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ON STABLE TYPE p BANACH SPACES

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RESUME

In the paper stable type p Banach spaces are considered. We extend a result known for independent random variables to the mixing random variables. A sufficient and necessary condition for stable type p Banach spaces is given.

Key words: Rademacher type p Banach space, a stable type p Banach space, mixing condition.

It is known that validity of limit theorems in Banach spaces depends on geometrical structure of Banach spaces (see for instance [1]-[11]). There are several types of Banach spaces. We will consider two types of Banach spaces.

Namely, we will consider Rademacher type p and a stable type p Banach spaces.

Let B be a separable Banach spaces with a norm $\|\cdot\|$ and $\{X_n,\ n \geq 1\}$ be a sequence of independent random variables with values in B. Now we will give a definition of Rademacher type p Banach space.

Definition 1. We say that B is Rademacher type $p\,(1 \leq p \leq 2)$ Banach space if for any finite collection of $B-$valued independent random variables $X_1,\ X_2,\ ...,\ X_n$ with $EX_i = 0,\ E\|X_i\|^p < \infty$ there exists a constant $C = C(p,\ B) > 0$ such that

$$E\left\|\sum_{i=1}^{n} X_i\right\|^p \leq C\sum_{i=1}^{n} E\|X_i\|^p.$$

Obviously any separable Banach space is Rademacher type 1 space. L_p, l_p spaces are Rademacher type 2 Banach spaces for $p \geq 2$ and L_p, l_p are Rademacher type p spaces if $1 \leq p \leq 2$. c_0 space is not Rademacher type p space for any $p \in (1,\ 2]$.

By θ we denote a real valued stable random variable with characteristic function

$$E\exp(it\theta) = \exp\left(-|t|^p\right)$$

$\{\theta_i,\ i \geq 1\}$ is a sequence of independent copies of θ.

Now we give a definition of stable type p Banach spaces.

Definition 2. We say that B is a stable type $p\,(0 < p \leq 2)$ Banach space if for each $q < p$ there exists a constant $C > 0$ such that for all integers n and any $x_1,\ x_2,\ ...,\ x_n \in B$ the following inequality holds

$$\left(E\left\|\sum_{i=1}^{n}\theta_i x_i\right\|^q\right)^{1/q} \leq C\left(\sum_{i=1}^{n}\|x_i\|^p\right)^{1/p}.$$

It is known that B is of stable type p if and only if for all sequences $x_1,\ x_2,\ ...$ of elements of B such that $\sum\limits_{i=1}^{\infty}\|x_i\|^p < \infty$ the series

$$\sum_{i=1}^{\infty}\theta_i x_i$$

converges almost surely, see [4].

If B is of stable type p, then for independent random variables ξ_1, ξ_2, ..., ξ_n with characteristic functions $\exp(-\alpha_i |t|^p)$, $i = 1, 2, ..., n$ and any x_1, x_2, ..., $x_n \in B$ the following inequality holds (see [4])

$$\left(E\left\|\sum_{i=1}^{n}\xi_i x_i\right\|^q\right)^{1/q} \leq C\left(\sum_{i=1}^{n}\alpha_i \|x_i\|^p\right)^{1/p}$$

where $q < p$ and $C = C(q) > 0$.

Some of above results were extended to the case of mixing random variables.

For the sequence $\{\theta_i,\ i \geq 1\}$ mixing coefficients are defined as following

$$\psi(k) = \sup\left\{\frac{|P(AB) - P(A)P(B)|}{P(A)P(B)}\ :\ A \in F_1^n,\ B \in F_{n+k}^{\infty},\ n \in N,\ P(A)P(B) > 0\right\}$$

where F_a^b is a $\sigma-$field generated by random variables θ_a, ..., θ_b.

We say that the sequence $\{\theta_i, i \geq 1\}$ is $\psi-$mixing, if $\lim\limits_{n\to\infty}\psi(n) = 0$.

Denote $S_n = \sum\limits_{i=1}^{n} X_i$.

We will give some of the results for $\psi-$mixing Banach space-valued random variables from [11].

Theorem 1 ([11]). The following statements are equivalent:

1) B is a stable type p Banach space.

2) For any sequence $\{x_i,\ i \geq 1\}$ of elements of B such that

$$\sum_{i=1}^{\infty}\|x_i\|^p < \infty$$

and any sequence $\{\theta_i,\ i \geq 1\}$ of $\psi-$mixing identically distributed $p-$stable random variables with characteristic function $f_{\theta_i}(t) = \exp(-|t|^p)$ and

$$\sum_{k=1}^{\infty}\psi(k) < \infty, \psi(1) < 1$$

the series

$$\sum_{i=1}^{\infty}\theta_i x_i$$

converges almost surely and in L_q $q < p$.

3) For any bounded sequence $\{x_n,\ n \geq 1\}$ of elements of B and any sequence $\{r_n,\ n \geq 1\}$ of $\psi-$mixing random variables with

$$P(r_n = \pm) = \frac{1}{2}, n = 1, 2, ...$$

$$\sum_{k=1}^{\infty}\psi(k) < \infty$$

the series

$$\sum_{n=1}^{\infty}\frac{r_n}{n^{1/p}}x_n$$

converges almost surely.

4) For any bounded sequence $\{x_n,\ n \geq 1\}$ of elements of B there exists a choice of $\varepsilon_n = \pm 1$ such that the series

$$\sum_{n=1}^{\infty}\frac{\varepsilon_n}{n^{1/p}}x_n$$

converges.

Theorem 2 ([11]). Let B be a stable type p $(1 \leq p \leq 2)$ Banach space and $\{X_i,\ i \geq 1\}$ be a sequence of symmetric, identically distributed $\psi-$mixing random variables with values in B. Assume that the following conditions hold

$$\sum_{k=1}^{\infty} \psi(k) < \infty,$$

$$\lim_{n\to\infty} nP\left(\|X_1\| > n^{1/p}\right) = 0.$$

Then as $n \to \infty$

$$\frac{1}{n^{1/p}} \sum_{i=1}^{n} X_i \to 0 \ \text{ in probability.}$$

MAIN RESULT.

Our goal is to give another sufficient and necessary condition for type p stability of Banach spaces. We will extend Theorem 1 of [7] to the case of mixing Banach space-valued random variables.

Let $\{X_i,\ i \geq 1\}$ be a sequence of $\psi-$mixing random variables with values in B a separable Banach space B.

Denote

$$\Lambda_p(X) = \sup_{t\,>0} t^p P\left(\|X\| > t\right).$$

The following is our main result.

Theorem 3. B is stable type p $(0 < p < 2)$ Banach space if and only if

$$\Lambda_p\left(\sum_{i=1}^{n} X_i\right) \leq C \sum_{i=1}^{n} \Lambda_p\left(X_i\right) \tag{1}$$

for all symmetric $\psi-$mixing $B-$valued random variables $X_1,\ X_2,\ ...,\ X_n$ such that

$$\Lambda_p\left(X_i\right) < \infty,\ i = \overline{1,\ n}, n \geq 1$$

and

$$\sum_{k=1}^{\infty} \psi(k) < \infty.$$

Proof of Theorem 3.

Necessity follows from Theorem 1 of [7] where this theorem was proved for independent Banach space-valued random variables. As independent random variables are $\psi-$mixing, the necessity part of the theorem follows from Theorem 1 of [7].

It remains to prove that in stable type p $(0 < p < 2)$ Banach B (1) holds for $X_1,\ X_2,\ ...,\ X_n$.

It is known (see for instance [4]) that if B is a stable type p $(0 < p < 2)$ Banach space then there exists $p' > p$ such that B will be Rademacher type p'. We will use this fact for the $Y_i = X_i\, I\left(\|X_i\| \leq 1\right)$.

As in [7] we have

$$P\left(\left\|\sum_{i=1}^{n} X_i\right\| > 1\right) \leq P\left(\left\|\sum_{i=1}^{n} X_i\right\| > 1,\ \max_{1\leq i\leq n} \|X_i\| \leq 1\right) + P\left(\max_{1\leq i\leq n} \|X_i\| > 1\right) \leq$$

$$P\left(\left\|\sum_{i=1}^{n} Y_i\right\| > 1\right) + \sum_{i=1}^{n} \Lambda_p\left(X_i\right).$$

Now we will use the following

Theorem 4 ([10]). Let B Rademacher type p' $(0 < p' < 2)$ Banach space. Then for any finite set of $\psi-$mixing $B-$valued random variables $X_1,\ X_2,\ ...,\ X_n$ such that

$$EX_i = 0, E\left\|X_i\right\|^{p'} < \infty,$$

$$\sum_{k=1}^{\infty}\psi(k)<\infty$$

there exists a constant $C\left(B,\ p',\ \psi\right)$ such that the following inequality holds

$$E\left\|\sum_{i=1}^{n}X_i\right\|^{p'}\leq C\left(B,\ p',\ \psi\right)\sum_{i=1}^{n}E\left\|X_i\right\|^{p'}.$$

Using above theorem and Markov inequality we have

$$P\left(\left\|\sum_{i=1}^{n}Y_i\right\|>1\right)+\sum_{i=1}^{n}\Lambda_p\left(X_i\right)\leq$$

$$E\left\|\sum_{i=1}^{n}Y_i\right\|^{p'}+\sum_{i=1}^{n}\Lambda_p\left(X_i\right)\leq$$

$$C\left(B,\ p',\ \psi\right)\sum_{i=1}^{n}E\left\|Y_i\right\|^{p'}+\sum_{i=1}^{n}\Lambda_p\left(X_i\right).$$

It is proved in [7] that

$$E\left\|Y_i\right\|^{p'}=\frac{p'}{p'-p}\Lambda_p\left(X_i\right).$$

Thus

$$P\left(\left\|\sum_{i=1}^{n}X_i\right\|>1\right)\leq\left(\frac{C\left(B,\ p',\ \psi\right)p'}{p'-p}+1\right)\sum_{i=1}^{n}\Lambda_p\left(X_i\right).$$

Replacing X_i by $\frac{X_i}{t}$ as in [7] we get

$$P\left(\left\|\sum_{i=1}^{n}X_i\right\|>t\right)\leq\frac{C}{t^p}\sum_{i=1}^{n}\Lambda_p\left(X_i\right).$$

The theorem is proved.

REFERENCES

1. Hoffmann-Jørgensen J. Sums of independent Banach space valued random variables, Studia Math. 1974, V.52, P.159-186.
2. Kuelbs J. and Mandrekar V. Domains of attraction of stable measures on a Hilbert space. Studia Math. 1974, V.59, P.149-162.
3. Ledoux M., Talagrand M. Probability in Banach Spaces: Isoperimetry and Processes. Springer, Berlin, 1991.
4. Marcus M. B. and Woyczynski W. A. Stable measures and central limit theorems in spaces of stable type. Transactions of the American Mathematical Society, July 1979, Volume 251, P.71-102.
5. Maurey B., Pisier G. Series de variables aleatoires vectorielles independantes et proprietes geometriques des espaces de Banach. Stud. Math. 1976, V.58, P.45-90.
6. Pisier G. Probabilistic methods in the geometry of Banach spaces. In: Probability and Analysis, Lectures Given at the 1st 1985 Session of the Centro Internazionale Matematico Estivo (C.I.M.E). Lecture Notes in Mathematics, 1986, vol. 1206, P.167-241. Springer, Berlin.
7. Rosinski J. Remarks on Banach spaces of stable type. Probab. Math. Statist. 1980, 1, P.67-71.

8. Woyczynski W. A. Geometry and martingales in Banach spaces, Part II: Independent increments. In: Kuelbs, J., Ney, P. (eds.) Probability on Banach Spaces. Advances in Probability and Related Topics, 1978, vol. 4, P.267-517. Dekker, New York.

9. Woyczynski W. A. Geometry and martingales in Banach spaces. CRC Press; 1st edition, September 30, 2020.

10. Sharipov O. Sh. On probabilistic characterization of Banach spaces by weakly dependent random elements. Uzb. Math. 1998, J. N.2, P.86-91, (in Russian).

11. Sharipov O. Sh., Kobilov U. X. Limit theorems for weakly dependent random variables with values in stable type p Banach spaces. Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences. 2023, Volume 6, Issue 2, P.106-119.

REZYUME

Maqolada turg‘un tip p Banax fazolari ko‘riladi. Biz bog‘liqsiz tasodifiy miqdorlar uchun ma’lum bo‘lgan natijalarni qorishmali tasodifiy miqdorlar uchun umumlashtiramiz. Turg‘un tip p Banax fazolari uchun zarur va yetarli shart keltirilgan.

Kalit so‘zlar: Rademaxer tip p Banax fazosi, turg‘un tip p Banax fazosi, qorishmalilik sharti.

РЕЗЮМЕ

В статье рассматриваются Банаховые пространства устойчивого типа p. Мы распространяем результаты, известные для независимых случайных величин, на случайные величины с перемешиванием. Приводится необходимое и достаточное условие для Банаховоых пространств устойчивого типа p.

Ключевые слова: Банахово пространство рademахеровского типа p, Банахово пространство устойчивого типа p, условие перемешивания.