphisms have four break points with trivial total product of jumps and irrational rotation numbers of bounded type is singular with respect to Lebegue measure on the circle.

Key words: circle homeomorphism, rotation number, invariant measure, break point.

In the present paper, we study invariant measures of circle homeomorphisms with break type of singularities.

Let f be an orientation preserving circle homeomorphism. It is well-known that any this kind of homeomorphism can be written as

$$f(x) = F(x)(mod\,1),\ x \in S^1,$$

where $F(x)$ - continuous, strictly increasing function on R^1 satisfying condition

$$F(x+1) = F(x) + 1,\, x \in R.$$

The function F is called a **lift** of the homeomorphism f. Let F be the lift with initial condition $F(0) \in [0,1)$. We denote by ρ_f the rotation number of the homeomorphism f (see [1]), that is,

$$\rho_f = \lim_{n\to\infty} \frac{F^n(x)}{n},\ x \in R^1.$$

Henceforth, $F^n(x)$ denotes the nth iteration of the function F. The limit ρ_f belongs to the interval $[0,1)$ and does not depend on the point x . The number ρ_f is the most important numerical characteristic of the homeomorphism f. Namely, if the rotation number ρ_f is irrational, then the homeomorphism f has a unique probability invariant measure μ_f. Moreover, there exists a continuous, non-decreasing function $\varphi : S^1 \to S^1$ such that $\varphi \circ f = f_{\rho_f} \circ \varphi$, where $f_{\rho_f} = x + \rho_f(mod\,1)$ (see [1]). Note that the semi-conjugation φ and the invariant measure μ_f are connected by the relation $\varphi(x) = \mu_f([0,x])$, $x \in S^1$. Because of this relation, invariant measure μ_f is absolutely continuous with respect to the Lebesgue measure ℓ if and only if φ is given by an absolutely continuous function.

The fundamental results in the problem of smoothness of the conjugacy were obtained by V.I. Arnold, J. Moser, M. Herman, J.C. Yoccoz, Ya.G. Sinai and K.M. Khanin, Y. Katznelson and D. Ornstein and others.

We formulate the last two important results in this area.

Theorem 1. (Katznelson-Ornstein,[2]). *Let f be an orientation preserving C^1- circle diffeomorphism. If f' is absolutely continuous, $\frac{f''}{f'} \in L_p$ for some $p > 1$ and the rotation number $\rho = \rho_f$ is of bounded type, then the invariant measure μ_f is absolutely continuous with respect to Lebesgue measure.*

Theorem 2. (Khanin-Sinai,[3]). *Let f be a $C^{2+\varepsilon}$ circle diffeomorphism for some $\varepsilon > 0$, and let the rotation number $\rho = \rho_f$ be a Diophantine number with exponent $\delta \in (0, \varepsilon)$, i.e., there is a constant $c(\rho)$ such that*

$$|\rho - \frac{p}{q}| \geq \frac{c(\rho)}{q^{2+\delta}},\ for\ any\ p,q \in Q.$$

Then the conjugating map φ belongs to $C^{1+\varepsilon-\delta}$.

One of the important class of circle homeomorphisms are homeomorphisms with break points, or the class of P -homeomorphisms.

Definition 1. *Let f be circle homeomorphisms with the lift F. If at the point $x_b \in S^1$ there exist one-sided positive derivatives $F'(x_b-0)$, $F'(x_b+0)$ and $F'(x_b-0) \neq F'(x_b+0)$, then $x = x_b$ is called break point of the homeomorphism f.*

The number $\sigma_f(x_b) = \frac{F'(x_b-0)}{F'(x_b+0)}$ is called **jump ratio** or **jump** of the homeomorphism f at the point $x = x_b$.

Definition 2. *An orientation preserving circle homeomorphism f with the lift F is called P -homeomorphism, if F satisfies the following conditions:*

1) F is differentiable on S^1 except at a finite or countable number of break points;

2) there exist constants $0 < c_1 < c_2 < +\infty$ such that

$$c_1 < F'(x_b - 0),\ F'(x_b + 0) < c_2,\ \ \forall x_b \in BP(f),$$

$$c_1 < F'(x) < c_2,\ \ \forall x \in S^1 \setminus \{BP(f)\},$$

where $BP(f)$ - set of all break points of f;

3) $\ln F'$ has bounded variation in S^1, i.e. $v(F) = var_{S^1} \ln F' < \infty$.

The regularity properties of invariant probability measures of circle homeomorphisms with break points differ from the properties in the case of circle diffeomorphisms. The piecewise-linear (PL) orientation preserving circle homeomorphisms with two break points are the simplest examples of P-homeomorphisms. The invariant measures of PL homeomorphisms were studied first by Herman in [4].

Theorem 3. (Herman). *A PL circle homeomorphisms with two break points and irrational rotation number has an invariant measure absolutely continuous with respect to Lebesgue measure if and only if its break points belong to the same orbit.*

General (non PL) circle homeomorphisms with one break point have been studied by Dzhalilov and Khanin in [5]. The character of their results for such circle maps is quite different from the one for $C^{2+\varepsilon}$-diffeomorphisms. The main result of [5] is the following:

Theorem 4. *Let f be a circle homeomorphisms with a single break point x_b. If the rotation number ρ_f of f is irrational and $f \in C^{2+\varepsilon}(S^1 \setminus \{x_b\})$ for some $\varepsilon > 0$, then the $f-$ invariant probability measure μ_f is singular with respect to Lebesgue measure ℓ.*

The invariant measures of circle homeomorphisms with two break points of "general type" , that is, which are not piecewise linear, were studied in [6], [7]. We state the main results of these papers.

Theorem 5. ([6]). *Suppose that a circle homeomorphism f with lift F satisfies the following conditions.*

1) The rotation number ρ_f is irrational of "bounded type", that is, the sequence of elements of the expansion of ρ_f into a continued fraction is bounded.

2) f has break points at two points b_1, b_2 of the circle that do not lie on the same trajectory.

3) The derivative $F'(x)$ exists on the set $S^1 \setminus \{b_1, b_2\}$ and satisfies Lipschitz conditions on every connected component of that set.

Then the $f-$invariant measure μ_f is singular with respect to Lebesgue measure ℓ.

Circle homeomorphisms with two break points but arbitrary irrational rotation number were studied in [7].

Theorem 6 ([7]). *Suppose that a circle homeomorphism f with lift F satisfies the following conditions.*

1) The rotation number ρ_f is irrational;

2) f has break points at points b_1, b_2 and the derivative $F'(x)$ is absolutely continuous on every connected component of the set $S^1 \setminus \{b_1, b_2\}$;

3) $F''(x) \in L_1(S^1, d\ell)$;

4) The product of the jumps at the break points is non-trivial, that is, $\sigma_1 \cdot \sigma_2 \neq 1$.

Then the $f-$ invariant probability measure μ_f is singular with respect to Lebesgue measure.

Now we formulate the main result of the paper of A.A. Dzhalilov, D. Mayer, U.A. Safarov [8].

Theorem 7. *Suppose that the lift $F(x)$ of circle homeomorphism f with irrational rotation number satisfies the following conditions:*

(1) *f has break points $b(1)$, $b(2)$, ..., $b(k) \in S^1$ and $F'(x)$ absolutely continuous function on each connected component of the set $S^1 \setminus \{b(i),\ i = \overline{1,k}\}$;*

(2) $\int\limits_{S^1} |F''(x)| d\ell < \infty$;

(3) $\prod\limits_{i=1}^{k} \sigma_i \neq 1$.

Then the f - invariant probability measure μ_f is singular with respect to Lebesgue measure ℓ on the circle S^1, i.e. there exists a set $A \subseteq S^1$ such that $\ell(A) = 1$ and $\mu_f(A) = 0$.

In the paper [9] author answered positively a question of whether it is possible for a circle diffeomorphisms with breaks to be smoothly conjugate to a rigid rotation in the case when its breaks are lying on pairwise distinct trajectories. An example constructed is a piecewise linear circle homeomorphisms with irrational rotation numbers of "unbounded type"that has four break points lying on distinct trajectories, and whose invariant measure is absolyutely continuous w.r.t. the Lebesgue measure.

Now we formulate our main result.

Theorem 8. *Let f be a P-homeomorphism with irrational rotation number ρ of "bounded type". Suppose that*

(*a*) *f has four break points $b^{(i)}$, $i = \overline{1,4}$, lying on pairwise distinct trajectories, with break jumps $\sigma_f(b^{(i)})$, $i = \overline{1,4}$, respectively and $f'(x)$ absolutely continuous function on each connected component of the set $S^1 \setminus \{b^{(i)},\ i = \overline{1,4}\}$;*

(*b*) $\int\limits_{S^1} |f''(x)| d\ell < \infty$;

(*c*) $\sigma_f(b^{(1)}) \neq \sigma_f(b^{(i)}), i = 2, 3, 4$;

(*d*) $\prod\limits_{i=1}^{4} \sigma_f(b^{(i)}) = 1$.

Then the f-invariant probability measure μ_f is singular w.r.t. Lebesgue measure ℓ on the circle S^1.

Sketch of the Proof of the Main Theorem. In this section, we provide a sketch of the proof for the main theorem, highlighting the critical ideas and techniques involved. While the full details are omitted, this outline aims to give a clear understanding of the underlying approach and the key steps that lead to the result. In the first step of the proof, we study cross-ratio of triple of intervals $([z_1, z_2], [z_2, z_3], [z_3, z_4])$ and its distortion. More precisely, we estimate cross-ratio distortion for two cases: (i) interval $[z_1, z_4]$ does not cover any of break points, that is, diffeomorphism in the interval $[z_1, z_4]$. (ii) interval $[z_1, z_4]$ covers some of the break points with either $[z_1, z_2]$ or $[z_3, z_4]$. In the case (i) cross ratio distortion tends to 1 as length of interval $[z_1, z_4]$ approaches 0. In the case (ii) cross ratio distortion tends to product of jumps of break points, covered by the interval $[z_1, z_4]$, as length of interval $[z_1, z_4]$ approaches 0.

In the step 2, we establish the existence of subsequence of dynamical partitions of the circle $\xi_n(x_0) = \{\Delta_i^{(n)}(x_0), 0 \leq i < q_{n-1}, \Delta_i^{(n-1)}(x_0), 0 \leq j < q_n\}$ that trajectories of break points separate at least two groups with nontrivial product of jumps.

In the step 3, we construct triple of interval $([z_1, z_2], [z_2, z_3], [z_3, z_4])$ that iteration of the interval $[z_1, z_4]$

under the $f^s(x)$, $s = q_n \vee q_{n-1}$ covers subset of break points with nontrivial product of jumps either iteration of $f^s([z_1, z_2])$ or $f^s([z_3, z_4])$ such that the break points sufficiently close to trajectory of either z_2 or z_3.

In the step 4, using the steps above we prove that invariant measure is singular w.r.t Lebegue measure. Let $\varphi(x) = \mu([0, x]),\ x \in S^1$ be conjugacy between f and f_ρ. Assume the contrary, that is, $\varphi(x)$ is absolutely continuous homeomorphism. Then $\exists x_0 \in S^1$ such that $\varphi'(x_0) > 0$. In the small neighbourhood of x_0 we construct triple of intervals satisfying condition of step 3. If we consider cross-ratio distortion of triple of interval $([z_1, z_2], [z_2, z_3], [z_3, z_4])$ w.r.t $f^{(s)},\ s = q_n \vee q_{n-1}$, then it differs from 1. On the other side, if we use the properties of conjugation map it must close to one. This contradiction proves the theorem.

Conclusion. The question of the absolute continuity and singularity of two probability measures is one of the important problems of modern probability theory. These results confirm that for circle homeomorphisms with four break points and with an irrational rotation number of bounded type, it is proved that the invariant probability measure is singular with respect to the Lebesgue measure.

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REZYUME

Ushbu maqolada sinish tipidagi maxsuslikka ega aylana gomeomorfizmlarining invariant o'lchovi o'rganilgan. Bunda to'rtta sinish nuqtalariga ega bo'lib, sinish kattaliklari ko'paytmasi trivial va bo'rish soni chegaralangan tipdagi irratsional bo'lgan aylana gomeomorfizmlarining invariant o'lchovi aylanada Lebeg o'lchoviga nisbatan singulyar bo'lishi isbotlangan.

Kalit so'zlar: aylana gomeomorfizmi, burish soni, invariant o'lchov, sinish nuqtasi.

РЕЗЮМЕ

В данной работе изучается инвариантная мера гомеоморфизмов окружности с особенностями типа излома. Доказано, что инвариантная мера гомеоморфизмов окружности с четырьмя точками излома, при тривиальном общем произведении скачков и иррациональном числом вращения ограниченного типа, является сингулярной относительно меры Лебега на окружности.

Ключевые слова: гомеоморфизм окружности, число вращения, инвариантная мера, точка излома.