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Development of a Model Based on the Ordinary Kriging Method for Surface Modeling and Reconstruction from Data Given at Scattered Nodess

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Abstract. This paper studies mathematical equations that can be used to reconstruct and analyze geological surfaces using the data provided at a point (scattered points). The work explains the working capabilities of the kriging technique in the reconstruction of surfaces, what the term signifies as a mathematical model, and how the technique is used in the computing process. The implementation of kriging method in the circumstances of irregularly spaced points of the space is discussed both mathematically and computationally and this example is compared to the case of grid nodes. The findings have revealed that when the points are not distributed consistently, the kriging technique has proven to be a natural and sound mathematical framework in the reconstruction of geological surfaces. This paper is dedicated to the justification of the efficiency of the kriging model in reconstruction and analysis of geological surfaces.

MSC (2020): 65D05; 62M30; 86A32; 41A63; 65D17.

Key words: scattered nodal points, spatial interpolation, ordinary kriging, semivariogram, surface reconstruction, spatial modeling, interpolation accuracy, mathematical model, function of two variables.

Introduction

The problem of how the values of the unknown intermediate points can be obtained based on known nodal points is of significant scientific and practical importance in interpolation issues and mathematical modeling problems. These issues are common in applied computations, numerical modelling, processing of spatial data, and reconstruction of surfaces of different types [1]. Specifically, the re-construction and analysis of a surface, based on the observation points, is of particular significance in the problems, associated with geological objects. This is due to the fact geological surfaces tend to be complex in nature, their proper definition demands the choice of suitable mathematical models [2]. In this sense, the issue of the construction of a continuous surface out of a set of points is the one among the significant steps in the reconstruction and analysis of the geological surfaces. Interpolation issues are the consideration of structured or grid type nodes in many situations. Spline techniques, and more particularly B-spline techniques, allow the creation of a surface smoothly and accurately in such a situation [3]. The reason is that the nodes are arranged along coordinate directions and it is convenient to build up mathematical models based on tensile products on their basis. This is why in most studies using B-spline methods has become a useful tool in cases where data is provided on a regular or irregular grid nodes with a step-size[4].

Nevertheless, these methods can work well only in situations where the data are pre-geometrically organized. In practice, however, the data are not always provided at grid nodes. Specifically, observation points on geological surfaces can be frequently observed in space in a disordered, haphazard location and can be not based on a pre-determined pattern in particular, when looking at the surface. The distance between the nodes is also different in these situations and the points are not a natural grid of rows and columns. Consequently, it becomes hard to employ classical grid-based methods of interpolation directly. Specifically, when using the

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model of B-splines based on the use of the tensor-product, it is assumed that the data must be in a form of a structure, which does not apply to scattered nodes. Hence, the issue of the reconstruction and analysis of geological surfaces built on the premises of non-continuous spatial points implies a special mathematical solution. In these circumstances mathematical models that consider the spatial dependence between points are particularly important.

One such model is kriging. Kriging is a geostatistical interpolation scheme which derives its primary concept on the premise that values found on points that are close together as far as space is concerned tend to be similar as opposed to those found on distant points [5]. Therefore, when it comes to estimating the value at an unknown point, the observed values are not the only important factor to take into consideration, but also the spatial layout. The biggest merit of the kriging model is that it is properly adapted to the work with the scattered points and the spatial autocorrelation in the estimation of values in the unknown points is taken into consideration [6]. This makes it a difference between it and the simple geometric interpolation methods. Reconstructed surface on the basis of kriging takes into consideration not only values in the known points, but also the statistical relations between them. This is why to data provided at sparse nodes, this method can be considered a natural, well-grounded, and efficient mathematical model of the reconstruction and analysis of geological surfaces. In this paper, mathematical models of the reconstruction and analysis of geological surfaces using data provided at scattered nodes are taken into consideration. Specifically, the practical utilities of the kriging method, the characteristics of the mathematical model, and its efficiency in circumstances of the sparse spatial locations are examined. Moreover, the methodological distinction between the case of scattered nodes and the case of structured grid nodes are also explained and the part played by the kriging model in the reconstruction of geological surfaces is justified.

Inference of data according to the kriging technique.

Let us assume that there are n observation points in the study area, and they are given in the form [6]

$$(x_1, y_1, z_1)\,,\,(x_2, y_2, z_2)\,,\,\dots\,,\,(x_n, y_n, z_n)$$

Here, (x_i, y_i) are the coordinates of the observation points, and is the value of the measured parameter at those points. Our aim is to determine the unknown value at the point (x_0, y_0). In the ordinary kriging method, this unknown value can be determined through the weighted sum of known values. The formula of the ordinary kriging method is as follows[6]:

$$\overline{Z}\left(x_0, y_0\right) = \sum_{i=1}^{N} \lambda_i Z\left(x_i, y_i\right)$$

In this case the λ_i values are kriging weights, and they are calculated through a special system that is founded on spatial dependence. Moreover, in the case of ordinary kriging, $\sum\limits_{i=1}^{N} \lambda_i = 1$ is a condition. This is what makes sure that the estimation is not biased. The Euclidean distances between the points are calculated in the practical implementation of the krige process. The Euclidean formula that is used to calculate the distance between two points is expressed as [7]

$$d_{ij} = \sqrt{\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2}$$

whereas the distance between the estimated point and a known point is expressed as

$$d_{i0} = \sqrt{\left(x_i - x_0\right)^2 + \left(y_i - y_0\right)^2}$$

It requires these distances in the following step to build the semivariogram that represents spatial dependence. In kriging, distance is no longer just a simple geometric value, but it is a way of gauging the extent of statistical similarity or difference between values. The primary aspect of kriging is the semivariogram. It explains spatial autocorrelation of the data i.e. it illustrates the extent to which the values vary with a higher distance. Theoretically, the semivariogram can be described as [8]

$$\gamma\left(h\right) = \frac{1}{2} Var\left(Z\left(s\right) - Z\left(s + h\right)\right)$$

In practical computation however, the semivariogram of experiment is applied. The calculation is done using the following formula [9]

$$\gamma(h) = \frac{1}{2N(h)} \sum_{(i,j)\in N(h)} (z_i - z_j)^2$$

Here, N(h) the number of pairs of points whose distance is approximately h. That is, each pair of points is placed in groups based on the distance, and the mean of the squared difference of values of each group is computed. By so doing semivariogram points are given. These are the points which are the spatial structure of the data. The points of experimental semivariogram tend to be jagged and hectic. To this end, kriging is no longer performed on the basis of them per se. Rather a theoretical variogram model is fitted on the experimental points. Practically, there is a high level of the application of spherical, exponential, Gaussian, and linear models. Precisely these kinds of semivariogram models are standard options of many software packages used to perform ordinary kriging [10]. The spherical model is written as follows:

$$\gamma(h) = \begin{cases} c_0 + c\left(\frac{3h}{2a} - \frac{1}{2}\left(\frac{h}{a}\right)^3\right), & 0 < h \le a \\ c_0 + c, & h > a \end{cases}$$

The exponential model is as follows:

$$\gamma(h) = c_0 + c\left(1 - e^{-\frac{h}{a}}\right)$$

The Gaussian model is:

$$\gamma(h) = c_0 + c\left(1 - e^{-\left(\frac{h}{a}\right)^2}\right)$$

The linear model is as follows:

$$\gamma(h) = c_0 + bh$$

In this formula, c_0 - is the nugget effect, $c_0 + c$ - is the sill, and a is the range parameter [11]. The nugget is a measure of error in measurement at a very fine scale or of random error at a small scale. The sill shows the higher level in which the semivariogram stabilizes. The range indicates the distance within which the spatial dependence of the points is actually maintained practically. When the distance between the points is larger than the range, then the values are almost independent of one another. One of the most widely applied models in cases of scattered data is the spherical one [12]. Once the semivariogram model of theory is chosen, ordinary kriging weights are determined using a special system of equations:

$$\sum_{j=1}^{n} \lambda_j \gamma(d_{ij}) + \mu = \gamma(d_{i0}), \quad i = 1, 2, \ldots, n$$

$$\sum_{j=1}^{N} \lambda_j = 1$$

μ in this case is the Lagrange multiplier, and this multiplier is added to consider the fact that the weight of the weights should add up to one [13]. After resolving this system weights $\lambda_1, \lambda_2, \ldots, \lambda_n$ are determined. The approximate value at the unknown point is then obtained using them as [14]:

$$\overline{Z}(x_0, y_0) = \sum_{i=1}^{N} \lambda_i Z(x_i, y_i)$$

Analysis and results

Here, the findings of surface reconstruction with the help of ordinary kriging method on the parameters of the data provided at the scattered nodes are presented. The computational experiment used the values developed based on 25 scattered points that were picked on a two-variable test function and an interpolation surface was plotted on them. Consider the following function as a selection function:

$$f(x, y) = \sin(x) \cdot e^{-\cos(y)} + \cos(y) \cdot e^{\sin(x)}$$

Let's select points from this function.

Then, using kriging, we reconstruct the surface in the entire considered domain and compare the results with the real function.

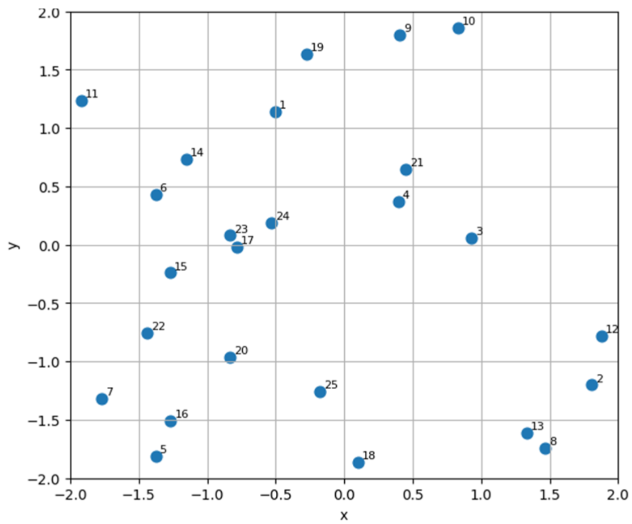

Рис. 1: The location of the selected points in the Oxy plane

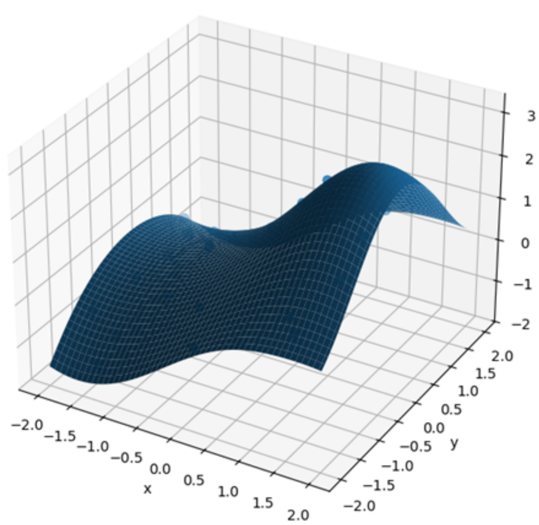

Рис. 2: Graph of the selected function

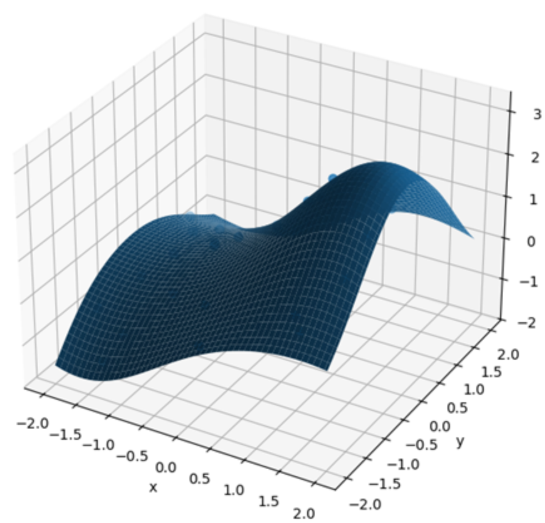

Рис. 3: Surface restored using the Kriging method based on selected points

Таблица 2: Comparison of the original function and Kriging interpolation results

x	y	f(x,y)	Kriging	Error
-2.0000	-2.0000	1.546220	-1.574491	0.028270
-2.0000	-1.1919	0.479171	-0.474588	0.004583
-2.0000	-0.3838	0.013735	-0.012677	0.026413
-2.0000	0.4242	0.011578	-0.050324	0.038532
-2.0000	1.2323	0.518625	-0.534890	0.016273
-2.0000	2.0000	1.546220	-1.128308	0.417912

-1.1919	-2.0000	1.572930	-1.541115	0.031816
-1.1919	-1.3919	0.495756	-0.496400	0.000644
-1.1919	-0.3838	0.011407	-0.002402	0.008995
-1.1919	0.4242	0.013565	-0.012620	0.000945
-1.1919	1.2323	0.535441	-0.516304	0.019137
-1.1919	2.0000	1.572930	-1.280070	0.292861
-0.3838	-2.0000	0.853915	-0.784247	0.069674
-0.3838	-1.1919	0.004355	-0.000882	0.003473
-0.3838	0.3838	0.489445	0.494018	0.004573
-0.3838	0.4242	0.478151	0.475187	0.002964
-0.3838	1.2323	0.040540	-0.037936	0.002407
-0.3838	2.0000	0.835915	-0.854694	0.017876
0.4242	-2.0000	0.004104	-0.043853	0.043850
0.4242	-1.1919	0.842616	0.832229	0.010380
0.4242	-0.3838	1.562313	1.587973	0.025660
0.4242	0.4242	1.540948	1.539491	0.001457
0.4242	1.2323	0.795477	0.832370	0.036893
0.4242	2.0000	0.004104	-0.059993	0.055991
1.2323	-2.0000	0.361279	0.319432	0.041847

Quality of interpolation was measured using the criteria of mean absolute error, root mean square error, root mean square error, maximum absolute error and coefficient of determination.

Comparison and analysis of interpolation performance.

In this experiment, the table of values determined by the test function given was used to restore the surface with the Kriging method based on the table of values. The accuracy of the reconstructed surface was evaluated using several criteria of errors. Specifically, the mean absolute error, root mean square error, root mean square error, the maximum absolute error and the coefficient of determination were determined. By means of these criteria, the closeness of the interpolation results to the actual function was examined. Let us assume that is the true value at the i-th point, and is the value obtained by interpolation. Then the average absolute error is calculated according to the following equation:

$$MAE = \frac{1}{N}\sum_{i=1}^{N} |z_i - \overline{z}_i|$$

This criterion demonstrates the average absolute discrepancy in the true and reconstructed values. The fact that Mae is small means that the total accuracy of the model is good. Based on the results of the calculation, the kriging method MAE amounted to MAE = 0.0219528535. This finding indicates that the values of the reconstruction with the help of kriging were not on average far apart in comparison with the true values of the functions. Thus, the model was accurate enough in the entire domain considered. The low value of MAE indicates that in data provided at sparsely spaced nodes, ordinary kriging method has the potential to offer consistent and accurate results. The mean squared error is calculated as follows:

$$MSE = \frac{1}{N}\sum_{i=1}^{N}(z_i - \overline{z}_i)^2$$

Since the errors are squared in MSE, larger deviations are evaluated more strongly. For this reason, this indicator reflects well the sensitivity of the model to uneven errors.

The kriging method MSE as per the results of the calculations was MSE =0.0024919208. As usual, in this criterion the squares of the errors are considered and thus the bigger deviations are considered more. The low value of MSE shows that the reconstructed surface of the kriging method is a good fit of the actual surface. Simultaneously, the same result indicates that extremely big errors were not common when interpolating.

Root of the mean squared error i.e. RSME is simply defined as:

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(z_i - \overline{z}_i)^2}$$

The criterion of RSME is quite easy to use in practice, since its unit of measurement is also a unit of the parameter being estimated. This is why RSME can be considered as one of the most significant indicators when the measurement of interpolation accuracy. Based on the results of the calculations, the kriging method RMSE = 0.0499191427. Given that the RSME value will be in the same units as the function being estimated, it is easier to interpret in real life. This finding demonstrates that the values that are obtained by kriging are almost similar to those of the actual values. That is, the mean squared deviation is roughly about 0.05, which is thought to be a small enough error in regards to the test function chosen.

The maximum absolute error is the difference between the result of the interpolation at the worst point and it is denoted as follows:

$$E_{\max} = \max_{1\le i\le N} |z_i - \overline{z}_i|$$

The criterion is particularly significant in the determination of the worst case, as it illustrates the magnitude of an error that can be made by the model at a particular point in the surface.

The results of the calculations revealed that it was equal to

$$E_{\max} = 0.4330520775$$

This indicator is very big compared to the other criteria which implies that there were some local points where significant deviations would have been realized during the interpolation process. That is, in spite of the fact that the overall performance is good, at some stages of the surface the kriging model became less close to the actual function. The main cause of this state of affairs is the sparsity of the nodes structure and lack of the data point density on certain regions. Thus, the supreme absolute error demonstrates that in the commonplace kriging outcome, local errors are not completely eradicated. To determine the overall level of fit of the model, the coefficient of determination i.e. the R^2 indicator was also estimated:

$$\overline{z} = \frac{1}{N}\sum_{i=1}^{N} z_i$$

is the arithmetic mean of the true values. R^2 being near to 1 means that the surface that is reconstructed is very close to the actual function. The results of the calculation were as follows: $R^2 = 0.9980582535$. The reconstructed surface using ordinary kriging was found to be a good reflector of the overall behavior of the true function since this value is very close to 1. Such a high value of R^2 is a high level of fit of the model. Thus, when the data is provided at separated nodes, ordinary kriging method was used to provide very accurate data in a global perspective.

Conclusion

Taken together, these results show that the ordinary kriging method works well for surface reconstruction based on data given at scattered nodes. In particular, the low values of MAE, MSE, and RMSE indicate that the model has high overall accuracy. The fact that the R^2 coefficient is very close to 1 further shows that the reconstructed surface agrees well with the true function. At the same time, the relatively larger maximum absolute error suggests that some local deviations are still present. Therefore, although ordinary kriging is an effective interpolation method for scattered spatial points, local errors may still occur in certain parts of the surface.

REFERENCES

1. Bakhromov, S. A.; Jumaev, J.; Kobilov, S.; Tukhtasinov, M. Analysis of the Construction of Local Interpolation Cubic Splines on the Basis of Detailed Data. *AIP Conference Proceedings* **2781** (2023), 020077.

2. Bakhromov, S. A.; Eshkuvatov, Z. K. Error Estimates for One Local Interpolation Cubic Spline with the Maximum Approximation Order of $O(h^3)$ for $W^1[a, b]$ Classes. *AIP Conference Proceedings* **2484** (2023), 030009.

3. Bakhromov, S. A. Digital Processing of Signals Ryabenky Cubic Spline Model. *AIP Conference Proceedings* **2781** (2023), 020015.

4. Bakhromov, S. Comparative Analysis of Methods for Approximation of Functions by Polynomial Splines. *AIP Conference Proceedings* **2781** (2023), 020069.

5. Maiti, P.; Mitra, D. Ordinary kriging interpolation for indoor 3D REM. *J. Ambient Intell. Humaniz. Comput.* **14** (2023), 13285–13299.

6. Kebede, H.; Demissie, Z.; Tadesse, H.; Eshetu, A. Spatial interpolation techniques comparison and evaluation: The case of ground-based gravity and elevation datasets of the central Main Ethiopian rift. *Heliyon* **10** (2024), e32806.

7. Boumpoulis, V.; Michalopoulou, M.; Depountis, N. Comparison between different spatial interpolation methods for the development of sediment distribution maps in coastal areas. *Earth Sci. Inform.* **16** (2023), 2069–2087.

8. Shao, C.; Xu, Y.; Chen, H.; Zheng, S.; Qin, X. Ordinary Kriging Interpolation Method Combined with FEM for Arch Dam Deformation Field Estimation. *Mathematics* **11** (2023), 1106.

9. Jang, D.; Kim, B.; Choi, H.; et al. Kriging Interpolation for Constructing Database of the Atmospheric Refractivity in Korea. *Remote Sens.* **16** (2024), 2379.

10. Dash, P. K.; et al. Exploring the Effect of Sampling Density on Spatial Interpolation Accuracy. *Land* **13** (2024), 1615.

11. Maniteja, M.; et al. Advancing Iron Ore Grade Estimation: A Comparative Study of Geo-Statistical and Machine Learning Interpolation Methods. *Minerals* **15** (2025), 131.

12. Zhang, J.; et al. Research on a coal seam modeling construction method based on ordinary kriging interpolation. *Sci. Rep.* (2024), doi: 10.1038/s41598-024-77114-1.

13. Zhang, J.; et al. Comparative Study on Different Interpolation Methods and Their Application in Spatial Distribution Analysis. *Sustainability* **16** (2024), 3545.

14. Liu, S.; et al. A combined spatial interpolation method of co-Kriging with machine learning for soil water-salt distribution evaluation. *J. Hydrol.* (2025).

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Affine shadowing of renormalizations for GIETs satisfying a certain Zygmund smoothness condition

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Abstract. Consider generalized interval exchange transformations (GIET) with irrational rotation number of periodic type. We will show that there exists a vector that shadows (with respect to accelerated height cocycle) the logarithm of mean non-linearity of renormalizations of GIET satisfying a certain Zygmund type smoothness conditions.

MSC (2020): 37E05, 37E10, 37E20, 37C15, 37A05.

Key words: interval exchange transformations, renormalizations, cocycle, rotation numbers.

Introduction

Generalized interval exchange transformations (GIETs) form an important class of one-dimensional dynamical systems that extend classical interval exchange transformations (IETs) by allowing the exchanged subintervals to be related by orientation-preserving homeomorphisms rather than rigid translations. They appear naturally as first return maps to Poincaré sections of smooth flows on surfaces, measured foliations, Teichmuller dynamics and piecewise smooth circle homeomorphisms. Despite their simple definition, these maps exhibit rich dynamical behavior and have been extensively studied over the past several decades.

The study of circle diffeomorphisms is a classical topic in dynamical systems, initiated by Poincare's invention of the rotation number followed by Denjoy's important distortion estimates and Arnold's introduction of KAM theory methods to the topic. Then the theory was further developed to establish the regularity of the map conjugating most minimal circle diffeomorphisms to their linear model. Extending these results to higher genus surfaces, and thus to generalized interval exchange transformations, has been the focus of investigations in the seminal works of Forni [6] and Marmi, Moussa and Yoccoz [9]-[10]. Similar to the classical Poincaré classification theorem for circle maps, a typical GIET with no periodic points is semi-conjugated to a standard interval exchange transformation and this semi-conjugacy is actually a conjugacy if and only if the GIET is minimal. Moreover, for a typical IET T , the set of GIETs smoothly conjugated to T defines a smooth submanifold of positive finite codimension [10]. As a continuation of these developments, the rigidity problem for GIETs (smoothness of conjugating map between two GIETs) has been investigated by Ghazouani and Ulcigrai in [7], and by Berk and Trujillo in [2].

A fundamental tool for analyzing one-dimensional dynamical systems is renormalization, which studies induced transformations on smaller subintervals and characterizes the asymptotic structure of the dynamics. The renormalization approach used in [8] is particularly natural in the spirit of Herman's theory. Within this framework, the regularity of a conjugacy can be deduced from the convergence properties of the renormalizations of sufficiently smooth circle diffeomorphisms. In particular, the renormalizations of a smooth circle diffeomorphism converge exponentially fast to a family of linear maps with slope one. This exponential convergence, together with a suitable condition on the rotation number (e.g., of Diophantine type), ensures the smoothness of the conjugacy. More generally, results on the regularity of conjugacies between two topologically equivalent maps can be obtained from the convergence properties of the renormalizations of the corresponding maps [2], [7].

In this paper we investigate affine shadowing properties of renormalizations for generalized interval exchange transformations with low regularity, assuming certain Zygmund-type smoothness conditions. Affine

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shadowing refers to the process by which the renormalizations of a given dynamical system can be approximated by a sequence of affine models determined by the combinatorial structure of the map. Understanding these approximations is important for analyzing the asymptotic geometry of renormalizations and for investigating rigidity properties of the underlying GIETs.

Preliminaries

Interval exchange transformation(IET). Let $I = [0,\, 1)$ be the unit interval. A *standard interval exchange transformation*, or simply an *interval exchange transformation* (IET), is a bijective, right-continuous function $T : I \to I$, with a finite number of discontinuities whose restriction to any subinterval of continuity is given by a translation. We say that T is an IET *on d intervals* if there exists a partition $\{I_\alpha\}_{\alpha\in\mathcal{A}}$, where the indexes belong to some finite alphabet $\mathcal{A}$ with $d \geq 2$ symbols such that T is continuous when restricted to I_α for each $\alpha \in \mathcal{A}$. Notice that T is simply exchanging the order of the intervals in the partition.

An IET T of d intervals can be encoded by a pair $(\lambda,\, \pi)$ corresponding to a *combinatorial datum* $\pi = (\pi_0,\, \pi_1)$, consisting of bijections $\pi_0,\, \pi_1 : \mathcal{A} \to 1, ..., d$ describing the order of the intervals before and after T is applied, and a *lengths vector* $\lambda = \{\lambda_\alpha\}_{\alpha\in\mathcal{A}}$ in the simplex $\Delta_d = \left\{\nu \in \mathbb{R}_+^{\mathcal{A}} : \sum_{\alpha\in\mathcal{A}} \nu_\alpha = 1\right\}$ which corresponds to the lengths of the intervals in the partition $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ associated to T. We always assume that the datum $\pi = (\pi_0,\, \pi_1)$ is *irreducible*, i.e.

$$\pi_1 \circ \pi_0^{-1}(\{1,\, 2,\, ..., k\}) = \{1,\, 2,\, ..., k\} \Rightarrow k = d.$$

We call $\pi_1 \circ \pi_0^{-1} : \{1,\, 2,\, ..., d\} \to \{1,\, 2,\, ..., d\}$ the *monodromy invariant* of π.

A combinatorial datum $\pi = (\pi_0,\, \pi_1)$ is said to be of *rotation type* if its monodromy invariant verifies

$$\pi_1 \circ \pi_0^{-1}(i) - 1 = i + k \pmod 1$$

for some $k \in \{1,\, 2,\, ..., d-1\}$ and for all $i \in \{1,\, 2,\, ..., d\}$. Similarly, we say that an IET is of *rotation type* if its combinatorial datum is of rotation type. Notice that any IET of rotation type induces a well-defined circle rotation on the circle $\mathbb{S}^1$.

Let $I_\alpha = [\ell_\alpha,\, r_\alpha]$ where ℓ_α and r_α are the left and right endpoints of I_α, respectively. We say that an IET $T = (\lambda,\, \pi)$ satisfies the *Keane condition* if $T^m(\ell_\alpha) \neq \ell_\beta$ for all $m \geq 1$ and all $\alpha,\, \beta \in \mathcal{A}$ with $\pi_0(\beta) \neq 1$. This condition is also called the infinite distinct orbit condition. Note that any IET verifying Keane's condition is irreducible and minimal.

Rauzy-Veech induction. Let $T = (\lambda,\, \pi)$ be an IET on d intervals. Denote $\alpha(\varepsilon) = \pi_\varepsilon^{-1}(d)$ for $\varepsilon = 0,\, 1$. The letters $\alpha(0)$ and $\alpha(1)$ correspond to the 'last' intervals in the partitions $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{T(I_\alpha)\}_{\alpha\in\mathcal{A}}$, respectively. If $\lambda_{\alpha(0)} \neq \lambda_{\alpha(1)}$, by comparing the lengths of these intervals, we define the *type* of T as $\varepsilon(\lambda,\, \pi) = 0$ if $\lambda_{\alpha(0)} > \lambda_{\alpha(1)}$, $\varepsilon(\lambda,\, \pi) = 1$ if $\lambda_{\alpha(0)} < \lambda_{\alpha(1)}$. The longest of these two intervals is sometimes referred to as the *winner* and the shortest as the *loser*. Notice that $\alpha(\varepsilon) := \alpha(\varepsilon,\, T)$ and $\alpha(1-\varepsilon) := \alpha(1-\varepsilon,\, T)$ correspond to the symbols of the winner and the loser intervals, respectively. We will sometimes refer to types 0 and 1 as *top* and *bottom*, respectively.

The *Rauzy-Veech induction* of T with type ε, which we denote by $\mathcal{RV}(T)$, is defined as the first return map of T to the subinterval $I^{(1)} = I \setminus I_{\alpha(1-\varepsilon)}$. The *Rauzy-Veech renormalization* of T is obtained by rescaling linearly $I^{(1)}$ to the unit interval I. The renormalized map is an IET with the same number of subintervals as T. This induction/renormalization procedure can be iterated infinitely many times if and only if T verifies the Keane's condition.

Denote by $\mathcal{G}_d$ the set of *irreducible combinatorial data* $\pi = (\pi_0,\, \pi_1)$ with d symbols. Also, denote X_d the set of IETs verifying Keane's condition. Given $\pi,\, \pi' \in \mathcal{G}_d$ the permutation π' is said to be a *successor* of π if there exist $\lambda,\, \lambda' \in \Delta_d$ such that $(\lambda',\, \pi') = \mathcal{RV}(\lambda,\, \pi)$. We denote this relation by $\pi \to \pi'$. Notice that any successor of an irreducible permutation is also irreducible. The relation '$\to$' defines an oriented graph structure on the set of irreducible permutations $\mathcal{G}_d$. We call *Rauzy classes* the connected components of the oriented graph $\mathcal{G}_d$ with respect to the successor relation.

GIETs and combinatorial rotation number. A generalized interval exchange transformation (GIET) is a piecewise smooth bijective, right-continuous function $f : I \to I$ with a finite number of discontinuities, whose derivative is non-negative and extends to the closure of any subinterval where the function is smooth. Similar to the case of IETs, to any GIET f, we can associate a partition $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ of I such that, for every $\alpha\in\mathcal{A}$, the restriction $f_{I_\alpha}: I_\alpha \to f(I_\alpha)$ is smooth, as well as a permutation π describing the order in which these intervals are exchanged.

Rauzy-Veech renormalization and Keane's condition, initially defined only for IETs, extend trivially to GIETs. Given a GIET f with associated permutation π and exchanged intervals $\{I_\alpha\}_{\alpha\in\mathcal{A}}$, we denote by $\mathcal{RV}(f)$ its Rauzy-Veech renormalization whenever it is well defined, that is, if $|I_{\alpha(0)}| \neq |f(I_{\alpha(1)})|$, where $|I|$ denotes the length of the interval I. We say that f is *infinitely renormalizable* if and only if $\mathcal{RV}^n(f)$ is well defined for all $n\in\mathbb{N}$. Similar to the IET setting, if f verifies Keane's condition, then it is infinitely renormalizable. The infinite path $\gamma(f)$ on the Rauzy diagram defined by infinitely renormalizable GIET f is called the *combinatorial rotation number* or simply the *rotation number* of f. An infinite path on the Rauzy diagram is called ∞-*complete* if each letter in $\mathcal{A}$ wins infinitely many times. We say that a GIET is *irrational* if it is infinitely renormalizable and its rotation number is irrational(∞-complete).

A GIET f is called of *periodic type* if it has no connections and its combinatorial rotation number $\gamma(T)$ is irrational and periodic, i.e. there exists a $p>0$ such that $\gamma_{n+p}=\gamma_n$ for every $n\in\mathbb{N}$. The minimal p with such property will be called the period of $\gamma(f)$. Throughout the paper, we will consider GIETs with irrational rotation number of periodic type.

The length cocycle. Given $T=(\lambda,\pi)$ such that $\lambda_{\alpha(0)}\neq\lambda_{\alpha(1)}$, we define the *Rauzy-Veech matrix* $A(T): \mathbb{R}^d\to\mathbb{R}^d$ associated to T as

$$A(T) = I_d + E_{\alpha(\varepsilon),\,\alpha(1-\varepsilon)}, \tag{1}$$

where I_d denotes the identity matrix on $\mathbb{R}^d$ and $E_{\alpha,\,\beta}$ is the matrix whose entries are 1 at the position $(\alpha,\,\beta)$ and 0 otherwise.

Given $T=(\lambda,\,\pi)$ verifying Keane's condition, denote

$$A^n(T) = A(T)\cdots A(\mathcal{RV}^{n-1}(T)),\quad A_{m,n}(T) = A(\mathcal{RV}^{n-1}(T))\cdots A(\mathcal{RV}^{m}(T)),\ n>m\geq 0.$$

Rauzy-Veech matrix encodes the change of the length vector after one step of Rauzy-Veech induction. Namely, the lengths vector $\lambda^{(n)}$ of $\mathcal{RV}^n(T)$ verifies

$$\lambda^{(n)} = A^n(T)^{-1}\lambda,\quad \lambda^{(n)} = A_{m,n}(T)^{-1}\lambda^{(m)},\ \text{ where }\ \lambda=\lambda(T)=\{\lambda_\alpha\}_{\alpha\in\mathcal{A}}.$$

The map

$$A^{-1}: X_d \longrightarrow SL(d,\,\mathbb{Z}),\qquad T\mapsto A(T)^{-1},$$

is a cocycle over $\mathcal{R}$, known as the *Rauzy-Veech cocycle* or *lengths cocycle*.

The height cocycle. Let $T=(\lambda,\,\pi)$ be an IET satisfying Keane's condition. Notice that the transformation $\mathcal{RV}^n(T)$ is defined as the first return map of T to some subinterval $I^{(n)}\subset I$. This interval admits a decomposition $I^{(n)}=\sqcup_{\alpha\in\mathcal{A}} I_\alpha^{(n)}$ such that the return time to $I^{(n)}$ on each subinterval $I_\alpha^{(n)}$ is constant. Denote by $q_\alpha^{(n)}$ the return time for the interval $I_\alpha^{(n)}$, that is $T^{q_\alpha^{(n)}}(x)\in I_\alpha^{(n)},\ \ x\in I_\alpha^{(n)}$.

Using the Rauzy-Veech matrix $A(T)$ given by equation (1), we define a cocycle over IETs, known as the *heights cocycle*, by

$$A^T: X_d \longrightarrow SL(d,\,\mathbb{Z}),\qquad T\mapsto A(T)^T,$$

that encodes the change of the return times vector after one step of Rauzy-Veech induction. The vector of return times $q^{(n)}=(q_\alpha^{(n)})_{\alpha\in\mathcal{A}}$ verifies

$$q^{(n)} = A^n(T)^T\overline{1},\qquad q^{(n)} = A_{m,n}(T)^T q^{(m)},\ \ n>m,$$

where $\overline{1}\in\mathbb{N}^{\mathcal{A}}$ is the vector whose entries are all equal to 1.

Dynamical partitions. Given an IET $T=(\pi,\lambda)$ verifying Keane's condition, we can associate a sequence of *dynamical partitions* and *Rohlin towers* as follows. We define the *dynamical partition* $\mathcal{P}^{(n)}$ of I *at level* n as

$$\mathcal{P}^{(n)} = \bigcup_{\alpha\in\mathcal{A}}\mathcal{P}_\alpha^{(n)},\ \text{ where }\ \mathcal{P}_\alpha^{(n)} = \left\{I_\alpha^{(n)},\,T(I_\alpha^{(n)}),\ldots,T^{q_\alpha^{(n)}-1}(I_\alpha^{(n)})\right\}.$$

One can verify that $\mathcal{P}^{(n)}$ is a partition of $[0,\,1)$ into subintervals and that, for each $\alpha\in\mathcal{A}$, the collection $\mathcal{P}_\alpha^{(n)}$ is a Rohlin tower of height $q_\alpha^{(n)}$. Notice that if $n>m$, then $\mathcal{P}^{(n)}$ is a refinement of $\mathcal{P}^{(m)}$.

Zorich acceleration. If T is of periodic type with period p, it is natural to consider the renormalization operator $\mathcal{R} = \mathcal{RV}^p$, viewed as an acceleration of $\mathcal{RV}$. In this case, for every $k \in \mathbb{N}$ we consider $\mathcal{R}^k(T) = \mathcal{RV}^{kp}(T)$.

Another important acceleration of $\mathcal{RV}$ is the *Zorich acceleration*, denoted by $\mathcal{Z}$. It is obtained by grouping together all consecutive elementary steps of the Rauzy-Veech induction that are of the same type (top or bottom). More precisely, the Zorich map is given by

$$\mathcal{Z}(T) = \mathcal{RV}^{z(T)}(T),$$

where $z(\pi, \lambda)$ is the largest integer such that $T, \mathcal{RV}(T), ..., \mathcal{RV}^{z(T)-1}(T)$ all have the same type. Using $z : X_d \to \mathbb{N}$ as the *accelerating map*, define the *accelerated lengths and heights cocycles*

$$B^{-1} : X_d \longrightarrow SL(d, \mathbb{Z}), \qquad B^T : X_d \longrightarrow SL(d, \mathbb{Z}),$$

by setting

$$B^{-1}(T) = A^{z(T)}(T)^{-1}, \qquad B^T(T) = A^{z(T)}(T)^T.$$

These cocycles are related to the transformation of lengths and heights under the action of $\mathcal{Z}$. Note that if T is of periodic type with period p, then the associated cocycle is given by $B(T) = A^p(T)$.

Affine interval exchange transformations An affine interval exchange transformation (AIET) is a GIET for which the restriction to each subinterval of continuity is a linear map. Notice that f will change the order of these intervals and linearly modify their lengths. Given an AIET f on d intervals with associated partition $\{I_\alpha\}_{\alpha\in\mathcal{A}}$, we define its *log-slope* as the logarithm of the slope of f in each interval of continuity, namely, the vector $\omega = (\omega_\alpha)_{\alpha\in\mathcal{A}}$, where $\omega_\alpha = \log Df(x),\ x \in I_\alpha$. The Rauzy-Veech induction and the Zorich acceleration extend naturally to the space of AIETs, as well as all the notions introduced above in the IET setting, such as combinatorial rotation number, dynamical partitions, incidence matrices, etc.

Height and length cocycles for GIET. It is important to note that while the height and length cocycles extend trivially to the GIET setting (since they depend only on the combinatorial rotation number), their roles differ. The height cocycle continues to represent the return times for the induced transformation. However, the length cocycle no longer describes the lengths of the intervals in the partition of the induced transformation.

We can define an equivalent of a length cocycle for GIETs. Let f be a GIET and let $T_0 = (\lambda, \pi)$ be an IET such that $\gamma(f) = \gamma(T_0)$. Let $A^{(n)} := A(\lambda, \pi) \cdots A(\mathcal{RV}^{n-1}(\lambda, \pi))$. For any $n \in \mathbb{N}$ consider a $d \times d$ matrix $A^{(n)}(f)$ defined as follows

$$A^{(n)}_{\alpha\beta}(f) := \sum_{i=1}^{A^{(n)}_{\alpha\beta}} \frac{|f^{m_i(\alpha,\beta)}(I_\alpha(\mathcal{RV}^n(f))|}{|I_\alpha(\mathcal{RV}^n(f)|},$$

where $m_i(\alpha, \beta)$ is the i-th return time of the interval $I_\alpha(\mathcal{RV}^n(f))$ to the interval $I_\beta(f)$ via f. The importance of the matrices defined above follows from the fact that

$$(|I_\alpha(f)|)_{\alpha\in\mathcal{A}} = A^{(n)}(|I_\alpha(\mathcal{RV}^n(f))|)_{\alpha\in\mathcal{A}}.$$

Main results

GIET with a certain Zygmund smoothness condition. Now we define a class of GIETs satisfying a Zygmund condition. Consider the function $\mathcal{Z}_\gamma : [0, 1) \to (0, +\infty)$, defined as

$$\mathcal{Z}_\gamma(x) = |\log x|^{-\gamma}, \quad \text{for} \quad x \in (0, 1)$$

and $\mathcal{Z}_\gamma(0) = 0$, where $\gamma > 0$.

Let $J = [a, b]$ be a finite interval and consider a differentiable function $K : J \mapsto K(J)$. Denote by $\Delta^2 K(\xi, \tau)$ the *second symmetric difference* of K on J, that is,

$$\Delta^2 K(\xi, \tau) = K'(\xi + \tau) + K'(\xi - \tau) - 2K'(\xi)$$

where $\xi \in J$ and $\tau \in [0, |J|/2]$ such that $\xi - \tau,\ \xi + \tau \in J$.

Suppose that there exists a constant $C > 0$ such that the following inequality holds:

$$\|\Delta^2 K(\cdot, \tau)\|_{L^\infty(J)} \le C\tau\mathcal{Z}_\gamma(\tau). \tag{2}$$

Note that the class of real valued functions satisfying (2) with $\mathcal{Z}_\gamma(\tau) \equiv 1$ is called the Zygmund class and denoted by Λ_* . The class Λ_* plays a key role to investigate trigonometric series. The class Λ_* was applied to the theory of circle homeomorphisms for the first time by Jun Hu and Sullivan. They extended the classical Denjoy's theorem to the class Λ_*. Generally speaking, the function satisfying (2) does not imply the boundedness of total variation of its and the reverse also is not true. In this work we study the GIETs satisfying (2) which have bounded variations.

Denote by $X_d^{1+\mathcal{Z}_\gamma}$ the set of GIETs f of d intervals that satisfy the following conditions:

(i) has irreducible combinatorics and has no connections;

(ii) has combinatorial irrational rotation number of periodic type;

(iii) derivatives Df have bounded variation and satisfy the inequality (2) on the closer of each interval of continuity.

Previous Results and the Statements of Main Theorems. We define a metric d_{C^r} on the space C^r of piecewise smooth homeomorphisms with d branches and with a fixed combinatorial data, where the distance $d_{C^r}(f,\, g)$ of any two maps f and g in this class, is given by

$$\max_{\alpha\in\mathcal{A}} \left\{ \|\Xi(f|_{I_\alpha(f)}) - \Xi(g|_{I_\alpha(g)})\|_{C^r} + \|I_\alpha(f) - I_\alpha(g)\| + \|f(I_\alpha(f)) - g(I_\alpha(g))\| \right\}$$

where Ξ is the zoom operator, which rescales any homeomorphism between two bounded intervals, via affine transformations, to a homeomorphism of the unit interval. More precisely, if $g : I \to J$ is an homeomorphism between two closed bounded intervals, then

$$\Xi(g) = A_1 \circ g \circ A_2,$$

where $A_1 : J \to [0,1],\ A_2 : [0,1] \to I$ are bijective orientation preserving homeomorphisms. Whenever necessary, we will use $D^m f$ instead of the m^{th} derivative of f.

Let $\mathcal{M}_N$ be a Möbius transformation $\mathcal{M}_N : [0,\,1] \mapsto [0,\,1]$ such that $\mathcal{M}_N(0) = 0$, $\mathcal{M}_N(1) = 1$ and

$$\mathcal{M}_N(x) = \frac{xN}{1 + x(N-1)}.$$

Note that if $\gamma > 1$ then second derivative of f exists on each continuity intervals of f. We define a new quantity as follows:

$$m_\alpha^{(n)} = \exp\left\{ -\sum_{i=0}^{q_\alpha^{(n)}-1} \int_{I_\alpha^{(n)}} \frac{f''(t)}{2f'(t)} dt \right\}.$$

Theorem 1.(see [1]) Let $f \in X_d^{1+\mathcal{Z}_\gamma},\ \gamma > 1$ be a GIET with combinatorial data of rotation type. Then there exists a constant $C = C(f) > 0$ such that for all $\alpha \in \mathcal{A}$ the following bounds hold:

$$\|\Xi(\mathcal{RV}^n(f))|_{I_\alpha^{(n)}} - M_{m_\alpha^{(n)}}\|_{C^1[0,1])} \le \frac{C}{n^\gamma},$$

$$\|\Xi(D^2\mathcal{RV}^n(f))|_{I_\alpha^{(n)}} - D^2 M_{m_\alpha^{(n)}}\|_{C^0[0,1])} \le \frac{C}{n^{\gamma-1}}.$$

Theorem 1 implies the following

Corollary 1. For a.e. $T = (\lambda,\, \pi) \in \mathcal{G}_d \times \Delta_d$ and for any GIET f of class $X_d^{1+\mathcal{Z}_\gamma},\ \gamma > 0$ with $\mathcal{N}(f) = 0$ and $\gamma(f) = \gamma(T)$, it holds

$$\max_{\alpha\in\mathcal{A}} \max_{x,y\in I_\alpha^n} \frac{Df^{q_\alpha^n}(x)}{Df^{q_\alpha^n}(y)} = 1 + O(\frac{1}{n^\gamma}).$$

Now we assume that f is a infinitely renormalizable generalized interval exchange map of periodic type. Thus, there exists a $p > 0$ such that the rotation number is periodic with period p, namely if $n = kp + r$ for some $k \in \mathbb{N}$ and $0 \le r < p$ then $\gamma(\mathcal{RV}^{kp+r}(f)) = \gamma(\mathcal{RV}^{kp+r}(f))$. In this case, we will use, as renormalization operator, the acceleration of Rauzy-Veech induction which corresponds to the period p of the rotation number $\gamma(f)$, namely the operator given by $\mathcal{R}^n(f) = (\mathcal{R}\mathcal{V}^p)^n(f)$. Consider the cocycle corresponding to $\mathcal{R}^n(T)$:

$$B(\mathcal{R}(f)) = A(f)\cdots A(\mathcal{R}\mathcal{V}^p(f)), \quad B_{m,n}(f) = B(\mathcal{R}^m(f))\cdots B(\mathcal{R}^m(f)),\ n > m \geq 0.$$

Let us write $\mathbb{R}^d = E^s \oplus E^c \oplus E^u$ for the splitting of $\mathbb{R}^d$ into respectively the stable space E^s, the central space E^c and the unstable space E^u for the action of A on $\mathbb{R}^d$ (corresponding to eigevectors with norm respectively smaller, equal and greater than 1). We will use the following hyperbolic properties of Rauzy-Veech cocycle restricted to the $K-$bounded combinatorics(which also hold in the periodic case).

Proposition 1(see [4]). For each K there exists $\mu = \mu(K) > 1$ and $C > 0$ with the following property

(1) For every $n \geq 1$ and $v \in E^u$, we have $\|^T B_{0,n} v\| \geq C_1 \mu^n \|v\|$.

(2) For every $n \geq 1$ and $v \in E^s$, we have $\|(^T B_{0,n})^{-1} v\| \geq C_1 \mu^n \|v\|$.

Proposition 2(see [4]). For all vectors $v \in E^c$ and for all $n \geq 1$, there exists $C > 0$ such that

$$C^{-1}\|v\| \leq \|^T B_{0,n} v\| \leq C\|v\|.$$

The following lemma can be easily verified.

Lemma 1. Assume $0 < \lambda < 1$ and $\gamma > 0$. Then

$$\sum_{i=0}^{n-1} \frac{\lambda^i}{(n-i)^\gamma} = O(\frac{1}{n^\gamma}), \qquad \sum_{i=0}^{n-1} \frac{\lambda^{n-i}}{(2n-i)^\gamma} = O(\frac{1}{n^\gamma}).$$

Let f be a GIET of class $X_d^{1+\mathcal{Z}_\gamma},\ \gamma > 1$ verifying $\mathcal{N}(f) = \int_0^1 D\log Df(x)dx = 0$. Define log-slope vectors $L^n = (L^n_\alpha)_{\alpha\in\mathcal{A}}$ of $\mathcal{R}^n(f)$ as

$$L^n_\alpha = \ln\left(\frac{1}{|I^{(n)}_\alpha|}\int\limits_{I^{(n)}_\alpha} Df^{q^n_\alpha}(s)ds\right),\ \ I^{(n)}_\alpha \in \mathcal{P}^{(n)}_b.$$

Now we state our main result in this article.

Theorem 2. Let $f \in X_d^{1+\mathcal{Z}_\gamma},\ \gamma > 1$ with $\mathcal{N}(f) = 0$. Then there exists a vector $\omega \in E^c(\tau,\lambda,\pi)$ such that

$$|\omega^n - L^n| = O(\frac{1}{n^\gamma}),$$

where

$$\omega^n =^T B_{0,n}(\tau,\lambda,\pi)\omega.$$

This theorem shows the existence of a vector that "shadows" the algorithm for the mean nonlinearity of the subsequent renormalizations $\mathcal{R}f$ of f, with respect to the accelerated height cocycle B associated with accelerated $\mathcal{R}$.

To prove this theorem we first show that the sequence $(L^n)_{n\geq 1}$ behaves as a pseudo-orbit with respect to the heights cocycle. The following lemma describes the relationship between L^n and L^{n+1}.

Lemma 2. We have

$$|L^{n+1} -^T B_{n,n+1} L^n| = O(\frac{1}{n^\gamma}),\ \ \gamma > 1.$$

Proof. For any $\alpha \in \mathcal{A}$ and $n \in \mathbb{N}$, let $x^n_\alpha \in I^{(n)}_\alpha(f)$ such that $L^n_\alpha = \ln Df^{q^n_\alpha}(x^n_\alpha)$. Let us fix $n \in \mathbb{N}$. Given $\alpha \in \mathcal{A}$, let

$$b^n_\alpha = \sum_{\beta\in\mathcal{A}} (^T B_{n,n+1})_{\alpha\beta}.$$

For any $\alpha \in \mathcal{A}$, we can express q^{n+1}_α uniquely as $q^{n+1}_\alpha = \sum_{i=1}^{b^n_\alpha} q^n_{\zeta_i(\alpha)}$, for some $\zeta_i(\alpha) \in \mathcal{A}$, such that

$$f^{h_i}(I^{n+1}_\alpha(f)) \subset I^n_{\zeta(\alpha)}(f), \quad \text{where,} \quad h_i = \sum_{j=1}^{i-1} q^n_{\zeta_j(\alpha)}, \tag{3}$$

for $i = 1, 2, ..., b^n_\alpha$. Notice that

$$({}^T B_{n,n+1} L^n)_\alpha = \sum_{\beta \in \mathcal{A}} ({}^T B_{n,n+1})_{\alpha\beta} L^n_\beta, \qquad \text{and} \qquad \#\{1 \le i \le b^n_\alpha : \zeta_i(\alpha) = \beta\} = ({}^T B_{n,n+1})_{\alpha\beta},$$

for any $\alpha, \beta \in \mathcal{A}$. A simple calculations shows that

$$(L^{n+1} -^T B_{n,n+1} L^n)_\alpha = \sum_{\beta \in \mathcal{A}} \sum_{\zeta_i(\alpha)=\beta} \ln \frac{Df^{q^n_\beta}(f^{h_i}(x^n_\alpha))}{Df^{q^n_\beta}(x^n_\beta)}, \tag{4}$$

for any $\alpha \in \mathcal{A}$. Corollary 1 and the relations (3)-(4) imply that

$$\frac{|L^{n+1} -^T B_{n,n+1} L^n|}{\|{}^T B_{n,n+1}\|} = O(\frac{1}{n^\gamma}), \;\; \gamma > 0.$$

Therefore, the claim is a direct consequence of the previous relation and the fact that $\|{}^T B_{n,n+1}\|$ is bounded.

We show that this pseudo-orbit is shadowed by the iterates of a vector under the height cocycle. For any $n \ge 0$, let us decompose L^n with respect to the Oseledet's splitting at $(\tau^n, \lambda^n, \pi^n)$ as $L^n = L^{n,s} \oplus L^{n,c} \oplus L^{n,u} \in E^s_n \oplus E^c_n \oplus E^u_n$, and define $v_n =^T B^{-1}_{0,n} L^{n,c}$.

Lemma 3. There exists $\omega \in E^c(\tau, \lambda, \pi)$ such that $\lim\limits_{n\to\infty} v_n = \omega$. Moreover,

$$|v_n - \omega| = O(\frac{1}{n^\gamma}), \;\; \gamma > 1.$$

Proof. By Lemma 2 and Proposition 2, we have

$$|v_{n+1} - v_n| = |{}^T B^{-1}_{0,n+1}(L^{n+1,c} -^T B^{-1}_{n,n+1} L^{n,c})| \le \|{}^T B^{-1}_{0,n+1}|_{E^c}\| |L^{n+1,c} -^T B^{-1}_{n,n+1} L^{n,c}| = O(\frac{1}{n^\gamma}).$$

Therefore, v_n converges. This completes the proof.

Lemma 4. The sequence $\{L^{n,s}\}_{n\ge1}$ satisfies $\|L^{n,s}\| = O(\frac{1}{n^\gamma}), \;\; \gamma > 1$.

Proof. By Proposition 1, for all j, $n \ge 0$ and for all $v \in E^s_j$ we have

$$\|{}^T B_{j,n+j} v\| \le \frac{1}{C \cdot \mu^n} \|v\|.$$

Proposition 2 implies that there exists constant $\mu > \widetilde{\mu} > 1$ such that for all $n \ge 0$ and for all $v \in E^s_j$, we have $\|{}^T B_{0,n} v\| \le \widetilde{\mu}^{-1} \|v\|$. By Lemma 1, we obtain

$$\|L^{n,s}\| \le \frac{1}{\widetilde{\mu}} \|L^{n-1,s}\| + C \cdot \frac{1}{n^\gamma}.$$

By iterating this estimate n times, we obtain

$$\|L^{n,s}\| \le \frac{1}{\widetilde{\mu}^n} \|L^{0,s}\| + C \cdot \sum_{i=0}^{n-1} \frac{1}{\widetilde{\mu}^i} \frac{1}{(n-i-1)^\gamma}.$$

Since $\widetilde{\mu} > 1$, last relation and Lemma 1 imply the result.

Lemma 5. The sequence $\{L^u_n\}_{n\ge1}$ satisfies $\|L^u_n\| = O(\frac{1}{n^\gamma}), \;\; \gamma > 1$.

Proof. The proof is analogous to that of Lemma 4, and we again use the adapted norm. For all $n \ge 0$, we have

$$\|L^{n+1,u}\| \ge \widetilde{\mu} \|L^{n,u}\| - C \cdot \frac{1}{n^\gamma}.$$

Applying this estimate k times, we obtain

$$\|L^{n+k,u}\| \geq \widetilde{\mu}^k \|L^{n,u}\| - C \cdot \sum_{j=0}^{k-1} \widetilde{\mu}^j \frac{1}{(n+k-j-1)^\gamma},$$

and therefore

$$\|L^{n,u}\| \leq \frac{1}{\widetilde{\mu}^k} \|L^{n+k,u}\| + C \cdot \sum_{j=0}^{k-1} \widetilde{\mu}^{j-k} \frac{1}{(n+k-j-1)^\gamma},$$

Taking $n = k$, we have

$$\|L^{n,u}\| \leq \frac{1}{\widetilde{\mu}^n} \|L^{2n,u}\| + C \cdot \sum_{j=0}^{n-1} \widetilde{\mu}^{j-n} \frac{1}{(2n-j-1)^\gamma}.$$

Since the sequence $\{L^{2n,u}\}$ is uniformly bounded and $\widetilde{\mu} > 1$, last relation and Lemma 1 imply the result.

Proof of Theorem 2. It follows from Lemma 3 that

$$\|\omega^n - L^{n,c}\| = \|^T B_{0,n}\omega - L^{n,c}\| = \|^T B_{0,n}(\omega -^T Q_{0,n}^{-1} L^{n,c})\| \leq$$

$$\leq \|^T B_{0,n}|_{E^c_{0,\infty}}\| \cdot \|\omega - v_n\| = O(\frac{1}{n^\gamma}).$$

Thus, the last relation, together with Lemmas 4 and 5, completes the proof of Theorem 2.

REFERENCES

1. H. Akhadkulov. Rauzy-Veech renormalizations of circle diffeomorphisms with several break points. Abstracts of International scientific-practical conference, Innovative development of science and education: new approach and research, October 5, 2024.

2. P. Berk, F. Trujillo. Rigidity for piecewise smooth circle homeomorphisms and certain GIETs, Advances in Mathematics, 441 (2024) 109560.

3. K. Cunha, D. Smania. Renormalization for piecewise smooth homeomorphisms on the circle, Ann. Inst. H.Poincare, Anal. Non Lineaire, 30(3), 441-462, (2013).

4. K. Cunha, D. Smania. Rigidity for piecewise smooth homeomorphisms on the circle, Advances in Mathematics, 250, 193-226, (2014).

5. A. Dzhalilov, K. Cunha and A. Begmatov. On the Renormalizations of Circle Homeomorphisms with Several Break Points. J Dyn Diff Equat. 34, 1919-1948 (2022).

6. G. Forni. Solutions of the cohomological equation for area-preserving flows on compact surfaces of higher genus. Ann. of Math. (2), 146(2): 295-344, 1997.

7. S. Ghazouani, C. Ulcigrai. A priori bounds for GIETs, affine shadows and rigidity of foliations in genus two. Publ.math.IHES. 138, 229–366 (2023).

8. K.M. Khanin, Y.G. Sinai, Smoothness of conjugacies of diffeomorphisms of the circle with rotations, translation of Usp. Mat. Nauk 44, 57–82 (1989)

9. S. Marmi, P. Moussa, and J.-C. Yoccoz: Affine interval exchange maps with a wandering interval, Proc. London Math. Soc. 100(3), 639-669, (2010).

10. S. Marmi, P. Moussa, J.-C. Yoccoz: Linearization of generalized interval exchange maps. Ann. of Math. 176, 1583-1646, (2012).

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Low Smoothness: Renormalizations of circle maps with rational rotation numbers again behave as Möbius functions

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Abstract. Consider a one-parameter family of circle maps $f_t = f_0 + t(\bmod 1)$, where f_0 is a circle homeomorphism with two break points. Suppose Df_0 satisfies a certain Zygmund type smoothness condition depending on a parameter $\gamma > 0$. We prove that the renormalizations of circle homeomorphisms from this family with rational rotation number of sufficiently large rank are approximated by Mobius functions in C^{1+L_1}-norm if $\gamma \in (1/2, 1]$ and in C^2-norm if $\gamma > 1$.

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Key words: circle maps, rotation number, break point, renormalization, Zygmund-type smoothness.

Introduction

The study of circle maps originates from the classical works of Poincaré and Denjoy. Their contributions laid the foundation for the modern study of dynamical systems on the circle. Poincaré (1885) noticed that the orbit structure of orientation-preserving diffeomorphism f is determined by some irrational mod 1 $\rho(f)$, called the rotation number of f. For orientation preserving homeomorphisms the rotation number, an average rotation rate, is a topological invariant that determines many properties of the dynamics. If the rotation number is rational then there is at least one periodic orbit, whilst if the rotation number is irrational then there are no periodic orbits and, provided the map is sufficiently smooth (at least C^2) all orbits are dense in the circle. The dynamics of circle maps are often described through their conjugacy classes. Two maps f and g are topologically conjugate if there is a homeomorphism h such that $h \circ f = g \circ h$. If h is a diffeomorphism then the maps are differentiably conjugate. Results on the regularity of h that go back to Denjoy state that if f is an orientation-preserving C^2-diffeomorphism of the circle with irrational rotation number ρ, then f is conjugate to the linear rotation. Since then the problem of smoothness of the conjugacy h of smooth diffeomorphisms has come to be very well understood by several authors [4]-[7]. For instance, Sinai and Khanin [7] introduced ideas that later evolved into the renormalization method, which studies asymptotical behaviour of the induced transformations on smaller subintervals of the circle.

An important generalization of circle diffeomorphisms consists of maps with break-type singularities, that is, circle homeomorphisms that are smooth away from finitely many points (the *breaks*), where the derivative has jump discontinuities. Circle diffeomorphisms with break points were introduced by Khanin and Vul [8], who also studied their renormalizations. Their results have many applications in various areas of one-dimensional dynamics, such as investigations of the invariant measures, nontrivial scalings and prevalence of periodic trajectories in one parameter families.

In one-parameter families of circle maps, rational rotation numbers typically correspond to mode-locking intervals, where the rotation number remains constant. Using the convexity of renormalizations of circle maps with a break and rational rotation numbers, Khanin and Vul [8] also showed that the Lebesgue measure of the set of parameters corresponding to rational rotation numbers is zero. This result was later generalized to the case of multiple breaks by Khmelev [9], who used renormalizations of circle homeomorphisms with rational rotation numbers and two break points.

The main contribution of this paper is to extend these results to a broader class of circle maps. More precisely, we study the asymptotic behavior of renormalizations of circle homeomorphisms with rational rotation

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numbers and two break points under a weaker smoothness assumption, namely a Zygmund-type condition depending on a parameter γ, rather than the $C^{2+\varepsilon}$-smoothness condition considered in [8] and [9].

Preliminaries and Notations

Define a class of circle maps satisfying a Zygmund condition. Consider the function $\mathcal{Z}_\gamma : [0, 1) \to (0, +\infty)$, defined as

$$\mathcal{Z}_\gamma(x) = |\log x|^{-\gamma}, \quad \text{for} \quad x \in (0, 1)$$

and $\mathcal{Z}_\gamma(0) = 0$, where $\gamma > 0$.

Let $J = [a, b]$ be a finite interval and consider a differentiable function $K : J \mapsto K(J)$. Denote by $\Delta^2 K(\xi, \tau)$ the *second symmetric difference* of K on J, that is,

$$\Delta^2 K(\xi, \tau) = K'(\xi + \tau) + K'(\xi - \tau) - 2K'(\xi)$$

where $\xi \in J$ and $\tau \in [0, |J|/2]$ such that $\xi - \tau, \xi + \tau \in J$.

Suppose that there exists a constant $C > 0$ such that the following inequality holds:

$$\|\Delta^2 K(\cdot, \tau)\|_{L^\infty(J)} \le C\tau\mathcal{Z}_\gamma(\tau). \tag{1}$$

Note that the class of real valued functions satisfying (1) with $\mathcal{Z}_\gamma(\tau) \equiv 1$ is called the Zygmund class and denoted by Λ_* . The class Λ_* plays a key role to investigate trigonometric series. The class Λ_* was applied to the theory of circle homeomorphisms for the first time by Jun Hu and Sullivan. They extended the classical Denjoy's theorem to the class Λ_*.

We recall two important facts concerning the smoothness of functions satisfying inequality (1).

Proposition 1([2]). *Let $K : \mathbb{R} \to \mathbb{R}$ be a 1-periodic and continuous function that satisfying (1) for some $\gamma \in (1/2, 1]$. Then K is absolutely continuous and $K' \in L_p([0, 1])$ for every $p > 1$.*

Proposition 2([2]). *Let $K : \mathbb{R} \to \mathbb{R}$ be 1-periodic continuous function that satisfies (1) for some $\gamma > 1$. Then K is of class C^1.*

In this work we study the circle maps with rational rotation numbers and satisfying (1). Consider one parameter family of the orientation preserving circle homeomorphisms

$$f_t(x) = f_0(x) + t \,(\mathrm{mod}\, 1), \quad x \in S^1, \quad t \in [0, 1). \tag{2}$$

where the initial lift f_0 satisfies the following conditions:

(a) f_0 is a continuous and strictly increasing on $\mathbb{R}^1$;

(b) $f_0(0) = 0, \ f_0(x + 1) = f_0(x) + 1, \ x \in \mathbb{R}^1$;

(c) there are two points $x_b^{(i)}$, $i = 1, 2$ such that one-sided derivatives $f_0'(x_b^i \pm 0) > 0$, $i = 1, 2$ exist and $f_0'(x_b^i - 0) \ne f_0(x_b^i + 0)$, $i = 1, 2$.

(d) $f_0 \in C^{1+Z_\gamma}\left(S^1 \setminus \{x_b^{(1)}, x_b^{(2)}\}\right), \ \gamma > 0.$

The points $x_b^{(i)}$, $i = 1, 2$ are called break points and the ratio $c_i = \sqrt{\frac{f_0'(x_b^i - 0)}{f_0'(x_b^i + 0)}}$, $i = 1, 2$ is called the jump ratio of f_t at $x_b^{(i)}$, $i = 1, 2$. Denote by ρ_t the rotation number of f_t, i.e. $\rho_t = \lim\limits_{n\to\infty} \frac{f_t^n(x)}{n}$ (mod1), $x \in R^1$. Here and later, g^n denotes the *nth* iteration of g.

Denoted by $\mathbb{H}^{1+Z_\gamma}$ the class of circle homeomorphisms f whose derivative f' has bounded variation and satisfying conditions (a)-(d) and the inequality (1). Note also that the class $\mathbb{H}^{1+\mathcal{Z}_\gamma}$ is bigger than $\mathbb{H}^{2+\varepsilon}$ for any positive γ and ε.

Note that for each rational number a the set $I(a) = \{\theta : \rho_\theta = a\}$ is nontrivial closed interval and $I(a)$ consists of only one point if a is irrational.

Let $\frac{p}{q} \in [0,1)$ be an arbitrary rational number of *rank* n, i.e. $\frac{p}{q} = [k_1,\, k_2,\, ...,\, k_n],\ k_n > 1$. Since the rank of $\frac{p}{q}$ equals n we write $p_n := p$ and $q_n := q$. Let us fix some $t \in I(\frac{p}{q})$ and denote $f = f_t$. Let $O_f(x_p; q_n) = \{f^j(x_p),\ j = 0,\, 1,\, ...,(q_n - 1)\}$ be a periodic orbit of f of period q_n. Denote by $\Delta_0 = [y_1; y_2]$ the closed interval formed by two consecutive points of orbit $O_f(x_p; q_n)$ that contains break point $x_b^{(1)}$. Also, denote $\Delta_i = f^i \Delta_0$. Then for some r we have $x_b^{(2)} \in \Delta_r$. Introduce the renormalized coordinate z on Δ_0 given by the formula $z = \frac{x-y_1}{y_2-y_1}$. Denote new coordinates of $x_b^{(1)},\ x_b^{(2)}$ and $f^{-r}(x_b^{(2)})$ on intervals $\Delta_0,\ \Delta_r$ and Δ_0 by

$$d_1 = \frac{x_b^{(1)} - y_1}{y_2 - y_1} = \frac{x_b^{(1)} - y_1}{|\Delta_0|},\ \ d_2 = \frac{x_b^{(2)} - f^r y_1}{f^r y_2 - f^r y_1} = \frac{x_b^{(2)} - f^r y_1}{|\Delta_r|},\ \ \widetilde{d}_2 = \frac{f^{-r} x_b^{(2)} - y_1}{y_2 - y_1} = \frac{f^{-r} x_b^{(2)} - y_1}{|\Delta_0|}.$$

Now, we define the function $\mathbf{f}_n$ corresponding to $f^{q_n} : [y_1,\, y_2] \to [y_1,\, y_2]$ in this new coordinates by

$$\mathbf{f}_n(z) = \frac{f^{q_n}(y_1 + z(y_2 - y_1)) - y_1}{y_2 - y_1},\ \ z \in [0,\, 1].$$

The map $\mathbf{f}_n(z)$ is called nth renormalization of f on the interval $[y_1; y_2]$.

We need the following notations:

$$f^l_{c,\,d}(z) = \frac{c^2 z}{1 + d(c^2 - 1)},\ \ f^r_{c,\,d}(z) = \frac{z + d(c^2 - 1)}{1 + d(c^2 - 1)},\ \ f^{l,r}_{c,\,d}(z) = \begin{cases} f^l_{c,\,d}(z), & z \in [0,\, d], \\ f^r_{c,\,d}(z), & z \in (d,\, 1]. \end{cases}$$

$$F_M(z) = \frac{z}{M(1 - z) + z},\ \ \ M_1 = M(1; r) = \exp\left(\sum_{i=1}^{r-1} \frac{f'(b_i) - f'(a_i)}{f'(b_i)} dy\right),$$

$$M_2 = M(r + 1; q_n) = \exp\left(\sum_{i=r+1}^{q_n-1} \frac{f'(b_i) - f'(a_i)}{f'(b_i)} dy\right),$$

where $a_i = f^i(y_1),\ b_i = f^i(y_2),\ 0 \le i \le q_n - 1$.

Define the function $G_{c_1,\, d_1,\, c_2,\, \widetilde{d}_2,\, M_1,\, d_2}(z) : [0,\, 1] \to [0,\, 1]$ as follows: if $0 \le d_1 \le \widetilde{d}_2$, then

$$G_{c_1,\, d_1,\, c_2,\, \widetilde{d}_2,\, M_1,\, d_2}(z) = \begin{cases} F_{\frac{c_1 c_2}{M_1}} \circ f^l_{c_2,\, d_2} \circ F_{M_1} \circ f^l_{c_1,\, d_1}(z) & \text{for} \quad 0 \le z < d_1, \\ F_{\frac{c_1 c_2}{M_1}} \circ f^l_{c_2,\, d_2} \circ F_{M_1} \circ f^r_{c_1,\, d_1}(z) & \text{for} \quad d_1 \le z < \widetilde{d}_2, \\ F_{\frac{c_1 c_2}{M_1}} \circ f^r_{c_2,\, d_2} \circ F_{M_1} \circ f^r_{c_1,\, d_1}(z) & \text{for} \quad \widetilde{d}_2 \le z \le 1. \end{cases}$$

If $\widetilde{d}_2 < d_1 \le 1$, then

$$G_{c_1,\, d_1,\, c_2,\, \widetilde{d}_2,\, M_1,\, d_2}(z) = \begin{cases} F_{\frac{c_1 c_2}{M_1}} \circ f^l_{c_2,\, d_2} \circ F_{M_1} \circ f^l_{c_1,\, d_1}(z) & \text{for} \quad 0 \le z < \widetilde{d}_2, \\ F_{\frac{c_1 c_2}{M_1}} \circ f^r_{c_2,\, d_2} \circ F_{M_1} \circ f^l_{c_1,\, d_1}(z) & \text{for} \quad \widetilde{d}_2 \le z < d_1, \\ F_{\frac{c_1 c_2}{M_1}} \circ f^r_{c_2,\, d_2} \circ F_{M_1} \circ f^r_{c_1,\, d_1}(z) & \text{for} \quad d_1 \le z \le 1. \end{cases}$$

In the sequel, to simplify the notation, we write G for $G_{c_1,\, d_1,\, c_2,\, \widetilde{d}_2,\, M_1,\, d_2}$. The following is the main result of this article.

Theorem 1. *Let $f \in \mathbb{H}^{1+Z_\gamma},\ \gamma > 0,\ t \in I(p_n/q_n)$ be circle homeomorphisms from the family (2) with rational rotation number $\rho_t = p/q$ of rank n. Then, there are constant $C > 0$ and natural number n_0 such that, for all $n \ge n_0$ the following inequalities hold:*

$$\|\mathbf{f}_n - G\|_{C^1([0,1]\setminus\{d_1,\, \widetilde{d}_2\})} \le \frac{C}{n^\gamma}, \quad \text{when} \quad \gamma > 1/2,$$

$$\|\mathbf{f}_n - G\|_{C^1([0,1]\setminus\{d_1,\, \widetilde{d}_2\})} \le \frac{C}{n^\gamma},\ \ \ \|\mathbf{f}''_n - G''\|_{C^0([0,1]\setminus\{d_1,\, \widetilde{d}_2\})} \le \frac{C}{n^{\gamma-1}}, \quad \text{when} \quad \gamma > 1.$$

Note that similar result is obtained by D. Khmelev [9] for the $C^{2+\varepsilon}$-smooth circle homeomorphisms with two break points.

Previous results and proof of the main theorem

Consider a circle homeomorphism f with two break points. Let $A=\left(\frac{p_1}{q_1},\, \frac{p_2}{q_2}\right)$ be Farey interval such that $\rho(f)\in A$. See [8] for the definition and the main properties of Farey intervals. Take an arbitrary point $x_0\in\mathbb{S}^1$ and consider orbit $\{x_i=f^i(x_0),\ 0\le i\le q_1+q_2\}$. We denote the intervals $[x_0,\, x_{q_1}]$ and $[x_{q_2},\, x_0]$ by $\Delta_0^{(1)}$ and $\Delta_0^{(2)}$. Also denote the images on these intervals under the action of f by $\Delta_i^{(1)}$ and $\Delta_j^{(2)}$:

$$\Delta_i^{(1)}=f^i\Delta_0^{(1)},\quad \Delta_j^{(2)}=f^j\Delta_0^{(2)}.$$

The following proposition was proved in [9].

Proposition 3.([9]) Suppose $\rho(f)\in\left(\frac{p_1}{q_1},\, \frac{p_2}{q_2}\right)$. The trajectory $\{x_i=f^i(x_0),\ 0\le i\le q_1+q_2\}$ forms a partition of the circle consisting of intervals

$$\Delta_i^{(1)},\ 0\le i\le q_2;\quad \Delta_j^{(2)},\ 0\le j\le q_1.$$

Denote $v=|\log\sigma_1^2|+|\log\sigma_2^2|+Var_{S^1}\log f',$ and $\lambda=(1+e^{-v})^{-1/2}<1$. The following lemma shows that the lengths of the intervals $\Delta_i^{(1)}$ and $\Delta_j^{(2)}$ are exponentially small.

Lemma 1.([9]) Assume that $\rho(f)\in\left[\frac{p_1}{q_1},\, \frac{p_2}{q_2}\right]$. Suppose the expression of $\frac{p_1}{q_1}$ to the continued fraction has length n: $\frac{p_1}{q_1}=[k_1,\, k_2,\, ...,k_n],\ k_n\ge 2$. Then

$$|\Delta_i^{(1)}|,\, |\Delta_j^{(2)}|\le Const\lambda^n,$$

for all $0\le i\le q_2$ and $0\le j\le q_1$.

In the case where the circle map f with rational rotation number $\frac{p_n}{q_n}:=\frac{p}{q}$ of rank n has a single break point x_b, we also denote by $\Delta_0^{(n)}=[y_1,\, y_2]$ the interval of the periodic trajectory containing break point x_b. Then renormalization map $\mathbf{f}_n$ corresponding to the return map $f^{q_n}:\Delta_0^{(n)}\to\Delta_0^{(n)}$ is represented as a composition $\mathbf{f}_n=F_2\circ F_1$ of two functions F_1 and F_2, corresponding to maps $f:\Delta_0^{(n)}\to\Delta_1^{(n)}$ and $f^{q_n-1}:\Delta_1^{(n)}\to\Delta_{q_n}^{(n)}=\Delta_0^{(n)}$. In this case the intervals $\Delta_i^{(n)},\ 1\le i\le q_n-1$ do not cover the break point x_b and consequently define the following quantity

$$m_n=\exp\left\{\sum_{i=1}^{q_n-1}\frac{f'(b_i)-f'(a_i)}{f'(b_i)}\right\},\quad\text{where}\quad a_i=f^i(y_1),\ b_i=f^i(y_2),\ 0\le i\le q_n-1.$$

In [1], it was shown that the functions F_1 and F_2 are approximated by $f_{c,\,d}^{l,\,r}$ and F_{m_n}, respectively. More precisely, we have the following

Lemma 2.([1]) *Let $f\in\mathbb{H}^{1+Z_\gamma},\ \gamma>0,\ \ t\in I(p_n/q_n)$ be circle homeomorphisms from the family (2) with rational rotation number $\rho_t=p/q$ of rank n and with a break point x_b. Then, there are constant $C>0$ and natural number n_0 such that, for all $n\ge n_0$ the following inequalities hold:*

$$\|F_2-F_{m_n}\|_{C^1([0,1]\setminus\{d_1\})}\le\frac{C}{n^\gamma},\quad\text{when}\quad\gamma>1/2,$$

$$\|F_2-F_{m_n}\|_{C^1([0,1]\setminus\{d_1\})}\le\frac{C}{n^\gamma},\quad\|F_2''-F_{m_n}''\|_{C^0([0,1]\setminus\{d_1\})}\le\frac{C}{n^{\gamma-1}},\quad\text{when}\quad\gamma>1.$$

Lemma 3.([1]) *Let $f\in\mathbb{H}^{1+Z_\gamma},\ \gamma>0,\ \ t\in I(p_n/q_n)$ be circle homeomorphisms from the family (2) with rational rotation number $\rho_t=p/q$ of rank n and with a break point x_b. Then, there are constant $C>0$ and natural number n_0 such that, for all $n\ge n_0$ the following inequalities hold:*

$$\|F_1-f^{l,\,r}\|_{C^1([0,1]\setminus\{d_1\})}\le C\lambda^n,\quad\text{when}\quad\gamma>1/2,$$

$$\|F_1-f^{l,\,r}\|_{C^2([0,1]\setminus\{d_1\})}\le C\lambda^n,\quad\text{when}\quad\gamma>1.$$

We now return to the case where the circle map f from the family (2) has two break points and rational rotation number $\frac{p}{q}$ of rank n. Assume that the intervals $\Delta_i^{(n)},\ i=r_1,...,r_2-1$ do not cover break points $x_b^{(1)}$ and $x_b^{(2)}$. Define renormalized coordinates of the map $f^{r_2-r_1}:\Delta_{r_1}^{(n)}\to\Delta_{r_2}^{(n)}$ in the new coordinates by

$$\mathcal{F}^{\,r_1,\,r_2}(z)=\frac{f^{r_2-r_1}((1-z)f^{r_1}y_1+zf^{r_1}y_2)-f^{r_2}y_1}{|\Delta_{r_2}^{(n)}|},\ \ z\in[0,\,1].$$

Denote

$$M(r_1;r_2)=\exp\left(\sum_{i=r_1}^{r_2-1}\int_{\Delta_i}\frac{f'(b_i)-f'(a_i)}{f'(b_i)}dy\right),\quad \text{where}\quad a_i=f^i(y_1),\ b_i=f^i(y_2),\ 0\le i\le q_n-1.$$

The following lemma is proved similarly to Lemma 2.

Lemma 4. *Let $f\in\mathbb{H}^{1+Z_\gamma},\ \gamma>0,\ t\in I(p_n/q_n)$ be circle homeomorphisms from the family (2) with rational rotation number $\rho_t=p/q$ of rank n and with two break points $x_b^{(1)}$ and $x_b^{(2)}$. Then, there are constant $C>0$ and natural number n_0 such that, for all $n\ge n_0$ the following inequalities hold:*

$$\|\mathcal{F}^{\,r_1,\,r_2}-F_{M(r_1;r_2)}\|_{C^1([0,1]\setminus\{d\})}\le\frac{C}{n^\gamma},\quad\text{when}\quad\gamma>0,$$

$$\|\mathcal{F}^{\,r_1,\,r_2}-F_{M(r_1;r_2)}\|_{C^1([0,1]\setminus\{d\})}\le\frac{C}{n^\gamma},\quad\|(\mathcal{F}^{\,r_1,\,r_2})''-F''_{M(r_1;r_2)}\|_{C^0([0,1]\setminus\{d\})}\le\frac{C}{n^{\gamma-1}},\quad\text{when}\quad\gamma>1.$$

Proof of Theorem 1. It is clear that the map $\mathbf{f}_n$ can be represented as

$$\mathbf{f}_n=\mathcal{F}^{\,r+1,\,q}\circ\mathcal{F}^{\,r,\,r+1}\circ\mathcal{F}^{\,1,\,r}\circ\mathcal{F}^{\,0,\,1}. \tag{2}$$

Then by Lemma 3 the map $\mathcal{F}^{\,0,\,1}$ is approximated by $f^{l,r}_{c_1,d_1}$ and $\mathcal{F}^{\,r,\,r+1}$ is approximated by $f^{l,r}_{c_2,d_2}$. On the other hand, Lemma 4 implies that $\mathcal{F}^{\,1,\,r}$ is close to F_{M_1} and $\mathcal{F}^{\,r+1,\,q}$ is close to F_{M_2}. To complete the proof of the theorem, we only need to compare M_1M_2 with c_1c_2.

By Propositions 1 and 2, the function f' is absolute continuous and $f''\in L_p$ for every $p>1$ in the case $\gamma\in(1/2,\,1]$ and it is differentiable in the case $\gamma>1$. Consequently,

$$\frac{f'(b_i)-f'(a_i)}{f'(b_i)}dy=\int_{\Delta_i}\frac{f''(y)}{f'(y)}dy+\int_{\Delta_i}\frac{f''(y)}{f'(y)}\left(\int_{a_i}^{y}\frac{f''(t)}{f'(t)}dt\right)dy.$$

By Lemma 1, $|\Delta_i^{(n)}|\le C\lambda^n$ for $i=0,1,...,(q_n-1)$, and therefore (sse also [3])

$$|\int_{\Delta_i}\frac{f''(y)}{f'(y)}\left(\int_{a_i}^{y}\frac{f''(t)}{f'(b_i)}dt\right)dy|=O(\lambda_1^n\int_{\Delta_i}|\frac{f''(y)}{f'(y)}dy|),\ \ \lambda_1=\lambda^{1-\frac{1}{p}}.$$

Now we estimate M_1M_2 as

$$\log M_1M_2=\sum_{i=1}^{q-1}\int\limits_{\Delta_i^{(n)}}\frac{f''(y)}{2f'(y)}dy-\int\limits_{\Delta_0^{(n)}}\frac{f''(y)}{2f'(y)}dy-\int\limits_{\Delta_1^{(n)}}\frac{f''(y)}{2f'(y)}dy+O(\lambda_1^n)=$$

$$=\log c_1c_2-\int\limits_{\Delta_0^{(n)}}\frac{f''(y)}{2f'(y)}dy-\int\limits_{\Delta_1^{(n)}}\frac{f''(y)}{2f'(y)}dy+O(\lambda_1^n).$$

Applying Lemma 1 again, we obtain

$$\left|\int\limits_{\Delta_0^{(n)}}\frac{f''(y)}{2f'(y)}dy+\int\limits_{\Delta_1^{(n)}}\frac{f''(y)}{2f'(y)}dy\right|=O(\lambda_1^n).$$

Hence, $M_1M_2 = c_1c_2 + O(\lambda_1^n)$. Moreover, M_1 and M_2 are bounded.

It can be easily verified that functions $f^l_{c,d}$, $f^r_{c,d}$ and F_M have the following useful properties:

$$F_M \circ F_N = F_{MN}, \;\; f^l_{c,d}(d) = f^r_{c,d}(d) = F_{1/c^2}(d),$$

for all M, N, c, d. These properties and relation (2), together with Lemmas 3 and 4 imply the assertions of the Theorem 1.

REFERENCES

1. H. Akhadkulov, T.Y. Ying, A.S. Begmatov, The behavior of renormalizations of circle maps with rational rotation numbersand with Zygmund conditions, Malaysian Journal of Fundamental and Applied Sciences, 2020, Vol. 16, No. 4, pp. 444-449.

2. H. Akhadkulov, M. Salmi Md Noorani and S. Akhatkulov, Renormalization of circle diffeomorphisms with a break-type singularity, Nonlinearity 30, pp. 2687-2717, (2017).

3. A.S. Begmatov, A.A. Dzhalilov and D.Mayer, Renormalizations of circle homeomorphisms with a single break point. Discrete and Continuous Dynamical Systems, Series- A, Volume 34, Number 11, 2014, pp. 4487-4513.

4. M. Herman, Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. Inst. Hautes Etudes Sci. Publ. Math., 49, 5-234 (1979).

5. Y. Katznelson and D. Ornstein, The differentiability of the conjugation of certain diffeomorphisms of the circle. Ergod. Theor. Dyn. Syst., 9, 643-680 (1989).

6. Y. Katznelson and D. Ornstein, The absolute continuity of the conjugation of certain diffeomorphisms of the circle. Ergod. Theor. Dyn. Syst., 9, 681-690 (1989).

7. K. Khanin and Ya. Sinai, Smoothness of conjugacies of diffeomorphisms of the circle with rotations. Russ. Math. Surv., 44, 69-99 (1989), translation of Usp. Mat. Nauk, 44, 57-82 (1989).

8. K.M. Khanin and E.B. Vul, Circle homeomorphisms with weak discontinuities. In Dynamical systems and statistical mechanics. American Mathematical Society, Providence, RI, 1991, pp. 57-98.

9. D.V. Khmelev, Rational rotation numbers for homeomorphisms with several break-type singularities. Ergodic Theory and Dynamical Systems. 2005; 25(2), pp. 553-592.

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Boundary Control Problem for the Heat Equation with a Nonlocal Boundary Condition

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Abstract. In this paper, we study a boundary control problem for the one-dimensional heat equation with a nonlocal boundary condition. The control is applied at one end of the domain, while the temperature at the opposite boundary is linked to the temperature at a fixed interior point. The control objective is formulated as an integral condition prescribing the average temperature of the rod. Using the method of separation of variables, the problem is reduced to a Volterra integral equation of the first kind. The existence of an admissible control function is proved by means of the Laplace transform method.

MSC (2020): 35K05; 35K15.

Key words: heat equation, initial-boundary problem, spectral problem, admissible control, rod.

Introduction

Control problems for the heat equation play a fundamental role in the theory of parabolic partial differential equations and have numerous applications in thermal processes. The foundations of optimal control theory for parabolic-type equations were developed by Fattorini and Friedman [1,2]. Later, Egorov [3] investigated control problems in infinite-dimensional spaces, extending Pontryagin's maximum principle to certain classes of equations in Banach spaces and establishing a bang-bang principle under appropriate conditions.

Boundary control problems for parabolic equations in multidimensional domains with piecewise smooth boundaries were studied in [4,5], where estimates for the minimum time required to reach a prescribed average temperature were obtained. Mathematical models of thermocontrol processes for parabolic equations were considered in [6], while control problems for the heat equation in three-dimensional domains were analyzed in [7].

Control problems in bounded one- and two-dimensional domains were investigated in [8–11]. In these works, estimates for the minimum time needed to achieve a given average temperature were derived, and the existence of a control function was established by means of the Laplace transform method.

Comprehensive treatments of optimal control theory for parabolic equations can be found in the monographs by Lions and Fursikov [12,13]. Numerical optimization methods and optimal control for second-order parabolic equations were studied in [14], and practical applications were presented in [15].

The aim of this work is to study the boundary control problem for the heat equation under fixed average temperature conditions. By applying the separation of variables method, the problem is reduced to a Volterra integral equation of the first kind. The solution of this integral equation is analyzed using the Laplace transform method, which allows us to precisely construct the control function and determine its existence.

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Statement of the Problem

In this paper, we consider the following heat equation in the domain $\Omega_T := (0,1) \times (0,\infty)$:

$$u_t(x,t) = u_{xx}(x,t), \quad (x,t) \in \Omega_T, \tag{1}$$

with a nonlocal boundary condition of Bitsadze-Samarskii type

$$u(0,t) = \nu(t), \quad u(1,t) = u(x_0,t), \quad t \geq 0, \quad x_0 \in (0,1), \tag{2}$$

and the initial condition

$$u(x,0) = 0, \quad 0 \leq x \leq 1. \tag{3}$$

Here $\nu(t)$ is a boundary control function applied at the left end of the rod, while $x_0 \in (0,1)$ is a fixed interior point.

From the physical point of view, equation (1) describes the heat conduction process in a thin homogeneous rod of unit length. The function $u(x,t)$ represents the temperature at position x and time t. The initial condition (3) means that the rod is initially at zero temperature. The boundary condition $u(0,t) = \nu(t)$ corresponds to an external heating or cooling device located at the left end of the rod. The function $\nu(t)$ plays the role of a time-dependent boundary heat source and is regarded as the control input.

The nonlocal condition $u(1,t) = u(x_0,t)$ indicates a thermal coupling between the right end of the rod and an interior point $x = x_0$. Physically, this may model a situation in which the temperature at the right boundary is required to match the temperature at some prescribed internal location, for example, due to a feedback mechanism, a thermal sensor placed at $x = x_0$, or a design constraint that equalises the temperature at two distinct points. Such nonlocal constraints appear in various engineering applications, including thermal control systems, thermal imaging, and in problems with internal monitoring or symmetry requirements. The presence of this nonlocal condition significantly influences the heat distribution along the rod and introduces mathematical challenges that differ from those of classical boundary-value problems.

If the control function $\nu(t) \in W_2^1(\mathbb{R}_+)$ satisfies the conditions $\nu(0) = 0$ and $|\nu(t)| \leq 1$ on the half-line $t \geq 0$, then we call it an *admissible control*.

We note that the regularity condition $\nu \in W_2^1(\mathbb{R}_+)$ will be justified later in Section 4 as a consequence of the solvability of the control problem.

We now formulate the main control problem considered in this paper.

Control Problem. *For the given function $\phi(t)$, find an admissible control $\nu(t)$ such that the solution $u(x,t)$ of the initial-boundary value problem (1)-(3) exists and satisfies the integral condition*

$$\int_0^1 u(x,t)\,dx = \phi(t), \quad t \geq 0. \tag{4}$$

Condition (4) has a clear physical interpretation. The integral $\int_0^1 u(x,t)\,dx$ represents the total (or average) thermal energy of the rod at time t. Therefore, the control objective is to regulate the boundary temperature $\nu(t)$ in such a way that the overall heat content of the rod follows a prescribed evolution $\phi(t)$.

This type of control problem naturally arises in thermal engineering, where it is often impossible or unnecessary to control the temperature at every point of the domain. Instead, one aims to achieve a desired average temperature profile by acting on the system through boundary inputs. The presence of the nonlocal constraint $u(1,t) = u(x_0,t)$ reflects additional physical coupling between the boundary and an interior point, which may arise from design specifications, sensor-based feedback, or symmetry requirements in thermal systems. This coupling introduces new mathematical challenges in the analysis and synthesis of the control process, distinguishing it from classical boundary control problems.

For any constant $M > 0$, we denote by $W(M)$ the set of functions $\phi \in W_2^2(-\infty,+\infty)$, $\phi(t) = 0$ for all $t \leq 0$ which satisfy the condition

$$\|\phi\|_{W_2^2(R_+)} \leq M.$$

We present the following main theorem.

Theorem 1. *There exists $M > 0$ such that for any function $\phi \in W(M)$ the solution $\nu(t)$ of the equation (4) exists and it satisfies condition $|\nu(t)| \leq 1$.*

Integral equation for control function

In this section, we show how the considered control problem can be reduced to a Volterra integral equation of the first kind. For this purpose, we first eliminate the nonhomogeneous boundary conditions and then solve the corresponding auxiliary problem by the Fourier method.

We consider the spectral problem

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < 1,$$

with the nonlocal boundary conditions

$$X(0) = 0, \quad X(1) = X(x_0), \quad x_0 \in (0, 1).$$

It is easy to see that for $\lambda \leq 0$ this problem admits only the trivial solution. Therefore, we restrict ourselves to the case $\lambda > 0$. In this case, the spectrum consists of two countable sequences of eigenvalues given by

$$\lambda_{k,1} = \left(\frac{(2k-1)\pi}{1+x_0}\right)^2, \qquad \lambda_{k,2} = \left(\frac{2k\pi}{1-x_0}\right)^2, \quad k \in \mathbb{N}, \quad x_0 \in (0,1).$$

The corresponding eigenfunctions have the form

$$X_{k,1}(x) = \sin\big(\sqrt{\lambda_{k,1}}\, x\big), \qquad X_{k,2}(x) = \sin\big(\sqrt{\lambda_{k,2}}\, x\big), \quad k \in \mathbb{N}.$$

The system of eigenfunctions $\big\{X_{k,1}(x),\, X_{k,2}(x)\big\}_{k=1}^{\infty}$ is orthogonal and complete in the space $L_2(0,1)$. This system will be used to construct the solution of the auxiliary problem by the Fourier method (see [16]).

Let B be an arbitrary Banach space and let $T > 0$. By $C([0,T] \to B)$ we denote the Banach space of all continuous mappings $u : [0,T] \to B$ equipped with the norm

$$\|u\| = \max_{0 \leq t \leq T} \|u(t)\|_B.$$

By $\widetilde{W}_2^1(\Omega)$ we denote the subspace of the Sobolev space $W_2^1(\Omega)$ consisting of functions whose trace on $\partial\Omega$ is equal to zero. Since $\widetilde{W}_2^1(\Omega)$ is closed in $W_2^1(\Omega)$, the sum of a series of functions from $\widetilde{W}_2^1(\Omega)$ converging in the $W_2^1(\Omega)$-norm also belongs to $\widetilde{W}_2^1(\Omega)$, where $\Omega = (0,1)$.

Definition 1. *By the solution of the problem (1)-(3) we mean a function $u(x,t)$, represented in the form*

$$u(x,t) = \nu(t) - w(x,t), \tag{5}$$

where the function $w(x,t)$ is a generalized solution from the class $C([0,T] \to \widetilde{W}_2^1(\Omega))$ of the following problem:

$$w_t(x,t) - w_{xx}(x,t) = \nu'(t),$$

with homogeneous initial and boundary conditions

$$w(0,t) = 0, \quad w(1,t) = w(x_0,t), \quad w(x,0) = 0.$$

We set

$$p_j := \|X_{k,j}\|_{L_2(0,1)} = \sqrt{\int_0^1 X_{k,j}^2 dx} > 0, \quad j = 1,2, \quad k \in \mathbb{N}.$$

Thus, we obtain (see [17])

$$w(x,t) = \sum_{j=1}^{2} \frac{1}{p_j} \sum_{k=1}^{\infty} a_{k,j} \left(\int_0^t e^{-\lambda_{k,j}(t-s)}\, \nu'(s)\, ds\right) \sin\big(\sqrt{\lambda_{k,j}}\, x\big), \tag{6}$$

where

$$a_{k,j} = \frac{1 - \cos\sqrt{\lambda_{k,j}}}{p_j \sqrt{\lambda_{k,j}}}, \quad j = 1,2, \quad k \in \mathbb{N}.$$

Note that the class $C([0,T] \to \widetilde{W}_2^1(\Omega))$ is a subset of the class $W_2^{1,0}(\Omega_T)$, which was considered in monograph [18] for defining a solution to the problem homogeneous boundary conditions (see the corresponding uniqueness theorem in Ch. III, Theorem 3.2, pp. 173-176). Therefore, the above introduced generalized solution is also a generalized solution in the sense of [18]. However, unlike a solution from the class $W_2^{1,0}(\Omega_T)$, which is guaranteed to have a trace for almost everywhere $t \in [0,T]$, a solution from a class $C([0,T] \to \widetilde{W}_2^1(\Omega))$ continuously depends on $t \in [0,T]$ in the metric $L_2(\Omega)$.

Lemma 1. *Let $\nu \in W_2^1(\mathbb{R}_+)$ and $\nu(0)=0$. Then the function*

$$u(x,t) = \sum_{j=1}^{2} \frac{1}{p_j} \sum_{k=1}^{\infty} a_{k,j}\, \lambda_{k,j} \left(\int_0^t e^{-\lambda_{k,j}(t-s)}\, \nu(s)\, ds \right) \sin\big(\sqrt{\lambda_{k,j}}\, x\big), \tag{7}$$

is the solution of the initial-boundary value problem (1)-(3).

Proof. Using representations (5) and (6), we rewrite the solution of problem (1)-(3) in the form

$$u(x,t) = \nu(t) - \sum_{j=1}^{2} \frac{1}{p_i} \sum_{k=1}^{\infty} a_{k,j} \left(\int_0^t e^{-\lambda_{k,j}(t-s)}\, \nu'(s)\, ds \right) \sin\big(\sqrt{\lambda_{k,j}}\, x\big).$$

We prove that the function $w(x,t)$ defined by series (6) belongs to the class $C([0,T] \to \widetilde{W}_2^1(\Omega))$. It is sufficient to show that the spatial derivative $w_x(\cdot,t)$ belongs to $L_2(0,1)$ for every $t \in [0,T]$ and depends continuously on t with respect to the $L_2(0,1)$-norm.

Differentiating series (6) with respect to x, we obtain

$$w_x(x,t) = \sum_{j=1}^{2} \frac{1}{p_j} \sum_{k=1}^{\infty} a_{k,j} \sqrt{\lambda_{k,j}} \left(\int_0^t e^{-\lambda_{k,j}(t-s)}\, \nu'(s)\, ds \right) \cos\big(\sqrt{\lambda_{k,j}}\, x\big).$$

By Parseval's equality, the $L_2(0,1)$-norm of $w_x(\cdot,t)$ is given by

$$\|w_x(\cdot,t)\|^2_{L_2(0,1)} = \sum_{j=1}^{2} \sum_{k=1}^{\infty} \frac{a_{k,j}^2\, \lambda_{k,j}}{p_j^2} \left(\int_0^t e^{-\lambda_{k,j}(t-s)}\, \nu'(s)\, ds \right)^2.$$

Using the Cauchy-Schwarz inequality, we estimate the integral term as

$$\left(\int_0^t e^{-\lambda_{k,j}(t-s)}\, \nu'(s)\, ds \right)^2 \le \int_0^t e^{-2\lambda_{k,j}(t-s)}\, ds \int_0^t |\nu'(s)|^2\, ds \le \frac{1}{2\lambda_{k,j}} \|\nu'\|^2_{L_2(0,T)}.$$

Substituting this estimate into the previous expression, we obtain

$$\|w_x(\cdot,t)\|^2_{L_2(0,1)} \le \frac{1}{2} \|\nu'\|^2_{L_2(0,T)} \sum_{j=1}^{2} \sum_{k=1}^{\infty} \frac{a_{k,j}^2}{p_j^2}.$$

Recalling that

$$a_{k,j} = \frac{1-\cos\sqrt{\lambda_{k,j}}}{p_j \sqrt{\lambda_{k,j}}},$$

and taking into account the asymptotic behavior $\lambda_{k,j} \sim k^2$ as $k \to \infty$, we conclude that the series

$$\sum_{j=1}^{2} \sum_{k=1}^{\infty} \frac{a_{k,j}^2}{p_j^2}$$

is convergent. Hence, there exists a constant $C>0$ such that

$$\|w_x(\cdot,t)\|^2_{L_2(0,1)} \le C\, \|\nu'\|^2_{L_2(0,T)}, \quad t \in [0,T].$$

Therefore, $w_x(\cdot,t) \in L_2(0,1)$ for all $t \in [0,T]$, and the mapping $t \mapsto w_x(\cdot,t)$ is continuous in the $L_2(0,1)$-norm. This implies that

$$w \in C([0,T] \to \widetilde{W}_2^1(\Omega)).$$

Finally, the fact that $w(x,t)$ is a generalized solution in the sense of the integral identity (3.5) of monograph [18] follows directly from Parseval's equality. The lemma is proved. □

Using the integral condition (4) and the solution (7), we can write

$$\phi(t) = \int\limits_0^1 u(x,t)\,dx = \sum_{j=1}^{2} \frac{1}{p_j} \sum_{k=1}^{\infty} a_{k,j}\,\sqrt{\lambda_{k,j}}\,(1-\cos\sqrt{\lambda_{k,j}}) \left(\int_0^t e^{-\lambda_{k,j}(t-s)}\,\nu(s)\,ds\right).$$

Set

$$K(t) = \sum_{j=1}^{2}\sum_{k=1}^{\infty} \Psi_{k,j}\,e^{-\lambda_{k,j}t}, \quad t>0, \tag{8}$$

where

$$\Psi_{k,j} = \frac{a_{k,j}}{p_j}\,\sqrt{\lambda_{k,j}}\,\big(1-\cos\sqrt{\lambda_{k,j}}\big), \quad j=1,2, \quad k\in\mathbb{N}.$$

Thus, we have the following Volterra integral equation of the first kind

$$\int\limits_0^t K(t-s)\,\nu(s)ds = \phi(t), \quad t>0. \tag{9}$$

Equation (9) is a Volterra integral equation of the first kind with a kernel $K(t)$ defined by (8). In order to investigate the solvability of this equation and the regularity properties of the control function $\nu(t)$, we first study the behaviour of the kernel $K(t)$ for $t>0$.

Lemma 2. *For the kernel function $K(t)$ defined by (8), the following estimate holds:*

$$K(t) \;\le\; \frac{C}{\sqrt{t}}, \quad 0<t\le 1,$$

where $C>0$ is a constant independent of t.

By definition, the kernel $K(t)$ has the representation

$$K(t) = \sum_{j=1}^{2}\sum_{k=1}^{\infty} \frac{a_{k,j}}{p_j}\,\sqrt{\lambda_{k,j}}\,\big(1-\cos\sqrt{\lambda_{k,j}}\big)\,e^{-\lambda_{k,j}t}.$$

Recall that the coefficients $a_{k,i}$ are given by

$$a_{k,j} = \frac{1-\cos\sqrt{\lambda_{k,j}}}{p_j\sqrt{\lambda_{k,j}}}, \quad j=1,2, \quad k\in\mathbb{N}.$$

Substituting this expression into the series for $K(t)$, we obtain

$$K(t) = \sum_{j=1}^{2}\sum_{k=1}^{\infty} \frac{(1-\cos\sqrt{\lambda_{k,j}})^2}{p_j^2}\,e^{-\lambda_{k,j}t}.$$

Using the elementary inequality

$$0 \le 1-\cos y \le 2, \quad y\in\mathbb{R},$$

we immediately get

$$(1-\cos\sqrt{\lambda_{k,j}})^2 \le 4.$$

Hence,

$$K(t) \le \sum_{j=1}^{2}\sum_{k=1}^{\infty} \frac{4}{p_j^2}\, e^{-\lambda_{k,j} t}.$$

Since the constants p_j do not depend on k, there exists a constant $C_1 > 0$ such that

$$K(t) \le C_1 \sum_{j=1}^{2}\sum_{k=1}^{\infty} e^{-\lambda_{k,j} t}.$$

Taking into account that the eigenvalues satisfy the asymptotic relation

$$\lambda_{k,j} \sim C_2 k^2, \quad k \to \infty,$$

we use the classical estimate

$$\sum_{k=1}^{\infty} e^{-ck^2 t} \le \frac{C_3}{\sqrt{t}}, \quad 0 < t \le 1.$$

Combining the above inequalities, we conclude that

$$K(t) \le \frac{C}{\sqrt{t}}, \quad 0 < t \le 1,$$

where $C > 0$ is a constant independent of t. The lemma is proved. □

Main result

We solve the Volterra integral equation (9) by means of the Laplace transform. Recall that the Laplace transform of the control function $\nu(t)$ is defined by

$$\widetilde{\nu}(p) = \int_0^{\infty} e^{-pt}\nu(t)\, dt, \quad \Re p > 0.$$

Applying the Laplace transform to both sides of equation (9) and using the convolution theorem, we obtain

$$\widetilde{\phi}(p) = \widetilde{K}(p)\, \widetilde{\nu}(p).$$

Hence,

$$\widetilde{\nu}(p) = \frac{\widetilde{\phi}(p)}{\widetilde{K}(p)}, \quad p = \xi + i\tau,\ \xi > 0,\ \tau \in \mathbb{R}.$$

By the inverse Laplace transform, the control function $\nu(t)$ can be written as

$$\nu(t) = \frac{1}{2\pi i}\int_{\xi - i\infty}^{\xi + i\infty} \frac{\widetilde{\phi}(p)}{\widetilde{K}(p)} e^{pt}\, dp = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\widetilde{\phi}(\xi + i\tau)}{\widetilde{K}(\xi + i\tau)} e^{(\xi + i\tau)t}\, d\tau. \tag{10}$$

Lemma 3. *For $\xi > 0$ and $\tau \in \mathbb{R}$ the estimate*

$$|\widetilde{K}(\xi + i\tau)| \ge \frac{C_\xi}{\sqrt{1+\tau^2}}$$

holds, where $C_\xi > 0$ is a constant depending only on ξ.

Proof. Using the definition of $K(t)$, we write

$$\widetilde{K}(p) = \int_0^{\infty} K(t) e^{-pt}\, dt = \sum_{j=1}^{2}\sum_{k=1}^{\infty} \frac{\Psi_{k,j}}{p + \lambda_{k,j}}.$$

For $p=\xi+i\tau$, we have

$$\widetilde{K}(\xi+i\tau)=\sum_{j=1}^{2}\sum_{k=1}^{\infty}\frac{\Psi_{k,j}(\xi+\lambda_{k,j})}{(\xi+\lambda_{k,j})^2+\tau^2}-i\tau\sum_{j=1}^{2}\sum_{k=1}^{\infty}\frac{\Psi_{k,j}}{(\xi+\lambda_{k,j})^2+\tau^2}.$$

Therefore,

$$\Re\widetilde{K}(\xi+i\tau)=\sum_{j=1}^{2}\sum_{k=1}^{\infty}\frac{\Psi_{k,j}(\xi+\lambda_{k,j})}{(\xi+\lambda_{k,j})^2+\tau^2},$$

$$\Im\widetilde{K}(\xi+i\tau)=-\tau\sum_{j=1}^{2}\sum_{k=1}^{\infty}\frac{\Psi_{k,j}}{(\xi+\lambda_{k,j})^2+\tau^2}.$$

Since

$$(\xi+\lambda_{k,j})^2+\tau^2\le((\xi+\lambda_{k,j})^2+1)(1+\tau^2),$$

we obtain

$$\frac{1}{(\xi+\lambda_{k,j})^2+\tau^2}\ge\frac{1}{1+\tau^2}\frac{1}{(\xi+\lambda_{k,j})^2+1}.$$

Consequently,

$$|\Re\widetilde{K}(\xi+i\,\tau)|=\sum_{j=1}^{2}\sum_{k=1}^{\infty}\frac{\Psi_{k,j}\,(\xi+\lambda_{k,j})}{(\xi+\lambda_{k,j})^2+\tau^2}\ge$$

$$\ge\frac{1}{1+\tau^2}\sum_{j=1}^{2}\sum_{k=1}^{\infty}\frac{\Psi_{k,j}\,(\xi+\lambda_{k,j})}{(\xi+\lambda_{k,j})^2+1}=\frac{C_{1,\xi}}{1+\tau^2},$$

and

$$|\Im\widetilde{K}(\xi+i\,\tau)|=|\tau|\sum_{j=1}^{2}\sum_{k=1}^{\infty}\frac{\Psi_{k,j}}{(\xi+\lambda_{k,j})^2+\tau^2}\ge$$

$$\ge\frac{|\tau|}{1+\tau^2}\sum_{j=1}^{2}\sum_{k=1}^{\infty}\frac{\Psi_{k,j}}{(\xi+\lambda_{k,j})^2+1}=\frac{C_{2,\xi}\,|\tau|}{1+\tau^2},$$

where

$$C_{1,\xi}=\sum_{j,k}\frac{\Psi_{k,j}(\xi+\lambda_{k,j})}{(\xi+\lambda_{k,j})^2+1},\quad C_{2,\xi}=\sum_{j,k}\frac{\Psi_{k,j}}{(\xi+\lambda_{k,j})^2+1}.$$

Hence,

$$|\widetilde{K}(\xi+i\tau)|^2\ge\frac{\min(C_{1,\xi}^2,C_{2,\xi}^2)}{1+\tau^2},$$

which proves the lemma. □

Then, proceed to the limit as $\xi\to 0$ from (10), we obtain the equality

$$\nu(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\widetilde{\phi}(i\,\tau)}{\widetilde{K}(i\,\tau)}\,e^{i\,\tau t}\,d\tau. \tag{11}$$

Lemma 4. ([11]) *Assume that $\phi\in W(M)$. Then for the imaginary part of the Laplace transform of function $\phi(t)$ the following inequality holds:*

$$\int_{-\infty}^{+\infty}|\widetilde{\phi}(i\,\tau)|\sqrt{1+\tau^2}\,d\tau\le C\,\|\phi\|_{W_2^2(R_+)},$$

where $C>0$ is a constant.

Now we prove Theorem 1.

Proof of Theorem 1. First of all, we prove that $\nu \in W_2^1(\mathbb{R}_+)$. Using lemmas 3 and 4, we get the estimate

$$\int_{-\infty}^{+\infty} |\widetilde{\nu}(\tau)|^2(1+|\tau|^2)\,d\tau \;=\; \int_{-\infty}^{+\infty} \left|\frac{\widetilde{\phi}(i\,\tau)}{\widetilde{K}(i\,\tau)}\right|^2 (1+|\tau|^2)\,d\tau \;\le$$

$$\le C_0 \int_{-\infty}^{+\infty} |\widetilde{\phi}(i\,\tau)|^2(1+|\tau|^2)^2\,d\tau \;=\; C_0\|\phi\|^2_{W_2^2(\mathbb{R})},$$

where $C_0 = \min(C_{1,0}, C_{2,0})$.

Besides, we have

$$|\nu(t)-\nu(s)| \;=\; \left|\int_s^t \nu'(y)\,dy\right| \;\le\; \|\nu'\|_{L_2}\,(t-s)^{1/2}.$$

From (11), lemmas 3 and 4, we can write

$$|\nu(t)| \le \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{|\widetilde{\phi}(i\,\tau)|}{|\widetilde{K}(i\,\tau)|}d\tau \le \frac{1}{2\pi C_0}\int_{-\infty}^{+\infty} |\widetilde{\phi}(i\,\tau)|\sqrt{1+\tau^2}d\tau \le$$

$$\le \frac{C}{2\pi C_0}\|\phi\|_{W_2^2(\mathbb{R}_+)} \le \frac{C\,M}{2\pi C_0} = 1,$$

where M is as follows:

$$M = \frac{2\pi C_0}{C}.$$

Conclusion

In this paper, a boundary control problem for the one-dimensional heat equation with a nonlocal boundary condition has been studied. The control objective was to steer the average temperature of the rod to a prescribed function. By applying the Fourier method, the problem was reduced to a Volterra integral equation of the first kind. Using the Laplace transform method, the existence of the admissible control function was proved and its regularity properties were established. The obtained results demonstrate that nonlocal boundary conditions lead to new analytical features that distinguish the problem from classical boundary control settings.

REFERENCES

1. Fattorini H.O., Time-optimal control of solutions of operational differential equations, *SIAM J. Control*, **2** (1965), pp. 49–65.

2. Friedman A., Optimal control for parabolic equations, *J. Math. Anal. Appl.*, **18** (1967), pp. 479–491.

3. Egorov Yu.V., Optimal control in Banach spaces, *Dokl. Akad. Nauk SSSR*, **150** (1967), pp. 241–244. (in Russian)

4. Albeverio S., Alimov Sh. A., On one time-optimal control problem associated with the heat exchange process, *Applied Mathematics and Optimization*, **57** (2008), pp. 58–68.

5. Alimov Sh.A., Dekhkonov F.N., On the time-optimal control of the heat exchange process, *Uzbek Mathematical Journal*, **2** (2019), pp. 4–17.

6. Alimov Sh.A., On the null-controllability of the heat exchange process, *Eurasian Mathematical Journal*, **2** (2011), pp. 5–19.

7. Dekhkonov F.N., On the control problem associated with the heating process, *Mathematical notes of NEFU*, **29** (2022), pp. 62–71.

8. Alimov Sh.A., Dekhkonov F.N., On a control problem associated with fast heating of a thin rod, *Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences*, **2** (2019), pp. 1–14.

9. Chen N., Wang Y. and Yang D., Time-varying bang-bang property of time optimal controls for heat equation and its applications, *Syst. Control Lett.*, **112** (2018), pp. 18–23.

10. Fayazova Z.K., Boundary control of the heat transfer process in the space, *Izv. Vyssh. Uchebe. Zaved. Mat.*, **12** (2019), pp. 82–90.

11. Dekhkonov F.N., Boundary control problem for the heat transfer equation associated with heating process of a rod, *Bulletin of the Karaganda University. Mathematics Series*, **110** (2023), pp. 63–71.

12. Lions J.L., *Contróle optimal de systèmes gouvernés par deséquations aux dérivées partielles*, Dunod Gauthier-Villars, Paris, 1968.

13. Fursikov A.V., *Optimal Control of Distributed Systems, Theory and Applications*, Translations of Math. Monographs, 187, Amer. Math. Soc., Providence, Rhode Island, 2000.

14. Altmüller A., Grüne L., *Distributed and boundary model predictive control for the heat equation*, Technical report, University of Bayreuth, Department of Mathematics, 2012.

15. Dubljevic S., Christofides P.D., *Predictive control of parabolic PDEs with boundary control actuation*, Chemical Engineering Science, **61** (2006), pp. 6239–6248.

16. Ashurov R. R., Kadirkulov B. J., Turmetov B. Kh., On the inverse problem of the Bitsadze-Samarskii type for a fractional parabolic equation, *Probl. Anal. Issues Anal.*, 2023, Issue 3, 20-40

17. Tikhonov A.N. and Samarsky A.A., *Equations of Mathematical Physics*, Nauka, Moscow. 1966.

18. Ladyzhenskaya O.A., Solonnikov V.A. and Uraltseva N.N., *Linear and quasi-linear equations of parabolic type*, Nauka, Moscow. 1967.

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Metallic Ricci soliton on the Sol^3 manifold

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Abstract. In this paper, we study the manifold Sol^3 endowed with its standard left-invariant Riemannian metric together with a polynomial structure. First, we recall the Lie group structure of Sol^3 and compute the main geometric objects associated with the standard metric, including an orthonormal frame, Lie brackets, the Levi-Civita connection, and the Ricci tensor. Then we introduce a polynomial structure satisfying a quadratic relation and investigate its interaction with the Ricci tensor. In particular, we prove that Sol^3 is not an Einstein manifold and that the Ricci operator commutes with the polynomial structure. Finally, we study a metallic Ricci soliton type equation associated with the polynomial structure and obtain an explicit family of solutions for the corresponding vector field. These results provide new examples of curvature structures compatible with polynomial tensor fields on solvable Lie groups.

MSC (2020): 53C30; 53C20; 22E25.

Key words: Sol^3 manifold, metallic structure, Einstein manifold, Ricci tensor, Ricci soliton, metallic Ricci soliton.

Introduction

The geometry Sol^3 is one of Thurston's eight model geometries [1]. It can be realized as a three-dimensional solvable and non-nilpotent Lie group equipped with a left-invariant Riemannian metric. Among homogeneous spaces, Sol^3 occupies a distinguished position because its geometric behavior differs essentially from that of spaces of constant sectional curvature.

A broad account of the geometries of 3-manifolds was given by P. Scott [2]. A projective interpretation of the eight three-dimensional homogeneous geometries, including Sol geometry, can be found in [3]. These foundational results show that Sol^3 provides a natural model for the study of homogeneous Riemannian manifolds with anisotropic geometric properties.

Geometric aspects of curves and surfaces in the homogeneous space Sol^3 are treated in [4]. Further properties of Thurston-type geometries were considered by Erjavec [5]. Together, these works indicate that Sol^3 is a rich geometric setting in which curvature phenomena differ substantially from the Euclidean, spherical, and hyperbolic cases.

Ricci solitons play a central role in differential geometry because they arise as self-similar solutions of the Ricci flow introduced by Hamilton [6]. They also appear as a natural generalization of Einstein metrics, and a broad survey of their geometric significance is available in [7]. In dimension three, homogeneous Ricci solitons were analyzed in detail in [8].

The case of the Lie group Sol^3 was examined in [9], where Ricci soliton structures were investigated. This confirms that Sol^3 is an appropriate framework for the study of Ricci-type equations and related curvature properties on solvable Lie groups.

Another active line of research concerns polynomial structures on Riemannian manifolds. Metallic structures and their basic geometric properties were studied in [10], while related symmetry properties for diagonal metrics were discussed in [11]. For Lie groups endowed with left-invariant metrics, curvature computations are closely connected with the classical formulas established by Milnor [12].

Recent works of A.S. Sharipov and collaborators also deal with curvature-related questions in differential geometry, including isometric properties of surfaces and existence problems for geometric objects with prescribed

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curvature conditions [13], [14]. These contributions further illustrate the continuing interest in curvature and their geometric applications.

Motivated by the above developments, in this paper we study the manifold Sol^3 endowed with its standard left-invariant metric together with a diagonal polynomial structure. We determine the corresponding orthonormal frame, Lie bracket relations, Levi-Civita connection, and Ricci tensor. Next, we examine the interaction between the Ricci operator and the polynomial structure, prove that Sol^3 is not an Einstein manifold, and obtain a decomposition of the Ricci tensor in terms of the tensors g and $g(JX,Y)$. Finally, we derive an explicit family of vector fields satisfying a metallic Ricci soliton type equation on Sol^3.

Preliminaries

In this section, we recall the basic definitions and formulas that will be used throughout the paper.

Definition 1. [2] The manifold Sol^3 is the Lie group on $\mathbb{R}^3$ with the group law

$$(x,y,z)\cdot(x',y',z') = \big(x+e^{-z}x',\ y+e^{z}y',\ z+z'\big).$$

Its matrix representation is given by

$$\begin{pmatrix} e^{-z} & 0 & x \\ 0 & e^{z} & y \\ 0 & 0 & 1 \end{pmatrix}.$$

The standard left-invariant Riemannian metric on Sol^3 is given by

$$g = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2.$$

We consider the orthonormal frame

$$E_1 = e^{-z}\frac{\partial}{\partial x}, \qquad E_2 = e^{z}\frac{\partial}{\partial y}, \qquad E_3 = \frac{\partial}{\partial z}. \tag{1}$$

With respect to the frame in (1), the Lie bracket relations are

$$[E_1,E_2]=0, \qquad [E_1,E_3]=E_1, \qquad [E_2,E_3]=-E_2. \tag{2}$$

Using the Koszul formula together with (2), we obtain the Levi-Civita connection

$$\nabla_{E_1}E_1=-E_3, \qquad \nabla_{E_1}E_3=E_1, \qquad \nabla_{E_2}E_2=E_3, \qquad \nabla_{E_2}E_3=-E_2, \tag{3}$$

and all remaining components vanish, that is, $\nabla_{E_i}E_j=0$.

From the above connection formulas, the Ricci tensor is diagonal in the orthonormal frame. Recall that the Ricci tensor is defined by

$$\mathrm{Ric}(X,Y)=\sum_{k=1}^{n} g(R(E_k,X)Y,E_k).$$

where any smooth vector fields X,Y. Therefore, we have

$$\mathrm{Ric}(E_1,E_1)=0, \qquad \mathrm{Ric}(E_2,E_2)=0, \qquad \mathrm{Ric}(E_3,E_3)=-2, \tag{4}$$

while

$$\mathrm{Ric}(E_i,E_j)=0, \qquad i\neq j.$$

From (3), a direct curvature computation yields

$$K(E_1,E_2)=1, \qquad K(E_1,E_3)=-1, \qquad K(E_2,E_3)=-1.$$

Hence, Sol^3 does not have constant sectional curvature.

Definition 2. [7] A Riemannian manifold (M,g) is called an Einstein manifold if there exists a constant $\lambda\in\mathbb{R}$ such that

$$\mathrm{Ric}=\lambda g. \tag{5}$$

Definition 3. [6] A Riemannian manifold (M, g) is called a Ricci soliton if there exist a smooth vector field V and a constant $\lambda \in \mathbb{R}$ such that

$$\mathcal{L}_V g + 2\mathrm{Ric} = 2\lambda g.$$

If $V = \nabla f$ for some smooth function f, then the Ricci soliton is called a gradient Ricci soliton.

Definition 4. [10] Let (M, g) be a Riemannian manifold. A $(1,1)$-tensor field J is called a metallic structure on the Riemannian manifold if

$$J^2 = pJ + qI. \tag{6}$$

Here p and q are positive integers, and I denotes the identity operator.

Let us denote the roots of the equation $x^2 = px + q$ by

$$\sigma_1 = \frac{p + \sqrt{p^2 + 4q}}{2}, \qquad \sigma_2 = \frac{p - \sqrt{p^2 + 4q}}{2}. \tag{7}$$

In what follows, we consider the tensor field J defined by

$$J(E_1) = \sigma_1 E_1, \qquad J(E_2) = \sigma_1 E_2, \qquad J(E_3) = \sigma_2 E_3, \tag{8}$$

on the othonormal frame of Sol^3 manifold.

For the metric, the relation

$$g(JX, Y) = g(X, JY)$$

holds for all vector fields X, Y.

Main Results

We now derive the principal results of the paper.

Proposition 1. The manifold Sol^3 endowed with the above metric is not an Einstein manifold.

Proof. we have (4) equalities. Now, assume that Sol^3 is an Einstein manifold. Then, by Definition 2, there exists a constant λ such that

$$\mathrm{Ric}(E_i, E_j) = \lambda g(E_i, E_j), \quad i, j = 1, 2, 3. \tag{9}$$

Since $\{E_1, E_2, E_3\}$ is an orthonormal frame, from (9) we obtain

$$\mathrm{Ric}(E_i, E_j) = \begin{cases} \lambda, & i = j, \\ 0, & i \neq j. \end{cases}$$

Comparing these equalities with (4), we obtain

$$\lambda = 0, \qquad \lambda = -2,$$

which is impossible. Therefore, Sol^3 is not an Einstein manifold. The proof is complete.

Proposition 2. On the manifold Sol^3, any diagonal left-invariant polynomial operator J of the form (8) commutes with the Ricci operator.

Proof. We shall prove that

$$\mathrm{Ric} \circ J = J \circ \mathrm{Ric}. \tag{10}$$

For the both Ric and J operators sufficient to verify equality (10) on the orthonormal basis $\{E_1, E_2, E_3\}$.

From (4), the Ricci operator is given by

$$\mathrm{Ric}(E_1) = 0, \qquad \mathrm{Ric}(E_2) = 0, \qquad \mathrm{Ric}(E_3) = -2E_3.$$

On the other hand, using by (8), we compare the two compositions on each basis vector.

For E_1, we obtain

$$\mathrm{Ric}(J(E_1)) = \mathrm{Ric}(\sigma_1 E_1) = \sigma_1 \mathrm{Ric}(E_1) = 0.$$

Also,

$$J(\mathrm{Ric}(E_1)) = J(0) = 0.$$

Hence

$$\mathrm{Ric}(J(E_1)) = J(\mathrm{Ric}(E_1)).$$

For E_2, similarly,

$$\mathrm{Ric}(J(E_2)) = \mathrm{Ric}(\sigma_1 E_2) = \sigma_1 \mathrm{Ric}(E_2) = 0,$$

and

$$J(\mathrm{Ric}(E_2)) = J(0) = 0.$$

Therefore,

$$\mathrm{Ric}(J(E_2)) = J(\mathrm{Ric}(E_2)).$$

For E_3, we have

$$\mathrm{Ric}(J(E_3)) = \mathrm{Ric}(\sigma_2 E_3) = \sigma_2 \mathrm{Ric}(E_3) = -2\sigma_2 E_3,$$

while

$$J(\mathrm{Ric}(E_3)) = J(-2E_3) = -2J(E_3) = -2\sigma_2 E_3.$$

Thus,

$$\mathrm{Ric}(J(E_3)) = J(\mathrm{Ric}(E_3)).$$

Since these equalities hold for each basis vector, relation (10) follows by linearity on the whole tangent space. After that $[\mathrm{Ric}, J] = 0$ for it is symmetric. Hence, the Ricci operator commutes with the polynomial structure J. The proposition is proved.

Proposition 3. Although the Ricci operator commutes with the metallic structure J, the Ricci tensor is not, in general, J-invariant. In other words,

$$\mathrm{Ric}(JX, JY) \neq \mathrm{Ric}(X, Y) \tag{11}$$

for any vector fields X and Y on Sol^3.

Proof. Indeed, let X and Y be arbitrary vector fields on Sol^3. With respect to the orthonormal frame $\{E_1, E_2, E_3\}$, we may write

$$X = x_1E_1 + x_2E_2 + x_3E_3, \qquad Y = y_1E_1 + y_2E_2 + y_3E_3,$$

where $x_1, x_2, x_3, y_1, y_2, y_3$ are smooth functions on the Sol^3 manifold.

From the form of the Ricci tensor, we know that

$$\mathrm{Ric}(E_1, E_1) = 0, \qquad \mathrm{Ric}(E_2, E_2) = 0, \qquad \mathrm{Ric}(E_3, E_3) = -2,$$

and

$$\mathrm{Ric}(E_i, E_j) = 0, \qquad i \neq j.$$

Using the bilinearity of the Ricci tensor, we compute

$$\mathrm{Ric}(X, Y) = \mathrm{Ric}(x_1E_1 + x_2E_2 + x_3E_3,\ y_1E_1 + y_2E_2 + y_3E_3).$$

Expanding this expression term by term, we see that every term vanishes except the one involving the E_3-components. Therefore,

$$\mathrm{Ric}(X, Y) = -2x_3y_3.$$

On the other hand, the metallic structure J on the Sol^3 is defined by (8). Hence,

$$JX = \sigma_1 x_1 E_1 + \sigma_1 x_2 E_2 + \sigma_2 x_3 E_3,$$

and similarly,

$$JY = \sigma_1 y_1 E_1 + \sigma_1 y_2 E_2 + \sigma_2 y_3 E_3.$$

Now we evaluate the Ricci tensor on the pair (JX, JY):

$$\mathrm{Ric}(JX, JY) = \mathrm{Ric}(\sigma_1 x_1 E_1 + \sigma_1 x_2 E_2 + \sigma_2 x_3 E_3,\ \sigma_1 y_1 E_1 + \sigma_1 y_2 E_2 + \sigma_2 y_3 E_3).$$

Again, since the Ricci tensor is zero on all components except the E_3-direction, all mixed terms disappear, and we obtain

$$\mathrm{Ric}(JX, JY) = -2\sigma_2^2 x_3 y_3.$$

Now comparing the two expressions, namely

$$\mathrm{Ric}(X, Y) = -2x_3 y_3, \qquad \mathrm{Ric}(JX, JY) = -2\sigma_2^2 x_3 y_3,$$

we conclude that

$$\mathrm{Ric}(JX, JY) = \mathrm{Ric}(X, Y)$$

if and only if

$$(\sigma_2^2 - 1)x_3 y_3 = 0.$$

Thus, this equality holds only in special cases, for example, when $\sigma_2^2 = 1$, i.e., when the structure is not metallic, or when at least one of the vectors X and Y has a vanishing E_3-component. In general, however, this condition is not satisfied.

Therefore, for arbitrary vector fields X and Y, one has in general

$$\mathrm{Ric}(JX, JY) \neq \mathrm{Ric}(X, Y).$$

This shows that the Ricci tensor is not J-invariant on Sol^3, even though the Ricci operator commutes with the metallic structure J.

Theorem 1. Sol^3 manifold is a metallic-Einstein type if and only if

$$\mathrm{Ric}(X, Y) = \alpha g(X, Y) + \beta g(JX, Y), \tag{12}$$

where

$$\alpha = -\frac{p + \sqrt{p^2 + 4q}}{\sqrt{p^2 + 4q}}, \qquad \beta = \frac{2}{\sqrt{p^2 + 4q}}.$$

Proof. We determine the constants α and β by comparing both sides of (12) on the orthonormal basis $\{E_1, E_2, E_3\}$. Since both sides of (12) are symmetric bilinear forms, it is sufficient to verify the equality on the basis vectors.

From the fact that $\{E_1, E_2, E_3\}$ is an orthonormal frame, we obtain

$$g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1,$$

and all remaining metrics vanish, that is, $g(E_i, E_j) = 0$. Substituting $X = Y = E_1 = E_2$ into (12), we obtain

$$\mathrm{Ric}(E_1, E_1) = \alpha g(E_1, E_1) + \beta g(JE_1, E_1),$$

Using relations (4) and (8), we get

$$\alpha + \beta\sigma_1 = 0.$$

In addition, this relation also holds for all non-diagonal cases.

Now, taking $X = Y = E_3$ in (12), we get

$$\mathrm{Ric}(E_3, E_3) = \alpha g(E_3, E_3) + \beta g(JE_3, E_3),$$

and therefore

$$\alpha + \beta\sigma_2 = -2.$$

Consequently, the unknown constants α and β satisfy the system

$$\begin{cases} \alpha + \beta\sigma_1 = 0, \\ \alpha + \beta\sigma_2 = -2. \end{cases}$$

Subtracting the first equation from the second one, we obtain

$$\beta(\sigma_2 - \sigma_1) = -2.$$

Using by (7),

$$\sigma_2 - \sigma_1 = -\sqrt{p^2 + 4q}.$$

Hence,

$$\beta = \frac{2}{\sqrt{p^2 + 4q}}.$$

Substituting this value into the relation $\alpha + \beta\sigma_1 = 0$, we obtain

$$\alpha = -\beta\sigma_1 = -\frac{2}{\sqrt{p^2 + 4q}} \cdot \frac{p + \sqrt{p^2 + 4q}}{2} = -\frac{p + \sqrt{p^2 + 4q}}{\sqrt{p^2 + 4q}}.$$

Thus, the Ricci tensor admits the decomposition (12). The proof is complete.

Now, we consider the metallic Ricci soliton. Recall that a metallic Ricci soliton is defined by the equation

$$\mathcal{L}_V g(X, Y) + \mathrm{Ric}(X, Y) = \lambda g(X, Y) + \mu g(JX, Y), \tag{13}$$

where V is a smooth vector field and $\lambda, \mu \in \mathbb{R}$.

Theorem 2. The metallic Ricci soliton admits the family of vector fields

$$V = \left(\left(\frac{\lambda + \mu\sigma_1}{2} - c_1\right) x + c_2\right) \frac{\partial}{\partial x} + \left(\left(\frac{\lambda + \mu\sigma_1}{2} + c_1\right) y + c_3\right) \frac{\partial}{\partial y} + c_1 \frac{\partial}{\partial z}, \tag{14}$$

where $c_1, c_2, c_3 \in \mathbb{R}$, provided that $\lambda + \mu\sigma_2 + 2 = 0$ on the Sol^3 manifold.

Proof. Let

$$V = v_1 E_1 + v_2 E_2 + v_3 E_3.$$

Substituting (4), (8), and the expression of $\mathcal{L}_V g$ into equation (13), we obtain the following system:

$$\begin{cases} 2(e^{-z}\dfrac{\partial v_1}{\partial x} + v_3) = \lambda + \mu\sigma_1, \\ 2(e^{z}\dfrac{\partial v_2}{\partial y} - v_3) = \lambda + \mu\sigma_1, \\ 2\dfrac{\partial v_3}{\partial z} - 2 = \lambda + \mu\sigma_2, \\ e^{-z}\dfrac{\partial v_2}{\partial x} + e^{z}\dfrac{\partial v_1}{\partial y} = 0, \\ e^{z}\dfrac{\partial v_3}{\partial y} + \dfrac{\partial v_2}{\partial z} + v_2 = 0, \\ e^{-z}\dfrac{\partial v_3}{\partial x} + \dfrac{\partial v_1}{\partial z} - v_1 = 0. \end{cases}$$

From the first, third and fifth equations we have

$$v_3 = c_1, \qquad c_1 \in \mathbb{R}, \qquad \lambda + \mu\sigma_2 + 2 = 0.$$

Next, the first two equations become

$$\frac{\partial v_1}{\partial x} = e^{z}\left(\frac{\lambda + \mu\sigma_1}{2} - c_1\right), \qquad \frac{\partial v_2}{\partial y} = e^{-z}\left(\frac{\lambda + \mu\sigma_1}{2} + c_1\right).$$

Integrating with respect to x and y, respectively, we obtain

$$v_1 = e^{z}\left(\left(\frac{\lambda + \mu\sigma_1}{2} - c_1\right) x + \phi_1(y, z)\right),$$

$$v_2 = e^{-z}\left(\left(\frac{\lambda+\mu\sigma_1}{2}+c_1\right)y+\phi_2(x,z)\right).$$

Since v_3 is constant, the last two equations imply

$$\frac{\partial v_1}{\partial z}=0, \qquad \frac{\partial v_2}{\partial z}=0.$$

Therefore, ϕ_1 and ϕ_2 do not depend on z, and hence

$$\phi_1=\phi_1(y), \qquad \phi_2=\phi_2(x).$$

Now we use the mixed relation:

$$e^{-z}\frac{\partial v_2}{\partial x}+e^{z}\frac{\partial v_1}{\partial y}=0.$$

Substituting the above expressions for v_1 and v_2, we get

$$e^{-z}\phi_2'(x)+e^{z}\phi_1'(y)=0.$$

Since this identity holds for every z, we must have

$$\phi_2'(x)=0, \qquad \phi_1'(y)=0.$$

Hence,

$$\phi_1=c_2, \qquad \phi_2=c_3, \qquad c_2,c_3\in\mathbb{R}.$$

Thus,

$$v_1=e^{z}\left(\left(\frac{\lambda+\mu\sigma_1}{2}-c_1\right)x+c_2\right), \qquad v_2=e^{-z}\left(\left(\frac{\lambda+\mu\sigma_1}{2}+c_1\right)y+c_3\right), \qquad v_3=c.$$

Substituting these expressions into the equality for V, we arrive at (14). The theorem is proved.

Conclusion

In this paper, we studied the manifold Sol^3 endowed with its standard left-invariant metric together with a diagonal polynomial structure. First, we recalled the Lie group structure of Sol^3, its standard metric, the associated orthonormal frame, the Lie bracket relations, and the Levi-Civita connection.

We then showed that the manifold Sol^3 is not Einstein and does not have constant sectional curvature. After introducing the polynomial structure J, we proved that the Ricci operator commutes with J. At the same time, we observed that the Ricci tensor is not J-invariant in general.

Next, we established that the Ricci tensor admits the decomposition

$$\mathrm{Ric}(X,Y)=\alpha g(X,Y)+\beta g(JX,Y),$$

where α and β are explicitly expressed through the parameters of the polynomial structure. Finally, we derived an explicit family of vector fields satisfying a metallic Ricci soliton type equation associated with J.

These results show that Sol^3 provides a natural framework for the study of polynomial structures and Ricci-type equations on solvable Lie groups. They may also be useful in further investigations of generalized Einstein conditions, geometric flows, and higher-dimensional analogues.

REFERENCES

1. Thurston, W. P. *Three-Dimensional Geometry and Topology*, Vol. 1. Princeton University Press, Princeton, 1997.

2. Scott, P. The geometries of 3-manifolds. *Bull. London Math. Soc.* **15** (1983), no. 5, 401–487.

3. Molnár, E. The projective interpretation of the eight 3-dimensional homogeneous geometries. *Beiträge Algebra Geom.* **38** (1997), no. 2, 261–288.

4. López, R. Differential geometry of curves and surfaces in the homogeneous space Sol_3. *Arab. J. Math.* **8** (2019), 143–160.

5. Erjavec, Z. Some geometric properties in Thurston geometries. *Rocky Mountain J. Math.* **45** (2015), no. 2, 445–463.

6. Hamilton, R. S. Three-manifolds with positive Ricci curvature. *J. Differential Geom.* **17** (1982), 255–306.

7. Cao, H.D. Recent progress on Ricci solitons. In: *Recent Advances in Geometric Analysis*, Adv. Lect. Math. **11** (2010), 1–38.

8. Baird, P.; Danielo, L. Three-dimensional Ricci solitons which project to surfaces. *J. Reine Angew. Math.* **608** (2007), 65–91.

9. Belarbi, L. Ricci solitons of the Sol_3 Lie group. *J. Indian Math. Soc.* **92** (2025), no. 2, 329–339.

10. Hretcanu, C. E.; CrásmIareanu, M. Metallic structures on Riemannian manifolds. *Rev. Un. Mat. Argentina* **54** (2013), no. 2, 15–27.

11. Blaga, A. M. On certain symmetries of $\mathbb{R}^3$ with a diagonal metric. *Filomat* **39** (2025), no. 24, 8417–8437.

12. Milnor, J. Curvatures of left invariant metrics on Lie groups. *Adv. Math.* **21** (1976), no. 3, 293–329.

13. Sharipov, A. S.; Topvoldiyev, F. F. On some properties of surfaces with isometric on sections. *Bull. Inst. Math.* **4** (2021), no. 1, 11–15.

14. Sharipov, A.; Keunimjaev, M. Existence and uniqueness of polyhedra with given values of the conditional curvature. *Int. Electron. J. Geom.* **16** (2023), no. 1, 160–170.

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Finiteness of Eigenvalues for Operator Matrices Arising in Quantum Mechanics

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Abstract. In this paper, an operator matrix of order three corresponding to a lattice system with a non-conserved number of particles not exceeding three, arising in quantum mechanics is considered as a linear, bounded, and self-adjoint operator acting in a Hilbert space. The essential spectrum of the considered third-order operator matrix is investigated. The two-particle and three-particle branches of the essential spectrum are identified. It is proved that, for an arbitrary value of the spectral parameter, the discrete spectrum of the operator matrix is finite.

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Key words: Hilbert space, quantum particle, operator matrix, spectral parameter, Birman-Schwinger principle, essential and discrete spectra.

Introduction and statement of problem

Block operator matrices are matrices where the entries are linear operators between Banach or Hilbert spaces [1]. An important subclass of such matrices is formed by Hamiltonians describing lattice systems in which the number of quasi-particles is not conserved. Depending on the model, the number of quasi-particles may be infinite, as in spin-boson models, or finite, as in so-called truncated spin-boson models. Block operator matrices of this type naturally occur in various areas of theoretical physics, including solid-state physics [2], quantum field theory [3], statistical physics [4], and quantum mechanics.

Currently, extensive research is being carried out on the essential spectrum of operator matrices and on the problem of determining the number of their eigenvalues. In particular, special attention is paid to the analysis of threshold phenomena for families of generalized Friedrichs models, to the description of the structure of the essential spectrum of families of third-order operator matrices [5], and to the identification of conditions under which the number of eigenvalues is finite or infinite [6, 7].

In this paper, a third-order operator matrix associated with a lattice system with a non-conserved number of particles not exceeding three is studied for a linear, bounded, and self-adjoint operator acting in a Hilbert space. It is proved that the discrete spectrum of this operator matrix is finite.

Let us state the problem.

Let $\mathbb{T}$ be the one-dimensional torus, $\mathcal{H}_0 := \mathbb{C}$ be the field of complex numbers, $\mathcal{H}_1 := L_2(\mathbb{T})$ be the Hilbert space of square integrable (complex) functions defined on $\mathbb{T}$ and $\mathcal{H}_2 := L_2^{\mathrm{s}}(\mathbb{T}^2)$ be the Hilbert space of square-integrable symmetric (complex) functions defined on $\mathbb{T}^2$. We denote by $\mathcal{H}$ the direct sum of $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_2$, that is, $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$. The spaces $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_2$ are called the zero-particle, one-particle and two-particle subspaces of the bosonic Fock space $\mathcal{F}_{\mathrm{b}}(L_2(\mathbb{T}))$ over $L_2(\mathbb{T})$, respectively, where

$$\mathcal{F}_{\mathrm{b}}(L_2(\mathbb{T})) := \mathbb{C} \oplus L_2(\mathbb{T}) \oplus L_2^{\mathrm{s}}(\mathbb{T}^2) \oplus L_2^{\mathrm{s}}(\mathbb{T}^3) \oplus \cdots .$$

Elements of the Hilbert space $\mathcal{H}$ are identified with ordered triples (f_0, f_1, f_2). The inner product of the elements $f = (f_0, f_1, f_2)$ and $g = (g_0, g_1, g_2)$ of $\mathcal{H}$ is given by

$$(f, g) = f_0 \cdot \overline{g_0} + \int_{\mathbb{T}} f_1(x)\overline{g_1(x)}dx + \int_{\mathbb{T}^2} f_2(x, y)\overline{g_2(x, y)}dxdy.$$

In the Hilbert space $\mathcal{H}$, we introduce an operator matrix $\mathcal{A}_\mu$ of the form

$$\mathcal{A}_\mu := \begin{pmatrix} A_{00} & \mu A_{01} & 0 \\ \mu A_{01}^* & A_{11} & \mu A_{12} \\ 0 & \mu A_{12}^* & A_{22} \end{pmatrix}, \qquad \mu > 0.$$

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Throughout this paper we suppose that the matrix entries $\mathcal{A}_{ij} : \mathcal{H}_j \to \mathcal{H}_i$, $i \leq j$, $i, j = 0, 1, 2$ are given by

$$A_{00} f_0 = a f_0, \quad A_{01} f_1 = \int_{\mathbb{T}} v(t) f_1(t) dt;$$
$$(A_{11} f_1)(x) = f_1(x), \quad (A_{12} f_2)(x) = \int_{\mathbb{T}} f_2(x,t) dt;$$
$$(A_{22} f_2)(x,y) = w(x,y) f_2(x,y), \quad f_i \in \mathcal{H}_i, \quad i = 0, 1, 2.$$

where $a \in \mathbb{R}$, the function $v(\cdot)$ is a real-valued continuous function on $\mathbb{T}$ and $w(\cdot, \cdot)$ is defined by

$$w(x,y) := \varepsilon(x) + \varepsilon(x+y) + \varepsilon(y), \quad \varepsilon(x) := 1 - \cos x$$

We remark that the operators A_{01}, A_{12} and A^*_{01}, A^*_{12} are called annihilation and creation operators respectively [8]. A trivial verification shows

$$A^*_{01} : \mathcal{H}_0 \to \mathcal{H}_1, \quad (A^*_{01} f_0)(x) = v(x) f_0, \quad f_0 \in \mathcal{H}_0;$$
$$A^*_{12} : \mathcal{H}_1 \to \mathcal{H}_2, \quad (A^*_{12} f_1)(x,y) = \frac{f_1(x) + f_1(y)}{2}, \quad f_1 \in \mathcal{H}_1.$$

Under these assumptions, the operator matrix $\mathcal{A}_\mu$ is bounded and self-adjoint.

Essential spectrum of the operator matrix $\mathcal{A}_\mu$

In this section, we introduce the basic concepts necessary to state the main results.

For any fixed $\mu > 0$ and $x \in \mathbb{T}$, we define the function $\Delta_\mu(x;\, \cdot)$ in $\mathbb{R} \setminus [E_1(x); E_2(x)]$ by the following condition

$$\Delta_\mu(x\,; z) := \begin{cases} 1 - z - I_\mu(x\,; z), & z < E_1(x), \\ 1 - z + I_\mu(x\,; z), & z > E_2(x) \end{cases}$$

$$I_\mu(x\,; z) := \frac{\pi \mu^2}{\sqrt{(3 - \cos x - z)^2 - 4 \cos^2 \frac{x}{2}}},$$

where the numbers $E_1(x)$ and $E_2(x)$ are determined by

$$E_1(x) := \min_{y \in \mathbb{T}} w(x,y) = 3 - \cos x - \sqrt{2 + 2\cos x};$$
$$E_2(x) := \max_{y \in \mathbb{T}} w(x,y) = 3 - \cos x + \sqrt{2 + 2\cos x}.$$

Suppose that σ_μ is the set of points $z \in \mathbb{R}$ for which the equation $\Delta_\mu(x;\, z) = 0$ has a solution for at least one point $x \in \mathbb{T}$. Let us introduce the notation

$$\Sigma_\mu := \sigma_\mu \cup [0; \tfrac{9}{2}].$$

$\sigma(\cdot)$, $\sigma_{\rm ess}(\cdot)$ and $\sigma_{\rm disc}(\cdot)$ are denoted by the spectrum, the essential spectrum, and the discrete spectrum of a bounded self-adjoint operator respectively.

The following theorem is about the elements of essential spectrum of the operator matrix $\mathcal{A}_\mu$.

Theorem 1. *The essential spectrum $\sigma_{\rm ess}(\mathcal{A}_\mu)$ of the operator matrix $\mathcal{A}_\mu$ coincides with the set Σ_μ, i.e., $\sigma_{\rm ess}(\mathcal{A}_\mu) = \Sigma_\mu$.*

Theorem 1 is proved using the Weyl criterion [9], the characteristic property of the Faddeev equation, and the analytic Fredholm theorem.

The sets σ_μ and $[0; \frac{9}{2}]$ are called two-particle and three-particle branches of the essential spectrum of the operator matrix $\mathcal{A}_\mu$.

Finiteness of the number of eigenvalues of $\mathcal{A}_\mu$

Following the classical definition introduced by Glazman [10], for any $\lambda \in \mathbb{R}$ and a bounded self-adjoint operator A acting in a Hilbert space H, we define the number $n(\lambda, A)$ as

$$n(\lambda, A) := \sup\Big\{\dim F \ : \ (Au, u) > \lambda, \ u \in F \subset H, \ \|u\| = 1\Big\}.$$

The quantity $n(\lambda, A)$ is infinite whenever $\lambda < \max \sigma_{\mathrm{ess}}(A)$.

If, on the other hand, $n(\lambda, A)$ is finite, then it coincides with the number of eigenvalues of the operator A that are greater than λ, counted according to their multiplicities.

Let $\tau_{\min}(\mathcal{A}_\mu)$ and $\tau_{\max}(\mathcal{A}_\mu)$ denote the lower and upper bounds of the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{A}_\mu)$ of the operator matrix $\mathcal{A}_\mu$, respectively, that is,

$$\tau_{\min}(\mathcal{A}_\mu) := \min \sigma_{\mathrm{ess}}(\mathcal{A}_\mu), \qquad \tau_{\max}(\mathcal{A}_\mu) := \max \sigma_{\mathrm{ess}}(\mathcal{A}_\mu).$$

By the definition of the quantity $N_{(a,b)}(\mathcal{A}_\mu)$, we have

$$N_{(-\infty,\, z)}(\mathcal{A}_\mu) = n\big(-z,\, -\mathcal{A}_\mu\big), \qquad z < \tau_{\min}(\mathcal{A}_\mu),$$

and

$$N_{(z,\, +\infty)}(\mathcal{A}_\mu) = n\big(z,\, \mathcal{A}_\mu\big), \qquad z > \tau_{\max}(\mathcal{A}_\mu).$$

Note that for any $x \in \mathbb{T}$ and $z < \tau_{\min}(\mathcal{A}_\mu)$ (resp. $z > \tau_{\max}(\mathcal{A}_\mu)$), the function $\Delta_\mu(x; z)$ (resp. $-\Delta_\mu(x; z)$) is positive and, therefore, admits a positive square root.

In the study of the discrete spectrum of the operator matrix $\mathcal{A}_\mu$, a fundamental role is played by the compact operator $\widehat{T}_\mu(z)$, $z \in \mathbb{R} \setminus [\tau_{\min}(\mathcal{A}_\mu), \tau_{\max}(\mathcal{A}_\mu)]$, acting in the space $\mathcal{H}_0 \oplus \mathcal{H}_1$ and given by

$$\widehat{T}_\mu(z) := \begin{pmatrix} \widehat{T}_{00}(\mu, z) & \widehat{T}_{01}(\mu, z) \\ \widehat{T}_{01}^*(\mu, z) & \widehat{T}_{11}(\mu, z) \end{pmatrix}.$$

where the matrix entries $\widehat{T}_{ij}(\mu, z) : \mathcal{H}_j \to \mathcal{H}_i$, $i \le j$, $i, j = 0, 1$, are defined as follows.

For $z < \tau_{\min}(\mathcal{A}_\mu)$,

$$\widehat{T}_{00}(\mu, z)\varphi_0 = (1 + z - a)\varphi_0, \qquad \widehat{T}_{01}(\mu, z)\varphi_1 = -\mu \int_{\mathbb{T}} \frac{v(t)\varphi_1(t)\, dt}{\sqrt{\Delta_\mu(t, z)}};$$

$$(\widehat{T}_{01}^*(\mu, z)\varphi_0)(x) = -\frac{\mu v(x)\varphi_0}{\sqrt{\Delta_\mu(x, z)}};$$

$$(\widehat{T}_{11}(\mu, z)\varphi_1)(x) = \frac{\mu^2}{2\sqrt{\Delta_\mu(x, z)}} \int_{\mathbb{T}} \frac{\varphi_1(t)\, dt}{\sqrt{\Delta_\mu(t, z)}\big(w(x, t) - z\big)}, \qquad \varphi_i \in \mathcal{H}_i, \ i = 0, 1.$$

For $z > \tau_{\max}(\mathcal{A}_\mu)$,

$$\widehat{T}_{00}(\mu, z)\varphi_0 = (1 + z - a)\varphi_0, \qquad (\widehat{T}_{01}(\mu, z)\varphi_1)(x) = -\mu \int_{\mathbb{T}} \frac{v(t)\varphi_1(t)\, dt}{\sqrt{-\Delta_\mu(t, z)}};$$

$$(\widehat{T}_{01}^*(\mu, z)\varphi_0)(x) = -\frac{\mu v(x)\varphi_0}{\sqrt{-\Delta_\mu(x, z)}};$$

$$(\widehat{T}_{11}(\mu, z)\varphi_1)(x) = -\frac{\mu^2}{2\sqrt{-\Delta_\mu(x, z)}} \int_{\mathbb{T}} \frac{\varphi_1(t)\, dt}{\sqrt{-\Delta_\mu(t, z)}\big(w(x, t) - z\big)}, \qquad \varphi_i \in \mathcal{H}_i, \ i = 0, 1.$$

The following lemma presents a modification of the well-known Birman-Schwinger principle for the operator matrix $\mathcal{A}_\mu$ (see. [11, 12]).

Lemma 1. *For all $z \in \mathbb{R} \setminus [\tau_{\min}(\mathcal{A}_\mu), \tau_{\max}(\mathcal{A}_\mu)]$, the operator $\widehat{T}_\mu(z)$ is compact and continuous with respect to z, and the following equalities hold:*

$$N_{(-\infty,\, z)}(\mathcal{A}_\mu) = n\big(1,\, \widehat{T}_\mu(z)\big), \quad \textit{for} \quad z < \tau_{\min}(\mathcal{A}_\mu),$$

$$N_{(z,+\infty)}(\mathcal{A}_\mu) = n\big(1, \widehat{T}_\mu(z)\big), \quad for \quad z > \tau_{\max}(\mathcal{A}_\mu).$$

Since $\tau_{\min}(\mathcal{A}_\mu) \in \sigma_\mu$, $(\tau_{\max}(\mathcal{A}_\mu) \in \sigma_\mu)$, there exists a point $x_1 \in \mathbb{T}$ $(x_1' \in \mathbb{T})$ such that

$$\Delta_\mu\big(x_1; \tau_{\min}(\mathcal{A}_\mu)\big) = 0 \quad \big(\Delta_\mu\big(x_1'; \tau_{\max}(\mathcal{A}_\mu)\big) = 0\big).$$

Since $\tau_{\min}(\mathcal{A}_\mu) < 0$, the function $\Delta_\mu(\cdot\,; \tau_{\min}(\mathcal{A}_\mu))$ is regular on $\mathbb{T}$. Therefore, the number of zeros of this function is finite.

Let

$$\{x \in \mathbb{T} : \Delta_\mu(x; \tau_{\min}(\mathcal{A}_\mu)) = 0\} = \{x_1, \ldots, x_n\},$$

and let k_j denote the multiplicity of the zero x_j for $j \in \{1, \ldots, n\}$.

Since $\tau_{\max}(\mathcal{A}_\mu) > \frac{9}{2}$, the function $\Delta_\mu(\cdot\,; \tau_{\max}(\mathcal{A}_\mu))$ is regular on $\mathbb{T}$. Hence, the number of zeros of this function is finite.

Let

$$\{x \in \mathbb{T} : \Delta(x; \tau_{\max}(\mathcal{A}_\mu)) = 0\} = \{x_1', \ldots, x_m'\},$$

and let $k_{j'}'$ denote the multiplicity of the zero $x_{j'}'$ for $j' \in \{1, \ldots, m\}$.

Since $\tau_{\min}(\mathcal{A}_\mu) < 0$, it follows that the difference

$$w(x, y) - z$$

is positive for all $x, y \in \mathbb{T}$ and $z \leq \tau_{\min}(\mathcal{A}_\mu)$. Hence, for all $z \leq \tau_{\min}(\mathcal{A}_\mu)$ the function

$$\big(w(\cdot, \cdot) - z\big)^{-1}$$

is analytic on $\mathbb{T}^2$. Therefore, there exists a number $\delta > 0$ such that for any $i, j \in \{1, \ldots, n\}$ and $z \leq \tau_{\min}(\mathcal{A}_\mu)$ the following representations hold.

For any $x \in \mathbb{T}$ and $y \in U_\delta(x_i)$,

$$\frac{\mu^2}{2\big(w(x, y) - z\big)} = \sum_{k=0}^{\lfloor k_i/2 \rfloor} a_{ik}^{(1)}(z; x)\,(y - x_i)^k + (y - x_i)^{\lfloor k_i/2 \rfloor + 1} b_i^{(1)}(z; x, y). \tag{1}$$

For any $y \in \mathbb{T}$ and $x \in U_\delta(x_i)$,

$$\frac{\mu^2}{2\big(w(x, y) - z\big)} = \sum_{k=0}^{\lfloor k_i/2 \rfloor} a_{ik}^{(2)}(z; y)\,(x - x_i)^k + (x - x_i)^{\lfloor k_i/2 \rfloor + 1} b_i^{(2)}(z; x, y). \tag{2}$$

For any $(x, y) \in U_\delta(x_i) \times U_\delta(x_j)$,

$$\begin{aligned}
\frac{\mu^2}{2\big(w(x, y) - z\big)} &= \sum_{k=0}^{\lfloor k_i/2 \rfloor} \sum_{r=0}^{\lfloor k_j/2 \rfloor} d_{ij}^{kr}(z)(x - x_i)^k (y - x_j)^r \\
&+ \sum_{k=0}^{\lfloor k_i/2 \rfloor} \sum_{r=\lfloor k_j/2 \rfloor + 1}^{\infty} d_{ij}^{(kr)}(z)(x - x_i)^k (y - x_j)^r \\
&+ \sum_{k=\lfloor k_i/2 \rfloor + 1}^{\infty} \sum_{r=0}^{\lfloor k_j/2 \rfloor} d_{ij}^{(kr)}(z)(x - x_i)^k (y - x_j)^r \\
&+ (x - x_i)^{\lfloor k_i/2 \rfloor + 1}(y - x_j)^{\lfloor k_j/2 \rfloor + 1} q_{ij}(z; x, y),
\end{aligned} \tag{3}$$

where $\lfloor m \rfloor$ denotes the integer part of m. For any $z \leq \tau_{\min}(\mathcal{A}_\mu)$, the numbers $d_{ij}^{(kr)}(z)$ are some real coefficients, the functions $a_{ik}^{(\alpha)}(z; \cdot)$, $\alpha = 1, 2$, $b_i^{(1)}(z; \cdot, \cdot)$, $b_i^{(2)}(z; \cdot, \cdot)$, and $q_{ij}(z; \cdot, \cdot)$ are analytic functions on $\mathbb{T}$, $\mathbb{T} \times U_\delta(x_i)$, $U_\delta(x_i) \times \mathbb{T}$, and $U_\delta(x_i) \times U_\delta(x_j)$, respectively.

The case $z > \tau_{\max}(\mathcal{A}_\mu)$ can be treated analogously.

Lemma 2. *Let $j \in \{1,\dots,n\}$ and $j' \in \{1,\dots,m\}$. Then there exist constants $C_1, C_2 > 0$ and $\delta > 0$ such that for all $x \in U_\delta(x_j)$ and $x \in U_\delta(x'_{j'})$ the following estimates hold:*

$$\frac{|x - x_j|^{\lfloor k_j/2\rfloor+1}}{\sqrt{\Delta_\mu(x;z)}} \le C_1, \quad z \le \tau_{\min}(\mathcal{A}_\mu), \tag{4}$$

$$\frac{|x - x'_{j'}|^{\lfloor k'_{j'}/2\rfloor+1}}{\sqrt{-\Delta_\mu(x;z)}} \le C_2, \quad z \ge \tau_{\max}(\mathcal{A}_\mu). \tag{5}$$

The proof of this lemma is analogous to the proofs of the corresponding lemmas in [7, 13].

Lemma 3. *Then, for every $z \in \mathbb{R} \setminus [\tau_{\min}(\mathcal{A}_\mu), \tau_{\max}(\mathcal{A}_\mu)]$, the operator $\widehat{T}_\mu(z)$ admits the representation*

$$\widehat{T}_\mu(z) = \widehat{T}^{(0)}_\mu(z) + \widehat{T}^{(1)}_\mu(z),$$

where the operator-valued function $\widehat{T}^{(0)}_\mu(\cdot)$ is continuous in the operator norm on the intervals $(-\infty, \tau_{\min}(\mathcal{A}_\mu)]$ and $[\tau_{\max}(\mathcal{A}_\mu), +\infty)$, and for all $z \in \mathbb{R} \setminus [\tau_{\min}(\mathcal{A}_\mu), \tau_{\max}(\mathcal{A}_\mu)]$ the operator $\widehat{T}^{(1)}_\mu(z)$ is finite-dimensional and does not depend on z.

Proof. Assume that $z \le \tau_{\min}(\mathcal{A}_\mu)$. Since the operators $\widehat{T}_{00}(\mu,z)$, $\widehat{T}_{01}(\mu,z)$, and $\widehat{T}^*_{01}(\mu,z)$ are one-dimensional and independent of z, it suffices to analyze the operator $\widehat{T}_{11}(\mu,z)$.

Let $\widehat{T}_{11}(\mu,z)$ denote the kernel of the integral operator $\widehat{T}_{11}(\mu,z)$

$$\widehat{T}_{11}(\mu;z;x,y) := \frac{\mu^2}{\sqrt{\Delta_\mu(x;z)}\,\big(w(x,y)-z\big)\,\sqrt{\Delta_\mu(y;z)}}.$$

In this case, $\tau_{\min}(\mathcal{A}_\mu) < 0$, and by using representations (1)–(4) we obtain

$$\widehat{T}_{11}(\mu,z) = \widehat{T}^0_{11}(\mu,z) + \widehat{T}^1_{11}(\mu,z),$$

where the kernels $\widehat{T}^0_{11}(\mu;z;x,y)$ and $\widehat{T}^1_{11}(\mu;z;x,y)$ of the integral operators $\widehat{T}^0_{11}(\mu,z)$ and $\widehat{T}^1_{11}(\mu,z)$, respectively, are given by

$$\begin{aligned}
\widehat{T}^0_{11}(\mu;z;x,y) &:= (1-\chi_{V_\delta}(x))(1-\chi_{V_\delta}(y))\widehat{T}_{11}(\mu;z;x,y) \\
&+ \frac{1-\chi_{V_\delta}(x)}{\sqrt{\Delta_\mu(x;z)}} \sum_{i=1}^{n} \frac{\chi_{V_\delta}(y)(y-x_i)^{\lfloor k_i/2\rfloor+1}}{\sqrt{\Delta_\mu(y;z)}}\, B^{(1)}_i(z;x,y) \\
&+ \frac{1-\chi_{V_\delta}(y)}{\sqrt{\Delta_\mu(y;z)}} \sum_{i=1}^{n} \frac{\chi_{V_\delta}(x)(x-x_i)^{\lfloor k_i/2\rfloor+1}}{\sqrt{\Delta_\mu(x;z)}}\, B^{(2)}_i(z;x,y) \\
&+ \chi_{V_\delta}(x)\chi_{V_\delta}(y) \sum_{i,j=1}^{n} \frac{(x-x_i)^{\lfloor k_i/2\rfloor+1}(y-x_j)^{\lfloor k_j/2\rfloor+1}}{\sqrt{\Delta_\mu(x;z)}\,\sqrt{\Delta_\mu(y;z)}}\, Q_{ij}(z;x,y);
\end{aligned}$$

$$\widehat{T}_{11}^{1}(\mu;z;x,y):=\frac{(1-\chi_{V_\delta}(x))\chi_{V_\delta}(y)}{\sqrt{\Delta_\mu(x;z)}\sqrt{\Delta_\mu(y;z)}}\sum_{i=1}^{n}\sum_{k=0}^{\lfloor k_i/2\rfloor}(y-x_i)^k\,a_{ik}^{(1)}(z;x)$$

$$+\frac{\chi_{V_\delta}(x)(1-\chi_{V_\delta}(y))}{\sqrt{\Delta_\mu(x;z)}\sqrt{\Delta_\mu(y;z)}}\sum_{i=1}^{n}\sum_{k=0}^{\lfloor k_i/2\rfloor}(x-x_i)^k\,a_{ik}^{(2)}(z;y)$$

$$+\frac{\chi_{V_\delta}(x)\chi_{V_\delta}(y)}{\sqrt{\Delta_\mu(x;z)}\sqrt{\Delta_\mu(y;z)}}\sum_{i,j=1}^{n}\Big(\sum_{k=0}^{\lfloor k_i/2\rfloor}\sum_{r=0}^{\lfloor k_j/2\rfloor}d_{ij}^{kr}(z)(x-x_i)^k\,(y-x_j)^r$$

$$+\sum_{k=0}^{\lfloor k_i/2\rfloor}\sum_{r=\lfloor k_j/2\rfloor+1}^{\infty}d_{ij}^{kr}(z)(x-x_i)^k\,(y-x_j)^r$$

$$+\sum_{k=\lfloor k_i/2\rfloor+1}^{\infty}\sum_{r=0}^{\lfloor k_j/2\rfloor}d_{ij}^{kr}(z)(x-x_i)^k\,(y-x_j)^r\Big),$$

where $V_\delta:=\bigcup_{i=1}^{n}U_\delta(x_i)$, $\chi_\Omega(\cdot)$ is the characteristic function of the set $\Omega\subset\mathbb{T}$.

$$B_i^{(1)}(z;x,y):=\begin{cases}b_i^{(1)}(z;x,y), & (x,y)\in\mathbb{T}\times U_\delta(x_i),\\ 0, & (x,y)\notin\mathbb{T}\times U_\delta(x_i),\end{cases}$$

$$B_i^{(2)}(z;x,y):=\begin{cases}b_i^{(2)}(z;x,y), & (x,y)\in U_\delta(x_i)\times\mathbb{T},\\ 0, & (x,y)\notin U_\delta(x_i)\times\mathbb{T},\end{cases}$$

$$Q_{ij}(z;x,y):=\begin{cases}q_{ij}(z;x,y), & (x,y)\in U_\delta(x_i)\times U_\delta(x_j),\\ 0, & (x,y)\notin U_\delta(x_i)\times U_\delta(x_j).\end{cases}$$

Applying Lemma 2, we obtain that the function $\widehat{T}_{11}^{0}(\mu;z;\cdot,\cdot)$ is square-integrable on $\mathbb{T}^2$ for $z\le\tau_{\min}(\mathcal{A}_\mu)$ and converges almost everywhere to $\widehat{T}_{11}^{0}(\mu;\tau_{\min}(\mathcal{A}_\mu);\cdot,\cdot)$ as $z\to\tau_{\min}(\mathcal{A}_\mu)-0$. Then by the Lebesgue dominated convergence theorem the operator $\widehat{T}_{11}^{0}(\mu,z)$ converges in the operator-norm to $\widehat{T}_{11}^{0}(\mu,\tau_{\min}(\mathcal{A}_\mu))$ as $z\to\tau_{\min}(\mathcal{A}_\mu)-0$.

The finite-dimensionality of the operator $\widehat{T}_{11}^{1}(\mu,\tau_{\min}(\mathcal{A}_\mu))$ follows directly from the definition of the function $\widehat{T}_{11}^{1}(\mu;z;x,y)$.

Setting

$$\widehat{T}_\mu^{(0)}(z):=\begin{pmatrix}0 & 0\\ 0 & \widehat{T}_{11}^{0}(\mu,z)\end{pmatrix},\qquad \widehat{T}_\mu^{(1)}(z):=\begin{pmatrix}\widehat{T}_{00}(\mu,z) & \widehat{T}_{01}(\mu,z)\\ \widehat{T}_{01}^{*}(\mu,z) & \widehat{T}_{11}^{1}(\mu,z)\end{pmatrix}.$$

The case $z>\tau_{\max}(\mathcal{A}_\mu)$ can be proved analogously. □

Theorem 2. *For any $\mu>0$, the operator $\mathcal{A}_\mu$ has only a finite number of eigenvalues lying to the left of $\tau_{\min}(\mathcal{A}_\mu)$ and to the right of $\tau_{\max}(\mathcal{A}_\mu)$.*

Proof. Using Weyl's inequality, we obtain

$$\begin{aligned}n(1,\widehat{T}_\mu(z))&\le n\Big(\tfrac{2}{3},\widehat{T}_\mu^{(0)}(z)\Big)+n\Big(\tfrac{1}{3},\widehat{T}_\mu^{(1)}(z)\Big)\\ &\le n\Big(\tfrac{1}{3},\widehat{T}_\mu^{(0)}(z)-\widehat{T}_\mu^{(0)}(\tau_{\min}(\mathcal{A}_\mu))\Big)+n\Big(\tfrac{1}{3},\widehat{T}_\mu^{(0)}(\tau_{\min}(\mathcal{A}_\mu))\Big)+n\Big(\tfrac{1}{3},\widehat{T}_\mu^{(1)}(z)\Big),\end{aligned}\tag{6}$$

for all $z<\tau_{\min}(\mathcal{A}_\mu)$.

According to Lemma 3, the operator $\widehat{T}_\mu^{(0)}(\tau_{\min}(\mathcal{A}_\mu))$ is compact and, consequently,

$$n\Big(\tfrac{1}{3},\widehat{T}_\mu^{(0)}(\tau_{\min}(\mathcal{A}_\mu))\Big)<\infty,$$

and moreover,

$$n\Big(\tfrac{1}{3},\widehat{T}^{(0)}_{\mu}(z)-\widehat{T}^{(0)}_{\mu}(\tau_{\min}(\mathcal{A}_{\mu})\Big)\to 0 \quad \text{as} \quad z\to\tau_{\min}(\mathcal{A}_{\mu})-0.$$

Since the operator $\widehat{T}^{(1)}_{\mu}(z)$ is finite-dimensional and the dimension of its range does not depend on z, $z<\tau_{\min}(\mathcal{A}_{\mu})$, there exists a constant $C>0$ such that for all $z<\tau_{\min}(\mathcal{A}_{\mu})$ the inequality

$$n\Big(\tfrac{1}{3},\widehat{T}^{(1)}_{\mu}(z)\Big)\le C<\infty$$

holds. Therefore, by inequality (6), we conclude that the number

$$n(1,\widehat{T}_{\mu}(z))$$

is finite for all $z<\tau_{\min}(\mathcal{A}_{\mu})$.

Now, Lemma 1 implies that

$$N_{(-\infty,\,z)}(\mathcal{A}_{\mu})=n(1,\widehat{T}_{\mu}(z)), \quad \text{for} \quad z<\tau_{\min}(\mathcal{A}_{\mu}),$$

and hence

$$\lim_{z\to\tau_{\min}(\mathcal{A}_{\mu})} N_{(-\infty,\,z)}(\mathcal{A}_{\mu})=N_{(-\infty,\,\tau_{\min}(\mathcal{A}_{\mu}))}(\mathcal{A}_{\mu})$$
$$\le n\Big(\tfrac{1}{3},\widehat{T}^{(0)}_{\mu}(\tau_{\min}(\mathcal{A}_{\mu}))\Big)+n\Big(\tfrac{1}{3},\widehat{T}^{(1)}_{\mu}(\tau_{\min}(\mathcal{A}_{\mu}))\Big)<\infty.$$

This proves that the operator $\mathcal{A}_{\mu}$ has only a finite number of eigenvalues lying to the left of $\tau_{\min}(\mathcal{A}_{\mu})$. The case $z>\tau_{\max}(\mathcal{A}_{\mu})$ can be treated analogously. □

REFERENCES

1. Tretter, C. *Spectral theory of block operator matrices and applications.* Imperial College Press, 2008.
2. Mogilner, A. I. Hamiltonians in solid state physics as multiparticle discrete Schrödinger operators: problems and results. *Advances in Sov. Math.*, **5** (1991), 139–194.
3. Friedrichs, K.O. *Perturbation of spectra in Hilbert space.* Amer. Math. Soc. Providence, Rhole Island, 1965.
4. Malishev, V. A.; Minlos, R. A. *Linear infnite-particle operators. Translations of Mathematical Monographs.* 143, AMS, Providence, RI, 1995.
5. Zhukov, Yu. V.; Minlos, R. A. Spectrum and scattering in the spin–boson model with no more than three photons. *Theoretical and Mathematical Physics*, **103**:1 (1995), 63–81.
6. Minlos R.A., Spohn H. The three-body problem in radioactive decay: the case of one atom and at most two photons Topics in Statistical and Theoretical Physics, AMS Transl.-Series 2, 177 1996, P. 159–193.
7. Muminov, M.; Neidhardt, H.; Rasulov, T.; On the spectrum of the lattice spin-boson Hamiltonian for any coupling: 1D case. *J. Math. Phys.*, **56** (2015), 053507.
8. K. O. Friedrichs, *Perturbation of Spectra in Hilbert Space*, American Mathematical Society, Providence, Rhode Island, 1965.
9. Reed, M.; Simon, B. *Methods of modern mathematical physics. IV: Analysis of Operators.* Academic Press, New York, 1979.
10. Glazman, I. M. *Direct methods of the qualitative spectral analysis of singular differential operators.* J.: IPS Trans., 1965.
11. Albeverio S.; Lakaev, S.N.; Rasulov T.H. On the spectrum of an Hamiltonian in Fock space. Discrete spectrum asymptotics. *J. Stat. Phys.*, **127**:2 (2007), 191220.

12. Albeverio, S.; Lakaev, S.N.; Muminov, Z.I. Schrödinger Operators on Lattices. The Efimov Effect and Discrete Spectrum Asymptotics. *Ann. Henri Poincare*, **5** (2004), 743–772.

13. Muminov, M. E.; Aliev, N. M. On the spectrum of a three-particle Schrödinger operator on a one-dimensional lattice. *Theoretical and Mathematical Physics*, **171**:3 (2012), 754–768.

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A non-local problem for the Barenblatt-Zheltov-Kochina type fractional equations

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Abstract. This paper investigates a non-local problem associated with the fractional-order Barenblatt–Zheltov–Kochina equation involving the Caputo fractional derivative. The non-local problem under consideration is reduced to two auxiliary problems, and the solution of the corresponding Cauchy problem is employed in the analysis. The existence and uniqueness theorems are established for both the initial-boundary value problem. The results obtained extend the theoretical framework of fractional differential equations and provide a foundation for further theoretical developments as well as potential practical applications.

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Key words: The Barenblatt–Zheltov–Kochina equation; the Caputo fractional derivative; the Fourier method; the Mittag–Leffler functions.

Introduction

Consider a separable Hilbert space H. Let $A : H \to H$ be a self-adjoint, positive, unbounded operator defined in the domain $D(A)$. We assume that A has a compact inverse A^{-1}. Denote by $\{v_k\}$ a complete orthonormal set of eigenfunctions and by $\{\lambda_k\}$ the associated set of positive eigenvalues. The eigenvalues can be ordered in a non-decreasing sequence, which means $0 < \lambda_1 \leq \lambda_2 \cdots \to +\infty$. The Caputo fractional derivative of order $\rho \in (0, 1)$ is defined as follows (see, for example [1])

$$D_t^\rho h(t) = \frac{1}{\Gamma(1-\rho)} \int_0^t \frac{h'(\xi)}{(t-\xi)^\rho} d\xi, \quad t > 0,$$

provided that the right-hand side exists. Let $\rho \in (0, 1)$ be a given constant and $C((a, b); H)$ be a set of continuous functions $u(t)$ of $t \in (a, b)$ with values in H.

Let us consider the following *non-local problem*:

$$\begin{cases} D_t^\rho u(t) + A(1 + \gamma D_t^\rho) u(t) = f, \quad \gamma > 0, \quad 0 < t \leq T; \\ u(\tau_0) = \alpha u(0) + \varphi, \quad 0 < \tau_0 \leq T, \end{cases} \tag{1}$$

here, $f, \varphi \in H$ are given functions. The parameters $\gamma > 0$, α is a constant, and τ_0 is a fixed point. When $\rho \in (0, 1)$ the first equation in (1) is called the Barenblatt–Zheltov–Kochina type fractional differential equation.

Definition 1. An absolutely continuous function $u(t) \in C([0, T]; H)$ that has properties $Au(t)$, $D_t^\rho u(t)$, $A(D_t^\rho u(t)), \in C((0, T]; H)$ and satisfies the conditions in (1) is called the solution of the non-local problem (1).

Let us provide an overview of the research conducted to date.

The Barenblatt–Zheltov–Kochina equation arises from the theory of fluid filtration in fissured porous media and other diffusion-type processes. The fractional version of this equation allows us to take into account memory effects and anomalous diffusion. Therefore, the study of boundary value problems for this equation is of both theoretical and practical interest.

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Barenblatt, Zheltov, and Kochina (see also [2]) developed the theory of unsteady filtration in fractured-porous media, which was later expanded and refined by numerous researchers ([3, 4, 5, 6, 7, 8, 9, 10, 11, 12]), who further elaborated its fundamental principles and governing equations.

The existence and uniqueness of a solution to the Cauchy problem for the Caputo derivative of the Barenblatt–Zheltov–Kochina type when $\gamma = 1$ was studied in [13].

In work [14], the Cauchy problem was studied in the case where $\gamma \geq 0$, the source term f is independent of time. The direct and inverse problems are considered for the cases $\gamma = 0$ and $\gamma > 0$. Later, the integral condition for a positive parameter was studied in [15].

If the operator D_t^ρ with $\rho \in (0,1)$ is replaced by the first-order time derivative $\frac{\partial}{\partial t}$, the equation reduces to the classical Rayleigh–Stokes equation.

In [16], a non-local problem for the Rayleigh–Stokes equation with the Riemann–Liouville fractional derivative was investigated. The non-local condition is given by

$$u(T) = \alpha u(0) + \varphi,$$

where $\alpha \in \{0, 1\}$ and $\gamma > 0$ is a constant. In particular, the case $\alpha = 1$ corresponds to a non-local problem. It was shown that when $\alpha = 0$, the associated backward problem is ill-posed. For certain values of α, the problem is well-posed; otherwise, additional restrictions on the function φ are imposed to guaranty well-posedness.

In [17], a non-local problem was studied for the Rayleigh–Stokes equation involving the Riemann–Liouville fractional derivative. In this work, the non-local condition is formulated as in (1). This paper studied a time non-local problem, instead of the initial condition, they considered the non-local condition. The results obtained were valid for the equation with the Laplace operator under the Dirichlet condition.

In [18], nonlocal boundary value problems with $\gamma = 0$ and condition (1) were studied for equations involving fractional derivative Caputo and Riemann–Liouville. The authors established the existence and uniqueness results and analyzed the solvability of the problem depending on the parameter α. It was shown that the type of fractional derivative affects the corresponding coercivity-type inequalities. In addition, inverse problems related to the determination of φ were considered.

Preliminaries

In this section, we introduce the order and domain of definition of the operator A, some properties of the Mittag–Leffler function, and orthogonality conditions. The tools that will be used throughout the paper.

Let the vector function (or the simple function) $h(t)$ be defined in the interval $[0, +\infty)$ with values in the Hilbert space H. Let ν be an arbitrary real number. We define the power of the operator A, acting in the Hilbert space H, according to the following rule

$$A^\nu h = \sum_{k=1}^{\infty} \lambda_k^\nu h_k v_k,$$

here h_k are the Fourier coefficients of a function $h \in H$ defined by $h_k = (h, v_k)$. The domain of this operator is structured as follows

$$D(A^\nu) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^{2\nu} |h_k|^2 < \infty\}.$$

For elements of $D(A^\nu)$ we introduce the norm

$$\|h\|_\nu^2 = \sum_{k=1}^{\infty} \lambda_k^{2\nu} |h_k|^2 = \|A^\nu h\|^2,$$

and together with this norm $D(A^\nu)$ turn into a Hilbert space.

Now we note some properties of the Mittag–Leffler function. Let $\rho > 0$ and σ be an arbitrary complex number. Denote by $E_{\rho,\sigma}(z)$ the two parametric Mittag–Leffler functions:

$$E_{\rho,\sigma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + \sigma)}.$$

If $\sigma = 1$, then we have one parametric Mittag–Leffler function:

$$E_{\rho,\,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + 1)} = E_\rho(z).$$

Lemma 1. If $t > 0$ then there is a constant $C > 0$ such that for any $\sigma \in \mathbb{C}$ one has (see, e.g., [19], p. 136)

$$|E_{\rho,\sigma}(-t)| \le \frac{C}{1+t}. \tag{2}$$

Lemma 2. The Mittag–Leffler function of the negative argument $E_\rho(-t)$ is a monotonically decreasing function for all $0 < \rho < 1$ and

$$0 < E_\rho(-t) < 1.$$

If $\alpha \in (0,\,1)$, there is $\lambda_0 > 0$ such that $E_\rho(-\lambda_0 \tau_0^\rho) = \alpha$. We suppose $\lambda_k = \lambda_0$ and let it be a multiple of p_0. We denote by

$$K_0 = \{k_0, k_0+1, k_0+2, \dots, k_0+p_0-1\},$$

the set of numbers k by $\lambda_k = \lambda_0$.

Main part

Let us state the main theorem of the section below.

Theorem 1. Let $\varphi \in D(A)$ and $f \in H$.

If $\alpha \in (0,\,1)$ or $\alpha \notin [0,\,1)$, but $\lambda_k \ne \lambda_0$ for all $k \ge 1$, then the problem (1) has a unique solution and this solution has the form

$$u(t) = \sum_{k=1}^{\infty} \left[\frac{\varphi_k - V_k(\tau_0)}{E_\rho(-\mu_k \tau_0^\rho) - \alpha} E_\rho(-\mu_k t^\rho) + V_k(t) \right] v_k, \tag{3}$$

where

$$V_k(t) = \frac{f_k}{1+\gamma\lambda_k} t^\rho E_{\rho,\rho+1}(-\mu_k t^\rho).$$

If $\alpha \in (0,\,1)$ and $\lambda_k = \lambda_0$, $k \in K_0$, we assume that the orthogonality conditions

$$(\varphi, v_k) = 0, (f, v_k) = 0, \ \text{ for all } \ t > 0, \ k \in K_0, \ K_0 = \{k_0, k_0+1, \dots, k_0+p_0-1\}, \tag{4}$$

are satisfied. The solution of problem (1) has the form

$$u(t) = \sum_{k \notin K_0}^{\infty} \left[\frac{\varphi_k - V_k(\tau_0)}{E_\rho(-\mu_k \tau_0^\rho) - \alpha} E_\rho(-\mu_k t^\rho) + V_k(t) \right] v_k + \sum_{k \in K_0} b_k E_\rho(-\mu_k t^\rho)\, v_k. \tag{5}$$

with arbitrary coefficients b_k, $k \in K_0$.

Proof. Now, to prove the main theorem, problem (1) is divided into two auxiliary problems. The corresponding theorems and their proofs are presented.

Let us take $u(t) = V(t) + W(t)$ and solve the following two auxiliary problems:

$$\begin{cases} D_t^\rho V(t) + A(1+\gamma D_t^\rho)V(t) = f, & 0 < t \le T; \\ V(0) = 0, \end{cases} \tag{6}$$

$$\begin{cases} D_t^\rho W(t) + A(1+\gamma D_t^\rho)W(t) = 0, & 0 < t \le T; \\ W(\tau_0) = \alpha W(0) + \psi, & 0 < \tau_0 \le T, \end{cases} \tag{7}$$

where a function $\psi \in H$ is given.

It is easy to check that the function $u(t) = W(t) + V(t)$ solves problem (7) if we set $\psi = \varphi - V(\tau)$, and $W(t)$ and $V(t)$ are solutions of the problems (6) and (7), respectively.

The problem (6) has been considered in [14]. Therefore, the corresponding theorem is presented without proof.

Theorem 2. Let $f \in H$. Then, there is a unique solution to the problem (6), which takes the following form:

$$V(t) = \sum_{k=1}^{\infty} \frac{f_k}{1+\gamma\lambda_k} t^\rho E_{\rho,\rho+1}(-\mu_k t^\rho) v_k. \tag{8}$$

Let us present the following theorem for the problem (7).

Theorem 3. Let $\psi \in H$.

If $\alpha \notin [0,\, 1)$ or $\alpha \in (0,\, 1)$, but $\lambda_k \neq \lambda_0$ for all $k \geq 1$, the problem (6) exists a unique solution. This solution takes the form of

$$W(t) = \sum_{k=1}^{\infty} \frac{\psi_k}{E_\rho(-\mu_k \tau_0^\rho) - \alpha} E_\rho(-\mu_k t^\rho)\, v_k. \tag{9}$$

If $\alpha \in (0,\, 1)$ and $\lambda_k = \lambda_0, k \in K_0$, that the orthogonality conditions

$$\psi_k = (\psi, v_k) = 0, \;\; k \in K_0, \;\; K_0 = \{k_0, k_0+1, k_0+2, \ldots, k_0+p_0-1\}, \tag{10}$$

are satisfied.

The equation

$$W(t) = \sum_{k \neq K_0} \frac{\psi_k}{E_\rho(-\mu_k \tau_0^\rho) - \alpha} E_\rho(-\mu_k t^\rho)\, v_k + \sum_{k \in K_0} b_k E_\rho(-\mu_k t^\rho)\, v_k, \tag{11}$$

with arbitrary coefficients b_k, $k \in K_0$, is the solution to the problem (7).

Proof. The auxiliary problem (7) is now solved. We search in the following form for a series:

$$W(t) = \sum_{k=1}^{\infty} T_k(t)\, v_k.$$

After substituting this expression into (7), we have the following problem:

$$\begin{cases} D_t^\rho T_k(t) + \mu_k T_k(t) = 0, & 0 < t \leq T; \\ T_k(\tau_0) = \alpha T_k(0) + \psi_k, \end{cases} \tag{12}$$

where $\mu_k = \dfrac{\lambda_k}{1+\gamma\lambda_k}$ and ψ_k are the Fourier coefficients of the function $\psi \in H$.

Let $T_k(0) = b_k$ be denoted. Then, given this initial condition, the unique solution to the differential equation (12) takes the following form: $T_k(t) = b_k E_\rho(-\mu_k t^\rho)$ (see, e.g., [20], p.174).

Using the non-local condition of (12), we can find the unknown numbers b_k using the following equation:

$$b_k E_\rho(-\mu_k \tau_0^\rho) = \alpha b_k + \psi_k. \tag{13}$$

For every $\alpha \geq 1$ and $\alpha < 0$, by Lemma 2, the Mittag-Leffler function has $E_\rho(-\mu_k \tau_0^\rho) \neq \alpha$. Consequently, using (13), we have

$$b_k = \frac{\psi_k}{E_\rho(-\mu_k \tau_0^\rho) - \alpha}, \;\; |b_k| \leq C_\alpha |\psi_k|, \;\; k \geq 1 \;\text{ and }\; \alpha \geq 1 \;\text{ or }\; \alpha < 0, \tag{14}$$

here and below, we will refer to a constant as C_α, which will rely on α, but not always on.

Assume that $0 < \alpha < 1$. Then, by Lemma 2, there exists a unique $\lambda_0 > 0$ such that $E_\rho(-\mu_0 \tau^\rho) = \alpha$. The estimate in (14) holds provided that $\lambda_k \neq \lambda_0$ for all $k \geq 1$, with some constant $C_\alpha > 0$.

Therefore, if $\alpha \notin [0,\, 1)$ or $\alpha \in (0,\, 1)$, $\lambda_k \neq \lambda_0$ for $k \geq 1$, then the solution to the problem (7) has the form (9).

Lastly, consider $\alpha \in (0; 1)$ and $\lambda_k = \lambda_0$ for $K_0 = \{k_0, k_0+1, k_0+2, \ldots, k_0+p_0-1\}$, where the number p_0 is a multiple of the eigenvalue λ_{k_0}. If the function ψ satisfies the orthogonality condition (10), then the non-local problem (12) has a solution. So, if $k \in K_0$, then b_k arbitrary numbers are solutions to the equation (13). For all other $k \notin K_0$, b_k are defined by the following equality:

$$b_k = \frac{\psi_k}{E_\rho(-\mu_k\tau_0^\rho) - \alpha}, \;\; |b_k| \le C_\alpha|\psi_k|, \quad k \notin K_0. \tag{15}$$

Thus, in this case, the solution to the problem (7) has the form (11).

We shall assume that the orthogonality condition (10) is satisfied, whenever $\alpha \in (0;1)$ and $\lambda_k = \lambda_0$.

Let us prove the uniqueness of the solution to problem (7). The solution to problem (12) with the condition $\psi_k = 0$ has been defined subject to the condition $T_k(\tau_0) = \alpha T_k(0)$.

If $\alpha \notin [0,\,1)$, then according to Lemma 2, this implies $b_k \equiv 0$, and consequently $T_k(t) \equiv 0$. By the completeness of the system $\{v_k\}$, it follows that $W(t) \equiv 0$.

Consider $\alpha \notin [0,\,1)$ or $\alpha \in (0,\,1)$, but for any $k \ge 1$, $\lambda_k \ne \lambda_0$. Then $T_k(t) = 0$ in this situation due to the completeness of the set of eigenfunctions $\{v_k\}$, we can deduce that $W(t) \equiv 0$. In this case, the problem (7) has a unique solution.

Assume that $\alpha \in (0,\,1)$ and $k \in K_0$, with $\lambda_k = \lambda_0$. Therefore, in this case, the solution to the problem (7) is not unique.

The partial sum of the series (9) is denoted by $S_n(t)$. Next

$$AS_n(t) = \sum_{k=1}^{n} \lambda_k \frac{\psi_k}{E_\rho(-\mu_k\tau_0^\rho) - \alpha} E_\rho(-\mu_k t^\rho)\, v_k.$$

The Parseval's equality allows us to write

$$\|AS_n(t)\|^2 = \sum_{k=1}^{n} \lambda_k^2 \left| \frac{\psi_k}{E_\rho(-\mu_k\tau_0^\rho) - \alpha} E_\rho(-\mu_k t^\rho) \right|^2.$$

Using estimates (2), (14) and (15)

$$\|AS_n(t)\|^2 \le C_\alpha \sum_{k=1}^{n} \lambda_k^2 \left| \frac{\psi_k}{1+\mu_k t^\rho} \right|^2 \le C_\alpha t^{-2\rho} \sum_{k=1}^{n} \lambda_k^2 |\psi_k|^2.$$

Hence, if $\psi \in D(A)$, then $Au(t) \in C((0,\,T];H)$.

Let us evaluate $D_t^\rho S_n(t)$. First, we find $D_t^\rho S_n(t)$

$$D_t^\rho S_n(t) = -(I+\gamma A)^{-1} AS_n(t) = -\sum_{k=1}^{n} \frac{\lambda_k}{1+\gamma\lambda_k} \frac{\psi_k}{E_\rho(-\mu_k\tau_0^\rho) - \alpha} E_\rho(-\mu_k t^\rho)\, v_k.$$

Applying Parseval's equality, estimates (2), (14) and (15)

$$\|D_t^\rho S_n(t)\|^2 = \sum_{k=1}^{n} \left| \frac{\mu_k\psi_k}{E_\rho(-\mu_k\tau_0^\rho) - \alpha} E_\rho(-\mu_k t^\rho) \right|^2 \le C_\alpha \sum_{k=1}^{n} \frac{|\psi_k|^2}{(1+\mu_k t^\rho)^2} \le C_\alpha t^{-2\rho} \sum_{k=1}^{n} |\psi_k|^2.$$

Let us evaluate $AD_t^\rho S_n(t)$. First, we find $AD_t^\rho S_n(t)$

$$A(D_t^\rho S_n(t)) = -A((I+\gamma A)^{-1} AS_n(t)) = -\sum_{k=1}^{n} \lambda_k\mu_k \frac{\psi_k}{E_\rho(-\mu_k\tau_0^\rho) - \alpha} E_\rho(-\mu_k t^\rho)\, v_k.$$

Applying Parseval's equality, estimates (2), (14), and (15)

$$A(D_t^\rho S_n(t)) = \left| \sum_{k=1}^{n} \lambda_k\mu_k \frac{\psi_k}{E_\rho(-\mu_k\tau_0^\rho) - \alpha} E_\rho(-\mu_k t^\rho) \right|^2$$

$$\le C_\alpha \sum_{k=1}^{n} \lambda_k^2 \frac{|\psi_k|^2}{(1+\mu_k t^\rho)^2} \le C_\alpha t^{-2\rho} \sum_{k=1}^{n} \lambda_k^2 |\psi_k|^2.$$

Hence, if $\psi \in D(A)$, then $A(D_t^\rho u(t)) \in C((0,\,T];H)$.

REFERENCES

1. Lizama, C. Abstract linear fractional evolution equations. In Handbook of Fractional Calculus with Applications, Vol. 2, De Gruyter, Berlin, 2019, 465–497.

2. Barenblatt, G. I.; Zheltov, Yu. P.; Kochina, I. N. On finiteness conditions in the mechanics of continuous media. Static problems of the theory of elasticity. *Prikl. Mat. Meh.* **24** (1960), no. 5, 316–322.

3. Barenblatt, G. I.; Zheltov, Yu. P. Fundamental equations of filtration of homogeneous liquids in fissured rocks. *Sov. Phys. Dokl.* **132** (1960), no. 5, 522–525.

4. Kochina, P. Ya. Development of research on the theory of filtration in the USSR (1917–1967). Nauka, Moscow, 1969.

5. Barenblatt, G. I.; Yentov, V. M.; Ryzhik, V. M. Movement of liquids and gases in natural reservoirs. Nedra, Moscow, 1984.

6. Basniev, K. S.; Kochina, I. N.; Maksimov, V. M. Underground hydromechanics. Nedra, Moscow, 1993.

7. Polubarinova-Kochina, P. Ya. The theory of groundwater movement. Nauka, Moscow, 1977.

8. Sviridyuk, G. A.; Shafranov, D. E. The Cauchy problem for the Barenblatt–Zheltov–Kochina equation on a smooth manifold. *Vest. ChelGU* **9** (2003), 171–177.

9. Sagadeeva, M. A.; Hasan, F. L. Bounded solutions of Barenblatt–Zheltov–Kochina model in quasi-Sobolev spaces. *Bull. South Ural State Univ. Ser. Math. Model. Program. Comput. Softw.* **8** (2015), no. 4, 138–144.

10. Fedorov, V. E.; Nagumanova, A. V. Inverse linear problems for a certain class of degenerate fractional evolution equations. *Results Sci. Technol. Ser. Modern Math. Appl. Thematic Rev.* **167** (2019), 97–111.

11. Umarov, Kh. G. Explicit solution of the mixed problem in an anisotropic half-space for the Barenblatt–Zheltov–Kochina equation. *Vladikavk. Mat. Zh.* **15** (2013), no. 1, 51–64.

12. Beshtoshov, M. Kh. Boundary-value problems for degenerating pseudoparabolic equations with Gerasimov–Caputo fractional derivative. *Izv. Vyssh. Uchebn. Zaved. Mat.* **10** (2018), 3–16.

13. Ashurov, R. R.; Fayziev, Yu. E.; Khushvaktov, N. Kh. Some problems for the Barenblatt–Zheltov–Kochina type fractional equations. *Bull. Inst. Math.* **5** (2022), no. 5, 97–104.

14. Ashurov, R. R.; Fayziev, Yu. E.; Khushvaktov, N. Kh. Forward and inverse problems for the Barenblatt–Zheltov–Kochina type fractional equations. *Lobachevskii J. Math.* **44** (2023), no. 7, 2567–2576.

15. Ashurov, R. R.; Fayziev, Yu. E.; Khushvaktov, N. Kh. Non-local problem in time for the Barenblatt–Zheltov–Kochina type fractional equations. *Lobachevskii J. Math.* **44** (2023), no. 12, 5164–5178.

16. Ashurov, R. R.; Vaisova, N. Z. Backward and non-local problems for the Rayleigh–Stokes equation. *Fractal Fract.* **6** (2022), 587.

17. Ashurov, R. R.; Muhiddinova, O. T.; Umarov, S. R. A non-local problem for the fractional-order Rayleigh–Stokes equation. *Fractal Fract.* **7** (2023), 490.

18. Ashurov, R. R.; Fayziev, Yu. E. On the non-local boundary value problems for time-fractional equations. *Fractal Fract.* **6** (2022), 41.

19. Dzherbashian, M. M. Integral transforms and representation of functions in the complex domain. Nauka, Moscow, 1966.

20. Pskhu, A. V. Fractional partial differential equations. Nauka, Moscow, 2005.

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The non uniform bounds of remainder term in CLT for the sum of functions of k-spacings

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Abstract. The paper establishes a non-uniform bound on the remainder term in the central limit theorem for sum of functions of disjoint uniform k-spacings, where the step size k may increase together with the sample size.

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Key words: Uniform spacings, central limit theorem, non-uniform bound, exponential distribution.

Introduction

Let $U_1, U_2, \ldots, U_{n-1}$ be an ordered sample of size $n-1$ from a uniform $[0,1]$ distribution, $U_0 = 0$, $U_n = 1$, $G_i^{(k)} = U_{ik} - U_{(i-1)k}$, $i = 1, 2, \ldots, N'$, $G_{N+1}^{(k)} = 1 - U_{N'k}$ their k-spacings, where $N' = \lfloor n/k \rfloor$ is the integer part of n/k. Let $N = N'$ if n/k is an integer and $N = N' + 1$ otherwise, and let $f_m(u)$, $m = 1, 2, \ldots, N$ be a set of measurable functions. We consider the statistics of the type

$$R_n(G) = \sum_{m=1}^{N} f_m\left(nG_m^{(k)}\right), \tag{1}$$

where $k = k(n)$ may increase to infinity jointly with n.

Obviously, all these random variables (r.v.s) depend on the sample size n. However, we omit here and below the corresponding suffix for notational simplicity.

Statistics of the type (1) are of great interest in several contexts, including hypothesis testing and reliability, circular data analysis where they play a pivotal role because they provide a maximal invariant under the rotation group, and spacings-based parameter estimation, just to name a few applications. There is a huge literature devoted to such statistics and their use; see [5], [3] and references therein. Given that these spacings are highly dependent random quantities with a Dirichlet distribution in finite samples, large-sample theory is the main avenue for studying such statistics.

Statistics of the form $R_n(G)$, which represent sums of functions of disjoint uniform k-spacings, are also employed in applied tasks such as testing for dispersive ordering and addressing problems related to random coverage of the circle. This is particularly significant when k increases with the sample size n, as it enables the construction of goodness-of-fit tests based on statistics of this type to verify hypotheses about the form of the distribution, discriminating alternatives at distances $(nk)^{-1/4}$ that can be arbitrarily close to $n^{-1/2}$ (see [3]).

In [2] a bound of the Berry-Esseen type was obtained, whereas in [6] a non-uniform bound on the remainder term is established in the case $k = 1$.

In this work we obtain a non-uniform estimate that holds for arbitrary integer $k = k(n)$, including the case when k increases together with the sample size n. Thus our result extends results of [2] and [6].

Most common and well-known examples of spacings tests of the form (1) are the so-called Greenwood statistic $G_N^2 = \sum_{m=1}^{N}(nG_m^{(k)} - k)^2$ and the Log-spacings statistic $M_N = \sum_{m=1}^{N} \log(nG_m^{(k)})$.

The main results of the present paper are given in Results. The proof of the assertions of Results is presented in Proofs.

In what follows, C, C_i are positive constants and D_t^k is the k-th order derivative.

Results

We suppose that the moments used below exist. Set $G = (G_1, \ldots, G_N)$, where $G_i = G_i^{(1)}$, $i = 1, 2, \ldots, n$, and let $Y_1, Y_2, \ldots$ be independent standard exponentially distributed r.v.s, $Y = (Y_1, \ldots, Y_n)$, $S_n = Y_1 + \cdots + Y_n$. Then $\mathcal{L}(nG) = \mathcal{L}(Y/S_n = n)$, where $\mathcal{L}(X)$ denotes the distribution of the random vector X. Put

$$Z_{m,k} = Y_{(m-1)k+1} + \cdots + Y_{mk}$$

where $m = 1, 2, \ldots, N'$,

$$Z_{N,k} = Z_{N'+1,k} = Y_{N'k+1} + \cdots + Y_n$$

if n/k is not an integer, and $Z_{N,k} = 0$ otherwise; $S_{N,k} = Z_{1,k} + \cdots + Z_{N,k}$;

$$R_n(Z) = \sum_{m=1}^{N} f_m(Z_{m,k}); \quad \rho = \operatorname{corr}(R_N(Z), S_{N,k});$$

$$g_m(u) = f_m(u) - Ef_m(Z_{m,k}) - (u-k)\rho\sqrt{\operatorname{Var} R_N(Z)/Nk}$$

$$T_N(G) = \sum_{m=1}^{N} g_m(nG_m^{(k)}), \quad T_N(Z) = \sum_{m=1}^{N} g_m(Z_{m,k}).$$

Note that $\sigma_N^2 = \operatorname{Var} T_N(Z) = (1-\rho^2)\operatorname{Var} R_N(Z)$ and

$$ET_N(Z) = 0, \quad \operatorname{cov}(T_N(Z), S_{N,k}) = 0. \tag{2}$$

From the definition of σ_N^2, it follows that $\sigma_N^2 = 0$ if and only if $f_m(u) = Cu + b_m$, $m = 1, 2, \ldots, N$, where constants b_m are arbitrary and C does not depend on m for all $m = 1, \ldots, N$. We suppose that $\sigma_N^2 > 0$ for all $N = 1, 2, \ldots$.

Put $\bar{g}_m = g_m(Z_{m,k})/\sigma_N$, $\bar{Z}_{m,k} = (Z_{m,k} - k)/\sqrt{Nk}$

$$\beta_{jN} = \sum_{m=1}^{N} E|\bar{g}_m|^j, \quad P_N(x) = P[T_N(G) < x\sigma_N], \quad \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left\{-\frac{t^2}{2}\right\} dt. \tag{3}$$

Theorem. There is a constant $C(s) > 0$ such that for arbitrary integer $s > 2$ and $n > s+1$

$$\Delta_N(x) = |P_N(x) - \Phi(x)| \le \frac{C(s)}{1+|x|^{s-2}}\left(\beta_{3N} + \beta_{sN}\right).$$

For the Greenwood statistic $f_m(u) = (u-k)^2$ we have $\sigma_N^2 = 2Nk(k+1)$.

Corollary 1. Applying the theorem for $s = 4$, a positive constant $C > 0$ exists such that, whenever $n > 5$,

$$\left|P\left(\frac{G_N^2 - EG_N^2}{\sigma_N} < x\right) - \Phi(x)\right| \le \frac{C}{1+|x|^2}\left(\frac{15k^3 + 222k^2 + 579k + 372}{Nk(k+1)^2}\right)^{1/2}$$

For the Log-spacings statistic $f_m(u) = \log u$ we have $\sigma_N^2 = N(\zeta(2,k) - k^{-1})$.

Corollary 2. From the theorem with $s = 4$, there is some $C > 0$ ensuring that for $n > 5$,

$$\left|P\left(\frac{M_N - EM_N}{\sigma_N} < x\right) - \Phi(x)\right| \le \frac{C}{1+|x|^2}\left(\frac{3k^{-2} - 2k^{-3} - 6k^{-1}\zeta(2,k) + 3\zeta^2(2,k) + 6\zeta(4,k)}{N(\zeta(2,k) - k^{-1})^2}\right)^{1/2}$$

The assertions in the corollaries can be obtained using the well-known inequality $\beta_{3,N} \le \beta_{4,N}^{1/2}$ and by direct calculation of $\beta_{4,N}$ for the corresponding statistics, which were computed in [3].

Proofs

For simplicity of computations, we suppose that n/k is an integer; the case when n/k is not an integer needs only quite clear additional computations.

Set $\varphi_N(t) = E\exp\{itT_N(G)/\sigma_N\}$. By Corollary 11.5 and Lemma 11.6 of [1] one has for arbitrary $T > 0$

$$(1+|x|^{s-2})\Delta_N(x) \le C_0 \max_{0\le k\le s} \int_{|t|\le T} \left| D_t^k(\varphi_N(t) - \exp\left\{-\frac{t^2}{2}\right\} \right| dt + \frac{C_1}{T} \tag{4}$$

where D_t^k denote k-th derivation. Denote

$$\psi_m(t,\tau) = E\exp\left\{it\bar{g}_m + i\tau\bar{Z}_{m,k}\right\}, \quad \Psi_N(t,\tau) = \prod_{m=1}^{N}\psi_m(t,\tau)$$

By equality (4) of [2] we have

$$\varphi_N(t) = \frac{1}{2\pi\sqrt{n}p_n(n)}\int_{-\infty}^{\infty}\Psi_N(t,\tau)d\tau. \tag{5}$$

Set

$$A(t,\tau) = \{(t,\tau) : |t| \le C_4\beta_{3N}^{-1}, \tau\in(-\infty,\infty)\}$$
$$A_1(t,\tau) = \left\{(t,\tau) : |t| \le \beta_{sN}^{-1/s},\ |\tau| \le N^{(s-2)/2s}\right\},$$
$$A_2(t,\tau) = \left\{(t,\tau) : |t| \le C_4\beta_{3N}^{-1},\ |\tau| \le \frac{1}{6}\sqrt{N}\right\},$$
$$A_3(t,\tau) = \left\{(t,\tau) : |t| \le C_4\beta_{3N}^{-1}, |\tau| > \frac{1}{6}\sqrt{N}\right\},$$

We need to choose C_4 sufficiently small for (16) to hold.

Using representation (5) for $\varphi_N(t)$ and taking into account inequality (4), we have

$$\begin{aligned}
J_k &:= \int_{|t|\le\beta_{3N}^{-1}} |D_t^k(\varphi_N(t) - \exp\{-t^2/2\})|\,dt \\
&\le \frac{1}{2\pi p_n(n)}\Bigg[\iint_{A_1(t,\tau)} |D_t^k(\Psi_N(t,\tau) - \exp\{-(t^2+\tau^2)/2\})|\,dt\,d\tau \\
&\quad + \iint_{A_2(t,\tau)-A_1(t,\tau)} |D_t^k\Psi_N(t,\tau)|\,dt\,d\tau + \iint_{A_3(t,\tau)} |D_t^k\Psi_N(t,\tau)|\,dt\,d\tau \\
&\quad + \iint_{A(t,\tau)-A_1(t,\tau)} |t|^k\exp\{-(t^2+\tau^2)/2\}\,dt\,d\tau\Bigg] \\
&\quad + \frac{1}{\sqrt{2\pi}}\left|\frac{1}{\sqrt{2\pi p_n(n)}} - 1\right| \iint_{A(t,\tau)} |t|^k\exp\{-(t^2+\tau^2)/2\}\,dt\,d\tau.
\end{aligned} \tag{6}$$

Let the symbols $J_{1k}, J_{2k}, J_{3k}, J_{4k}$ be summands in the brackets, correspondingly, and J_{5k} be the outside of the summand bracket on the right hand side of (6).

Lemma 1. 1. If $(t,\tau)\in A_1(t,\tau)$ then for each $k : 0\le k\le s$ there is a constant $C_6(s,k)$ such that

$$|D_t^k(\Psi_N(t,\tau) - \exp\{-(t^2+\tau^2)/2\})| \le C_6(s,k)\,(\beta_{3N}+\beta_{sN})$$
$$\left(1+|t|^{3(s-2)+k} + |\tau|^{3(s-2)+k}\right)\exp\{-(t^2+\tau^2)/4\}.$$

2. If $(t,\tau)\in A_2(t,\tau)$, then for $k = 0,1,\ldots,s$

$$|D_t^k\Psi_N(t,\tau)| \le (t^2+\tau^2)^k\exp\{-(t^2+\tau^2)/4\}.$$

Proof. Let $P_r(t,\tau), r=1,2,\dots$ be a well-known polynomials on the theory of asymptotical expansion of a characteristic functions of a sum of independent r.v.s. (see, [1], the functions $\tilde{P}_r(iBt,\{\chi_\nu\})$, p.52, 82). From Theorem 9.11 ([1]) and properties (2) it follows that there is constant $C_7(s,k)$ such that for each $k: 0\le k\le s,\dots$ and $(t,\tau)\in A_1(t,\tau)$ the inequality

$$\left|D_t^k\left(\Psi_N(t,\tau)-\exp\left\{-\frac{t^2+\tau^2}{2}\right\}\left(1+\sum_{r=1}^{s-3}\frac{P_r(t,\tau)}{N^{r/2}}\right)\right)\right| \\ \le C_7(s,k)\left(\beta_{sN}+N^{-(s-2)/2}\right)\left(1+(t^2+\tau^2)^{3(s-2)+k}\right)\exp\{-(t^2+\tau^2)/4\} \tag{7}$$

holds true. A same reasoning as in proof Lemma 9.5 ([1], p.71) give us that

$$|N^{-r/2}P_r(t,\tau)|\le C_8(r)(\beta_{r+2,N}+N^{-r/2})(1+(t^2+\tau^2)^{3r-k}) \tag{8}$$

The inequalities (7) and (8) and $\beta_{kN}\le\beta_{3N}+\beta_{sN}, 3\le k\le s$, implies part 1 of Lemma 1.

Put $Q_r=(q_1,\dots,q_r)$ is an r subset of the set $L=(1,\dots,N), r\ge 0, Q_0=\emptyset$, and (Q_r) is collection of all Q_r. It is easy to see that

$$|D_t^k\Psi_N(t,\tau)|\le\sum_{r=0}^{k}C(r,k)\sum_{(Q_r)}\prod_{i\in L-Q_r}|\Psi_{iN}(t,\tau)|\prod_{q\in Q_r}|D_t^{\gamma_q}\Psi_{qN}(t,\tau)|, \tag{9}$$

where $\gamma_{q_1},\dots,\gamma_{q_r}$ are non negative integers such that $\gamma_{q_1}+\dots+\gamma_{q_r}=k$. We have

$$|\psi_m(t,\tau)|^2\le 1-E(t\bar{g}_m+\tau\bar{Z}_{m,k})^2+\frac{2}{3}E|t\bar{g}_m+\tau\bar{Z}_{m,k}|^3 \\ \le\exp\{-E(t\bar{g}_m+\tau\bar{Z}_{m,k})^2+\frac{2}{3}E|t\bar{g}_m+\tau\bar{Z}_{m,k}|^3\}. \tag{10}$$

Thereto, using inequality between moments of r.v.s, we get

$$\exp\left\{E(t\bar{g}_m+\tau\bar{Z}_{m,k})^2-\frac{2}{3}E|t\bar{g}_m+\tau\bar{Z}_{m,k}|^3\right\}\le e^{1/3}$$

since $\max_{y\ge 0}(y^2-\frac{2}{3}y^3)\le\frac{1}{3}$. Hence, recollecting (3) and that $|a+b|^3\le 4(|a|^3+|b|^3)$, we obtain

$$\prod_{i\in L-Q_r}|\psi_i(t,\tau)|\le e^{r/3}\exp\{-(t^2+\tau^2)/4\}. \tag{11}$$

Obviously

$$|D_t\psi_m(t,\tau)|\le E|\bar{g}_m|, \tag{12}$$

and

$$|D_t^k\psi_m(t,\tau)|\le E|\bar{g}_m|^k\le E\bar{g}_m^2+E|\bar{g}_m|^s \tag{13}$$

Putting $d_m=\max\{E|\bar{g}_m|, E\bar{g}_m^2+E|\bar{g}_m|^s\}$ we get

$$\sum_{(Q_r)}\prod_{q\in Q_r}|D_t^{\gamma_q}\psi_q(t,\tau)|\le\left(\sum_{m=1}^{N}d_m\right)^r\le C(s)\max(1,|t|+|\tau|)^r, \tag{14}$$

The second part of Lemma 1 follows from (9), (11), and (14).

Using Lemma 1 we find that

$$|J_{1k}+J_{2k}|\le C(s)(\beta_{sN}+N^{-(s-2)/2})\le C(s)(\beta_{3N}+\beta_{sN}), \tag{15}$$

since, from the definition in (3), one has the lower bound $N^{-(s-2)/2}\le\beta_{sN}$.

With the aid of the inequality $x < \exp(x-1)$ we have for any $m = 1, \ldots, N$ and $(t,\tau) \in A_3(t,\tau)$

$$\begin{aligned}
&|\psi_m(t,\tau)| = \left|E\exp\{i\tau\bar{Z}_{m,k}\}(\exp\{it\bar{g}_m\}-1)\right| \le \left|E\exp\{i\tau\bar{Z}_{m,k}\}\right| \\
&+|t|E|\bar{g}_m| \le \exp\left\{-\left(1-\left|E\exp\{i\tau\bar{Z}_{m,k}\}\right|\right)+|t|E|\bar{g}_m|\right\} \\
&\le \exp\left\{-2C_0+|t|E|\bar{g}_m|\right\}
\end{aligned} \tag{16}$$

because $\left|E\exp\{i\tau\bar{Z}_{m,k}\}\right| = \left(1+\tau^2/(N)\right)^{-k/2}$ and $|\tau| > \frac{1}{6}\sqrt{N}$. Using (12), and that $|D_t\psi_m(t,\tau)| \le E|\bar{g}_m|$ we get

$$\sum_{(Q_r)}\left(\prod_{q\in Q_r}|D_t^{\gamma_q}\psi_q(t,\tau)|\right) \le \left(\sum_{m=1}^{N}\left(E|\bar{g}_m|+E\bar{g}_m^2+E|\bar{g}_m|^s\right)\right)^r \le C(r)N^{r/2}. \tag{17}$$

From (9), (16), (17) and (6) we have

$$J_{3k} \le C(k)N^{(k+1)/2}\exp\{-C_0N\} \tag{18}$$

because $N > s+1$, and $|t| \le C_4\beta_{3N}^{-1} \le C_0\sqrt{N}$. For J_{4k} the obvious estimate

$$|J_{4k}| \le C(\beta_{sN}+N^{-(s-2)/2}) \le C(\beta_{3N}+\beta_{sN}) \tag{19}$$

is true.

Since $p_n(n) = n^{n-1}(n-1)!^{-1}\exp(-n)$ then with the aid of Stirling's formula we obtain

$$\left|\frac{1}{\sqrt{2\pi\sqrt{n}p_n(n)}}-1\right| \le \frac{C}{n}. \tag{20}$$

Hence the bound

$$|J_{5k}| \le \frac{C}{n} \le C(\beta_{3N}+\beta_{sN}) \tag{21}$$

is true (noting that $\beta_{3N} \ge 1/\sqrt{N}$ and $N \approx n/k$).

In (6) we replace summands by their bounds from (8), (15), (18), (19) and (21). Then we find

$$|J_k| \le C\left(\beta_{3N}+\beta_{sN}\right). \tag{22}$$

Putting in (4) $T = C_4\beta_{3N}^{-1}$ and using (22) we complete the proof of the theorem.

REFERENCES

1. Bhattacharya R.N., and Rao R. R. Normal Approximation and Asymptotic Expansions. New York, Wiley. (1976).
2. Mirakhmedov, S. A. Lower estimation of the remainder term in the CLT for a sum of the functions of k-spacings. *Statist. Probab. Lett.* **73** (2005), no. 4, 411–424.
3. Mirakhmedov, S. M. Jammalamadaka, S. R. Higher-order expansions and efficiencies of tests based on spacings. *J. Nonparametr. Stat.* **25** (2013), no. 2, 339–359.
4. Holst, L. Asymptotic normality of sum-functions of spacings. *Ann. Probab.* **7** (1979), 1066–1072.
5. Pyke, R. Spacings (with discussion). *J. Roy. Statist. Soc. Ser. B* **27** (1965), 395–449.
6. Mirakhmedov, S.; Kalandarov, U. The non-uniform bounds of remainder term in CLT for the sum of functions of uniform spacings. *Turkish J. Math.* **22** (1998), no. 1, 53–60.

7. Petrov, V. V. Sums of independent random variables *Science*. Moscow. 1972.

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Modeling and Analysis of Rayleigh-Type Surface Waves in Elastic Solids with Double Porosity

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Abstract. This investigation examines Rayleigh-type surface waves in an isotropic, homogeneous elastic half-space possessing a dual-porosity structure. The surface is considered stress-free. From the general analysis, the frequency equations for elastic media with single porosity are recovered as a limiting case. Numerical solutions of the derived equations are obtained. Graphical representations for copper material illustrate the dependence of Rayleigh wave speed and attenuation coefficient on wave number.

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Key words: dual porosity; elastic wave propagation; surface waves; frequency equation.

Introduction

Rayleigh-type surface waves play a fundamental role in elastodynamics and seismology, as they propagate along free surfaces with amplitudes that decay exponentially with depth [2]. Due to their strong localization near the surface, Rayleigh waves are responsible for a significant portion of ground motion observed during seismic events and are therefore of central importance in earthquake engineering, near-surface geophysics, and nondestructive material characterization [2,3].

Classical Rayleigh wave theory, originally established for homogeneous, isotropic elastic half-spaces, has been extensively extended to account for additional physical effects such as thermal fields, diffusion processes, electromagnetic coupling, rotation, and material anisotropy. In particular, thermoelastic and diffusion-related extensions have demonstrated that coupled fields can significantly modify dispersion and attenuation characteristics of surface waves, as shown in a series of studies by Kumar and co-authors for thermoelastic diffusion and microstructured media [4,5].

Similarly, Abd-Alla and collaborators investigated Rayleigh wave propagation under magnetic, thermal, rotational, and viscoelastic effects, revealing substantial deviations from classical elastic behavior in granular and orthotropic media [8,9].

In parallel with these developments, the modeling of porous and fractured materials has emerged as a critical challenge in solid and geophysical mechanics. Classical single-porosity theories, while effective for relatively uniform pore distributions, are often inadequate for materials characterized by complex multiscale pore networks, such as fractured rocks, reservoir formations, and certain engineered composites. To address this limitation, dual-porosity theories were introduced, accounting for two interacting pore systems-typically representing matrix porosity and fissures or micro-macro pore structures. Early foundational contributions in this area include the seepage and consolidation models of Barenblatt and Biot, which laid the groundwork for modern poroelastic formulations [10,11].

More rigorous continuum-mechanical frameworks for materials with double porosity were subsequently developed and analyzed. In particular, the works of Iesan and Quintanilla established thermodynamically consistent field equations for elastic and thermoelastic solids with double porosity, while Svanadze provided

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fundamental solutions, uniqueness theorems, and stability analyses for elastic, viscoelastic, and thermoelastic media possessing dual pore structures [12,13].

These studies demonstrated that the presence of interacting pore systems introduces additional internal variables and coupling mechanisms, which can substantially alter wave propagation behavior [14].

Despite these advances, the investigation of Rayleigh-type surface waves in elastic solids with double porosity remains comparatively limited. Surface waves in such media are of particular interest because they are highly sensitive to near-surface microstructural features, including pore connectivity, fissure density, and fluid-solid interactions. Recent studies on surface wave propagation in elastic and poroelastic half-spaces, including seismic excitation models and harmonic wave formulations, have highlighted the importance of refined theoretical models for accurately capturing near-surface wave phenomena [15,16].

Moreover, modern applications such as synthetic accelerogram generation, baseline correction in seismic modeling, and stability analysis of dynamic half-space problems further emphasize the need for advanced wave models that incorporate complex material structures [17,18].

Motivated by these considerations, the present study investigates Rayleigh-type surface wave propagation in a homogeneous, isotropic elastic half-space endowed with a dual-porosity structure. Based on the continuum model of materials with double porosity, the governing field equations are formulated and reduced to a coupled system describing mechanical displacements and porosity fields. By imposing stress-free boundary conditions at the surface, an explicit frequency (secular) equation for Rayleigh waves is derived. Numerical solutions of this equation are then obtained to analyze the dispersion and attenuation characteristics of surface waves, with particular emphasis on the influence of wave number and phase velocity [19,20].

The results demonstrate that dual porosity induces pronounced oscillatory behavior in both phase velocity and attenuation, distinguishing the response of double-porosity media from that of classical single-porosity or purely elastic solids. These findings provide new insight into surface wave behavior in fractured and multiscale porous materials and may contribute to improved interpretation of seismic data, nondestructive evaluation techniques, and wave-based characterization of complex elastic media [21,22].

Governing Relations

Based on the model by Iesan and Quintanilla [14], the constitutive and field equations for a dual-porosity elastic solid, in the absence of body forces, are:

Stress-Porosity Relations

$$\begin{aligned} t_{ij} &= \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + b\,\delta_{ij}\phi + d\,\delta_{ij}\psi, \\ \sigma_i &= \alpha\,\phi_{,i} + b_1\,\psi_{,i}, \\ \tau_i &= b_1\,\phi_{,i} + \gamma\,\psi_{,i}. \end{aligned} \tag{1}$$

Momentum Balance

$$\mu\nabla^2 u_i + (\lambda+\mu)u_{j,ji} + b\phi_{,i} + d\psi_{,i} = \rho\ddot{u}_i, \tag{2}$$

Porosity Evolution Equations

$$\begin{aligned} \alpha\nabla^2\phi + b_1\nabla^2\psi - bu_{r,r} - \alpha_1\phi - \alpha_3\psi = \kappa_1\ddot{\phi}, \\ b_1\nabla^2\phi + \gamma\nabla^2\psi - du_{r,r} - \alpha_3\phi - \alpha_2\psi = \kappa_2\ddot{\psi}, \end{aligned} \tag{3}$$

Where: $\lambda, \mu-$ Lam constants; $\rho-$ density; u_i- displacements; $t_{ij}-$ stress; κ_1, κ_2- equilibrated inertia; $\phi, \psi-$ porosity fields (pores and fissures); σ_i, τ_i- associated equilibrated stresses; $b, d, b_1, \gamma, \alpha_i-$ constitutive constants; $\delta_{ij}-$Kronecker delta; dot indicates time derivative. The operators are:

$$\nabla = \hat{i}\partial_{x_1} + \hat{j}\partial_{x_2} + \hat{k}\partial_{x_3}, \;\; \nabla^2 = \partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3}.$$

Problem Statement

Consider a half-space $x_3 \geq 0$. The x_1 -axis is the propagation direction, x_3 points inward. All fields are independent of x_2.

Wave Solution

Consider a half-space $x_3 \geq 0$. The x_1 -axis is the propagation direction, x_3 points inward. All fields are independent of x_2. Introduce dimensionless variables (primed):

$$\begin{array}{l} x'_i = \frac{\omega_1}{c_1} x_i, \;\; u'_i = \frac{\omega_1}{c_1} u_i, \;\; t'_{ij} = \frac{t_{ij}}{\lambda}, \\ \phi' = \frac{\kappa_1 \omega_1^2}{\alpha_1} \phi, \;\; \psi' = \frac{\kappa_1 \omega_1^2}{\alpha_1} \psi, \;\; t' = \omega_1 t, \\ \sigma'_1 = \frac{c_1}{\alpha \omega_1} \sigma_i, \;\; \tau'_1 = \frac{c_1}{\alpha \omega_1} \tau_1, \end{array} \tag{4}$$

with $c_1^2 = (\lambda + 2\mu)/\rho$ and $\omega_1 = \lambda / \kappa_1$.

Applying (4) to (2)-(3) and dropping primes yields:

$$\begin{array}{l} \left(\frac{\lambda+\mu}{\rho c_1^2}\right) \frac{\partial e}{\partial x_1} + \frac{\mu}{\rho c_1^2} \nabla^2 u_1 + a_1 \frac{\partial \phi}{\partial x_1} + a_2 \frac{\partial \psi}{\partial x_1} = \frac{\partial^2 u_1}{\partial t^2}, \\ \left(\frac{\lambda+\mu}{\rho c_1^2}\right) \frac{\partial e}{\partial x_3} + \frac{\mu}{\rho c_1^2} \nabla^2 u_3 + a_1 \frac{\partial \phi}{\partial x_3} + a_2 \frac{\partial \psi}{\partial x_3} = \frac{\partial^2 u_3}{\partial t^2}, \\ a_3 \nabla^2 \phi + a_4 \nabla^2 \psi - a_5 e - a_6 \phi - a_7 \psi = \frac{\partial^2 \phi}{\partial t^2}, \\ a_8 \nabla^2 \phi + a_9 \nabla^2 \psi - a_{10} e - a_{11} \phi - a_{12} \psi = \frac{\partial^2 \psi}{\partial t^2}, \end{array} \tag{5}$$

where the dimensionless coefficients a_1 to a_{12} are combinations of material constants, and $e = u_{1,1} + u_{3,3}$.

Displacements are expressed via potentials:

$$u_1 = \frac{\partial \phi_1}{\partial x_1} - \frac{\partial \psi_1}{\partial x_3}, \quad u_3 = \frac{\partial \phi_1}{\partial x_3} + \frac{\partial \psi_1}{\partial x_1}, \tag{6}$$

Substituting (6) into (5) gives:

$$\begin{array}{l} \left(\nabla^2 - \partial_t^2\right) \phi_1 + a_1 \phi + a_2 \psi = 0 \\ -a_5 \nabla^2 \phi_1 + \left(a_3 \nabla^2 - a_6 - \partial_t^2\right) \phi + \left(a_4 \nabla^2 - a_7\right) \psi = 0 \\ -a_{10} \nabla^2 \phi_1 + \left(a_8 \nabla^2 - a_{11}\right) \phi + \left(a_9 \nabla^2 - a_{12} - \partial_t^2\right) \psi = 0 \end{array} \tag{7}$$

and an equation for ψ_1 :

$$a_{12} \nabla^2 - \partial_t^2) \psi_1 = 0. \tag{8}$$

Wave Solution

Assume harmonic propagation:

$$[\phi_1, \phi, \psi, \psi_1] = [\phi_1^*, \phi^*, \psi^*, \psi_1^*] \exp\left[i\xi(x_1 - ct)\right]. \tag{9}$$

Here ξ is wave number, c is phase velocity, and $\omega = \xi c$. Inserting (9) into (7) leads to a system whose solvability condition provides the characteristic equation:

$$E_1 \frac{d^6}{dz^6} + E_2 \frac{d^4}{dz^4} + E_3 \frac{d^2}{dz^2} + E_4 = 0 \tag{10}$$

with coefficients E_i depending on a_i, ξ, and c. For ψ_1^*:

$$\left(\frac{d^2}{dx_3^2} - \zeta_4^2\right) \psi_1^* = 0, \;\; \zeta_4^2 = \xi^2 \left(1 + \frac{\phi^2}{a_{12}}\right). \tag{11}$$

For surface waves decaying with depth $(x_3 \to \infty)$:

$$(\phi_1, \phi, \psi) = \sum_{i=1}^{3} (1, r_i, s_i) B_i \exp(-m_i x_3 + i\xi(x_1 - ct)), \tag{12}$$

$$\psi_1 = B_4 \exp(-m_4 x_3 + i\xi(x_1 - ct)) \tag{13}$$

where $m_4 = \zeta_4$, B_i are amplitudes, and r_i, s_i are coupling constants derived from the system.

Surface Conditions and Frequency Equation

At the free surface $x_3 = 0$:

$$t_{33} = 0, \ t_{31} = 0, \ \ \sigma_3 = 0, \ \ \tau_3 = 0. \tag{14}$$

In dimensionless form:

$$\begin{aligned} t_{33} &= p_1 u_{3,3} + u_{1,1} + p_2\phi + p_3\psi, \\ t_{31} &= p_4\left(u_{3,1} + u_{1,3}\right), \\ \sigma_3 &= p_5\phi_{,3} + \bar{p}_6\psi_{,3}, \\ \tau_3 &= p_6\phi_{,3} + \bar{p}_7\psi_{,3}, \end{aligned} \tag{15}$$

with constants p_1 to p_7 defined from material parameters. Applying solutions (12)-(13) to conditions (14) yields a 4×4 linear system for B_j :

$$\sum_{j=1}^{4} Q_{ij}B_j = 0, \quad i = 1, ..., 4. \tag{16}$$

The matrix elements Q_{ij} involve m_j, ξ, p_k, r_j, s_j. Non-trivial solutions require:

$$\det[Q_{ij}] = 0, \tag{17}$$

which is the frequency (secular) equation for Rayleigh waves in the dual-porosity medium.

Single-Porosity Limit: When $b_1, \gamma, \alpha_3, \alpha_2, d \to 0$, equation (17) reduces to the known result for a material with single porosity.

Numerical Analysis and Graphs

Copper material constants:

$$\begin{aligned} &\lambda = 7.76 \times 10^{10} Nm^{-2}, \ \ \mu = 3.86 \times 10^{10} Nm^{-2}, \ \ \rho = 8.954 \times 10^{3} Kgm^{3}, \\ &\alpha = 1.3 \times 10^{-5} N, \quad \alpha_1 = 1.65 \times 10^{10} Nm^{-2}, \quad \alpha_2 = 1.96 \times 10^{10} Nm^{-2}, \\ &\alpha_3 = 1.86 \times 10^{10} Nm^{-2}, \ \ \gamma = 0.19 \times 10^{-5} N, \ \ b_1 = 0.12 \times 10^{-5} N, \\ &d = 0.49 \times 10^{10} Nm^{-2}, \quad \kappa_1 = 0.1456 \times 10^{-12} Nm^{-2}s^2, \ \ \kappa_2 = 0.1546 \times 10^{-12} Nm^{-2}s^2, \\ &b = 0.4 \times 10^{10} Nm^{-2}, \quad \omega_1 = 1 \times 10^{11} s^{-1}, \ \ t = 0.1s. \end{aligned}$$

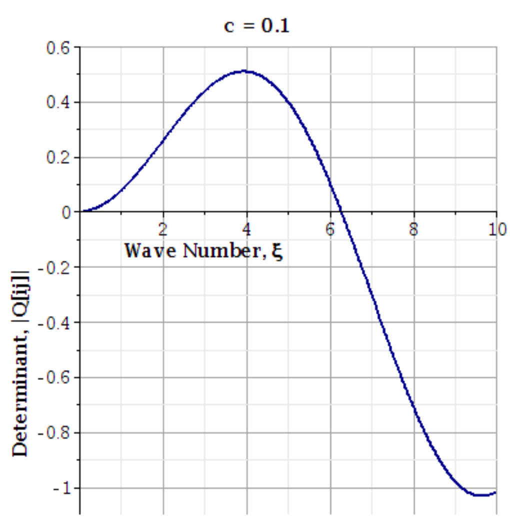

Figure 1. Determinant of the secular equation for Rayleigh waves corresponding to different values of the parameter ξ at $c = 0.1$.

Figure 1 plots the determinant of (17) versus wave number ξ for fixed $c = 0.1$. The response oscillates harmonically, with amplitude growing with ξ.

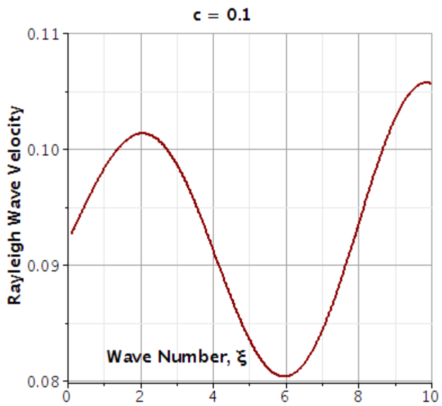

Figure 2. Rayleigh wave velocity for different values of the parameter ξ at $c = 0.1$.

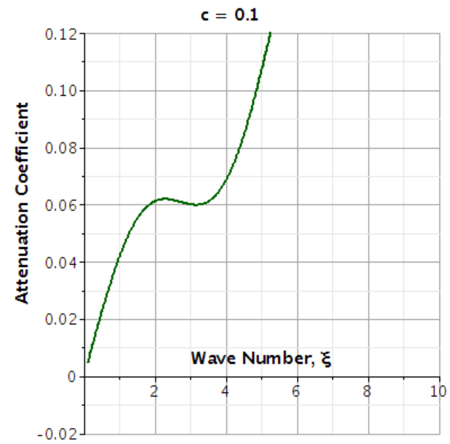

Figure 3. Attenuation coefficient for different values of the parameter ξ at $c = 0.1$.

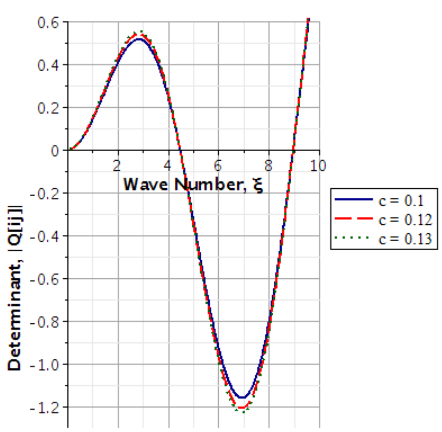

Figure 4. Determinant of the Rayleigh wave secular equation for different values of c with respect to the parameter ξ.

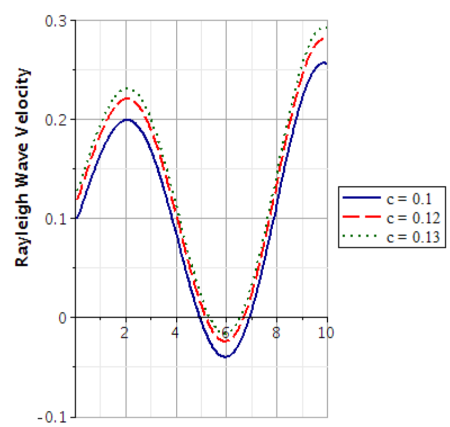

Figure 5. Rayleigh wave velocity for different values of c with respect to the parameter ξ.

Figures 2 and 3 show phase velocity and attenuation coefficient versus ξ for $c = 0.1$. Both exhibit oscillatory dependence, with increasing oscillation amplitude at higher ξ.

Figures 4-5 present comparisons for different phase velocities ($c = 0.1, 0.12, 0.13$). The determinant (Fig. 4) maintains harmonic variation with larger magnitude at higher ξ. Phase velocity (Fig. 5) and attenuation.

These results demonstrate that dual porosity introduces significant oscillatory dispersion and attenuation, strongly dependent on wave number and phase velocity.

Conclusions

This study has presented a rigorous theoretical and numerical investigation of Rayleigh-type surface wave propagation in a homogeneous, isotropic elastic half-space characterized by a dual-porosity structure. By employing a continuum mechanics framework for materials with interacting pore systems, the governing coupled field equations were systematically derived and reduced to a tractable form suitable for surface wave analysis.

A closed-form frequency (secular) equation for Rayleigh waves was explicitly obtained under stress-free boundary conditions. Unlike classical elastic or single-porosity models, the derived dispersion relation incorporates additional coupling mechanisms associated with the interaction between matrix porosity and fissures. This formulation provides a clear mathematical foundation for assessing how multiscale pore structures influence surface wave behavior.

Numerical simulations revealed that dual porosity induces pronounced oscillatory dispersion and attenuation characteristics with respect to the wave number and phase velocity. In particular, both the Rayleigh wave speed and attenuation coefficient exhibit non-monotonic, wave-number-dependent behavior, with oscillation amplitudes increasing at higher phase velocities. These features are absent in conventional elastic models and demonstrate that internal pore interactions play a decisive role in controlling near-surface wave dynamics.

The results obtained in this work significantly advance the understanding of surface wave propagation in fractured and multiscale porous media. From a practical perspective, the proposed model offers a physically consistent tool for interpreting surface wave measurements in geophysics, seismic hazard assessment, and nondestructive evaluation of porous and composite materials. Moreover, the derived framework establishes a solid basis for further extensions to more complex settings, including anisotropic, thermo-poroelastic, and fluid-saturated layered media.

In summary, this study provides both a novel theoretical formulation and quantitative insight into Rayleigh wave propagation in dual-porosity elastic solids, thereby contributing a meaningful and original advancement to the broader field of wave propagation in complex continua.

REFERENCES

1. Biot, M. A. Theory of three-dimensional consolidation. *J. Appl. Phys.* **12** (1941), 155–164.
2. Svanadze, M. Fundamental solution for consolidation with double porosity. *J. Mech. Behav. Mater.* **16** (2005), 123–130.
3. Svanadze, M. Viscoelasticity for materials with double porosity. *Discrete Contin. Dyn. Syst. B* **19** (2014), no. 7, 2335–2352.
4. Svanadze, M. Uniqueness in thermoelasticity with double porosity. *Meccanica* **49** (2014), 2099–2108.
5. Kumar, R.; Kansal, T. Rayleigh waves in transversely isotropic thermoelastic diffusion. *J. Eng. Phys. Thermophys.* **82** (2009), 1199–1210.
6. Kumar, R.; et al. Rayleigh waves in microstretch thermoelastic diffusion half-space. *Latin American J. Solids Struct.* **11** (2014), 299–311.
7. Kumar, R.; Gupta, V. Rayleigh waves in thermoelastic medium with mass diffusion. *Canad. J. Phys.* **93** (2015), 1–11.
8. Abd-Alla, A. N.; Abo-Dahab, S. M. Rayleigh waves in magneto-thermo-viscoelastic solid. *Appl. Math. Comput.* **149** (2004), 861–877.

9. Abd-Alla, A. M.; et al. Magneto-thermoelastic Rayleigh waves in granular medium. *J. Vib. Control* **17** (2011), no. 1, 115–128.

10. Abd-Alla, A. M.; et al. Rayleigh waves in magneto-thermo-viscoelastic granular medium. *Math. Probl. Eng.* **2011** (2011), Art. ID 1–47.

11. Abd-Alla, A. M.; et al. Rayleigh waves in rotating orthotropic half-space. *J. Mech. Sci. Technol.* **26** (2012), no. 9, 2815–2823.

12. Abd-Alla, A. M.; et al. Propagation in magneto-thermo-elastic orthotropic half-space. *J. Vib. Control* **19** (2013), no. 9, 1395–1420.

13. Abd-Alla, A. M.; et al. Rotational effect on waves in fibre-reinforced viscoelastic media. *J. Mech. Sci. Technol.* (2015), in press.

14. Iesan, D.; Quintanilla, R. Thermoelastic materials with double porosity. *J. Therm. Stresses* **37** (2014), 1017–1036.

15. Kholmurodov, A.; Matanov, M.; Quzratov, M. Propagation of harmonic plane waves in an elastic half-space. *AIP Conf. Proc.* **3244** (2024), no. 1, Art. 040001.

16. Kholmurodov, A. E.; Matanov, M. C. Seismic excitation model of half-space propagation of Rayleigh waves. *Problems Comput. Appl. Math.* **62** (2024), no. 6, 45–56 (in Russian).

17. Nagao, T. The impact of the ground irregular sedimentary structure on H/V spectrum of Rayleigh waves. *Eng. Technol. Appl. Sci. Res.* **13** (2023), no. 4, 5785–5795.

18. Matanov, M. Development of synthetic accelerograms and baseline correction methods for deterministic modeling of seismic wave propagation. *Digital Transform. Artif. Intell.* **3** (2025), no. 2, 100–108.

19. Charshamievich, M. M. Stability and accuracy of the hybrid method for dynamic half-space models under sinusoidal impulsive surface loads. *Int. Sci. J.* **2** (2025), no. 1, 15–20 (in Russian).

20. Matanov, M. Harmonic fundamental solutions and boundary integral equations for wave propagation in elastic, scalar and poroelastic media. *Acta NUUz* **2** (2025), no. 2.1, 80–87.

21. Kholmurodov, A. A comprehensive theoretical framework for surface wave propagation in layered liquid–porous systems with incompressible saturated constituents. *Int. J. Appl. Math.* **38** (2025), no. 12s, 2388–2402.

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On the existence of a solution to the inverse source problem for the Hopf equation

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Abstract. In this paper, we consider a one-dimensional inverse source problem for the Hopf equation. The problem consists in determining the unknown solution and the time-dependent source term from the given initial condition and an additional overdetermination condition at a fixed spatial point. By reducing the original inverse problem to a loaded equation and applying the fixed-point method, we establish the existence of a solution in the class of functions of finite smoothness. The proof is based on a priori estimates, the Schauder fixed-point theorem for smooth data, and a limit transition argument using weak-* compactness for the general case. As a result, sufficient conditions for the solvability of the inverse source problem are obtained

MSC (2020): 35R30

Key words: Hopf equation, inverse source problem, existence theorem, loaded equation, fixed-point method, Schauder theorem, Banach-Alaoglu theorem.

Introduction

The study of wave propagation in media for which the intrinsic pressure difference can be neglected, that is, in pressureless media, is of both theoretical and applied interest. A direct mathematical model of such media is given by the equations of gas dynamics in which the pressure term is formally set equal to zero. From the applied point of view, pressureless media arise in the description of various physical phenomena, including multiphase flows, the motion of dispersed media, dust particles, droplets, cumulative processes, and granular materials. Various examples of gas-dynamic problems involving pressureless media can be found in classical monographs and research papers [1-4].

From the mathematical point of view, pressureless media also generate a wide class of nontrivial problems [5-6]. In particular, the theory of classical solutions for the corresponding systems has been extensively developed up to the moment when singularities arise. It has been shown that singularities may occur on manifolds of different dimensions. At the same time, earlier studies demonstrated, at the physical level of rigor, that solutions for pressureless media may still preserve their meaning even after the formation of singularities. In such situations, new types of discontinuous solutions appear, in which strong density concentrations are formed on hypersurfaces of different codimensions. These phenomena are also closely related to the mathematical modeling of two-phase and multiphase flows [7-10].

In the present paper, we consider a one-dimensional inverse source problem for the Hopf equation. More precisely, we study the problem of determining the pair $u(t,x)$ and $(g(t)$ from the equation

$$u_t + uu_x = f(x)g(t),$$

subject to the initial condition

$$u|_{t=0} = u_0(x),$$

and the additional condition

$$u|_{x=0} = \varphi(t).$$

Here the function $f(x)$ is assumed to be known and satisfies the normalization condition $f(0) = 1$. The compatibility condition $\varphi(0) = u_0(0)$ is also imposed.

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To investigate this inverse problem, we first eliminate the unknown function $g(t)$ by means of the additional condition and reduce the original problem to a loaded equation for the derivative $v = u_x$. After that, the solvability of the resulting problem is established by the fixed-point method. For sufficiently smooth input data, the existence of a solution is obtained by applying the Schauder fixed-point theorem together with suitable a priori estimates. Then, by approximating the initial data with smooth functions and using weak-* compactness arguments based on the Banach–Alaoglu theorem, we pass to the limit and obtain a solution in the class of functions of finite smoothness.

Thus, the main result of the paper is an existence theorem for the one-dimensional inverse source problem for the Hopf equation in the class of finitely smooth functions.

Inverse source problem for the Hopf equation

Let us consider a one-dimensional dynamic inverse problem of determining the function $u(t,x)$, $g(t)$, if the inhomogeneous Hopf equation and the following conditions are satisfied:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = f(x) \cdot g(t), \quad t > 0, \quad x \in R, \tag{1}$$

$$u|_{t=0} = u_0(x), \quad x \in R \tag{2}$$

$$u|_{x=0} = \varphi(t), \quad t > 0 \tag{3}$$

The function $f(x)$ is given. Without loss of generality, we can assume that $f(0) = 1$. Let us assume that the agreement condition is satisfied

$$\varphi(0) = u_0(0) \tag{4}$$

Using (3), we eliminate the unknown function $g(t)$ from (1) and obtain the Cauchy problem for the loaded equation. For this purpose, we formally differentiate with respect to x both parts of (1) and introduce the notation $v = u_x$. Thus, we arrive at the following problem for the loaded equation [11]

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v^2 = F(t,x) + G(t,x) \cdot v(t,0), \tag{5}$$

$$v(0,x) = {u_0}'(x), \tag{6}$$

$$\frac{\partial u}{\partial x} = v, \tag{7}$$

$$u(t,0) = \varphi(t), \tag{8}$$

where $F\,(t,\,x) = f'(x) \cdot \varphi'(t)$ and $G\,(t,\,x) = f'(x) \cdot \varphi(t)$.

Existence theorem

Theorem 1. Let the functions $u_0(x), f(x) \in C^1[-1,1]$, $\varphi(t) \in C^1[0,1]$, and let $f(0) = 1$. Then a solution to problem (1)–(4) exists and

$$u(t,x),\ u_t(t,x),\ u_x(t,x) \in L_\infty(\Omega), \qquad g(t) \in L_\infty(0,t_0).$$

Proof. First, we prove the solvability of problem (5)-(8). To do this, we use the fixed point method. First, we assume that the input data $u_0(x), f(x)$ and $\varphi(t)$ are sufficiently smooth functions.

Fixed point

Let be $z(t,x),\ h(t)$ some functions. Instead of system (5), (6) we consider the system

$$\frac{\partial v}{\partial t} + z\frac{\partial v}{\partial x} + v^2 = F\,(t,\,x) + G\,(t,\,x) \cdot h(t), \tag{9}$$

$$v(0,x) = {u_0}'(x), \tag{10}$$

And then another problem

$$\frac{\partial u}{\partial x} = v, \tag{11}$$

$$u(t,0) = \varphi(t). \tag{12}$$

The sets Z and Ω will depend on some positive parameters M, V , δ, τ, which will be specified later. Let's put it this way

$$M = 1 + 2\left(\sup_{|x|<1} |u_0(x)| + \sup_{0<t<1} |\varphi(t)|\right),$$

$$V = 1 + 2\sup_{|x|<1} |u_0'(x)|.$$

Next, for some quantities $0 < \delta < 1$ and $0 < \tau < \delta/M$ we assume

$$\Omega_{\delta,\tau} = \{(t,x)|\, 0 < t < \tau,\ |x| < \delta - tM\}.$$

The domain $\Omega_{\delta,\tau}$ is a trapezoid with base $(-\delta,\delta)$ and height τ. Further, $Z \subset C(\bar{\Omega}_{\delta,\tau}) \times C[0,\tau]$ is the set of such pairs $(z(t,x),\ h(t)\ )$ that

$$\|z(t,x)\|_{C(\bar{\Omega}_{\delta,\tau})} \le M,$$

$$\|z_x(t,x)\|_{C(\bar{\Omega}_{\delta,\tau})} \le V,$$

$$z(t,0) = \varphi(t),$$

$$\|h(t)\|_{C[0,\tau]} \le V.$$

Let us consider the system (9)-(12) with coefficients. Let us put

$$F_0 = \sup_{|x|<1,0<t<1} |F(t,x)|, \quad G_0 = \sup_{|x|<1,0<t<1} |G(t,x)|.$$

The problem for $v(t,x)$ is equivalent to a nonlinear system

$$\dot{v}(t,y) + v^2(t,y) = (F + Gh)(t,y), \tag{13}$$

$$\dot{y}(t,x) = z(t,y), \tag{14}$$

$$v(0,x) = {u_0}'(x), \tag{15}$$

$$y(0,x) = x. \tag{16}$$

Locally in time this system is solvable and for small t, the estimate holds

$$|\dot{v}| \le v^2 + F_0 + G_0 V.$$

Since $|{u_0}'(x)| < V/2$, then on some interval $(0,t_0)$ the estimate $|v(t,x)| < V$ holds, which means

$$|v(t,x)| \le V/2 + \tau\left(V^2 + F_0 + G_0 V\right).$$

We choose τ so small that

$$\tau\left(V^2 + F_0 + G_0 V\right) < V/2.$$

Then $\tau \le t_0$. This means that system (13)-(16) is solvable in the domain $(0,\tau) \times (-\delta,\delta)$ and

$$|v(t,x)| \le V.$$

Further, the following estimate holds:

$$|y(t,x) - x| \le Mt.$$

This means that

$$y(t,-\delta) \le -\delta + Mt, y(t,\delta) \ge \delta - Mt.$$

By the theorem on the continuous dependence of the solution on the parameter, we conclude that the values $y(t,x)$ cover the interval $(-\delta+Mt, \delta-Mt)$ when x runs through the interval $(-\delta,\delta)$.

In addition, differentiability with respect to the parameter takes place, which means that this covering is univalent. Consequently, system (9)-(12) is solvable in the domain Ω. In addition, the estimate holds

$$|u(t,x)| \leq |\varphi(t)| + \delta V \leq M/2 + \delta V.$$

We choose the value δ so that

$$\delta V \leq M/2.$$

Then inside Ω

$$|u(t,x)| \leq M.$$

Thus, for the given choice of parameters δ,τ , system (9)-(12) is solvable in the domain Ω , the functions $v(t,x)$, $u(t,x)$, $u_x(t,x)$ are continuous and the inequalities are satisfied

$$\|u(t,x)\|_{C(\bar{\Omega}_{\delta,\tau})} \leq M,$$

$$\|u_x(t,x)\|_{C(\bar{\Omega}_{\delta,\tau})} \leq V,$$

$$u(t,0) = \varphi(t),$$

$$\|v(t,0)\|_{C[0,\tau]} \leq V.$$

Therefore, the operator L is well defined and $L(Z) \subset Z$.

Now we will show that the operator L is completely continuous. To do this, we will show $|v_x(t,x)| \leq N$ that for some constant N. It is at this point that additional smoothness of the input data is required. We will prove this statement later. For now, we assume that such an estimate has already been obtained. Then

$$|v_t(t,x)| \leq MN + V^2 + F_0 + VG_0 = N_1.$$

Hence, $v(t,0)$ is Lipschitz with constant N_1. Further,

$$|u_{xx}(t,x)| = |v_x(t,x)| \leq N,$$

and

$$|u_t(t,x)| \leq |\varphi'(t)| + \delta N_1.$$

These estimates show that pairs $(u(t,x), v(t,0))$ have some excess smoothness, and therefore the set of such pairs is compactly embedded in Z. Consequently, by the Schauder theorem [12] the operator has a fixed point. Thus, problem (5)-(8) is solvable, provided that the input functions are smooth. We will show that in this case the function $u(t,x)$is a solution to problem (1)-(4). Indeed, from the construction of the solution it follows that the trace $v(t,0)$ is defined. Then

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - f(x)g(t)\right) = 0,$$

$$g(t) = \varphi'(t) + \varphi(t)v(t,0).$$

Hence,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - f(x)g(t) = H(t),$$

with some function $H(t)$. Putting in the last equality $x=0$, we obtain $H(t) \equiv 0$. So, the function $u(t,x)$ is a solution to problem (1)-(4). Now we can get rid of the assumption about the increased smoothness of the input data. We only needed it to evaluate $v_x(t,x)$ and prove the COMPACTNESS of the operator L.

To do this, we note that the following estimates hold:

$$\|u(t,x)\|_{C(\bar{\Omega}_{\delta,\tau})} \leq M,$$

$$\|u_x(t,x)\|_{C(\bar{\Omega}_{\delta,\tau})} \leq V,$$

$$\|u_x(t,0)\|_{C[0,\tau]} \le V.$$

In addition, from equation (1) we easily obtain

$$g(t) = \varphi'(t) + \varphi(t)v(t,0),$$

$$|u_t| \le MV + |f(x)| \left(|\varphi'(t)| + |\varphi(t)|\, V\right).$$

Moreover, the quantities M, V and the domain Ω depend only on the norms of the functions $u_0(x)$, $u_0'(x)$, $\varphi(t)$, $\varphi'(t)$, $f'(x)$.

Consider a sequence of smooth functions $u_0^\tau(x)$, $\varphi^\tau(t)$, $f^\tau(x)$. For them we solve problem (5)-(8) and obtain a solution to problem (1)-(4) $\{u^\tau\}$, $\{g^\tau\}$. In this case

$$u^\tau(t,x), u_t^\tau(t,x), u_x^\tau(t,x) \in L_\infty(\Omega), g^\tau(t) \in L_\infty(0,t_0).$$

Since $(L_1(\Omega))' = L_\infty(\Omega)$ by the Banach-Alaoglu theorem [13] the functions $u_t^\tau(t,x)$, $u_x^\tau(t,x)$, $g^\tau(t)$ *- weakly converge to the corresponding functions $u_t(t,x)$, $u_x(t,x)$, $g(t)$. And the functions $u^\tau(t,x)$ converge strongly in $C(\Omega)$ to $u(t,x)$.

After this, it only remains to pass to the limit in system (1)-(4).

Estimate $v_x(t,x)$.

So, let the input data be smooth. Let us differentiate equation (9) with respect to and denote by $w(t,x) = v_x(t,x)$

$$\frac{\partial w}{\partial t} + z\frac{\partial w}{\partial x} + z_x w + 2vw = F_x(t,x) + G_x(t,x)h(t), \tag{17}$$

There is an inequality on the characteristics

$$|\dot{w}| \le 3V\,|w| + F_1 + G_1 V.$$

Hence (by Gronwall's lemma) when $t < \tau$

$$|w(t,x)| \le \left((F_1 + G_1 V)\,\tau + |w(0,x)|\right) e^{3V\tau} \le \tilde{N}.$$

The value $\tilde{N}$ depends on some additional derivatives of the input data, but this is not important. The main thing is that such a value exists. **Theorem 1 is proven.**

Corollary. Let the conditions of Theorem 1 be satisfied. Then the solution to problem (1)-(4) $u(t,x)$ is Lipschetz.

REFERENCES

1. Chernyi, G. G. Introduction to Hypersonic Flow. *Academic Press, New York–London,* 1961.

2. Sedov, L. I. Similarity and Dimensional Methods in Mechanics. *Academic Press, New York–London*, 1959.

3. Stanyukovich, K. P. The Unsteady Motion of a Continuous Medium. *Nauka, Moscow,* 1971. (In Russian).

4. Ovsyannikov, L. V. Isobaric gas motions. *Differ. Equ.* **30** (1994), no. 10, 1792–1799. (In Russian).

5. Chupakhin, A. P. On barochronic gas movements. *Dokl. RAS* **352** (1997), no. 5, 624–626. (In Russian).

6. Chupakhin, A. P. Barochronic Gas Movements. *Author's abstract of diss. doc. phys.-math. sciences, Lavrentyev Institute of Hydrodynamics, Siberian Branch of the Russian Academy of Sciences, Novosibirsk,* 1999. (In Russian).

7. Kraiko, A. N. About the surfaces of discontinuity in a medium devoid of its own pressure. *J. Appl. Math. Mech.* **43** (1979), no. 3, 500–510. (In Russian).

8. Kraiko, A. N.; Sulajmanova, S. M. Two-fluid flows of a mixture of gas and solid particles with the “sheets” and “filaments” that arise when flow impermeable surfaces. *J. Appl. Math. Mech.* **47** (1983), no. 4, 619–630. (In Russian).

9. Kraiko, A. N. Mathematical models for describing gas and foreign particle flows and non-stationary filtration of liquids and gases in porous media. *Vestn. SUSU. Ser. Mathematical Modelling and Programming* **7** (2014), no. 1, 34–48. (In Russian).

10. Nemtsev, M. Yu.; Menshov, I. S.; Semenov, I. V. Numerical modeling of dynamic processes in a medium of finely dispersed solid particles. *Matem. Model.* **34** (2022), no. 8, 73–96. (In Russian).

11. Kozhanov, A. I. Nonlinear loaded equations and inverse problems. *Comput. Math. Math. Phys.* **44** (2004), no. 4, 722–744. (In Russian).

12. Kantorovich, L. V.; Akilov, G. P. *Functional Analysis*. Nauka, Moscow, 1977. (In Russian).

13. Dunford, N.; Schwartz, J. T. *Linear Operators. Part I: General Theory*. Interscience, New York, 1958.

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A Simple Pursuit-Evasion Problem for Differential Games with Time Delay

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Abstract. In this article, pursuit-evasion differential games with simple motion and time delay are studied under geometric constraints on the controls of both players. Depending on the initial states of the players and the parametric values involved in the control constraints, the problem is analyzed accordingly. To solve the pursuit problem, a parallel pursuit strategy ($\Pi-$strategy) is proposed, which ensures the best possible convergence of the players, and its structure is examined with respect to the parameters. For the considered class of differential games, sufficient conditions for the solvability of both the pursuit and evasion problems are obtained.

MSC (2020): 49N79; 49N70; 91A24.

Key words: differential game, pursuit problem, evasion problem, $\Pi-$strategy, resolving function, control, time delay, convergence function, guaranteed time.

Introduction

The study of differential games was initiated by the American mathematician R. Isaacs. His research was published in the form of a monograph [1] in 1965, in which a large number of examples were considered, while theoretical issues were addressed only partially. Since the 1960s, differential games have become one of the main directions of scientific research, and their fundamental results were obtained by Isaacs [1], Pontryagin [2], Krasovskii [3], Petrosyan [4], and others.

In IsaacsвЂ™ monograph [1], a number of game-theoretic problems were considered, discussed in detail, and proposed for further investigation. A simplified analytical solution of this problem in the half-plane was presented by Isaacs in [1].

In the case where the controls of both players are subject to geometric constraints, this game was studied in considerable detail by Petrosyan [4], based on the approximation of measurable controls by the most effective piecewise-constant controls implementing a strategy of parallel approach. Later, this approach to control in differential pursuit games became known as the $\Pi-$strategy.

The strategy proposed in [4] for a simple pursuit game with geometric constraints served as a starting point for the development of pursuit methods in games with several pursuers [5]. Subsequently, for cases where the controls of both players are subject to integral, Gronwall-type, or mixed constraints, the game was investigated in the works of Samatov [5вЂ“6].

In [11], it is proved that under the stated conditions, as well as an additional condition imposed on the parameters of the game, the pursuit can be completed in an arbitrarily small neighborhood of the terminal set. To ensure the completion of the game, an ε-positional pursuit strategy is constructed.

Based on the fundamental approaches to the theory of differential games developed by Pontryagin [2] and Krasovskii [3], a differential game is viewed as a control problem from the standpoint of either the pursuer or the evader. According to this approach, the game is reduced either to a pursuit (approach) problem or to an evasion (escape) problem.

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The principal method for solving pursuit and evasion problems consists in constructing optimal strategies for the players and determining the value of the game. Papers [7-9] are devoted to the study of differential games of simple motion, in which the existence of the value of the game was established by means of optimal strategies of the players.

In [10], pursuit game problems described by a system of delay differential equations under integral constraints on the players controls are studied. The proposed scheme is based on the ideas of the resolving function method. Modifications of the methods (namely, the first and the so-called third methods) of pursuit are proposed for the case where integral constraints are imposed on the players' controls. Sufficient conditions are obtained for the possibility of completing the pursuit in finite time.

In [12], the pursuit problem for neutral-type differential-difference equations is investigated. A new sufficient condition for the solvability of this problem is obtained, and a model example illustrating the result is presented.

In [13], issues of strong and weak invariance of a constant multivalued mapping are studied for a boundary value problem of heat conduction in the presence of time delay. In this setting, the control parameter appears in the right-hand side of the equation, and its control action has an impulsive character, which is represented by the Dirac delta function. The obtained conditions differ from previously known results established for control problems with delay.

In the present paper, pursuitвЂ“evasion problems are considered under simple motion of the players with delay. Geometric constraints are imposed on the playersвЂ™ controls. To solve the pursuit problem, a $\Pi-$strategy of the pursuer is proposed. Furthermore, a sufficient condition for the realizability of pursuit is formulated, and the guaranteed pursuit time is determined.

Problem Formulation

Let an object P, called the *pursuer*, chase another object E, called the *evader*, in the space $\mathbb{R}^n$.

Denote the state vector of the pursuer by x, and the state vector of the evader by y. In the present work, we consider a pursuitвЂ“evasion problem whose dynamical properties are described by the equations [15]:

$$P: \qquad \dot{x}(t) = x(t-h) + u(t), \tag{1}$$

$$E: \qquad \dot{y}(t) = y(t-h) + v(t), \tag{2}$$

where $x(t), y(t), u(t), v(t) \in \mathbb{R}^n, t \geq 0, n \geq 1$ and h is the time delay.

The initial conditions for system (1)вЂ“(2) are given respectively by

$$x_0(t) = \varphi(t)x_0, \qquad y_0(t) = \varphi(t)y_0,$$

where $\varphi(t)$ is an absolutely continuous function satisfying

$$\varphi(t) > 0 \quad \text{for all } t \in [-h, 0],$$

and $x_0 \neq y_0$.

The set of all measurable functions $u(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$ (the controls of player P) satisfying the condition

$$|u(t)| \leq \rho \quad \text{for } t \geq 0, \tag{3}$$

is denoted by U.

Similarly, the set of all measurable functions $v(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$ (the controls of player E) satisfying the condition

$$|v(t)| \leq \sigma \quad \text{for } \ t \geq 0, \tag{4}$$

is denoted by V.

In the theory of differential games, inequalities of the form (3) and (4) are called *geometric constraints* (briefly, $G-$constraints). In these definitions, ρ and σ are nonnegative numerical parameters.

Definition 1. A measurable function $u(\cdot)$ (respectively, $v(\cdot)$) satisfying condition (3) (respectively, condition (4)) is called an *admissible control* of the pursuer (respectively, the evader) belonging to class U (respectively, class V). The pair of admissible control classes (U, V) thus defines a differential game.

Introduce the notation [15]

$$\xi(t,\varphi(\cdot)) = \eta(t)\varphi(0) + \int_{-h}^{0} \eta(t-h-s)\varphi(s)ds,$$

where

$$\eta(t) = \begin{cases} 0 & \text{for } t<0, \\ \sum\limits_{j=0}^{\nu} \frac{(t-hj)^j}{j!} & \text{for } \nu h \le t \le (\nu+1)h,\ \nu = 0,1,2,.... \end{cases}$$

We present several necessary properties of the function $\eta(t)$ [16вЂ“18].

Property 1. *Integrability:*

$$\int_{t_1}^{t_2} \eta(s), ds = \eta(s+h)\Big|_{t_1}^{t_2}.$$

Property 2. *Differentiability:*

$$\eta'(t) = \eta(t-h).$$

By virtue of equations (1)-(2), each pair $(x_0(\cdot), u(\cdot))$, consisting of an initial position $x_0(\cdot)$ and a control function $u(\cdot) \in U$ (respectively, $(y_0(\cdot), v(\cdot))$, with $v(\cdot) \in V$), generates a trajectory given by the formulas

$$x(t) = \xi(t,\varphi(\cdot))x_0 + \int_0^t \eta(t-s)u(s)ds, \tag{5}$$

$$y(t) = \xi(t,\varphi(\cdot))y_0 + \int_0^t \eta(t-s)v(s)ds, \tag{6}$$

respectively.

The main objective of the pursuer P is to capture the evader E, that is, to achieve the equality

$$x(t^*) = y(t^*)$$

at some time $t^* > 0$. The objective of the evader is to ensure that the inequality

$$x(t) \neq y(t)$$

holds for all $t \ge 0$.

It is well known that control functions of the pursuer P alone are insufficient for solving the pursuit problem, since they depend only on the time parameter t, $t \ge 0$. Therefore, appropriate types of controls should be formulated in terms of strategies. There exist various approaches to defining this concept. For our purposes, the following formulation is sufficient.

First, we introduce the following notation:

$$z(t) = x(t) - y(t), \quad t \ge 0,$$

$$z_0(t) = x_0(t) - y_0(t) = \varphi(t)(x_0 - y_0) = \varphi(t)z_0, \quad t \in [-h, 0].$$

In what follows, it is assumed that the initial functions $x_0(\cdot)$ and $y_0(\cdot)$ are specified such that

$$x_0(t) \neq y_0(t), \quad t \in [-h, 0],$$

that is, $x_0 \neq y_0$.

Definition 2. A mapping $\mathbf{u} : \mathbb{R}^n \times V \to U$ is called a strategy of the pursuer if the following conditions are satisfied:

a) For every $v(\cdot) \in V$, the inclusion $\mathbf{u}(z_0(\cdot), v(\cdot)) \in U$ holds on some time interval $[0, T]$. The function $\mathbf{u}(z_0(\cdot), v(\cdot)), \quad t \ge 0$, is called the realization of the strategy corresponding to $v(\cdot) \in V$.

b) If for $v_1(\cdot), v_2(\cdot) \in V$ the equality $v_1(t) = v_2(t)$ holds almost everywhere on $[0, T]$, then $u_1(t) = u_2(t)$ holds almost everywhere on $[0, T]$, where $u_i(\cdot) = \mathbf{u}(z_0(\cdot), v_i(\cdot)), \quad i = 1, 2$.

Definition 3. A strategy $\mathbf{u} = \mathbf{u}(z_0(\cdot), v(\cdot))$ is called a parallel pursuit strategy (or P-strategy) if, for every $v(\cdot) \in V$, the solution of the Cauchy problem

$$\dot{z}(t) = z(t-h) + \mathbf{u}(z_0(\cdot), v(t)) - v(t),\ t \geq 0, \quad z(t) = z_0(t),\ t \in [-h, 0], \tag{7}$$

can be represented in the form

$$z(t) = \Lambda_G(t, v(\cdot))\xi(t, v(\cdot))z_0, \qquad \Lambda_G(0, v(\cdot)) = 1,$$

where $\Lambda_G(t, v(\cdot))$ is a scalar function continuous in t, $t \geq 0$. The function $\Lambda_G(t, v(\cdot))$ will hereafter be called the convergence function in the pursuit problem.

Definition 4. A $\Pi-$strategy is said to be winning for the pursuer on the time interval $[0, T]$ if, for any $v(\cdot) \in V$, the following conditions hold:

a) there exists a time $t^* \in [0, T]$ such that $z(t^*) = 0$;

b) $\mathbf{u}(z_0(\cdot), v(\cdot)) \in U$ on the interval $[0, T]$. The number T is called the guaranteed pursuit (or capture) time.

Let us now consider the game (U, V) from the viewpoint of the evader.

Definition 5. A control $\mathbf{v}^*(\cdot) \in V$ is called winning for the evader in the game (U, V) if, for every $u(\cdot) \in U$, the solution $z(t)$ of the Cauchy problem

$$\dot{z}(t) = z(t-h) + u(t) - \mathbf{v}^*(t),\ t \geq 0, \quad z(t) = z_0(t),\ t \in [-h, 0], \tag{8}$$

satisfies the inequality $z(t) \neq 0$ for all $t \geq 0$.

The present work is devoted to solving the following problems under the assumption that the players' controls satisfy constraints (3) and (4), respectively.

Problem 1. *Pursuit Problem:* Construct a $\Pi-$strategy for the pursuer and determine the guaranteed capture time in the game (U, V).

Problem 2. *Evasion Problem:* Construct a strategy for the evader and estimate the variation of the distance between the players.

Solution of the Pursuit Problem

In many parametric mathematical problems, analytical results explicitly depend on parameters that are regarded as constants. These parameters make it possible to formulate conditions for the solvability of the problem. In this section, necessary and sufficient conditions for the solvability of the pursuit problem in the game (U, V) are established.

If the pursuer and the evader choose admissible controls $u(\cdot) \in U$ and $v(\cdot) \in V$, respectively, then, according to equation (7), the solution takes the form

$$z(t) = \xi(t, \varphi(\cdot))z_0 + \int_0^t \eta(t-s)\big(u(s) - v(s)\big)ds. \tag{9}$$

For the pursuer, it is insufficient to use only open-loop (program) strategies depending solely on time t. Therefore, by analogy with [4], the pursuer's strategy in the considered problem may depend on the current value of the evader's control $v(t)$, as well as on the given parameters z_0 and ρ.

It is assumed that at time t, the pursuer knows the initial data $x_0, y_0, \varphi(\cdot)$, the constants ρ, σ, the current time t, and the current value of the evaderвЂ™s control $v(t)$.

Definition 6. Let $\rho \geq \sigma$. In the game (U, V), the function

$$\mathbf{u}(z_0(\cdot), v) = v - \lambda_G(z_0(\cdot), v)e_0, \tag{10}$$

is called the parallel pursuit strategy (briefly, the $\Pi-$strategy) of the pursuer, where

$$\lambda_G(z_0(\cdot), v) = \langle v, e_0\rangle + \sqrt{\langle v, e_0\rangle^2 + \rho^2 - |v|^2}, \qquad e_0 = \frac{z_0}{|z_0|}, \tag{11}$$

and $\langle v, e_0\rangle$ denotes the scalar product of the vectors v and e_0 in $\mathbb{R}^n$. The function $\lambda_G(z_0(\cdot), v)$ is commonly referred to as the *resolving function*.

The main properties of strategy (10) and the resolving function (11) are presented below.

Lemma 1. Strategy (10) is well-defined and continuous for all $v \in V$, and throughout the pursuit game the equality

$$|\mathbf{u}(z_0(\cdot), v)| = \rho$$

holds.

Lemma 2. The resolving function $\lambda_G(z_0(\cdot), v)$ is well-defined and nonnegative for all $v \in V$. Moreover, it satisfies the following bounds:

$$\rho - \sigma \le \lambda_G(z_0(\cdot), v) \le \rho + \sigma.$$

Proof. The minimum and maximum values of the resolving function $\lambda_G(z_0(\cdot), v)$ for arbitrary $v \in V$ are determined as follows:

$$\min_{v\in V} \lambda_G(z_0(\cdot), v) = \lambda_G(z_0(\cdot), v)\big|_{v=-\sigma e_0} = \langle -\sigma e_0, e_0\rangle + \sqrt{\langle -\sigma e_0, e_0\rangle^2 + \rho^2 - |-\sigma e_0|^2} = \rho - \sigma,$$

$$\max_{v\in V} \lambda_G(z_0(\cdot), v) = \lambda_G(z_0(\cdot), v)\big|_{v=\sigma e_0} = \langle \sigma e_0, e_0\rangle + \sqrt{\langle \sigma e_0, e_0\rangle^2 + \rho^2 - |\sigma e_0|^2} = \rho + \sigma.$$

Therefore,

$$\rho - \sigma \le \lambda_G(z_0(\cdot), v) \le \rho + \sigma.$$

□

Definition 7. If $\rho > \sigma$, then the scalar function

$$\Lambda_G(t, v(\cdot)) = 1 - \frac{1}{|z_0|\xi(t)} \int_0^t \eta(t-s)\lambda_G\big(z_0(\cdot), v(s)\big)ds \tag{12}$$

is called the players' approach (convergence) function in the game (U, V).

Lemma 3. Let $\rho > \sigma$. Then:

a) Function (12) is bounded for all $t \in [0, T]$ as follows:

$$\underline{\Lambda}_G(t, v(\cdot)) \le \Lambda_G(t, v(\cdot)) \le \overline{\Lambda}_G(t, v(\cdot)), \tag{13}$$

where

$$\underline{\Lambda}_G(t, v(\cdot)) = 1 - \frac{\rho+\sigma}{|z_0| \cdot \min\limits_{-h\le s\le 0} \varphi(s)} \left(1 - \frac{\eta(h)}{\eta(t+h)}\right),$$

$$\overline{\Lambda}_G(t, v(\cdot)) = 1 - \frac{\rho-\sigma}{|z_0| \cdot \max\limits_{-h\le s\le 0} \varphi(s)} \left(1 - \frac{\eta(h)}{\eta(t+h)}\right).$$

b) For all $v(\cdot) \in V$, function (12) is monotonically decreasing with respect to the variable t, $t \ge 0$.

Proof. Let $\rho > \sigma + |z_0| \cdot \max\limits_{-h\le s\le 0} \varphi(s)$.

a) From Property 1, Lemma 2, and formulas (5) and (12), we obtain the following inequality:

$$\Lambda_G(t, v(\cdot)) = 1 - \frac{1}{|z_0|\xi(t)} \int_0^t \eta(t-s)\lambda_G\big(z_0(\cdot)v(s)\big)ds \le$$

$$\le 1 - \frac{\rho-\sigma}{|z_0|\left(\eta(t)\varphi(0) + \int_{-h}^0 \eta(t-h-s)\varphi(s)ds\right)} \int_0^t \eta(t-s)ds \le$$

$$\le 1 - \frac{\rho-\sigma}{|z_0| \cdot \max\limits_{-h\le s\le 0} \varphi(s)} \cdot \frac{\int_0^t \eta(t-s)ds}{\eta(t) + \int_{-h}^0 \eta(t-h-s)ds} =$$

$$= 1 - \frac{\rho - \sigma}{|z_0| \cdot \max\limits_{-h \le s \le 0} \varphi(s)} \cdot \frac{\int_0^t \eta(s)ds}{\eta(t) + \int_{t-h}^t \eta(s)ds} =$$

$$= 1 - \frac{\rho - \sigma}{|z_0| \cdot \max\limits_{-h \le s \le 0} \varphi(s)} \cdot \frac{\eta(t+h) - \eta(h)}{\eta(t+h)} = \overline{\Lambda}_G(t, v(\cdot)).$$

The lower estimate of $\Lambda_G(t, v(\cdot))$ is obtained analogously to the upper estimate.

b) From Property 2 and formula (6), we obtain

$$\frac{d\overline{\Lambda}_G(t, v(\cdot))}{dt} = -\frac{\rho - \sigma}{|z_0| \cdot \max\limits_{-h \le s \le 0} \varphi(s)} \cdot \frac{\eta(h)\eta(t)}{\eta^2(t+h)} < 0.$$

Therefore, the function $\overline{\Lambda}_G(t, v(\cdot))$ is strictly decreasing. Consequently, by inequality (13), it follows that the function $\Lambda_G(t, v(\cdot))$ is also monotonically decreasing with respect to t. □

Theorem 1. Let

$$\rho > \sigma + |z_0| \cdot \max_{-h \le s \le 0} \varphi(s)$$

in the game (U, V). Then the $\Pi-$strategy (10) is winning for the pursuer, and the guaranteed time T is given by

$$T = \eta^{-1}\left(\frac{(\rho - \sigma)\eta(h)}{\rho - \sigma - |z_0| \cdot \max\limits_{-h \le s \le 0} \varphi(s)}\right) - h.$$

Proof. Consider the function

$$\overline{\Lambda}_G(t, v(\cdot)) = 1 - M\left(1 - \frac{\eta(h)}{\eta(t+h)}\right), \qquad M = \frac{\rho - \sigma}{|z_0| \cdot \max\limits_{-h \le s \le 0} \varphi(s)}.$$

From the assumption of the theorem, it follows that $M > 1$.

By the previously established properties of the function $\eta(t)$, the function $\overline{\Lambda}_G(t, v(\cdot))$ is continuous and monotonically decreasing with respect to t on the half-line $[0, +\infty)$. Moreover,

$$\overline{\Lambda}_G(0, v(\cdot)) = 1, \qquad \lim_{t \to +\infty} \overline{\Lambda}_G(t, v(\cdot)) = 1 - M < 0.$$

Therefore, there exists a unique time $T > 0$ such that

$$\overline{\Lambda}_G(T, v(\cdot)) = 0.$$

Substituting this condition, we obtain

$$1 - M\left(1 - \frac{\eta(h)}{\eta(T+h)}\right) = 0,$$

which implies

$$\frac{\eta(h)}{\eta(T+h)} = 1 - \frac{1}{M}.$$

Since $M > 1$, the right-hand side is positive, and therefore

$$\eta(T+h) = \frac{M\eta(h)}{M-1}.$$

Because the function $\eta(t)$ is strictly increasing on $[0, +\infty)$, it admits an inverse function η^{-1}. Hence,

$$T = \eta^{-1}\left(\frac{M\eta(h)}{M-1}\right) - h.$$

Substituting the expression for M, we finally obtain

$$T = \eta^{-1}\left(\frac{(\rho-\sigma)\eta(h)}{\rho-\sigma-|z_0|\cdot\max\limits_{-h\le s\le 0}\varphi(s)}\right) - h.$$

From formula (13) and Definition 4, it follows that there exists a time $t^* \in [0,T]$ such that $z(t^*) = 0$. This completes the proof of Theorem 1. □

Solution of the Evasion Problem

In this section, we propose an admissible strategy of the evader that guarantees evasion in the game.

Definition 8. The control function

$$v^*(t) = -\sigma e_0, \qquad t \ge 0, \tag{14}$$

is called the evader's strategy in the game (U, V).

Theorem 2. Let $\rho \le \sigma$. Then control (14) is winning for the evader in the game (U, V), and moreover, $|z(t)| \ge |z_0|\,\eta(h)\min\limits_{-h\le s\le 0}\varphi(s)$ for all $t \ge 0$.

Proof. Assume that $\rho \le \sigma$, and that the pursuer chooses a control $u(\cdot) \in U$, while the evader applies strategy (14). Then, from formula (9), we obtain

$$|z(t)| = \left|\xi(t,\varphi(\cdot))z_0 + \int_0^t \eta(t-s)\big(u(s)+\sigma e_0\big)ds\right| \ge$$

$$\ge \left|\xi(t,\varphi(\cdot))z_0 + \sigma e_0\int_0^t \eta(t-s)ds\right| - \left|\int_0^t \eta(t-s)u(s)ds\right| \ge$$

$$\ge |z_0|\xi(t,\varphi(\cdot))\left(1+\frac{\sigma}{|z_0|\xi(t,\varphi(\cdot))}\int_0^t \eta(t-s)ds\right) - \rho\int_0^t \eta(t-s)ds =$$

$$= |z_0|\xi(t,\varphi(\cdot))\left(1-\frac{\rho-\sigma}{|z_0|\xi(t,\varphi(\cdot))}\int_0^t \eta(t-s)ds\right) \ge$$

$$\ge |z_0|\xi(t,\varphi(\cdot)) \ge |z_0|\,\eta(t+h)\cdot\min_{-h\le s\le 0}\varphi(s) \ge |z_0|\,\eta(h)\cdot\min_{-h\le s\le 0}\varphi(s) > 0$$

for all $t \ge 0$.

Thus, the inequality

$$|z(t)| \ge |z_0|\,\eta(h)\cdot\min_{-h\le s\le 0}\varphi(s)$$

holds for all $t \ge 0$, which proves that control (14) is winning for the evader. Theorem 2 is proved. □

REFERENCES

1. Isaacs, R. Differential Games. *NY: John Wiley and Sons,* (1965), 385 p.

2. Pontryagin, L.S. Selected Works. *Moscow: MAKS Press,* (2004), 551 p. (in Russian)

3. Krasovskii, N.N.; Subbotin, A.I. Game-Theoretical Control Problems. *NY: Springer,* (2011), 517 p.

4. Petrosjan, L.A. Differential Games of Pursuit. *Ser. Optim, vol. 2. Singapore: World Scientific Publ.,* (1993), 340 p. DOI: 10.1142/1670

5. Azamov, A.A.; Samatov, B.T. The P-strategy: analogies and applications. *In: Proc. The Fourth Int.Conf. Game Theory and Management (GTM 2010), June 28-30, 2010, St. Petersburg, Russia,* **4** (2010), 33–47.

6. Samatov, B.T.; Soyibboev, U.B. Differential game with a lifeline for the inertial movements of players. *Ural Math. J.* **7** (2021), no. 2, 94–109. DOI: 10.15826/umj.2021.2.007

7. Mamadaliyev, N. A.; Mustapokulov, Kh. Ya. Construction of P-strategies in a simple pursuitвЂ“evasion game with impulse control. *Acta NUUz* **2/2.1.** (2024), 164–175. (in Russian)

8. Mustapokulov, Kh. Ya. PursuitвЂ“Evasion Problems for Second-Order Differential Games with Impulse Control. *Acta NUUz* **2/1.1.** (2025), 178–184. (in Russian)

9. Mamadaliyev, N. A.; Mustapokulov, Kh. Ya. A Linear Differential Pursuit Game with Impulse Controls. *Acta NUUz* **2/2.** (2025), 171–178. (in Russian)

10. Mamadaliyev, N. A.; Mustapokulov, Kh. Ya.; Abdualimova, G. M. The method of resolving functions for solving a pursuit problem with integral constraints on player controls. *Vestnik Udmurtskogo Universiteta. Matematika. Mekhanika. Komp'yuternye Nauki* **33** (2025), no. 1, 103–118. (in Russian) https://doi.org/10.35634/vm230107

11. Tukhtasinov M.; Mustapokulov, Kh. Ya. $\varepsilon-$Positional Strategies In Theory Of Differential Pursuit Games And Invariance Of Constant Multivalued Mappings In Heat-Conductivity Problems. *Journal of Mathematical Sciences* **265** (2022), no. 1, 117–128. DOI: 10.1007/s10958-022-06049-7.

12. Mamadaliyev, N. A.; Mustapokulov, Kh. Ya. Pursuit Problem for Differential-Difference Equations of Neutral Type. *Journal of Mathematical Sciences* **277** (2023), no. 3, 438–448. DOI 10.1007/s10958-023-06846-8

13. Mamadaliyev, N. A.; Mustapokulov, Kh. Ya. On the Invariance of a Constant Multivalued Mapping in a Heat Conduction Problem with Impulse Control in the Presence of Time Delay. *Bulletin of the Institute of Mathematics* **6** (2023), no. 6, 90–98. (in Russian)

14. Bellman, R.; Cooke, K. Differential-Difference Equations. *Moscow: Mir.* (1967), 254 p.

15. Hale, J. Theory of Functional Differential Equations. *Moscow: Mir.* (1984), 421 p.

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Surface theory in four-dimensional Galilean space

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Abstract. This paper develops several fundamental aspects of the theory of surfaces in four-dimensional Galilean space. The first and second fundamental forms of a surface are introduced and used to define the normal curvature, principal curvatures, mean curvatures, and total curvature. The principal curvatures are characterized as extremal values of the normal curvature. Derivative formulas for surfaces are established, leading to relations that express the coefficients of the second fundamental form in terms of the coefficients of the first fundamental form and their partial derivatives. As a consequence, the mean and total curvatures are represented without explicit use of the coefficients of the second fundamental form.

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Key words: Galilean space, surface theory, principal curvatures, mean curvature, second-order curvature, total curvature, fundamental forms, derivative formulas.

Introduction

Galilean geometry is a type of geometry in spaces with a special metric, and it is widely used in various fields of mathematics, mechanics, and theoretical physics. With the development of non-Euclidean geometry, interest in Galilean geometry has also grown. The first scientific works devoted to Galilean space appeared in the 1950s. The scientists who founded this field were R.G. Bukharaev and E.E. Khatipova, who laid the fundamental foundations of the geometry of spaces with broken metrics. The general theory of projective metric spaces, both metric and non-metric, is systematically presented in B. A. Rozenfeld's classic monograph "Non-Euclidean Spaces" [1]. Among the first to address the problems of "full" geometry in Galilean space were A. Artikbayev and Pankina [2],[3]. The theoretical foundations of three-dimensional Galilean space are described in detail in Roschel's monograph [4]. The subsequent development of "full" geometry is reflected in the monograph by A. Artikbayev and D.D. Sokolov [5]. The axiomatic construction of the geometry of the Galilean plane and space was studied by A.I. Dolgorev [6], while the solution of differential geometry problems and classical problems in this space was investigated by I.A. Dolgorev [7]. The above-mentioned studies are primarily from the late 20th century.

The theory of surfaces in Galilean space has been studied by authors such as Aydin, Dede, and Yoon, and the theory of surfaces in four-dimensional Euclidean and non-Euclidean spaces has been investigated by Y. Aminov and Dolgorev [8],[9],[10]. Ismoilov investigated application of isotropic geometry methods to solving Monge–Ampere equation in differential geometry problems [11]. Sharipov and Keunimjaev studied existence and uniqueness problems of polyhedra determined by conditional curvature values [12]. In three-dimensional Galilean space, A. Artikbayev and B. Sultonov conducted research on parabolic surface theory [13]. Dolgarev primarily worked on two-dimensional surface theory in four-dimensional space. The fundamental concepts of four-dimensional Galilean space are presented in the work [14] by A. Artikbayev and A. Nurbayev.

Preliminaries

In G_4 space, we take the surface equation in the following form for convenience:

$$\vec{r}(t,u,v) = t\vec{e} + x(t,u,v)\vec{i} + y(t,u,v)\vec{j} + z(t,u,v)\vec{k} \tag{1}$$

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In Euclidean geometry, the first quadratic form I is expressed as follows:

$$I = Adt^2 + 2Bdtdu + 2Cdtdv + Edu^2 + 2Fdudv + Gdv^2 \tag{2}$$

However, in Galilean geometry, its form is somewhat simplified with respect to the scalar product. For the surface (1) in G_4 space, the first quadratic form of the surface is determined by the formula [15]:

$$I = \begin{cases} dt^2 & \text{if } t \neq \text{const}, \\ Edu^2 + 2Fdudv + Gdv^2 & \text{if } t = \text{const} \end{cases} \tag{3}$$

Here

$$\begin{aligned} &E = \overrightarrow{r_u}^2 = {x'_u}^2 + {y'_u}^2 + {z'_u}^2, \quad F = \overrightarrow{r_u}\,\overrightarrow{r_v} = x'_u x'_v + y'_u y'_v + z'_u z'_v, \\ &G = \overrightarrow{r_v}^2 = {x'_v}^2 + {y'_v}^2 + {z'_v}^2 \end{aligned} \tag{4}$$

It is important to note that in determining some coefficients of the first quadratic form, there are components that are not involved in the product. We denote them as follows, and they are determined by the following formulas:

$$\begin{aligned} A &= \overrightarrow{r'_t}^2 = {x'_t}^2 + {y'_t}^2 + {z'_t}^2, \\ B &= \overrightarrow{r'_t}\,\overrightarrow{r'_u} = x'_t x'_u + y'_t y'_u + z'_t z'_u, \\ C &= \overrightarrow{r'_t}\,\overrightarrow{r'_v} = x'_t x'_v + y'_t y'_v + z'_t z'_v \end{aligned} \tag{5}$$

For a given surface, we introduce the unit normal vector in the following form:

$$\overrightarrow{n} = \frac{\overrightarrow{r'_u} \times \overrightarrow{r'_v}}{\left\|\overrightarrow{r'_u} \times \overrightarrow{r'_v}\right\|} = \frac{\begin{vmatrix} y'_u & z'_u \\ y'_v & z'_v \end{vmatrix} \overrightarrow{e_2} - \begin{vmatrix} x'_u & z'_u \\ x'_v & z'_v \end{vmatrix} \overrightarrow{e_3} + \begin{vmatrix} x'_u & y'_u \\ x'_v & y'_v \end{vmatrix} \overrightarrow{e_4}}{\sqrt{EG - F^2}} = \frac{n_2\overrightarrow{e_2} + n_3\overrightarrow{e_3} + n_4\overrightarrow{e_4}}{W} \tag{6}$$

Here

$$n_2 = \begin{vmatrix} y'_u & z'_u \\ y'_v & z'_v \end{vmatrix}, \quad n_3 = -\begin{vmatrix} x'_u & z'_u \\ x'_v & z'_v \end{vmatrix}, \quad n_4 = \begin{vmatrix} x'_u & y'_u \\ x'_v & y'_v \end{vmatrix}, \quad W = \sqrt{EG - F^2} \quad [13]$$

The second quadratic form of the surface is defined by the introduced normal vector. The general form of the second quadratic form is determined by the following equality [16]:

$$\mathrm{II} = \left(\vec{n}\, d^2 r\right) = Pdt^2 + Ldu^2 + Ndv^2 + 2Qdtdu + 2Rdtdv + 2Mdudv \tag{7}$$

Its coefficients are determined as follows:

$$\begin{cases} P = \overrightarrow{r''_{tt}} \cdot \overrightarrow{n} = \dfrac{x''_{tt}n_2 + y''_{tt}n_3 + z''_{tt}n_4}{W} \\ Q = \overrightarrow{r''_{tu}} \cdot \overrightarrow{n} = \dfrac{x''_{tu}n_2 + y''_{tu}n_3 + z''_{tu}n_4}{W} \\ R = \overrightarrow{r''_{tv}} \cdot \overrightarrow{n} = \dfrac{x''_{tv}n_2 + y''_{tv}n_3 + z''_{tv}n_4}{W} \\ L = \overrightarrow{r''_{uu}} \cdot \overrightarrow{n} = \dfrac{x''_{uu}n_2 + y''_{uu}n_3 + z''_{uu}n_4}{W} \\ M = \overrightarrow{r''_{uv}} \cdot \overrightarrow{n} = \dfrac{x''_{uv}n_2 + y''_{uv}n_3 + z''_{uv}n_4}{W} \\ N = \overrightarrow{r''_{vv}} \cdot \overrightarrow{n} = \dfrac{x''_{vv}n_2 + y''_{vv}n_3 + z''_{vv}n_4}{W} \end{cases} \tag{8}$$

Curvature of a curve on a surface

We determine the principal curvatures of the surface through the extremal values of the normal curvatures at a given point on the surface. It is well known that the normal curvature is defined as follows:

$$k_n = \frac{\mathrm{II}}{\mathrm{I}}.$$

In Galilean geometry, based on the first quadratic form presented above, we determine the normal curvature by dividing it into two parts.

First, for the case where $t \neq \text{const}$, we have:

$$k_n = \frac{\mathrm{II}}{\mathrm{I}} = \frac{Pdt^2 + Ldu^2 + Ndv^2 + 2Qdtdu + 2Rdtdv + 2Mdudv}{dt^2}$$

We define the principal curvatures of the surface as the extremal values of the normal curvature k_n. To achieve this, by taking derivatives with respect to the directions dt, du, dv, setting them to zero, and performing certain transformations, we arrive at the following system:

$$\begin{cases} Pdt + Qdu + Rdv = kdt \\ Qdt + Ldu + Mdv = 0 \\ Rdt + Mdu + Ndv = 0 \end{cases} \Rightarrow \begin{cases} (P-k)dt + Qdu + Rdv = 0 \\ Qdt + Ldu + Mdv = 0 \\ Rdt + Mdu + Ndv = 0 \end{cases} \tag{9}$$

For this system to have a non-trivial (non-zero) solution, its principal determinant must be equal to zero. In this work, we consider the case where $LN - M^2 \neq 0$ for the given surface.

$$\begin{vmatrix} P-k & Q & R \\ Q & L & M \\ R & M & N \end{vmatrix} = 0 \quad \Rightarrow \quad k_1 = \frac{\begin{vmatrix} P & Q & R \\ Q & L & M \\ R & M & N \end{vmatrix}}{LN - M^2} \tag{10}$$

If $t = \text{const}$, then according to equality (4), the normal curvature becomes:

$$k_n = \frac{II}{I} = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}$$

Since $dt = 0$ if $t = \text{const}$, the expression simplifies accordingly. To find the extremal values of k_n, by taking derivatives with respect to the directions du, dv, setting them to zero, and performing certain transformations, we arrive at the following system:

$$\begin{cases} (L-kE)du + (M-kF)dv = 0 \\ (M-kF)du + (N-kG)dv = 0 \end{cases} \tag{11}$$

For system (11) to have a non-trivial solution, the principal determinant must be equal to zero:

$$\begin{vmatrix} L-kE & M-kF \\ M-kF & N-kG \end{vmatrix} = 0$$

From this, the following relations are derived:

$$k_2 + k_3 = \frac{LG - 2MF + NE}{EG - F^2}, \qquad k_2 \cdot k_3 = \frac{LN - M^2}{EG - F^2} \tag{12}$$

Based on the definitions above, the first and second mean curvatures, as well as the total (Gaussian) curvature of the surface, can be determined as follows:

Mean curvature

$$3H_1 = k_1 + k_2 + k_3$$

Second-order curvature

$$H_2 = k_1k_2 + k_1k_3 + k_2k_3$$

Total curvature

$$K = k_1 k_2 k_3$$

Using relations (10) and (12), the following formulas for H_1, H_2, and K can be derived:

$$3H_1 = \frac{\begin{vmatrix} P & Q & R \\ Q & L & M \\ R & M & N \end{vmatrix}}{LN - M^2} + \frac{LG - 2MF + NE}{EG - F^2} \tag{13}$$

$$H_2 = \frac{\begin{vmatrix} P & Q & R \\ Q & L & M \\ R & M & N \end{vmatrix}}{LN - M^2} \cdot \frac{LG - 2MF + NE}{EG - F^2} + \frac{LN - M^2}{EG - F^2} \tag{14}$$

$$K = \frac{\begin{vmatrix} P & Q & R \\ Q & L & M \\ R & M & N \end{vmatrix}}{EG - F^2} \tag{15}$$

Main part

Derivative formulas for the surface in G_4 space.

To study the surface (1) defined by the vector-valued function $\vec{r} = \vec{r}(t, u, v)$, we establish the derivation formulas. The second-order partial derivatives of the function are expressed as a linear combination of the vectors $\{\mathbf{r}_u, \mathbf{r}_v, \mathbf{n}\}$. To simplify the notation, we introduce the following conventions:

$$\vec{r}_t' = r_t, \quad \vec{r}_u' = r_u, \quad \vec{r}_v' = r_v, \quad \vec{n} = n.$$

The derivation formulas are then given by the following systems of equations:

$$\begin{cases} r_{tt} = \Gamma_{11}^2 \mathbf{r}_u + \Gamma_{11}^3 \mathbf{r}_v + b_{11}\mathbf{n} \\ r_{tu} = \Gamma_{12}^2 \mathbf{r}_u + \Gamma_{12}^3 \mathbf{r}_v + b_{12}\mathbf{n} \\ r_{tv} = \Gamma_{13}^2 \mathbf{r}_u + \Gamma_{13}^3 \mathbf{r}_v + b_{13}\mathbf{n} \\ r_{uu} = \Gamma_{22}^2 \mathbf{r}_u + \Gamma_{22}^3 \mathbf{r}_v + b_{22}\mathbf{n} \\ r_{uv} = \Gamma_{23}^2 \mathbf{r}_u + \Gamma_{23}^3 \mathbf{r}_v + b_{23}\mathbf{n} \\ r_{vv} = \Gamma_{33}^2 \mathbf{r}_u + \Gamma_{33}^3 \mathbf{r}_v + b_{33}\mathbf{n} \end{cases} \tag{16}$$

$$\begin{cases} n_t = \Gamma_{14}^2 \mathbf{r}_u + \Gamma_{14}^3 \mathbf{r}_v \\ n_u = \Gamma_{24}^2 \mathbf{r}_u + \Gamma_{24}^3 \mathbf{r}_v \\ n_v = \Gamma_{34}^2 \mathbf{r}_u + \Gamma_{34}^3 \mathbf{r}_v \end{cases} \tag{17}$$

By taking the scalar product of each equation in the system with the vector $\mathbf{n}$, the coefficients can be easily determined as follows:

$$b_{11} = P, \quad b_{12} = Q, \quad b_{13} = R, \quad b_{22} = L, \quad b_{23} = M, \quad b_{33} = N$$

To determine the remaining Γ_{ij}^k coefficients ($k = 2, 3;\ i, j = 1, 2, 3$), we take the scalar product of each equality in system (16) with the basis vectors $\{\mathbf{r}_u, \mathbf{r}_v\}$. For instance, considering the first equation:

$$\begin{cases} r_{tt} \cdot r_u = \Gamma_{11}^2 E + \Gamma_{11}^3 F = B_t - \frac{1}{2} A_u \\ r_{tt} \cdot r_v = \Gamma_{11}^2 F + \Gamma_{11}^3 G = C_t - \frac{1}{2} A_v \end{cases}$$

From this system, the solutions for Γ_{11}^2 and Γ_{11}^3 are found as:

$$\Gamma_{11}^2 = \frac{\begin{vmatrix} B_t - \frac{1}{2}A_u & F \\ C_t - \frac{1}{2}A_v & G \end{vmatrix}}{W^2}, \qquad \Gamma_{11}^3 = \frac{\begin{vmatrix} E & B_t - \frac{1}{2}A_u \\ F & C_t - \frac{1}{2}A_v \end{vmatrix}}{W^2}.$$

The remaining coefficients are determined in a similar manner:

$$\Gamma_{12}^2 = \frac{\begin{vmatrix} E_t & F \\ C_u - B_v + F_t & G \end{vmatrix}}{2W^2}, \quad \Gamma_{12}^3 = \frac{\begin{vmatrix} E & E_t \\ F & C_u - B_v + F_t \end{vmatrix}}{2W^2},$$

$$\Gamma_{13}^2 = \frac{\begin{vmatrix} B_v - C_u + F_t & F \\ G_t & G \end{vmatrix}}{2W^2}, \quad \Gamma_{13}^3 = \frac{\begin{vmatrix} E & B_v - C_u + F_t \\ F & G_t \end{vmatrix}}{2W^2},$$

$$\Gamma_{22}^2 = \frac{\begin{vmatrix} E_u & F \\ 2F_u - E_v & G \end{vmatrix}}{2W^2}, \quad \Gamma_{22}^3 = \frac{\begin{vmatrix} E & E_u \\ F & 2F_u - E_v \end{vmatrix}}{2W^2},$$

$$\Gamma_{23}^2 = \frac{\begin{vmatrix} E_v & F \\ G_u & G \end{vmatrix}}{2W^2}, \quad \Gamma_{23}^3 = \frac{\begin{vmatrix} E & E_v \\ F & G_u \end{vmatrix}}{2W^2},$$

$$\Gamma_{33}^2 = \frac{\begin{vmatrix} 2F_v - G_u & F \\ G_v & G \end{vmatrix}}{2W^2}, \quad \Gamma_{33}^3 = \frac{\begin{vmatrix} E & 2F_v - G_u \\ F & G_v \end{vmatrix}}{2W^2}.$$

The Γ_{ij}^k coefficients determined above are expressed solely through the coefficients A, B, C, E, F, G and their derivatives; notably, the coefficients of the second quadratic form are not involved.

We perform similar operations for system (17). By considering the following identities:

$$n_t \cdot r_t = -P, \quad n_t \cdot r_u = n_u \cdot r_t = -Q, \quad n_t \cdot r_v = n_v \cdot r_t = -R,$$
$$n_u \cdot r_u = -L, \quad n_u \cdot r_v = n_v \cdot r_u = -M, \quad n_v \cdot r_v = -N,$$

we take the scalar product of each equation in system (17) with the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$. From the resulting equalities, the coefficients Γ_{i4}^k of system (17) are determined. From the first equation, the following system is obtained:

$$\begin{cases} n_t \cdot r_u = \Gamma_{14}^2 E + \Gamma_{14}^3 F = -Q \\ n_t \cdot r_v = \Gamma_{14}^2 F + \Gamma_{14}^3 G = -R \end{cases}$$

The solutions to this system are given by:

$$\Gamma_{14}^2 = \frac{\begin{vmatrix} F & Q \\ G & R \end{vmatrix}}{W^2}, \qquad \Gamma_{14}^3 = \frac{\begin{vmatrix} Q & E \\ R & F \end{vmatrix}}{W^2}.$$

The remaining coefficients are determined using the same method. Their values are given by:

$$\Gamma_{24}^2 = \frac{\begin{vmatrix} F & L \\ G & M \end{vmatrix}}{W^2}, \quad \Gamma_{24}^3 = \frac{\begin{vmatrix} L & E \\ M & F \end{vmatrix}}{W^2},$$

$$\Gamma_{34}^2 = \frac{\begin{vmatrix} F & M \\ G & N \end{vmatrix}}{W^2}, \quad \Gamma_{34}^3 = \frac{\begin{vmatrix} M & E \\ N & F \end{vmatrix}}{W^2}.$$

If the vector-valued function $\mathbf{r} = \mathbf{r}(t,u,v)$ is continuous in the domain under consideration and possesses mixed partial derivatives up to the third order, then according to Schwarz's theorem, the following identities must hold [17]:

$$\begin{aligned}
&(r_{tt})_u = (r_{tu})_t, \quad (r_{tt})_v = (r_{tv})_t,\\
&(r_{tu})_v = (r_{tv})_u, \quad (r_{uu})_t = (r_{tu})_u,\\
&(r_{vv})_t = (r_{tv})_v, \quad (r_{uv})_t = (r_{tu})_v,\\
&(r_{uu})_v = (r_{uv})_u, \quad (r_{uv})_v = (r_{vv})_u
\end{aligned} \tag{18}$$

From the first identity, $(\mathbf{r}_{tt})_u - (\mathbf{r}_{tu})_t = 0$, and by utilizing systems (16) and (17), the following expression can be derived:

$$\begin{aligned}
&\left[(\Gamma^2_{11})_u + \Gamma^2_{11}\Gamma^2_{22} + \Gamma^3_{11}\Gamma^2_{23} + P\Gamma^2_{24} - (\Gamma^2_{12})_t - \Gamma^2_{12}\Gamma^2_{12} - \Gamma^3_{12}\Gamma^2_{13} - Q\Gamma^2_{14}\right] r_u +\\
&+ \left[(\Gamma^3_{11})_u + \Gamma^2_{11}\Gamma^3_{22} + \Gamma^3_{11}\Gamma^3_{23} + P\Gamma^3_{24} - (\Gamma^3_{12})_t - \Gamma^2_{12}\Gamma^3_{12} - \Gamma^3_{12}\Gamma^3_{13} - Q\Gamma^3_{14}\right] r_v +\\
&+ \left[\Gamma^2_{11}L + \Gamma^3_{11}M + P_u - \Gamma^2_{12}Q - \Gamma^3_{12}R - Q_t\right] n = 0
\end{aligned}$$

Due to the linear independence of the vectors $\mathbf{r}_u, \mathbf{r}_v$, and $\mathbf{n}$ in the above equality, it follows that their respective coefficients must vanish. By setting these coefficients to zero and performing several transformations, we derive the following equations:

$$\begin{aligned}
&P\Gamma^2_{24} - Q\Gamma^2_{14} = (\Gamma^2_{12})_t + \Gamma^2_{12}\Gamma^2_{12} + \Gamma^3_{12}\Gamma^2_{13} - (\Gamma^2_{11})_u - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^3_{11}\Gamma^2_{23} = \alpha_{11}\\
&P\Gamma^3_{24} - Q\Gamma^3_{14} = (\Gamma^3_{12})_t + \Gamma^2_{12}\Gamma^3_{12} + \Gamma^3_{12}\Gamma^3_{13} - (\Gamma^3_{11})_u - \Gamma^2_{11}\Gamma^3_{22} - \Gamma^3_{11}\Gamma^3_{23} = \alpha_{12}\\
&P_u - Q_t = \Gamma^2_{12}Q + \Gamma^3_{12}R - \Gamma^2_{11}L - \Gamma^3_{11}M = \beta_1
\end{aligned}$$

In a similar manner, the remaining 21 expressions for α_{lt} and β_i ($l = \overline{1,8};\ t = \overline{1,2}$) can be determined from the other seven identities. Consequently, a total of 24 equations are obtained from the eight identities.

Result. Since the terms α_{lt} in the derived equations are expressed linearly through the Christoffel symbols Γ^k_{ij} ($k = 2,3;\ i,j = 1,2,3$) and their partial derivatives, they can be determined without involving the coefficients of the second quadratic form. To define these terms, it is sufficient to obtain the coefficients A, B, C, E, F, G and their respective partial derivatives.

Lemma. *The ratios of the coefficients of the second quadratic form of the surface are fully expressed through the coefficients $A, B, C, E, F,$ and G, as well as their partial derivatives.*

Proof. By substituting the values of Γ^k_{i4} into the equations derived from the identities, we isolate the six relations necessary for the proof of the Lemma:

$$\begin{cases}
P\Gamma^2_{24} - Q\Gamma^2_{14} = \dfrac{PFM - PLG - QFR + Q^2G}{W^2} = \alpha_{11}\\
P\Gamma^3_{24} - Q\Gamma^3_{14} = \dfrac{PFL - PEM + QER - Q^2F}{W^2} = \alpha_{12}\\
P\Gamma^2_{34} - R\Gamma^2_{14} = \dfrac{PFN - PGM + QGR - R^2F}{W^2} = \alpha_{21}\\
L\Gamma^3_{14} - Q\Gamma^3_{24} = \dfrac{-E(LR - QM)}{W^2} = \alpha_{42}\\
M\Gamma^3_{14} - Q\Gamma^3_{34} = \dfrac{-E(MR - QN)}{W^2} = \alpha_{62}\\
L\Gamma^3_{34} - M\Gamma^3_{24} = \dfrac{-E(LN - M^2)}{W^2} = \alpha_{72}
\end{cases} \tag{19}$$

The system of equations in (19) can be rewritten in the following form:

$$\begin{cases} F(PM-QR)-G(PL-Q^2)=\alpha_{11}W^2 \\ -E(PM-QR)+F(PL-Q^2)=\alpha_{12}W^2 \\ F(PN-R^2)-G(PM-QR)=\alpha_{21}W^2 \\ QM-RL=\dfrac{\alpha_{42}W^2}{E} \\ QN-RM=\dfrac{\alpha_{62}W^2}{E} \\ LN-M^2=\dfrac{-\alpha_{72}W^2}{E} \end{cases}$$

From the given system, we determine the values of the expressions involving only the coefficients of the second quadratic form:

$$\begin{cases} PM-QR=\begin{vmatrix} -G & \alpha_{11} \\ F & \alpha_{12} \end{vmatrix}=a \\ PL-Q^2=\begin{vmatrix} \alpha_{11} & F \\ \alpha_{12} & -E \end{vmatrix}=b \\ PN-R^2=\begin{vmatrix} -G & \alpha_{21} \\ F & \alpha_{22} \end{vmatrix}=c \\ QN-RM=\dfrac{\alpha_{62}W^2}{E}=d \\ QM-RL=\dfrac{\alpha_{42}W^2}{E}=e \\ LN-M^2=\dfrac{-\alpha_{72}W^2}{E}=l \end{cases} \tag{20}$$

As seen from this system of equations, the terms $a, b, c, d, e,$ and l are all expressed in terms of the coefficients A, B, C, E, F, G and their partial derivatives. Using the established system (20), the following ratios can be constructed:

$$\begin{aligned} &\frac{Q}{L}=\frac{ae-bd}{e^2-bl}; \quad \frac{L}{M}=\frac{e^2-bl}{de-al}; \quad \frac{M}{N}=\frac{de-al}{d^2-cl}; \\ &\frac{N}{P}=\frac{d^2-cl}{a^2-bc}; \quad \frac{P}{R}=\frac{a^2-bc}{ce-ad}. \end{aligned} \tag{21}$$

In conclusion, the mutual ratios of the coefficients of the second quadratic form are expressed through the coefficients A, B, C, E, F, G and their partial derivatives. The Lemma is proved. □

Theorem 1. *The second-order curvature of the surface, defined by $H_2 = k_1k_2 + k_1k_3 + k_2k_3$, is not expressible through the coefficients of the first quadratic form alone; however, there exists a representation of this curvature in which the coefficients of the second quadratic form do not explicitly appear.*

Proof. To prove the theorem, we must demonstrate that the second-order curvature of the surface, expressed as:

$$H_2=\frac{\begin{vmatrix} P & Q & R \\ Q & L & M \\ R & M & N \end{vmatrix}}{LN-M^2}\cdot\frac{LG-2MF+NE}{EG-F^2}+\frac{LN-M^2}{EG-F^2}$$

can be rewritten without the coefficients $P, Q, R, L, M,$ and N. We must show that while the coefficients $E, F,$ and G are insufficient to represent this equality, the additional coefficients $A, B,$ and C (which do not appear in the standard first quadratic form) are required.By expanding the determinant in the expression for H_2, we divide both the numerator and the denominator of the first fraction by P^2:

$$H_2=\frac{P(LN-M^2)-Q(QN-MR)+R(QM-LR)}{LN-M^2}\cdot\frac{LG-2MF+NE}{EG-F^2}+\frac{LN-M^2}{EG-F^2}$$

$$H_2 = \frac{(LN-M^2) - \frac{Q}{P}(QN-MR) + \frac{R}{P}(QM-LR)}{\frac{L}{P}\frac{N}{P} - \left(\frac{M}{P}\right)^2} \cdot \frac{\frac{L}{P}G - 2\frac{M}{P}F + \frac{N}{P}E}{EG-F^2} + \frac{LN-M^2}{EG-F^2} \tag{22}$$

According to the Lemma and the system of equations (20), H_2 can be expressed as follows:

$$\begin{aligned} H_2 &= \frac{l - \frac{ae-bd}{a^2-bc}d + \frac{ce-ad}{a^2-bc}e}{\frac{e^2-bl}{a^2-bc}\frac{d^2-cl}{a^2-bc} - \left(\frac{de-al}{a^2-bc}\right)^2} \cdot \frac{\frac{e^2-bl}{a^2-bc}G - 2\frac{de-al}{a^2-bc}F + \frac{d^2-cl}{a^2-bc}E}{EG-F^2} + \frac{l}{EG-F^2} \\ &= \frac{1}{EG-F^2} \cdot \left\{ \frac{\left[l(a^2-bc) - d(ae-bd) + e(ce-ad)\right]\left[G(e^2-bl) - 2F(de-al) + E(d^2-cl)\right]}{(e^2-bl)(d^2-cl)-(de-al)^2} + l \right\} \\ &= \frac{l^2 - G(e^2-bl) + 2F(de-al) - E(d^2-cl)}{l(EG-F^2)} \end{aligned}$$

Thus, the expression for the second-order curvature takes the following form:

$$H_2 = \frac{l^2 - G(e^2-bl) + 2F(de-al) - E(d^2-cl)}{l(EG-F^2)}. \tag{23}$$

In the derived formula (23), the expressions $a, b, c, d, e,$ and l consist solely of the coefficients E, F, G and $A, B, C,$ along with their derivatives. The theorem is proved.The aforementioned Theorem 1 is of significant importance for surface theory. If, under certain conditions, the terms in H_2 containing $A, B,$ and C vanish, it follows that the second-order curvature of the surface belongs to its intrinsic geometry.

Theorem 2. *The total curvature (Gaussian curvature) of the surface is determined by the following formula:*

$$K = \pm\frac{\sqrt{\begin{vmatrix} a & b & e \\ d & e & l \\ c & a & d \end{vmatrix}}}{EG-F^2}$$

Proof. For a given surface, the total curvatureвЂ”defined as the product of the principal curvaturesвЂ”is determined by equation (15). Certain coefficients of the second quadratic form can be expressed as follows:

$$\begin{aligned} P^2 &= \frac{P^2(LN-M^2)}{LN-M^2} = \frac{LN-M^2}{\frac{L}{P}\frac{N}{P} - \left(\frac{M}{P}\right)^2} = \frac{l}{\frac{e^2-bl}{a^2-bc}\frac{d^2-cl}{a^2-bc} - \left(\frac{de-al}{a^2-bc}\right)^2} \\ &= \frac{(a^2-bc)^2}{2ade - e^2c - bd^2 + bcl - a^2l} = \frac{\begin{vmatrix} a & b \\ c & a \end{vmatrix}^2}{\begin{vmatrix} a & b & e \\ d & e & l \\ c & a & d \end{vmatrix}} \end{aligned}$$

$$Q^2 = \frac{Q^2(LN-M^2)}{LN-M^2} = \frac{(ae-bd)^2}{2ade - e^2c - bd^2 + bcl - a^2l} = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix}^2}{\begin{vmatrix} a & b & e \\ d & e & l \\ c & a & d \end{vmatrix}}$$

$$R^2 = \frac{R^2(LN-M^2)}{LN-M^2} = \frac{(ce-ad)^2}{2ade - e^2c - bd^2 + bcl - a^2l} = \frac{\begin{vmatrix} c & a \\ d & e \end{vmatrix}^2}{\begin{vmatrix} a & b & e \\ d & e & l \\ c & a & d \end{vmatrix}}$$

By utilizing these relations, the formula for the total curvature can be derived as follows:

$$K=\frac{\begin{vmatrix}P & Q & R\\ Q & L & M\\ R & M & N\end{vmatrix}}{EG-F^2}=\frac{P(LN-M^2)-Q(QN-MR)+R(QM-LR)}{EG-F^2}$$
$$\pm\frac{\begin{vmatrix}a & b\\ c & a\end{vmatrix}l-\begin{vmatrix}a & b\\ d & e\end{vmatrix}d+\begin{vmatrix}c & a\\ d & e\end{vmatrix}e}{\sqrt{\begin{vmatrix}a & b & e\\ d & e & l\\ c & a & d\end{vmatrix}}}=\frac{}{EG-F^2}=\pm\frac{\sqrt{\begin{vmatrix}a & b & e\\ d & e & l\\ c & a & d\end{vmatrix}}}{EG-F^2}$$

The theorem is proved. □

In surface theory, several fundamental equations can be derived from the system of equations (18). In particular, using system (20), the following relation can be established:

$$\frac{\begin{vmatrix}P & Q & R\\ Q & L & M\\ R & M & N\end{vmatrix}}{EG-F^2}=\pm\frac{\sqrt{\begin{vmatrix}a & b & e\\ d & e & l\\ c & a & d\end{vmatrix}}}{EG-F^2} \tag{24}$$

The right-hand side of this equation is not expressed solely through the coefficients of the first quadratic form. However, it is important to note that the coefficients of the second quadratic form do not appear in this expression either. Instead, the right-hand side is fully defined by the first quadratic form, its derivatives, and the additional coefficients A, B, and C, along with their derivatives. This equation serves as the analogue of the Gauss equation in the G_4 space.

Theorem 3. *The mean curvature of a surface in Galilean space is determined by the following identity:*

$$3H_1=\frac{G(e^2-bl)-2F(de-al)+E(d^2-cl)}{(a^2-bc)(EG-F^2)}-\frac{1}{l} \tag{25}$$

Proof. The mean curvature of the given surface is defined as shown in (13). Based on the previously established Lemma and Theorems, the following derivation can be constructed:

$$3H_1=\frac{\begin{vmatrix}P & Q & R\\ Q & L & M\\ R & M & N\end{vmatrix}}{LN-M^2}+\frac{LG-2MF+NE}{EG-F^2}=\frac{-1}{l}+\frac{\frac{e^2-bl}{a^2-bc}G-2\frac{de-al}{a^2-bc}F+\frac{d^2-cl}{a^2-bc}E}{EG-F^2}$$
$$=\frac{G(e^2-bl)-2F(de-al)+E(d^2-cl)}{(a^2-bc)(EG-F^2)}-\frac{1}{l}$$

The theorem is proved. □

Conclusion

In this study, several fundamental aspects of surface theory in four-dimensional Galilean space were investigated. The first and second quadratic forms of a surface were introduced, and their roles in defining the normal curvature and the principal curvatures were analyzed. Based on these concepts, formulas for the principal curvatures, mean curvature, second-order curvature, and total curvature of the surface were derived.

A system of derivative formulas describing the behavior of the surface was also established. By analyzing the relationships obtained from these formulas and applying the identities derived from the Schwarz theorem, it was shown that the ratios of the coefficients of the second quadratic form can be expressed through the coefficients of the first quadratic form and their partial derivatives. As a consequence, new representations for the mean curvature, second-order curvature, and total curvature were obtained in which the coefficients of the second quadratic form do not explicitly appear.

The obtained results contribute to the development of the differential geometry of surfaces in Galilean spaces and may be useful for further investigations of geometric properties of multidimensional non-Euclidean spaces.

REFERENCES

1. Rosenfeld B. A. *Non-Euclidean Spaces*. Moscow: Nauka, 1969. 548 pp.

2. Artikbayev A., Sultanov B. M., Ismoilov Sh. Sh. *Geometry of Semi-Euclidean Spaces: Isotropic and Galilean*. Tashkent: Tashkent State Transport University, 2023. 249 pp.

3. Pankina N. E. On the Theory of Surfaces in Three-Dimensional Galilean Space. *Geometry of Embedded Manifolds*. Moscow, 1980. 71–77.(in russian)

4. Roschel O. *Die Geometrie des Galileischen Raumes*, Habilitationsschrift, Institut für Math. und Angew. Geometrie, Leoben, 1984. 114–118.

5. Artikbayev A.; Sokolov D. D. *Geometry in General in the Flat Space-Time*. Tashkent: Fan, 1991. 180 pp.(in russian)

6. Dolgariev I. A. Finding a surface in the 3-dimensional Galilean space by its quadratic forms. *Izvestiya Vysshikh Uchebnykh Zavedeni: Povolzhsky Region*. 2006. No. 5 (26). 51–60.(in russian)

7. Dede M. On parallel ruled surfaces in Galilean space. *Kragujevac Journal of Mathematics*. 2016. Vol. 40(1). 47–59.

8. Yoon D. W. Some Classification of Translation Surfaces in Galilean 3-Space. *Int. Journal of Math. Analysis*. 2012. Vol. 6. No. 28. 1355–1361.

9. Dolgariev A. I. Surfaces of 4-Dimensional Space-Time Galilean. Complete Curvature of Surfaces. *Izvestiya vysshikh uchebnykh zavedeniy. Povolzhsky region. Physics and Mathematics*. 2008. No. 3. 3–19.

10. Aminov Yu. A. *Geometry of Submanifolds*. Kyiv: Naukova Duma, 2002. 468 pp. ISBN 966-00-0761-0.

11. Ismoilov Sh. Sh. Application of isotropic geometry to the solution of the MongeвЂ“Ampere equation. *Bulletin of the Karaganda University. Mathematics Series*, 2025, No. 4(120), 134–147.

12. Sharipov A.; Keunimjaev M. Existence and Uniqueness of Polyhedra with Given Values of the Conditional Curvature. *International Electronic Journal of Geometry*, 2023, Vol. 16, No. 1, 160–170.

13. Artykbaev A.; Sultanov B. M. Research of parabolic surface points in Galilean space. *Bulletin of National University of Uzbekistan: Mathematics and Natural Sciences*. 2019. Vol. 2. Issue 4. 231–245.(in russian)

14. Artykbaev A.; Nurbaev A. R. The indicatrix of the Surface in Four dimensional Galilean space. *Mathematics and Statistics*. 2020. 8(3):306–310.

15. Artikbaev A.; Nurbaev A. R.; Sultonov B. M. Theory of Surfaces in Four-Dimensional Galilean Space. *Proceedings of Science and Technology. Modern Math. and its Applications*. 2020.

16. Nurbaev A. The total curvature of a surface in a four-dimensional Galilean space. *Bulletin of the Institute of Mathematics*. 2023. Vol. 6. No. 1.

17. Fichtenholz G. M. *A Course of Differential and Integral Calculus*: In 3 Vol. Vol. 1. 10th ed. St. Petersburg: Lan' Publishing House, 2023. 608 pp.(in russian)

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Existence conditions for periodic solutions for differential equations with piecewise constant arguments of mixed type

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Abstract. This paper presents a way for locating n-periodic solutions to first-order differential equations with piecewise constant arguments of mixed type. The conditions for the existence of n-periodic solutions are thoroughly described, and an explicit formula for these solutions is derived. Additionally, an example is provided, illustrating a case where the problem admits infinitely many solutions.

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Key words: First order differential equations, piecewise constant arguments, periodic solution.

Introduction

Differential equations with piecewise constant arguments (DEPCAs) arise from extending the theory of functional differential equations with continuous arguments to those with discontinuous arguments. This extension holds significant applied interest since DEPCA, as particular cases, include impulsive and loaded equations used in control theory, as well as models similar to those in biomedical research. In [1], Cooke and Wiener introduced a novel differential equation of alternately retarded and advanced types, demonstrating that all equations with piecewise constant delays share characteristics with those studied in [2]. These equations are closely related to impulsive and loaded equations, especially discrete argument difference equations, and exhibit structural similarities to certain "sequential-continuous" disease dynamic models [3].

The DEPCAs are often classified as hybrid systems and can be used to model certain harmonic oscillators with almost periodic forcing [4], [5]. For a comprehensive survey on ordinary and partial differential equations with piecewise constant arguments, we refer the reader to [6], [7]. Functional differential equations with deviated arguments serve as mathematical models for systems where changes in state depend on either past history or future states. DEPCA also emerge when certain terms in a differential equation are replaced by their piecewise constant approximations. This approach finds applications in impulsive or loaded differential equations in control theory, as well as in the stabilization of systems with discrete (sampled) control [7], [8].

In [10], the delay differential equation

$$x'(t)+p_0(t)x(t)+\sum_{i=1}^{m}p_i(t)x(t-\tau_i)=0, \quad t\geq 0,$$

was considered, establishing approximations of delay differential equation solutions via those of delay differential equations with piecewise constant arguments.

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The study of DEPCA of mixed type was initiated by S. M. Shah and J. Wiener [11] in 1983, followed by G. Ladas [12] in 1988. They observed that changes in the sign of the argument deviation not only introduced interesting periodic properties but also added complexity to the asymptotic and oscillatory behavior of solutions. Naturally, this led to efforts to study the oscillatory and stability properties of DEPCA of mixed type with general deviation arguments.

Criteria for the existence of oscillatory solutions of DEPCA have been explored by numerous authors [2], [3], [4], [9], [13], [14], [15], [16]. It is also important to understand the additional conditions required for the stability of oscillatory solutions. These issues have been addressed in the broader field of differential equations. For example, A. R. Aftabizadeh and J. Wiener examined the oscillatory properties of solutions to first-order linear DEPCA of retarded and advanced types in [14].

$$x'(t) + a(t)x(t) + p(t)x([t]) = 0,$$

$$x'(t) + a(t)x(t) + q(t)x([t+1]) = 0,$$

where $a(t)$, $p(t)$ and $q(t)$ are continuous on $\mathbb{R}_+ = [0, \infty)$, and $[\cdot]$ is the greatest-integer function.

Authors in [17], [18], [19], [20] have streamlined the problem of n-periodic solvability to a set of n linear equations. Leveraging established properties of linear systems in algebra, they systematically delineated all conditions necessary for the existence of n-periodic solutions and furnished explicit formulas for solving these equations.

In this paper, we investigate the existence conditions for periodic solutions of a linear differential equation with piecewise constant arguments of mixed type of the form:

$$T'(t) = a(t)T(t) + b(t)T([t]) + c(t)T([t+1]), \quad t > 0, \tag{1}$$

subject to the initial condition:

$$T(0) = T_0, \tag{2}$$

where the functions $a(t)$, $b(t)$, and $c(t)$ are nonzero, n-periodic, and continuous on $\mathbb{R}_+ = [0, \infty)$, with n being a positive integer. We identify the conditions that either admit a unique periodic solution or an unlimited number of periodic solutions to the initial value issue (1)–(2).

Existence condition of a periodic solution

Let us define the concept of a solution for the DEPCA of mixed type as given by equations (1)–(2).

Definition. *A function $T(t)$ is called a solution of the DEPCA of mixed type (1)–(2) if the following conditions are satisfied:*

i) *$T(t)$ is continuous on $\mathbb{R}_+$.*

ii) *The derivative $T'(t)$ exists at each point $[t] \in \mathbb{R}_+$, with the possible exception of the points $[t] \in \mathbb{R}_+$, where the one-side derivatives exist.*

iii) *Equation (1) is satisfied for T on each interval $(k, k+1)$, $k \in \mathbb{N}$, and it holds for the right derivative at the points k, $k \in \mathbb{Z}$.*

The following assumption will be used throughout the paper. Let us denote:

$$M(i,t) = e^{\int_i^t a(m)dm}\left(1 + \int_i^t b(s)e^{-\int_i^s a(r)dr}ds\right), \quad t > i, \quad i = 0, 1, 2, ...,$$

$$P(i,t) = e^{\int_i^t a(m)dm}\int_i^t c(s)e^{-\int_i^s a(r)dr}ds, \quad t > i, \quad i = 0, 1, 2, ...$$

The following theorem provides a representation formula for the solutions of the DEPCA of mixed type (1)–(2) for $t > 0$.

Theorem 1. *Let $a(t), b(t),$ and $c(t)$ be continuous function on $[0, \infty)$, and suppose that*

$$P(i, i+1) \neq 1, \quad i = 0, 1, 2, ...$$

Then, the solution of the DEPCA of mixed type (1) and (2) is well defined for all $t \geq 0$ and given by

$$T(t) = \frac{M(n,t)}{1 - P(n,t)} \left(\prod_{i=0}^{n-1} \frac{M(i,i+1)}{1 - P(i,i+1)} \right) T_0, \quad t \in [n, n+1), \quad n = 0, 1, 2, \dots. \tag{3}$$

Proof. Suppose that the function $T(t)$ is a solution of the DEPCA of mixed type (1) and (2) on the interval $k \leq t < k+1$, then we have

$$T'(t) = a(t)T(t) + b(t)T(k) + c(t)T(k+1). \tag{4}$$

The solution of(1) and (2) in $t \in [0, 1)$ has the form

$$T(t) = T_0 \frac{\exp\left[\int_0^t a(m)dm\right] (1 + \int_0^t b(s) e^{-\int_0^s a(r)dr} ds)}{1 - \exp\left[\int_0^t a(m)dm\right] \int_0^t c(s) \exp\left[-\int_0^s a(r)dr\right] ds} = T_0 \frac{M(0,t)}{1 - P(0,t)} \quad \text{for} \quad t \in [0, 1).$$

Then, for $t \to 1 - 0$, we have

$$T(1) = \lim_{t \to 1-0} T(t) = T_0 \frac{M(0,1)}{1 - P(0,1)}.$$

For $t \in [1, 2)$, the function $T(t)$ has the form

$$T(t) = T_1 \frac{\exp\left[\int_0^t a(m)dm\right] \left(1 - \int_1^t b(s) \exp\left[-\int_1^s a(r)dr\right] ds\right)}{1 - \exp\left[\int_1^t a(m)dm\right] \int_1^t c(s) \exp\left[-\int_1^s a(r)dr\right] ds}$$

or

$$T(t) = T_0 \frac{M(0,1)}{1 - P(0,1)} \frac{M(1,t)}{1 - P(1,t)} \quad \text{for} \quad t \in [1, 2).$$

Let the function

$$T(t) = T(0) \prod_{i=0}^{k-2} \Big(\frac{M(i,i+1)}{1 - P(i,i+1)} \Big) \frac{M(k-1,t)}{1 - P(k-1,t)} \quad \text{for} \quad t \in [k-1, k), \quad k = 3, 4, ...$$

be solution of (1) and (2) in $[k-1, k)$ and

$$T(k) = \lim_{t \to k-0} T(t) = T_0 \frac{M(k-1,k)}{1 - P(k-1,k)}.$$

Then, integrating (4), we get the function

$$T(t) = T(0) \prod_{i=0}^{k-1} \Big(\frac{M(i,i+1)}{1 - P(i,i+1)} \Big) \frac{M(k,t)}{1 - P(k,t)} \quad \text{for} \quad t \in [k, k+1), \quad k = 3, 4, ...$$

Theorem 2. *Let $a(t)$, $b(t)$ and $c(t)$ be n-periodic continuous functions, and assume $P(i, i+1) \neq 1, \quad i = 0, 1, 2, ...$. Then the solution for the DEPCA of mixed type (1)–(2) is n-periodic if and only if*

$$\prod_{i=0}^{n-1} \frac{M(i,i+1)}{1 - P(i,i+1)} = 1.$$

Proof. Let $T(t)$ be a n-periodic solution for the DEPCA of mixed type (1)–(2). Then $T(n) = T_0$. From (3), when $t = n$ we obtain the condition: $\prod_{i=0}^{n-1} \frac{M(i,i+1)}{1-P(i,i+1)} = 1$.

Conversely, suppose that $\prod_{i=0}^{n-1} \frac{M(i,i+1)}{1-P(i,i+1)} = 1$. We will show that $T(n+t) = T(t)$ for all $t \in \mathbb{R}_+$. Let $t \in [k, k+1)$, which implies that $t+n \in [n+k, n+k+1)$, where k is an integer number. Then, we have

$$T(t+n) = \frac{M(n+k,t+n)}{1-P(n+k,t+n)} \left(\prod_{i=0}^{n+k-1} \frac{M(i,i+1)}{1-P(i,i+1)} \right) T_0 \quad \text{for} \quad t+n \in [n+k, n+k+1).$$

Changing the variables $r' = \rho + n$, $s = s' + n$ in the integral, we get:

$$M(n+k,t+n) = e^{-\int_{n+k}^{t+n} a(r')dr'} \left(1 + \int_{n+k}^{t+n} b(s) e^{\int_{n+k}^{s} a(r)dr} ds \right),$$

$$P(n+k,t+n) = e^{-\int_{n+k}^{t+n} a(r')dr'} \int_{n+k}^{t+n} c(s) e^{\int_{n+k}^{s} a(r)dr} ds.$$

After the change of variables, we obtain:

$$M(n+k,t+n) = e^{-\int_{k}^{t} a(\rho)d\rho} (1 + \int_{k}^{t} b(s') e^{\int_{n+k}^{s'+n} a(r)dr} ds'),$$

$$P(n+k,t+n) = e^{-\int_{k}^{t} a(\rho)d\rho} \int_{k}^{t} c(s') e^{\int_{n+k}^{s'+n} a(r)dr} ds'.$$

Then, by changing the variable $r = r' + n$, we obtain:

$$M(n+k,t+n) = e^{-\int_{k}^{t} a(\rho)d\rho} \left(1 + \int_{k}^{t} b(s') e^{\int_{k}^{s'} a(r')dr'} ds' \right),$$

$$P(n+k,t+n) = e^{-\int_{k}^{t} a(\rho)d\rho} \int_{k}^{t} c(s') e^{\int_{k}^{s'} a(r')dr'} ds',$$

i.e. $\frac{M(n+k,t+n)}{1-P(n+k,t+n)} = \frac{M(k,t)}{1-P(k,t)}$ for all $t+n \in [n+k, n+k+1)$. By applying this equation, we present the subsequent calculations.

$$\begin{aligned} \prod_{i=0}^{n+k-1} \frac{M(i,i+1)}{1-P(i,i+1)} &= \prod_{i=0}^{n-1} \frac{M(i,i+1)}{1-P(i,i+1)} \prod_{i=n}^{n+k-1} \frac{M(i,i+1)}{1-P(i,i+1)} = \prod_{i=n}^{n+k-1} \frac{M(i,i+1)}{1-P(i,i+1)} \\ &= \prod_{j=0}^{k-1} \frac{M(j+n,j+n+1)}{1-P(j+n,j+n+1)} = \prod_{j=0}^{k-1} \frac{M(j,j+1)}{1-P(j,j+1)}. \end{aligned}$$

Therefore

$$\begin{aligned} T(t+n) &= T(0) \prod_{i=0}^{n+k-1} \frac{M(i,i+1)}{1-P(i,i+1)} \frac{M(n+k,t+n)}{1-P(n+k,t+n)} \\ &= T(0) \prod_{i=0}^{k-1} \frac{M(i,i+1)}{1-P(i,i+1)} \frac{M(k,t)}{1-P(k,t)} = T(t) \quad \text{for} \quad t \in [k, k+1). \end{aligned}$$

Hence, $T(n+t) = T(t)$ for $t \in [k, k+1)$.

Theorem 3. *Assuming $M(i,i+1) = 0$ and $P(i,i+1) = 1$, then the Cauchy problem for the DEPCA of mixed type (1) and (2) admits an unlimited number of solutions for $i \in \mathbb{N} \cup \{0\}$. Specifically, the issue has an infinite number of n-periodic solutions and a unique one-periodic solution, $n = 2, 3, \ldots$.*

The demonstration of the theorem closely resembles the proof provided for Theorem 2 (1) in [20].

Illustrative examples

We will introduce appropriate example in this section. The example will show the usefulness of our theoretical results.

Example 1. Suppose $a(t) = a \in \mathbb{R}$, $b(t) = \beta \cos 2\pi t$, and $c(t) = \gamma \cos 2\pi t$, where $a \neq 0$, $\beta = \frac{(a^2+4\pi^2)e^a}{a(1-e^a)}$, and $\gamma = \frac{a^2+4\pi^2}{a(e^a-1)}$. Then, we have

$$M(i,t) = e^{a(t-i)} + \frac{\beta}{a^2+4\pi^2}\Big(2\pi \sin 2\pi t - a\cos 2\pi t - e^{a(t-i)}(2\pi \sin 2\pi i - a \cos 2\pi i)\Big) \quad \text{for} \quad t > i,$$

$$P(i,t) = \frac{\gamma}{a^2+4\pi^2}\Big(2\pi \sin 2\pi t - a\cos 2\pi t - e^{a(t-i)}(2\pi \sin 2\pi i - a \cos 2\pi i)\Big) \quad \text{for} \quad t > i.$$

It can be readily verified that the DEPCA of mixed type (1) and (2) satisfies the conditions outlined in Theorem 3. The function

$$T_2(t) = \begin{cases} M(0,t)T_0 + P(0,t)T_1, & t \in [0,1), \\ M(1,t)T_1 + P(1,t)T_0, & t \in [1,2), \end{cases}$$

is a 2-periodic solution of the DEPCA of mixed type (1) and (2), where T_1 is an arbitrary constant. As a result, the DEPCA of mixed type (1) and (2) admits infinitely many 2-periodic solutions. This result is further demonstrated by the simulations presented in Figs. 1 and 2.

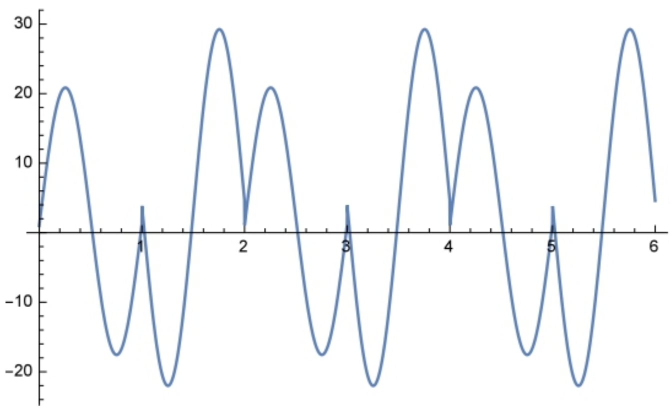

Рис. 4: The 2-periodic solution $T_2(t)$ for the DEPCA of mixed type with the parameters $T_1 = 4$ and $a = \frac{1}{6}$.

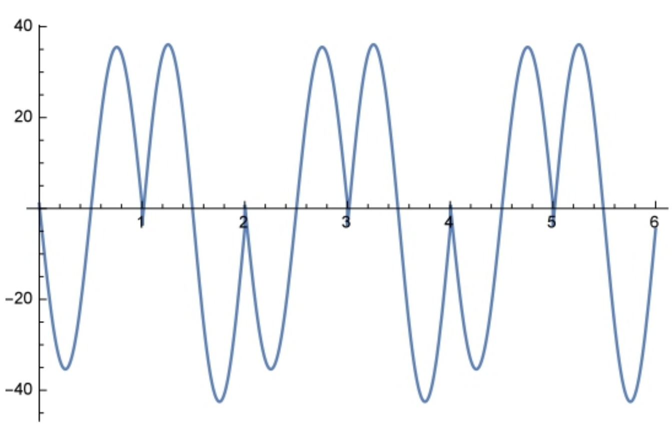

Рис. 5: The 2-periodic solution $T_2(t)$ for the DEPCA of mixed type with the parameters $T_1 = -4$ and $a = \frac{1}{6}$.

REFERENCES

1. Cooke K. L, Wiener J. An equation alternately of retarded and advanced type , Proc. Amer. Math. Soc. **99**, 1984. P. 726–732.

2. Cooke K. L, Wiener J. Retarded differential equations with piecewise constant delays, J. Math. Anal. Appl. **99**, 1984. P. 265–297.

3. Busenberg S, Cooke K. L. Models of vertically transmitted diseases with sequentialcontinuous dynamics, Nonlinear Phenomena in Mathematical Sciences (V. Lakshmikantham, ed.) Academic Press, New York, 1982. P. 179–187.

4. Hale J. Lunel K. V. Introduction to Functional Differential Equations, Springer, New York, 1993.

5. Hino Y, Naito T, Minh N, Shin V. Almost Periodic Solutions of Differential Equations in Banach Spaces. Stability and Control: Theory, Method and Applications. vol. 15. Taylor & Francis. London. 2002.

6. K.L. Cooke ,Wiener J. A survey of differential equations with piecewise continuous arguments., Delay Differential Equations and Dynamical Systems. vol. 1475, 1991. P. 1-15. Springer. Berlin. Germany.

7. Wiener J. Generalized Solutions of Functional Differential Equations, World Scientific. Singapore. 1993.

8. Pinto M, Robledo G. Controllability and observability for a linear time varying system with piecewise constant delay, Acta Appl. Math. 2014. P. 1-24.

9. Veloz T, Pinto M. Existence, computability and stability for solutions of the diffusion equation with general piecewise constant argument, J. Math. Anal. Appl. 426(1). 2015. P. 330–339.

10. Györi I. On approximation of the solutions of delay differential equations by using piecewise constant arguments, Int. J. Math. Math. Sci. **14**(1), 1991. P. 111–126.

11. Shah S. M, Wiener J. Advanced differential equations with piecewise constant argument deviations, Int. J. Math. Math. Sci. **6**:4, 1983. P. 671–703.

12. Ladas G. Oscillations of equations with piecewise constant mixed arguments, Differential Equations and Applications II, Ohio University Press, Athens, 1988, P. 64–69.

13. Wiener J, Heller W. Oscillatory and periodic solutions to a diffusion equation of neutral type, Int. J. Math. Math. Sci. **22**(2). 1999. P. 313–348.

14. Aftabizadeh A. R, Wiener J. Oscillatory properties of first order linear functional differential equations, Appl Anal. **20**, 1985. P. 165–187.

15. Aftabizadeh A. R, Wiener J, Jian-Ming Xu. Oscillatory and periodic solutions of delay differential equations with piecewise constant argument, Proc. Amer. Math. Sot. **99**, 1987. P. 673–679.

16. Wiener J, Debnath L. Boundary value problems for the diffusion equation with piecewise continuous time delay, Internat. J. Math. and Math. Sci. **20**(1). 1997. P. 187–195.

17. Muminov M. I, Radjabov T. A. Existence conditions for 2-periodic solutions to a non-homogeneous differential equations with piecewise constant argument, Examples and Counterexamples. V. 5. 2024. P. 100145.

18. Muminov M. I, Radjabov T. A. On existence conditions for periodic solutions to a differential equation with constant argument, Nanosystems: Phys. Chem. Math. V. 13. **5**.2022. P. 491–497.

19. Radjabov T. A. Existence conditions for periodic solutions to a non-homogeneous differential equations with piecewise constant argument. Uzbek Mathematical Journal. - Volume 67, Issue 3, 2023. P. 139-144. DOI: 10.29229/uzmj.2023-3-14

20. Muminov M. I, Radjabov T. A. Existence of Solutions of a Boundary Value Problem for a Diffusion Equation with Piecewise Constant Arguments,Differential Equations **61** (1), 2025. P. 16–28 https://doi.org/10.1134/S0012266125010033.

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Local derivations on algebra of measurable operators affiliated with commutative real von Neumann algebras

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Abstract. Local derivations on algebra of measurable operators affiliated with commutative real von Neumann algebras are considered. A necessary and sufficient condition for the existence of a non-trivial (non-inner) derivation (local derivation) on an algebra of measurable operators affiliated with commutative real von Neumann algebra has been found. It turns out that such a condition is the not-atomicity of the lattice of projectors of the algebra.

MSC (2020): 46L10; 17A30; 17B30; 17B40; 17B56.

Key words: von Neumann algebras, derivations, local derivations.

Introduction

As is well known, the theory of derivations and local derivations on bounded and unbounded operator algebras, in particular, the algebras of all measurable (local and τ- measurable) operators on a Hilbert space with respect to von Neumann algebras, has been fairly well studied. Key achievements in this area are associated with names such as Sh.A.Ayupov, K.K.Kudaybergenov, V.I.Chilin, A.F.Ber, F.A.Sukochev and others. However, the real analogue of some results has not yet been described, i.e., local derivations on the algebras of all measurable (local and τ- measurable) operators with respect to real von Neumann algebras, have not been considered. In this article we will look at commutative real von Neumann algebra with a faithful normal semifinite trace.

Using the result of V.I.Chilin, A.F.Ber and F.A.Sukochev obtained for regular algebras, we obtained a real analogue of the results Sh.A.Ayupov, K.K.Kudaybergenov obtained for commutative (complex) von Neumann algebra. Namely, it is proved that in commutative real von Neumann algebra A with a faithful normal semifinite trace τ, the following conditions are equivalent:

- the lattice $P(\mathcal{A})$ of projections in $\mathcal{A}$ is not atomic;
- the algebra $S(\mathcal{A})$ (resp. $S(\mathcal{A},\tau)$) admits a non-inner derivation;
- the algebra $S(\mathcal{A})$ (resp. $S(\mathcal{A},\tau)$) admits a non-zero local derivation;
- the algebra $S(\mathcal{A})$ (resp. $S(\mathcal{A},\tau)$) admits a local derivation which is not a derivation,

where $S(\mathcal{A})$ (resp. $S(\mathcal{A},\tau)$) is the algebra of all measurable (resp. τ-measurable) operators with respect to A.

Preliminaries

Let A be an algebra. A linear operator $D : A \to A$ is called a *derivation* if it satisfies the identity $D(xy) = D(x)y + xD(y)$, for all $x, y \in A$. Each element $a \in A$ defines a derivation D_a on A given as $D_a(x) = ax - xa$, $x \in A$. Such derivations D_a are said to be *inner derivations.* A linear map $\delta : A \to A$ is called a *local derivation*, if for every $x \in A$, there exists a derivation $\delta_x : A \to A$ such that $\delta(x) = \delta_x(x)$.

Let $B(H)$ be the *-algebra of all bounded linear operators on a Hilbert space H and let $\mathbb{1}$ be the identity operator on H. A W^**-algebra* is a weakly closed *-subalgebra of $B(H)$, containing $\mathbb{1}$. A *real* W^**-algebra* is

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a weakly closed real *-subalgebra $R \subset B(H)$ with the identity which satisfies the equality $R \cap iR = \{0\}$. W*-algebras are also known as *von Neumann algebras*.

Let M' is a set which consists of all bounded linear operators on the Hilbert space H, that commute with every element of M. That is $M' = \{y \in B(H) : xy = yx, \forall x \in M\}$. M' is called the commutant of M. The set of all projections in M we denote by $P(M)$. If $u(\mathcal{D}) \subseteq \mathcal{D}$ for any unitary $u \in M'$ then a linear subspace $\mathcal{D}$ in H is said to be affiliated with M (denoted as $\mathcal{D}\eta M$). A linear operator $x : \mathcal{D}(x) \to H$ is said to be affiliated with M (denoted as $x\eta M$) if $\mathcal{D}(x)\eta M$ and $u(x(\xi)) = x(u(\xi))$ for all $\xi \in \mathcal{D}(x)$ and for every unitary $u \in M'$, which domain is $\mathcal{D}(x)$ of x is a linear subspace of H. If $\mathcal{D}\eta M$ and there exists a sequence of projections $\{p_n\}_{n=1}^{\infty}$ in $P(M)$ such that $p_n \uparrow \mathbb{1}$, $p_n(H) \subseteq \mathcal{D}$ and $p_n^* = \mathbb{1} - p_n$ is finite in M for all $n \in \mathbb{N}$, then a linear subspace $\mathcal{D}$ in H is said to be strongly dense in H with respect to M. If $x\eta M$ and $\mathcal{D}(x)$ is strongly dense in H, then a closed linear operator x in the Hilbert space H is said to be measurable with respect to M. Let us denote by $S(M)$ all linear operators measurable with respect to M on H.

For a von Neumann algebra M, we typically define a trace on its set of positive elements, denoted as M_+. A *trace* on M is a function $\tau : M_+ \to [0, \infty]$ that satisfies the following three conditions for all $x, y \in M_+$ and scalar $\lambda \geq 0$: i) $\tau(x+y) = \tau(x)+\tau(y)$, ii) $\tau(\lambda x) = \lambda\tau(x)$ (with the convention $0\cdot\infty = 0$) and iii) $\tau(xx^*) = \tau(x^*x)$ for every $x \in M$. Traces are rarely "just"traces; they are usually categorized by how well-behaved they are:

- *faithful*: $\tau(x) = 0$ implies $x = 0$. This means the trace doesn't "miss"any non-zero operators.
- *normal*: If a directed increasing net of operators x_i converges to x, then $\tau(x_i)$ converges to $\tau(x)$. This is a continuity requirement essential for the von Neumann setting.
- *finite*: $\tau(\mathbb{1}) < \infty$. If $\tau(\mathbb{1}) = 1$, it is often called a *tracial state*.
- *semifinite*: For every non-zero $x \in M_+$, there exists a non-zero $y \leq x$ such that $\tau(y) < \infty$.

Let M be a von Neumann algebra with semifinite trace τ. If $x\eta M$ and $\mathcal{D}(x)$ is τ-dense in H, i.e. for any $\epsilon > 0$ there is a projection $p \in M$ such that $p(H) \subseteq \mathcal{D}(x)$ and $\tau(p^\perp) < \epsilon$, then a closed linear operator x is said to be τ-measurable with respect to M. Let $S(M, \tau)$ is a set of all τ-measurable operators on H with respect to M. It is worth mentioning that $S(M, \tau) = S(M)$ if the trace τ is finite. All the above concepts are similarly defined for real von Neumann algebras.

Main part.

Below we discuss the problem of existence of local derivations which are not derivations on the algebras $S(\mathcal{R})$ and $S(\mathcal{R}, \tau)$ in the case where the real von Neumann algebra $\mathcal{R}$ is commutative. In many places, we use the concepts and results of the work [3], which considers a regular algebra over an arbitrary field K, in particular $K = \mathbb{C}$ or $K = \mathbb{R}$.

Let A be a commutative algebra with the unit $\mathbb{1}$ over the field $\mathbb{C}$ or $\mathbb{R}$. We denote by ∇ the set

$$\nabla = \{e \in A : e^2 = e\}$$

of all idempotents in A. For $e, f \in \nabla$ we set $e \leq f$ if $ef = e$. With respect to this partial order, the lattice operations

$$e \vee f = e + f - ef, \qquad e \wedge f = ef,$$

and the complement $e^\perp = \mathbb{1} - e$, the set ∇ forms a Boolean algebra.

A non-zero element q from the Boolean algebra ∇ is called an *atom* if

$$0 \neq e \leq q,\ e \in \nabla \quad \Rightarrow \quad e = q.$$

If for any non-zero $e \in \nabla$ there exists an atom q such that $q \leq e$, then the Boolean algebra ∇ is said to be *atomic*.

An algebra A is called *regular* (in the sense of von Neumann) if for any $a \in A$ there exists $b \in A$ such that

$$a = aba.$$

Further, we shall always assume that A is a unital commutative regular algebra over $\mathbb{C}$ or $\mathbb{R}$, and that ∇ is the Boolean algebra of all its idempotents. In this case, given any element $a \in A$, there exists an idempotent

$e \in \nabla$ such that $ea = a$, and if $ga = a$, $g \in \nabla$, then $e \leq g$. This idempotent is called the *support* of a and denoted by $s(a)$.

Suppose that μ is a strictly positive countably additive finite measure on the Boolean algebra ∇ of idempotents from A and consider the metric

$$\rho(a,b) = \mu(s(a-b)), \qquad a, b \in A.$$

From now on we shall assume that (A, ρ) is a complete metric space (cf. [2,3]).

Example 3.1 The most important example of a complete commutative regular algebra (A, ρ) is the algebra $A = L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$ of all (classes of equivalence of) measurable complex (or real) functions on a measure space (Ω, Σ, μ), where μ is a finite countably additive measure on Σ, and

$$\rho(a,b) = \mu(s(a-b)) = \mu\big(\{\omega \in \Omega : \ a(\omega) \neq b(\omega)\}\big)$$

(see for details [1, Lemma] and [3, Example 2.5]).

Remark 3.2 If (Ω, Σ, μ) is a general localizable measure space, i.e. the (not finite in general) measure μ has the finite sum property, then the algebra $L^0(\Omega, \Sigma, \mu)$ is a unital regular algebra, but $\rho(a,b) = \mu(s(a-b))$ is not a metric in general. But one can represent Ω as a union of pairwise disjoint measurable sets with finite measures and thus this algebra is a direct sum of commutative regular complete metrizable algebras from the above example.

Following [3] we call an element $a \in A$ *finitely valued* (respectively, *countably valued*) if

$$a = \sum_{k=1}^{n} \alpha_k e_k,$$

where $\alpha_k \in \mathbb{C}$ or $\alpha_k \in \mathbb{R}$, $e_k \in \nabla$, $e_k e_j = 0$, $k \neq j$, $k, j = 1, \ldots, n$, $n \in \mathbb{N}$ (respectively,

$$a = \sum_{k=1}^{\omega} \alpha_k e_k,$$

where $\alpha_k \in \mathbb{C}$ or $\alpha_k \in \mathbb{R}$, $e_k \in \nabla$, $e_k e_j = 0$, $k \neq j$, $k, j = 1, \ldots, \omega$, and ω is a natural number or ∞; in the latter case the convergence of the series is understood with respect to the metric ρ).

We denote by $K(\nabla)$ (respectively, $K_c(\nabla)$) the set of all finitely valued (respectively, countably valued) elements in A. It is known that

$$\nabla \subset K(\nabla) \subset K_c(\nabla),$$

both $K(\nabla)$ and $K_c(\nabla)$ are regular subalgebras in A, and moreover the closure of $K(\nabla)$ in (A, ρ) coincides with $K_c(\nabla)$ (see [3], Proposition 2.8).

Now let D be a derivation on the given regular commutative algebra A. By [3], Proposition 2.3 we have that

$$s(D(a)) \leq s(a), \qquad \forall a \in A,$$

and $D|_{\nabla} = 0$. Therefore by the definition, each local derivation Δ on A satisfies the following two conditions:

$$s(\Delta(a)) \leq s(a), \qquad \forall a \in A, \quad (1)$$

$$\Delta|_{\nabla} \equiv 0. \quad (2)$$

This means that (1) and (2) are necessary conditions for a linear operator Δ to be a local derivation on the algebra A. We are going to show that these two conditions are in fact also sufficient.

First we recall some further notions from the paper [3]. Let B be a unital subalgebra in the algebra A. An element $a \in A$ is called:

- *algebraic with respect to* B, if there exists a polynomial $p \in B[x]$ (i.e. a polynomial in x with coefficients from B) such that $p(a) = 0$;

• *integral with respect to* B, if there exists a unitary polynomial $p \in B[x]$ (i.e. the coefficient of the largest degree of x in $p(x)$ is equal to $\mathbb{1} \in B$) such that $p(a) = 0$;

• *transcendental with respect to* B, if a is not algebraic with respect to B;

• *weakly transcendental with respect to* B, if $a \neq 0$ and for any non-zero idempotent $e \leq s(a)$ the element ea is not integral with respect to B.

In the paper [2] the following two lemmas are proved using the results of paper [3]. Note that the fields K do not play a role in the proofs of these lemmas.

Lemma 3.3 [2, Lemma 3.3] Given any element $a \in A$ there exists an idempotent $e \in \nabla$ such that

(i) ea is integral with respect to $K_c(\nabla)$, moreover in this case $ea \in K_c(\nabla)$;

(ii) $e^{\perp}a$ is weakly transcendental with respect to $K_c(\nabla)$, if $e \neq \mathbb{1}$.

The following lemma is the crucial step for the proof of the main results in this section.

Lemma 3.4 [2, Lemma 3.4] Each linear operator on the algebra A satisfying the conditions (1) and (2) is a local derivation on A.

The following is the main result concerning the existence of local derivations on commutative regular algebras.

Theorem 3.5 Let A be a unital commutative regular algebra over $\mathbb{C}$ or $\mathbb{R}$ and let μ be a finite strictly positive countably additive measure on the Boolean algebra ∇ of all idempotents of A. Suppose that A is complete in the metric

$$\rho(a, b) = \mu(s(a - b)), \qquad a, b \in A.$$

Then the following conditions are equivalent:

(i) $K_c(\nabla) \neq A$;

(ii) the algebra A admits a non-zero derivation;

(iii) the algebra A admits a non-zero local derivation;

(iv) the algebra A admits a local derivation which is not a derivation.

The implications $(i) \Leftrightarrow (ii)$ are proved in [3], Theorem 3.2. The implication $(ii) \Rightarrow (iii)$ is trivial because any derivation is a local derivation.

To prove $(iii) \Rightarrow (iv)$ we need the following lemma.

Lemma 3.6 Let A be a commutative regular algebra over the field K, where $K = \mathbb{R}$ or $K = \mathbb{C}$. If D is a derivation on A, then D^2 is a derivation if and only if $D = 0$.

Proof. For $K = \mathbb{C}$ the lemma is proved in [2, Lemma 3.6]. Let us prove it for $K = \mathbb{R}$. Let R be a commutative regular real algebra and $A = R + iR$. As is known, the regularity of the ring is preserved during its complexification. Therefore A is a commutative regular algebra. Let $D : R \to R$ be a derivation on R. Let us extend a derivation D on A as $\overline{D}(x + iy) = D(x) + iD(y)$. If $D = 0$ then $\overline{D} = 0$ and by [2, Lemma 3.6] we have $\overline{D}^2$ is a derivation. Since

$$\overline{D}^2(x + iy) = \overline{D}(D(x) + iD(y)) = D^2(x) + iD^2(y) = \overline{D^2}(x + iy)$$

then $\overline{D}^2 = \overline{D^2}$. Hence $\overline{D^2}$ is also a derivation. Therefore D^2 is a derivation.

Now let D^2 be a derivation. Then $\overline{D^2}$ is also a derivation, hence $\overline{D}^2$ is a derivation. By [2, Lemma 3.6] we have $\overline{D} = 0$, hence we obtain $D = 0$. Proof is complete.

Now $(iii) \Rightarrow (iv)$ follows. Let $a \in A \setminus K_c(\nabla)$. By Lemma 3.3 there exists an idempotent $e \in \nabla$ such that $ea \in K_c(\nabla)$ and $b = e^{\perp}a$ is weakly transcendental with respect to $K_c(\nabla)$. By [3, Proposition 3.7, Theorem 3.1] there exists a derivation D on A such that $D(b) = b$. Consider $\Delta = D^2$. Then Δ satisfies (1) and (2), hence by Lemma 3.4 Δ is a local derivation. Moreover,

$$\Delta(b) = D(D(b)) = D(b) = b \neq 0,$$

so $\Delta \neq 0$. By Lemma 3.6 Δ is not a derivation.

Finally, $(iv) \Rightarrow (i)$ follows from [3, Theorem 3.2].

Corollary 3.7 Let (Ω, Σ, μ) be a finite measure space and let $L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$. The following conditions are equivalent:

(i) the Boolean algebra of all idempotents from $L^0(\Omega)$ is not atomic;

(ii) $L^0(\Omega)$ admits a non-zero derivation;

(iii) $L^0(\Omega)$ admits a non-zero local derivation;

(iv) $L^0(\Omega)$ admits a local derivation which is not a derivation.

Proof. This follows easily from Theorem 1.3, Example 3.1, [2, Corollary 3.7] and [3, Corollary 3.7].

Theorem 3.8 Let $\mathcal{A}$ be a commutative complex or real von Neumann algebra with a faithful normal semifinite trace τ. The following conditions are equivalent:

(i) the lattice $P(\mathcal{A})$ of projections in $\mathcal{A}$ is not atomic;

(ii) the algebra $S(\mathcal{A})$ (respectively $S(\mathcal{A}, \tau)$) admits a non-inner derivation;

(iii) the algebra $S(\mathcal{A})$ (respectively $S(\mathcal{A}, \tau)$) admits a non-zero local derivation;

(iv) the algebra $S(\mathcal{A})$ (respectively $S(\mathcal{A}, \tau)$) admits a local derivation which is not a derivation.

In the complex case, this theorem was proved in [2, Theorem 3.8]. In the real case, it is proved by similar arguments using the above results.

Remark 3.9 For general (non-commutative) complex or real von Neumann algebras the above conditions are not equivalent. Some implications remain valid, but others fail.

REFERENCES

1. Ayupov Sh. A., Derivations on algebras of measurable operators, Dokl. Akad.Nauk R.Uzbekistan, 3 (2000) 14-17.

2. Albeverio S., Ayupov Sh. A. , Kudaybergenov K. K., Nurjanov B. O. Local Derivations on Algebras of Measurable Operators. Communications in Contemporary Mathematics 13(4):643-657.

3. Ber A. F., Chilin V. I., Sukochev F. A., Non-trivial derivation on commutative regular algebras, Extracta Math., 21 (2006) 107-147.

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Properties of quasitraces on real C*-algebras

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Abstract. Quasitraces play a significant role in the study of structural properties of C*-algebras, especially in connection with finiteness and classification problems. In this paper, quasitraces on real C*-algebras are studied and some of their main properties are established. The behavior of quasitraces under matrix amplifications is investigated, and the conditions ensuring their extension to 2-quasitraces are considered. In addition, the relationship between quasitraces on real C*-algebras and their complexifications is analyzed. The obtained results provide a basis for understanding tracial-type functionals in the real conditions and complement existing results known for complex C*-algebras.

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Key words: real C*-algebras, quasitraces, center-valued quasitraces, AW*-algebras, dimension functions.

Introduction

As is known, traces play an important role in understanding the structure of C*-algebras, and moreover, they are very important in the theory of von Neumann algebra. For example, it is well known and very important that the II_1 factors admit a unique tracial state. But not every C*-algebra possesses a trace. There is a generalization of traces to quasitraces. In 1982, B.Blackadar and D.Handelman defined an analogue of the trace, called a quasitrace. Despite the fact that this concept does not completely replace a trace, they and some researchers managed to obtain an analogue of the results available for traces.

In this paper, a real analogue of a (center-valued) quasitrace is given, and its connection with the quasitrace of the enveloping C*-algebra is found. Similar to the complex case, some interesting properties of a quasitrace and the corresponding metric for real C*-algebras are obtained.

Preliminaries

A *W*-algebra* is a weakly closed *-subalgebra of the algebra of all bounded linear operators *B(H)* on a complex Hilbert space H, containing the identity operator. A *real W*-algebra* is a weakly closed *-subalgebra R with the identity operator, satisfying the condition $R \cap iR = \{0\}$. A *C*-algebra* is a Banach *-algebra over the complex numbers, in which the norm satisfies the equality $\|aa^*\| = \|a\|^2$ for all elements a. A *real C*-algebra* is a Banach *-algebra over the reals, where $\|xx^*\| = \|x\|^2$, and the element formed by adding the identity operator to the square of any element x is invertible. *Baer *-rings* are *-rings in which the right annihilator of any subset can be expressed as a principal right ideal generated by a projection. (Real) C*-algebras with a Baer *-ring are called *(real) AW*-algebras* (for more details see [1]). Every W*-algebra is an AW*-algebra, but not all AW*-algebras can be represented as W*-algebras. Factors are W*-algebras with trivial center and are classified into types I_n, I_∞, II_1, II_∞, and III (see, e.g., [2]). Analogous notions for AW*-algebras can be found in [1]. Any W*- or AW*-algebra can be uniquely decomposed along its center into these factors. For every element in an

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AW*-algebra, the right and left support projections exist, and they describe annihilation properties. Spectral projections for self-adjoint elements correspond to intervals such as (λ, ∞) and play a fundamental role in describing spectral properties within the algebra (see [Remark 1.5] in [3]).

Definition 1. [1] Let A be a unital C*-algebra. A *quasitrace* τ on A is a function $\tau : A \to \mathbb{C}$ that satisfies:

(i) $\tau(x^*x) = \tau(xx^*) \geq 0$, for $x \in A$;

(ii) $\tau(a + ib) = \tau(a) + i\tau(b)$, for $a, b \in A_h$, where $A_h = \{a = a^*, a \in A\}$;

(iii) τ is linear on any abelian C*-subalgebra B_c of A.

Furthermore, τ is called a n-quasitrace $(n \geq 2)$ if there exists a 1-quasitrace τ_n on $M_n(A) = A \otimes M_n(\mathbb{C})$ such that *(iv)* $\tau(x) = \tau_n(x \otimes e_{11}), \quad x \in A,$

We present the definition of quasitrace in the complex case given above in Real case. This is one of the main definitions of our article.

Definition 2. Let R be a unital real C*-algebra. A quasitrace τ on R is a function $\tau : R \to \mathbb{R}$ that satisfies:

(i') $\tau(x^*x) = \tau(xx^*) \geq 0$, for $x \in R$;

(ii') $\tau(a + b) = \tau(a)$, for $a \in R_h$, $b \in R_k$, where $R_k = \{b = -b^*, b \in R\}$;

(iii') τ is linear on any abelian C*-subalgebra B of R.

The center $\mathcal{Z}(R)$ of a real AW* algebra R is the set of elements in R that commute with all other elements of R. Formally, it is defined as:

$$\mathcal{Z}(R) = \{a \in R \mid ab = ba \text{ for all } b \in R\}.$$

In other words, $a \in \mathcal{Z}(R)$ if a commutes with every element B in R.

A center-valued quasitrace on C*-algebras.

Definition 3. Let N be a unital complex (or real) C^*-algebra and $\mathcal{Z}(R)$ is the center of R. A center-valued quasitrace is the map $T : R \to \mathcal{Z}(R)$ that satisfies:

(a) $T(x^*x) = T(xx^*) \geq 0$;

(b) $T(a + b) = T(a)$, for $a \in R_h$ and $b \in R_k$;

(c) T is linear on commutative real C*-subalgebras of R;

The following theorem is a real analogue of Theorem 1.27 given in [3], the proof of which is given in [6].

Theorem 1. [[6], Theorem 4.] Let R be a finite real AW*-algebra. Then, there exists a unique center-valued quasitrace $T : R \to \mathcal{Z}(R)$ with the following properties:

1. $T(x^*x) = T(xx^*) \geq 0$, for all $x \in N$;
2. $T(a + b) = T(a)$, for $a \in N_h$ and $b \in N_k$;
3. T is linear on commutative real C*-subalgebras of N;
4. $T(x^*x) = 0$ if and only if $x = 0$;
5. $T|_{P(N)} = D$, where D is the center-valued dimension function from the previous theorem;
6. $T(hx) = hT(x)$, for all self-adjoint $h \in \mathcal{Z}(N)$ and for all $x \in N$;
7. $T|_{\mathcal{Z}(N)} = \mathrm{id}_{\mathcal{Z}(N)}$;
8. T is order-preserving on N_h;

9. T is continuous in norm, in particular, $\|T(x) - T(y)\| \leq 2\|x - y\|$, for all $x, y \in N$.

Theorem 2. Let R be a unital real AW*-algebra, and let $A = R + iR$ be its enveloping AW*-algebra. If $\overline{T}$ is a center-valued quasitrace on A, then the map $T : R \to \mathcal{Z}(R)$ defined by

$$T(a + b) = \overline{T}(a), \quad \text{where } a \in R_h,\, b \in R_k \tag{26}$$

is a center-valued quasitrace on R.

Proof. 1. Assume $x \in R$ and express it as $x = a + b$, where $a \in R_h$ and $b \in R_k$. From equation (1),

$$T(x^*x) = T(a^2 - b^2 + ab - ba) = \overline{T}(a^2 - b^2),$$

where $a^2 - b^2 \in R_h$ and $ab - ba \in R_k$. Similarly, for xx^*, we have

$$T(xx^*) = T(a^2 - b^2 + ba - ab) = \overline{T}(a^2 - b^2).$$

Thus, $T(x^*x) = T(xx^*)$. Since $a^2 - b^2 \geq 0$, it follows that $T(x^*x) = \overline{T}(a^2 - b^2) \geq 0$. 2. By equation (1), $T(a + b) = \overline{T}(a) = T(a + 0) = T(a)$. 3. Let B be an abelian real C*-subalgebra of R and let $B_c = B + iB$ be its complexification, which is an abelian C*-subalgebra of A. By Definition 1, the quasitrace $\overline{T}$ is linear on B_c. We now show that T is linear on B. Let $\lambda \in Z(R)$. Then, λ can be represented as $\lambda = \lambda_1 + \lambda_2$, where $\lambda_1 \in R_h, \lambda_2 \in R_k$. $x, y \in B$, with $x = a + b$ and $y = c + d$, where $a, c \in R_h$ and $b, d \in R_k$. Since $\lambda x + y = (\lambda_1 a + c + \lambda_2 b) + (\lambda_1 b + d + \lambda_2 a)$ where $(\lambda_1 a + c + \lambda_2 b) \in R_h; (\lambda_1 b + d + \lambda_2 a) \in R_k$, we compute

$$T(\lambda x + y) = T(\lambda_1 a + c + \lambda_2 b) + (\lambda_1 b + d + \lambda_2 a) = \overline{T}(\lambda_1 a + c + \lambda_2 b).$$

By $a, c \in R_h \subset B_c$ and the linearity of $\overline{T}$ on B_c,

$$\overline{T}(\lambda_1 a + c + \lambda_2 b) = \lambda_1 \overline{T}(a) + \lambda_2 \overline{T}(b) + \overline{T}(c) + \lambda_2 \overline{T}(a) - \lambda_2 \overline{T}(a) = (\lambda_1 + \lambda_2)\overline{T}(a) + 0 + \overline{T}(c) - 0 =$$

$$(\lambda_1 + \lambda_2)T(a + b) + T(c + d) = \lambda T(x) + T(y).$$

This confirms that T is linear on B. Consequently, T is a quasitrace on R. The theorem is proved.

Theorem 3. If T is a center-valued quasitrace on R, then the map $\overline{T} : A \to \mathcal{Z}(A)$ defined by

$$\overline{T}(x + iy) = T(x) + iT(y), \quad \text{where } x, y \in R, \tag{2}$$

is a center-valued quasitrace on A.

Proof. 1. Recall that $A = R + iR$, and that a is embedded into $M_2(A)$ as $x \mapsto e_{11} \otimes x$. The mapping $\pi : M_2(A) \to M_2(\mathbb{C}) \otimes A$ is defined by

$$\pi([a_{ij}]) = \sum_{i,j=1}^{2} e_{ij} \otimes a_{ij}$$

and is a *-isomorphism. Let $x = c + id$, where $c, d \in R$. Then,

$$\overline{T}(x^*x) = \overline{T}((c + id)^*(c + id)) = \overline{T}(c^*c + d^*d + i(c^*d - d^*c)).$$

Since $c^*d - d^*c \in R_k$, by applying the property $T(a + b) = T(a)$ (where $a \in R_h$ and $b \in R_k$), we get:

$$\overline{T}(c^*c + d^*d + i(c^*d - d^*c)) = T(c^*c + d^*d).$$

Since $T(x) = T(x \otimes e_{11})$, we have:

$$\overline{T}(x^*x) = T(c^*c + d^*d) = T((c^*c + d^*d) \otimes e_{11}) = T\begin{pmatrix} c^*c + d^*d & 0 \\ 0 & 0 \end{pmatrix}.$$

This can be rewritten as:

$$T\begin{pmatrix} c^* & d^* \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix} = T\begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix}\begin{pmatrix} c^* & d^* \\ 0 & 0 \end{pmatrix}.$$

Finally, we have:

$$T\begin{pmatrix} cc^* & cd^* \\ dc^* & dd^* \end{pmatrix} = T(cc^* + dd^*) = \overline{T}((c+id)(c+id)^*) = \overline{T}(xx^*),$$

which is what we needed to prove.

2. Linearity of $\overline{T}$ on A_h: Let $x, y \in A_h$, and $x = a + ib$, $y = c + id$. Since $x = x^*$, it follows that $a = a^*$, $b^* = -b$, i.e., $a \in R_h$, $b \in R_k$. Similarly, $c \in R_h$, $d \in R_k$. Since $T(b) = T(d) = 0$, we obtain:

$$\overline{T}(x + iy) = \overline{T}(a - d + i(b + c)) = T(a - d) + iT(b + c) = T(a) + iT(c).$$

Therefore,

$$\overline{T}(x + iy) = \overline{T}(x) + i\overline{T}(y).$$

3. Abelian C*-subalgebra B_c and linearity of $\overline{T}$: Let B_c be an abelian C*-subalgebra of the AW*-algebra a. Since $A = R + iR$, for all $x \in B_c$, there exist $a, b \in R$ such that $x = a + ib$, and thus $B_c = B_1 + iB_2$, where $B_1, B_2 \subset R$.

a) Since $B_c \ni 0 = 0 + i0$, $B_c \ni \epsilon = \epsilon + i0$, and $B_c \ni i\epsilon$, we have $0, \epsilon \in B_i, i = 1, 2$. If $B_c = iB_2$, then for $\lambda \in \mathbb{C}$, $\lambda = \lambda_1 + i\lambda_2$, $x = ia, y = ib$, where $a, b \in B_2$, we have:

$$\overline{T}(\lambda x + y) = \overline{T}((\lambda_1 + i\lambda_2)ia + ib) = \overline{T}(-\lambda_2 a + i(\lambda_1 a + b)).$$

This simplifies to:

$$T(-\lambda_2 a) + iT(\lambda_1 a + b) = -\lambda_2 T(a) + i\lambda_1 T(a) + iT(b).$$

Factoring terms:

$$i(\lambda_1 + i\lambda_2)T(a) + iT(b) = \lambda(0 + iT(a)) + (0 + iT(b)).$$

Hence:

$$\overline{T}(\lambda x + y) = \lambda\overline{T}(x) + \overline{T}(y),$$

which proves that $\overline{T}$ is linear in this case.

b) Let $x = a + ic$, $y = b + id$, where $a, b \in B_1$ and $c, d \in B_2$. Then:

$$xy = ab - cd + i(cb + ad).$$

Consequently, $ab - cd \in B_1$ and $cb + ad \in B_2$. For $c = d = 0$, we have $ab \in B_1$, which shows B_1 is an algebra. Similarly, B_2 is also an algebra. When $d = 1$ and $b = 0$, we find $a \in B_2$, so $B_1 \subset B_2$. Conversely, it can also be shown that $B_2 \subset B_1$. Thus, $B_1 = B_2$, and $B_c = B + iB$. Since $x^* = a^* - ib^* \in B_c$, it follows that $a^*, b^* \in B$. Therefore, B is a real *-subalgebra.

c) For additivity:

$$\overline{T}(x + y) = \overline{T}(a + ib + c + id) = T(a + c) + iT(b + d),$$

which simplifies to:

$$T(a) + T(c) + iT(b) + iT(d) = \overline{T}(a + ib) + \overline{T}(c + id) = \overline{T}(x) + \overline{T}(y).$$

For homogeneity:

$$\overline{T}(\lambda x) = \overline{T}((\lambda_1 + i\lambda_2)(a + ib)) = \overline{T}(\lambda_1 a - \lambda_2 b + i(\lambda_1 b + \lambda_2 a)).$$

This simplifies to

$$T(\lambda_1 a - \lambda_2 b) + iT(\lambda_1 b + \lambda_2 a) = (\lambda_1 + i\lambda_2)(T(a) + iT(b)) = \lambda\overline{T}(x).$$

Thus, $\overline{T}$ is linear on B_c. The theorem is proved.

4. Properties of 1- and 2- quasitraces on real C*-algebras.

We have proven above the theorems about the relationship between T-center-valued quasitrace on R and $\overline{T}$-center-valued quasitrace on A. Similarly, we can introduce the dependence of the quasitrace τ in R and the quasitrace $\overline{\tau}$ in A as follows:

Theorem 7. [7] Let R be unital real C*-algebra and $A = R + iR$ be complexification of R.

1. If $\overline{\tau}$ is a quasi-trace on a C*-algebra $A = R + iR$, then its restriction to a real C*-algebra R is defined as
$$\tau(a+b) = \overline{\tau}(a) \tag{3}$$
$a \in R_h, b \in R_k$ is a quasi-trace on R.

2. If τ is a quasi-trace on a real C*-algebra R, then its extension $\overline{\tau}$ to $A = R + iR$, defined as
$$\overline{\tau}(x+iy) = \tau(x) + i\tau(y) \tag{4}$$
is a quasi-trace on A, where $x, y \in R$.

Corollary 6. [6] Let R be a real AW *-algebra. Then R is finite iff there is a family of faithful quasi-traces , i.e., a family $(\tau_i)_{i\in I}$ of quasi-traces with $\tau_i(x^*x) = 0$ for $i \in I$ if and only if $x = 0$.

Corollary is proven in [6] as Corollary 1.

Lemma 7. Let τ be a 1-quasitrace on an real AW*-algebra R. Then τ is order-preserving on R_{sa}. Furthermore, τ is continuous, in particular, $|\tau(x) - \tau(y)| \le \|x - y\|$ for all $x, y \in M$.

Proof. The idea of the proof is the same as for the center-valued quasitrace in [[6], Theorem 4]. We use Lemma 1.24 in [3]: Let $a, b \in R_{sa}$ be self-adjoint elements with $a \le b$ and $\lambda > 0$. Let $E_{(\lambda,\infty)}(a)$ be the spectral projection defined as $E_{(\lambda,\infty)}(a) := RP((a - \lambda 1)_+) = LP((a - \lambda 1)_+)$, where $RP(x)$ (resp. $LP(x)$) is the right (resp. left) support projection of x. Then $E_{(\lambda,\infty)}(a) \le E_{(\lambda,\infty)}(b)$. So, there exists $v \in R$ such that $E_{(\lambda,\infty)}(a) = v^*v$ and $vv^* \le E_{(\lambda,\infty)}(b)$, so we can compute
$$\tau(E_{(\lambda,\infty)}(a)) = \tau(v^*v) = \tau(vv^*) \le \tau(E_{(\lambda,\infty)}(b)),$$
where we obtain the last inequality from the fact that vv^* and $E_{(\lambda,\infty)}(b)$ commute, so $0 \le \tau(E_{(\lambda,\infty)}(b) - vv^*) = \tau(E_{(\lambda,\infty)}(b)) - \tau(vv^*)$. We again get the inequality $\tau(a) \le \tau(b)$ by integrating over the spectral projections. The proof of the continuity of τ is then the same as for the center-valued quasitrace. The proof of Lemma 7 is completed.

Theorem 8. [[6], Theorem 6.] Let τ be a 1-quasitrace on a real finite AW*-algebra R, and let T be the center-valued quasitrace constructed in [6] Theorem 4. Then τ is uniquely expressible in the form $\tau = \varphi \circ T$ for a positive functional φ on $Z(R)$.

Corollary 9. [[6], Corollary 2.] Let τ be a 1-quasitrace on a finite AW*-algebra M, then τ is a n-quasitrace for every $n \in \mathbb{N}$.

Definition 10. [3] A rank function D is a map $D : A \to [0,1]$ such that

(i) D is normalized, that is, $\sup_{a\in A} D(a) = 1$.

(ii) For all $a, b \in A$ with $a \perp b$, we have $D(a+b) = D(a) + D(b)$.

(iii) For all $a \in A$: $D(a) = D(a^*a) = D(aa^*) = D(a^*)$.

(iv) For all positive elements $0 \le a \le b$: $D(a) \le D(b)$.

(v) For all $a, b \in A$ with $a \precsim b$, say that a is Cuntz sub-equivalent to b, which means that there exist sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ of elements in A such that $(x_n b y_n)_{n\in\mathbb{N}}$ converges in norm to a, we get $D(a) \le D(b)$.

A dimension function is a map $D : \bigcup_{n\in\mathbb{N}} M_n(A) \to [0,\infty)$ that satisfies properties (i)-(v) above. A rank (dimension) function is called *subadditive* if $D(a+b) \le D(a) + D(b)$ for all $a, b \in A$; *weakly subadditive* if $D(a+b) \le D(a) + D(b)$ for all positive commuting $a, b \in A$.

Theorem 11. Let R be a real C^*-algebra. Denote by $QT(R)$ the family of all quasitraces on R and by $F(R)$ the collection of lower semi-continuous, weakly additive rank-type functions defined on R. There is a natural embedding of $QT(R)$ in the set $F(R)$. Furthermore, 2-quasitraces correspond exactly to the lower semi-continuous subadditive dimension functions.

Proof. Let $A = R+iR$ and $F(A)$ be weakly additive lower semi-continuous rank functions on a C* algebra A. By Theorems 1 and 2 [7], there is a one-to-one correspondence between sets $QT(R)$ and $QT(A)$, and by Theorem 2.8 [3], there is a one-to-one correspondence between sets $QT(A)$ and $F(A)$. Since for any $\overline{D} \in F(A)$ we have $D \in F(R)$, where $D(x) := \overline{D}(x)$, $x \in R$, the set $F(A)$ can be embedded in $F(R)$. The second part of the statement follows from Theorem 2.8 [3]. $\square$

We need an induced quasitrace on quotient C*-algebras, so we need the following theorem. This is Proposition 3.3 in [8] and uses Theorem 1.1.17 in [1].

Theorem 12. Let τ be a 2-quasitrace on a C*-algebra A. We define the kernel of τ:

$$I_\tau := \{a \in A \mid \tau(a^*a) = 0\}.$$

Then I_τ is a two-sided closed ideal of A, and there exists a quasitrace τ_1 on A/I_τ such that

$$\tau(a) = \tau_1(\pi(a))$$

for all $a \in A$ where $\pi : A \to A/I_\tau$ is the quotient map.

Proof. Let, by Theorem 11, D_τ be the corresponding dimension function to τ, and then define $\ker(D_\tau) := \{x \in R : D_\tau(x) = 0\}$. Theorem 1.1.17 in [BH82] shows that $\ker(\overline{D_{\overline{\tau}}}) = \{a \in A : \overline{D_{\overline{\tau}}}(x) = 0\}$ is a two-sided closed ideal in A and that $\overline{D_{\overline{\tau}}}$ induces a lower semi-continuous dimension function $\tilde{\overline{D}}_{\overline{\tau}}$ on $A/\ker(\overline{D_{\overline{\tau}}})$. It is easy to see that $\ker(D_\tau)$ is also a two-sided closed ideal in R. and that D_τ induces a lower semi-continuous dimension function $\tilde{D}_\tau$ on $R/\ker(D_\tau)$. We note that $I_\tau = \ker D_\tau$, and the quasitrace corresponding to $\tilde{D}_\tau$ is exactly the desired quasitrace τ_1.

The idea of the proof can be schematically illustrated as follows:

$$\begin{tikzcd}
QT(A) \arrow[r, leftrightarrow] & F(A) \arrow[d, bend right] & \bar{\tau} \arrow[r] \arrow[d] & \bar{D} \arrow[d] \\
QT(R) \arrow[u] & F(R) & \tau & D
\end{tikzcd}$$

From Theorems 11 and 12 follows the following result

Theorem 13. Let τ be a 2-quasitrace on a C*-algebra A. Then there exists a finite AW*-algebra M, a 2-quasitrace τ' on M, and a unital *-homomorphism $\theta : A \to M$ such that $\tau = \theta \circ \tau'$.

Corollary 14. Let τ be a 2-quasitrace on a C*-algebra A.

1. τ is an n-quasitrace for every $n \in \mathbb{N}$.
2. τ is order-preserving on A_{sa}.
3. τ is continuous.
4. τ is bounded.

Proof. (i): The quasitrace τ is of the form $\tau = \theta \circ \tau'$ for a *-homomorphism $\theta : A \to M$ and an n-quasitrace τ' on M. Since θ is completely positive, it follows that τ is also an n-quasitrace. (ii): Again, τ and θ are both order-preserving, and so is τ. (iii): This again follows from (ii) or again from the fact that τ is the composition of continuous maps. (iv): From Remark I.1.19(b) in [1], we know that dimension functions are bounded, and then it follows from the fact that $\tau(a) \leq D_\tau(a)$ for all positive $a \in A$ with $\|a\| \leq 1$.

Remark From now on, when we write quasitrace, we will always mean 2-quasitraces on A. If A is a unital C*-algebra, we write $q(A)$ for the set of normalized quasitraces on A.

We want to state this last corollary without the proof. It is a combination of Theorem II.4.4 and Proposition II.4.5 in [1].

Corollary 15. If R is unital real C*-algebra, then $QT(R)$ is a compact convex set. Furthermore, $QT(R)$ is a simplex, and the set $T(R)$ of normalized traces is a closed face in $QT(R)$.

REFERENCES

1. Blackadar B. and D. Handelman, Dimension functions and traces on C*-algebra, J. Funct. Anal. 45, 297–340 (1982)
2. Murray F. J. and von Neumann J., On rings of operators IV, Ann. Math. **44**, 716–808 (1943).
3. Fehlker F, Quasitrace and AW*-bundles, Monster, 20–25 (2018)
4. Sakai S., *C*-algebras and W*-algebras*, Springer, Berlin (1971).
5. Berberian S. K., *Baer *-rings*, Springer, Berlin (1972).
6. Rakhimov A. and Rakhmonova N., The center-valued quasitraces on AW*-algebras, AIP Conf. Proc. **3244**, 020036 (2024), doi:10.1063/5.0241470.
7. Kim D. I. and Rakhimov A. A., Quasitraces on real C*- and AW*-algebras, Uzbek Math. J. **69**(3), 121–125 (2025), doi:10.29229/uzmj.2025-3-12.
8. Haagerup U. Quasitraces on exact C*-algebras are traces. C. R. Math. Acad. Sci. Soc. R. Can. 36(2-3), 67–92 (2014)

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Boundary Value Problem for the Gellerstedt Equation in an Unbounded Domain

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Abstract. The Tricomi boundary value problem for the Gellerstedt equation is investigated in a domain where the elliptic part is the first quadrant of the plane. The unique solvability of the problem under consideration is established using the method of integral equations. The problem is equivalently reduced to solving a singular integral equation with a Cauchy kernel. By regularizing this equation using the Carleman-Vekua method, an explicit solution is obtained.

MSC (2020): 35M10; 35M12.

Key words: mixed-type equation, unbounded domain, unique solvability, singular integral equation, index of an equation.

Introduction.

The first fundamental studies of equations of mixed elliptic-hyperbolic type were done by F. Tricomi [1]. Following Tricomi's work on mixed-type equations, S. Gellerstedt [2] investigated boundary value problems in which the values of the desired solution in the hyperbolic part of the domain under consideration are prescribed on two internal characteristics of the equation.

The Gellerstedt problem has important applications in the field of transonic gas dynamics. In [3], the Tricomi boundary value problem for the Tricomi equation was studied in a half-strip and a quarter-plane. The Frankl boundary value problem for the Lavrent'ev equation in an unbounded domain was considered in [4]. Works [5-7] are devoted to the study of nonlocal problems for mixed-type equations.

In the elliptic part of the mixed domain, the Tricomi problem for the Gellerstedt equation in an unbounded domain was investigated [8] by employing solutions of the generalized Neumann problem.

In the present paper, we investigate the Tricomi boundary value problem for an equation of mixed elliptic-hyperbolic type in an unbounded domain by utilizing solutions of the Dirichlet problem.

Problem statement.

Consider the following mixed-type equation

$$\operatorname{sgn} y|y|^m u_{xx} + u_{yy} = 0, \qquad m > 0, \tag{1}$$

in the domain $D : (x > 0)$, bounded by the y axis for $y > 0$ and by the following characteristic of equation (1)

$$\Gamma : \quad x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 0.$$

Introduce the notation: D^+ and D^- are the elliptic and hyperbolic parts of the mixed domain D , respectively; I is the semi-infinite interval $0 < x < +\infty$ on the line $y = 0$. For $y > 0$, let $R = \sqrt{x^2 + \frac{4}{(m+2)^2} y^{m+2}}$.

Problem T. Find a function $u = u(x, y)$, which possesses the following properties:

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1) $u(x,y) \in C(\overline{D}) \cap C^1(\overline{D} \setminus I) \cap C^2(D \setminus I)$ and satisfies equation (1) in $D^+ \cup D^-$;

2)
$$\lim_{R\to\infty} u(x,y) = 0, \qquad x \geq 0, \quad y \geq 0; \tag{2}$$

3) $u(x,y)$ satisfies the boundary conditions
$$u(0,y) = \varphi(y), \qquad y \geq 0, \tag{3}$$
$$u(x,y)|_\Gamma = \psi(x), \quad x \in [0,\infty), \tag{4}$$
and the conjugation condition
$$\lim_{y\to -0} u_y = \lim_{y\to +0} u_y, \qquad x \in I. \tag{5}$$

Here, $\psi(x)$ and $\varphi(y)$ are given functions satisfying the Holder conditions of orders δ_1 and δ_2, respectively, on $[0,\infty)$.

Theorem. Let $y^{\frac{3m}{4}}\varphi(y) \in L(0,\infty)$, $\varphi(\infty) = 0$, $\psi(\infty) = 0$. Then problem T is uniquely solvable.

Proof. It is known [7] that the solution to the Dirichlet problem for equation (1) satisfying conditions (2), (3) and condition $u(x,0) = \tau(x)$, $x \in [0,\infty)$ can be represented in the form
$$u(x,y) = k_2 y \int_0^\infty \tau(t) \left(\left[(t-x)^2 + \frac{4}{(m+2)^2} y^{m+2} \right]^{\beta-1} - \left[(t+x)^2 + \frac{4}{(m+2)^2} y^{m+2} \right]^{\beta-1} \right) dt$$
$$+ \frac{2}{m+2} y^{\frac{1}{2}} \int_0^\infty t^{\frac{2m+1}{2}} \varphi(t) dt \int_0^\infty s e^{-sx} J_{\frac{1-2\beta}{2}} \left(\frac{2st^{\frac{m+2}{2}}}{m+2} \right) J_{\frac{1-2\beta}{2}} \left(\frac{2sy^{\frac{m+2}{2}}}{m+2} \right) ds, \tag{6}$$
where
$$k_2 = \frac{1}{4\pi} \left(\frac{4}{m+2} \right)^{2-2\beta} \frac{\Gamma^2(\beta)}{\Gamma(2-2\beta)}, \qquad \beta = \frac{m}{2(m+2)}.$$
Differentiating equality (6) with respect to y, we obtain
$$\frac{\partial u}{\partial y} = k_2 \int_0^\infty \tau(t) \frac{\partial}{\partial y} y \left(\left[(t-x)^2 + \frac{4y^{m+2}}{(m+2)^2} \right]^{\beta-1} - \left[(t+x)^2 + \frac{4y^{m+2}}{(m+2)^2} \right]^{\beta-1} \right) dt + \frac{\partial \Phi(x,y)}{\partial y}, \tag{7}$$
where
$$\Phi(x,y) = \frac{2}{m+2} y^{\frac{1}{2}} \int_0^\infty t^{\frac{2m+1}{2}} \varphi(t) dt \int_0^\infty s e^{-sx} J_{\frac{1-2\beta}{2}} \left(\frac{2st^{\frac{m+2}{2}}}{m+2} \right) J_{\frac{1-2\beta}{2}} \left(\frac{2sy^{\frac{m+2}{2}}}{m+2} \right) ds.$$
From (7) it is easy to show that
$$\frac{\partial}{\partial y} y \left(\left[(x-t)^2 + \frac{4y^{m+2}}{(m+2)^2} \right]^{\beta-1} - \left[(x+t)^2 + \frac{4y^{m+2}}{(m+2)^2} \right]^{\beta-1} \right)$$
$$= \frac{m+2}{2} \frac{\partial}{\partial t} \left((x-t) \left[(x-t)^2 + \frac{4y^{m+2}}{(m+2)^2} \right]^{\beta-1} + (x+t) \left[(x+t)^2 + \frac{4y^{m+2}}{(m+2)^2} \right]^{\beta-1} \right). \tag{8}$$
Taking identity (8) into account, we rewrite equality (7) in the form
$$\frac{\partial u}{\partial y} = k_2 \frac{m+2}{2} \int_0^\infty \tau(t) \frac{\partial}{\partial t} \left((x-t) \left[(x-t)^2 + \frac{4}{(m+2)^2} y^{m+2} \right]^{\beta-1} \right.$$
$$\left. + (x+t) \left[(x+t)^2 + \frac{4}{(m+2)^2} y^{m+2} \right]^{\beta-1} \right) dt + \frac{\partial \Phi(x,y)}{\partial y}. \tag{9}$$

In the integral on the right-hand side of (9) performing integration by parts, without loss of generality assuming $\tau(0)=0, \tau(\infty)=0$, we obtain

$$\frac{\partial u}{\partial y}=-k_2\frac{m+2}{2}\int\limits_0^\infty \tau'(t)\left((x-t)\left[(x-t)^2+\frac{4}{(m+2)^2}y^{m+2}\right]^{\beta-1}\right.$$

$$\left.+(x+t)\left[(x+t)^2+\frac{4}{(m+2)^2}y^{m+2}\right]^{\beta-1}\right)dt+\frac{\partial\Phi(x,y)}{\partial y}. \tag{10}$$

Passing to the limit as $y\to+0$ in (10), we have

$$\nu(x)=-k_2\frac{m+2}{2}\int\limits_0^\infty \tau'(t)\left((x-t)|x-t|^{2\beta-2}+(x+t)^{2\beta-1}\right)dt+\Phi_0(x),\quad x\in(0,\infty), \tag{11}$$

where

$$\Phi_0(x)=\lim_{y\to+0}\frac{\partial\Phi(x,y)}{\partial y}=\frac{2}{(m+2)^{\frac{1-2\beta}{2}}\Gamma\left(\frac12-\beta\right)}\int\limits_0^\infty\varphi^+(t)t^{\frac{2m+1}{2}}dt\int\limits_0^\infty s^{\frac{3-2\beta}{2}}e^{-sx}J_{\frac{1-2\beta}{2}}\left(\frac{2st^{\frac{m+2}{2}}}{m+2}\right)ds.$$

Formula (11) gives the first functional relation between $\tau(x)$ and $\nu(x)$, extended to I from the elliptic part D^+ of the domain D.

Using the solution of the Cauchy problem for equation (1) in the domain D^- [7]

$$u(x,y)=\gamma_1\int\limits_0^1\tau\left[x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}}(2t-1)\right]t^{\beta-1}(1-t)^{\beta-1}dt$$

$$+\left(\frac{4}{m+2}\right)^{1-2\beta}\gamma_2 y\int\limits_0^1\nu\left[x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}}(2t-1)\right]t^{-\beta}(1-t)^{-\beta}dt,$$

where $\gamma_1=\dfrac{\Gamma(2\beta)}{\Gamma^2(\beta)}$, $\gamma_2=\dfrac12\left(\dfrac{4}{m+2}\right)^{2\beta}\dfrac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}$, by virtue of condition (4) we have

$$\psi(x)=\gamma_1\Gamma(\beta)x^{1-2\beta}D_{0x}^{-\beta}x^{\beta-1}\tau(x)-\gamma_2\Gamma(1-\beta)D_{0x}^{\beta-1}x^{-\beta}\nu(x), \tag{12}$$

where $D_{0x}^{-\beta}$ is the Riemann–Liouville fractional integration operator [6,8].

Applying the operator

$$D_{0x}^{1-\beta}f(x)=\frac{d}{dx}D_{0x}^{-\beta}f(x)=\frac{1}{\Gamma(\beta)}\frac{d}{dx}\int\limits_0^x\frac{f(t)dt}{(x-t)^{1-\beta}}$$

to both sides of equality (12) and taking into account that $D_{0x}^{1-\beta}D_{0x}^{\beta-1}f\equiv D^0f$, we obtain

$$\gamma_2\Gamma(1-\beta)x^{-\beta}\nu(x)=\gamma_1\Gamma(\beta)D_{0x}^{1-\beta}x^{1-2\beta}D_{0x}^{-\beta}x^{\beta-1}\tau(x)-D_{0x}^{1-\beta}\psi(x). \tag{13}$$

It is easy to show that

$$D_{0x}^{1-\beta}x^{1-2\beta}D_{0x}^{-\beta}x^{\beta-1}\tau(x)=x^{-\beta}D_{0x}^{1-2\beta}\tau(x).$$

Then from equality (13) we have

$$\nu(x)=\gamma D_{0x}^{1-2\beta}\tau(x)+\Psi_1(x), \tag{14}$$

where $\gamma = \frac{\gamma_1 \Gamma(\beta)}{\gamma_2 \Gamma(1-\beta)}$, $\quad \Psi_1(x) = -\frac{x^\beta D_{0x}^{1-\beta}\psi(x)}{\gamma_2\Gamma(1-\beta)}$.

Equality (14) is the second functional relation between $\tau(x)$ and $\nu(x)$, extended to I from the hyperbolic domain D^- of D.

According to (5) eliminating $\nu(x)$ from equalities (11) and (14), we obtain

$$\gamma D_{0x}^{1-2\beta}\tau(x) + \Psi_1(x)$$
$$= -k_2\frac{m+2}{2}\int_0^\infty \tau'(t)\left((x-t)|x-t|^{2\beta-2} + (x+t)^{2\beta-1}\right)dt + \Phi_0(x), \quad x \in (0,\infty). \tag{15}$$

Applying the operator $\Gamma(1-2\beta)D_{0x}^{2\beta-1}$ to both sides of equality (15), we get

$$\gamma\Gamma(1-2\beta)\tau(x) + \Gamma(1-2\beta)D_{0x}^{2\beta-1}\psi_1(x)$$
$$= -k_2\frac{m+2}{2}\Gamma(1-2\beta)D_{0x}^{2\beta-1}\int_0^\infty \tau'(t)\left((x-t)|x-t|^{2\beta-2} + (x+t)^{2\beta-1}\right)dt$$
$$+\Gamma(1-2\beta)D_{0x}^{2\beta-1}\Phi_0(x), \quad x \in (0,\infty). \tag{16}$$

In equation (16), it is easy to show that

$$\Gamma(1-2\beta)D_{0x}^{2\beta-1}\int_0^\infty \tau'(t)(x-t)|x-t|^{2\beta-2}dt = \frac{\pi(1-\cos 2\pi\beta)\tau(x)}{\sin 2\pi\beta} - \int_0^\infty \left(\frac{t}{x}\right)^{2\beta-1}\frac{\tau(t)dt}{t-x}, \tag{17}$$

$$\Gamma(1-2\beta)D_{0x}^{2\beta-1}\int_0^\infty \tau'(t)(x+t)^{2\beta-1}dt = \int_0^\infty \left(\frac{t}{x}\right)^{2\beta-1}\frac{\tau(t)dt}{t+x}. \tag{18}$$

Substituting (17) and (18) into (16), we obtain

$$\left(\gamma\Gamma(1-2\beta) + k_2\frac{(m+2)\pi}{2\sin 2\pi\beta}(1-\cos 2\pi\beta)\right)\tau(x)$$
$$-k_2\frac{m+2}{2}\int_0^\infty \left(\frac{x}{t}\right)^{1-2\beta}\left(\frac{1}{t-x} - \frac{1}{t+x}\right)\tau(t)dt = \Phi_1(x), \qquad x \in (0,\infty), \tag{19}$$

where $\Phi_1(x) = \Gamma(1-2\beta)D_{0x}^{2\beta-1}(\Phi_0(x) - \Psi_1(x))$.

After straightforward calculations, equality (19) takes the following form

$$\tau(x) - \lambda\int_0^\infty \left(\frac{x}{t}\right)^{1-2\beta}\left(\frac{1}{t-x} - \frac{1}{t+x}\right)\tau(t)dt = \Phi_1(x), \quad x \in (0,\infty), \tag{20}$$

where $\lambda = \dfrac{\cos\pi\beta}{\pi(1+\sin\pi\beta)}$.

In equation (20), by letting $x^{2\beta-1}\tau(x) = \rho(x)$, $x^{2\beta-1}\Phi_1(x) = \Phi_2(x)$, we get

$$\rho(x) - \lambda\int_0^\infty \left(\frac{1}{t-x} - \frac{1}{t+x}\right)\rho(t)dt = \Phi_2(x), \quad x \in (0,\infty). \tag{21}$$

In equation (21), we make the change of variables $t^2 = \dfrac{s}{1-s}$, $x^2 = \dfrac{\xi}{1-\xi}$, then it takes the form

$$\rho\left(\sqrt{\frac{\xi}{1-\xi}}\right) - \lambda\int_0^1 \rho\left(\sqrt{\frac{s}{1-s}}\right)\frac{\sqrt{\xi}\sqrt{1-\xi}ds}{\sqrt{s}\sqrt{1-s}(s-\xi)} = \Phi_2\left(\sqrt{\frac{\xi}{1-\xi}}\right). \tag{22}$$

Multiplying both sides of equation (22) by $\dfrac{1}{\sqrt{\xi(1-\xi)}}$ and denoting $\dfrac{1}{\sqrt{\xi(1-\xi)}}\rho\left(\sqrt{\frac{\xi}{1-\xi}}\right) = \mu(\xi)$, $\dfrac{1}{\sqrt{\xi(1-\xi)}}\Phi_2\left(\sqrt{\frac{\xi}{1-\xi}}\right) = \Phi_3(\xi)$, we obtain the singular integral equation with the Cauchy kernel

$$\mu(\xi) - \lambda\int\limits_0^1 \frac{\mu(s)ds}{s-\xi} = \Phi_3(\xi), \quad \xi \in (0,1). \tag{23}$$

We seek the solution of equation (23) in the class of $h(1)$ functions. $\mu(\xi) \in H(0,1)$ bounded as $\xi \to 1$ and having an infinity of order less than $1-2\beta$ at the point $\xi = 0$. The index χ of equation (23) in this class is equal to zero.

Using the result of work [9], we write the explicit form of the solution to equation (23)

$$\mu(\xi) = \frac{1+\sin(\beta\pi)}{2}\Phi_3(\xi) + \frac{\cos(\pi\beta)}{2\pi}\left(\frac{\xi}{1-\xi}\right)^{-\frac{1}{4}(1-2\beta)}\int\limits_0^1\left(\frac{s}{1-s}\right)^{\frac{1}{4}(1-2\beta)}\frac{\Phi_3(s)ds}{s-\xi}.$$

As a result, returning to the original variables and functions, we have

$$\tau(x) = \frac{1+\sin(\beta\pi)}{2}\Phi_1(x) + \frac{\cos(\pi\beta)}{2\pi}\int\limits_0^\infty\left(\frac{x}{t}\right)^{\frac{1}{2}(1-2\beta)}\left(\frac{1}{t-x} - \frac{1}{t+x}\right)\Phi_1(t)dt.$$

The proof of the theorem is finished.

REFERENCES

1. Tricomi, F. O. O lineynykh uravneniyakh smeshannogo tipa. M.: Gostekhizdat, 1947, 192 c.
2. Gellerstedt, S. Quelques problemes mixtes pour l'equation $y^m z_{xx} + z_{yy} = 0$. *Ark. Mat. Astron. Fys.* **26A** (1938), 3, 1–32.
3. Sevostyanov, G. D. Kraevaya zadacha Trikomi dlya polupolosi i chetverti ploskosti. *Volzhsk. Mat. Sb.* **3** (1965), 312–320.
4. Fleisher, N. M. Ob odnoy zadache Franklya dlya uravneniya Lavrenteva v sluchae neogranichennoy oblasti. *Izv. VUZ. Matematika* (1966), 6, 152–156.
5. Ruziev, M. Kh. A nonlocal problem for a mixed-type equation with a singular coefficient in an unbounded domain. *Russian Mathematics* **54** (2010), no. 11, 36–43.
6. Kumykova, S. K.; Shardanova, M. A. The Bitsadze–Samarskii problem for a mixed-type equation in an unbounded domain. *Uspekhi sovremennogo estestvoznaniya* (2015), no. 1, 80–83.
7. Ruziev, M. Kh.; Yuldasheva, N. T. On a certain boundary value problem for the Gellerstedt equation with a singular coefficient. *Izvestiya VUZ. Matematika* (2024), no. 12, 57–70.
8. Smirnov, M. M. Uravneniya smeshannogo tipa. M., 1984, 304 c.
9. Muskhelishvili, N. I. Singulyarnye integral'nye uravneniya. M., 1962, 600 c.

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The dynamics of p-adic Potts model with an external field

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Abstract. This paper explores the dynamical systems approach to constructing translation-invariant p-adic quasi Gibbs measures for the Potts model with an external field. We conduct a rigorous analysis of the fixed points associated with the corresponding nonlinear operator. Our results establish the existence of three distinct fixed points; we prove that one serves as an attracting fixed point, while the dynamical behavior of the remaining two is contingent upon specific parameter constraints.

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Key words: p-adic number, p-adic Potts model with an external field, fixed point, attracting, repelling, indefferent.

Introduction

The Potts model is a fundamental model in statistical mechanics that extends the Ising model by allowing spins to take more than two distinct states [10]. Owing to its rich mathematical structure and wide applicability to various physical systems, the Potts model has attracted considerable attention in recent years [12, 14]. In particular, the works [3, 4, 13] have significantly advanced the theoretical understanding of Potts-type models.

p-adic probability theory has emerged as a novel framework in theoretical physics, arising naturally in models formulated over p-adic number fields, such as p-adic string theory originally proposed by I. Volovich [16]. In [2], a comprehensive theory of stochastic processes with values in p-adic and more general non-Archimedean fields was developed. These stochastic processes are characterized by probability distributions taking values in non-Archimedean fields. Moreover, a non-Archimedean analog of the Kolmogorov extension theorem was established, which allows the construction of broad classes of stochastic processes from finite-dimensional distributions. This theoretical framework provides a solid foundation for studying and developing problems in statistical mechanics within the context of p-adic probability theory.

In the context of statistical mechanics on trees, p-adic dynamical systems naturally arise from recursive equations corresponding to Gibbs and quasi Gibbs measures on the Cayley tree. Such an approach was initiated in the works of Albeverio and Rozikov [1], where p-adic Gibbs measures for the Potts model on the Cayley tree were studied. The associated recursive operators were investigated as p-adic dynamical systems, and their fixed points were shown to correspond to translation-invariant Gibbs measures. Further developments were obtained by Mukhamedov, Ganikhodjaev and Rozikov (see, for example, [6, 7]), where p-adic Ising and Potts models on the Cayley tree were considered. In [11], studied the existence of multiple fixed points of the dynamical systems for the p-adic Potts model with an external field. In these works, We conducted [11] and analyzed attracting or repelling character of fixed points.

Preliminaries

Let $\mathbb{Q}$ denote the field of rational numbers and let p be a fixed prime number. Every nonzero rational number $x \in \mathbb{Q}$ can be uniquely represented in the form

$$x = p^r \frac{n}{m},$$

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where $r, n \in \mathbb{Z}$, m is a positive integer, and $(n,p) = (m,p) = 1$. The integer r is called the *p-adic order* of x. The *p-adic norm* of x is defined by

$$|x|_p = \begin{cases} p^{-r}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

The norm $|\cdot|_p$ is non-Archimedean; that is, it satisfies the strong triangle inequality

$$|x+y|_p \leq \max\{|x|_p, |y|_p\}, \qquad \text{for all } x, y \in \mathbb{Q}.$$

We note that the following essential properties are relevant to the non-Archimedenity of the norm:
i) if $|x|_p \neq |y|_p$, then $|x \pm y|_p = \max\{|x|_p, |y|_p\}$;
ii) if $|x|_p = |y|_p$, then $|x-y|_p \leq |x|_p$.
The completion of the field $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$ yields the p-adic field $\mathbb{Q}_p$. Every nonzero p-adic number $x \in \mathbb{Q}_p$ admits a unique representation in the canonical form

$$x = p^{\gamma(x)}\left(x_0 + x_1 p + x_2 p^2 + \cdots\right),$$

where $\gamma(x) \in \mathbb{Z}$ and the coefficients x_j are integers satisfying $x_0 > 0$, $0 \leq x_j \leq p-1$ for all $j \geq 1$. With this representation, the p-adic norm of x is given by $|x|_p = p^{-\gamma(x)}$.

Let $a \in \mathbb{Q}_p$, $a \neq 0$, $a = p^{\gamma(a)}(a_0 + a_1 p + a_2 p^2 + \ldots)$, $0 \leq a_j \leq p-1, j \in \mathbb{N}$, $a_0 > 0$.

Lemma 1.[15] *The equation $x^2 = a$ has a solution in $x \in \mathbb{Q}_p$ iff the followings hold:*
i) $\gamma(a)$ is even;
ii) a_0 is a quadratic residue modulo p if $p \neq 2$; the equality $a_1 = a_2 = 0$ hold if $p = 2$.

The symbol "o"is introduced in [8], which simplify certain calculations. Essentially, these symbols help us write calculations in our work more concisely. To understand their meanings: for a given p-adic number x, $o[x]$ refers to a p-adic number whose norm satisfies $|x|_p < |o[x]|_p$. For example, if $x = 1 + p + p^3$, we write $o[1] = x - 1$ or $o[p^2] = x - 1 - p$.

For any $a \in \mathbb{Q}_p$ and $r > 0$, we denote

$$B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p < r\},$$

and the set

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}, \ \mathbb{Z}_p^* = \mathbb{Z}_p \backslash p\mathbb{Z}_p.$$

$\mathbb{Z}_p$ is called the set of p- adic integers, $\mathbb{Z}_p^*$ is called the set of p- adic units. Note that the p-adic exponential is defined by the series

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for $x \in B(0, \frac{1}{2})$ if $p = 2$ and $x \in B(0,1)$ if $p \neq 2$.

Put

$$\mathcal{E}_p = \left\{x \in \mathbb{Q}_p : |x - 1|_p < p^{-1/(p-1)}\right\}.$$

A more thorough explanation of p-adic calculus and p-adic mathematical physics is provided in [5,15].

Main part

In [11], we studied translation invariant p-adic quasi Gibbs measures for the Potts model with an external field on the Cayley tree of order two. To find such measures it was need to solve the following equation:

$$h = f_{\theta,\eta}(h)$$

where

$$f_{\theta,\eta}(h) = \left(\frac{(\theta + q - 2)h + \eta}{(q-1)h + \theta\eta}\right)^2 \tag{27}$$

Consequently, the existence of a phase transition is ensured by the result proved in [11], which states that the equation admits three distinct nontrivial solutions if $p > 3$, $|\, q\, |_p = 1$, $(1-q)^{\frac{p-1}{2}} \equiv 1 (\bmod\, p)$.

This section is devoted to the analysis of the fixed-point behavior of the function (27). Under the simplifying assumptions that $p > 3$, $|\, q\, |_p = 1$, $(1-q)^{\frac{p-1}{2}} \equiv 1 (\bmod\, p)$, it has been established [11] that there exist three translation-invariant p-adic Gibbs measures, denoted as μ_0, μ_1, and μ_2 which corresponding directly to the fixed points of the function $f_{\theta,\eta}$. Specifically, these fixed points are initiated by

$$x_0 = 1 + o[1]$$

and

$$x_{1,2} = \overline{1-q} + o[1], \tag{28}$$

where $\overline{1-q}$ is inverse of $1-q$ modulo p.

Let $f : U \to U$ be an analytic function. For a given point $x_0 \in U$, define the sequence

$$x_n = f^n(x_0),$$

where

$$f^n(x) = \underbrace{f \circ \cdots \circ f}_{n \text{ times}}(x)$$

denotes the n-th iterate of f.

We recall some standard terminology from the theory of dynamical systems. A point $x_0 \in U$ is called a *fixed point* of f if $f(x_0) = x_0$. The set of all fixed points of f is denoted by $\mathrm{Fix}(f)$.

A fixed point x_0 is called an *attracting fixed point* if there exists a neighborhood $V(x_0)$ of x_0 such that

$$\lim_{n\to\infty} f^n(x) = x_0 \quad \text{for all } x \in V(x_0).$$

The *basin of attraction* of an attracting fixed point x_0 is defined by

$$A(x_0) = \{x \in \mathbb{C}_p \mid f^n(x) \to x_0 \text{ as } n \to \infty\}.$$

A fixed point x_0 is called a *repelling fixed point* if there exists a neighborhood $V(x_0)$ of x_0 such that

$$|f(x) - x_0|_p > |x - x_0|_p \quad \text{for all } x \in V(x_0),\ x \neq x_0.$$

Let $x^{(0)}$ be a fixed point of an analytic function $f(x)$, and let

$$\lambda = \frac{d}{dx} f\big(x^{(0)}\big).$$

The fixed point $x^{(0)}$ is called *attractive* if $0 \le |\lambda|_p < 1$, *indifferent* if $|\lambda|_p = 1$, and *repelling* if $|\lambda|_p > 1$ [9].

Theorem 1. Let $x_0 \in \mathrm{Fix} f_{\theta,\eta}$. Then x_0 is attracting point.

Proof. Let x_i, $i = 0, 1, 2$ be the fixed points of the function $f_{\theta,\eta}$. Then we have

$$f'_{\theta,\eta}(x_i) = \frac{2x_i\eta(\theta-1)(\theta-1+q)}{((q-1)x_i + \theta\eta)((\theta+q-2)x_i + \eta)}.$$

Using this, we get

$$f'_{\theta,\eta}(x_0) = \frac{2\eta(\theta-1)(\theta-1+q) + o[\theta-1]}{(q+\theta\eta-1)(\theta-1+\eta-1+q+o[\theta\eta-1])}.$$

Since $|\theta-1|_p = |\eta-1|_p < 1$ and $|q|_p = 1$, it follows that $\left|f'_{\theta,\eta}(x_0)\right|_p < 1$. Therefore, the fixed point x_0 is attracting.

Theorem is proved.

Theorem 2. Let x_1, x_2 be the fixed points of the function $f_{\theta,\eta}$. Then the following assertions hold:

1. If $|\theta-1|_p \geq |\eta-1|_p$ or $|\theta-1|_p < |\eta-1|_p < \sqrt{|\theta-1|_p}$, then fixed points x_1, x_2 are repelling;
2. If $|\eta-1|_p = \sqrt{|\theta-1|_p}$, then fixed points x_1, x_2 are indifferent points;
3. If $\sqrt{|\theta-1|_p} < |\eta-1|_p$, then fixed points x_1, x_2 are attracting points;

Proof. From (28), we have

$$f'_{\theta,\eta}(x_{1,2}) = \frac{2\eta(\theta-1)(\theta-1+q)+o[\theta-1]}{\big(\theta-1+(1-q)(\eta-1)\big)(\theta\eta-1+o[\theta\eta-1])}.$$

We consider the following cases:
Case 1. Let $|\theta-1|_p \geq |\eta-1|_p$. Then we have

$$|f'_{\theta,\eta}(x_{1,2})|_p = \left|\frac{\theta-1}{(\theta-1)^2}\right|_p > 1.$$

It yields that fixed points x_1, x_2 are repelling.
Case 2. Let $|\theta-1|_p < |\eta-1|_p < \sqrt{\theta-1}$. Then we have

$$|f'_{\theta,\eta}(x_{1,2})|_p = \left|\frac{\theta-1}{(\eta-1)^2}\right|_p > 1.$$

Hence, fixed points x_1, x_2 are repelling.
Case 3. Let $|\sqrt{\theta-1}|_p = |\eta-1|_p$. Then we have

$$|f'_{\theta,\eta}(x_{1,2})|_p = \left|\frac{\theta-1}{(\eta-1)^2}\right|_p = 1.$$

It means that fixed points x_1, x_2 are indefferent.
Case 4. Let $|\sqrt{\theta-1}|_p < |\eta-1|_p$. Then we have

$$|f'_{\theta,\eta}(x_{1,2})|_p = \left|\frac{\theta-1}{(\eta-1)^2}\right|_p < 1.$$

It means that fixed points x_1, x_2 are attracting.
Theorem is proved.

Remark 1. Note that the dynamical system associated with the p-adic Potts model was studied in [9]. In that work, the authors established the existence of one attracting and two repelling fixed points. In contrast to the results obtained in [9], the inclusion of an external field in our model leads to essentially different dynamical behavior.

REFERENCES

1. Albeverio S.; RozikovU. A. p-adic Gibbs measures for Potts model on the Cayley tree, *Indagationes Mathematicae*, **16** (2005), 317–338.

2. Derrida B.; Seze De.L.; Itzykson C. Fractal structure of zeros in hierarchical models, *J. Stat. Phys.* **33**, (1983), 559-569.

3. Ganikhodjaev N.N.; Mukhamedov F.M.; and Rozikov U.A. Existence of a phase transition for the Potts p-adic model on the set Z. *Theor. Math. Phys.* **130**, (2002), 425–431.

4. Ganikhodjaev N. N.; Rozikov U. A. The Potts model with countable set of spin values on a Cayley tree, *Lett. Math. Phys.* **75**, (2006), 99–109.

5. Khrennikov A. Yu.; Nilson M. p-Adic Deterministic and Random Dynamical Systems, Kluwer.Dordreht 2004, pp. 270.

6. Mukhamedov F.; Rozikov U. A. On Gibbs measures of p-adic Ising model on the Cayley tree, *Indagationes Mathematicae*, **15** (2004), 85–100.

7. Mukhamedov F.; Ganikhodjaev N. N. On dynamical systems generated by p-adic Potts model on a Cayley tree, *Mathematical Physics, Analysis and Geometry*, **14** (2011), 1–17.

8. Mukhamedov F.; Khakimov O. On equation $x^k = a$ over Q_p and its applications. *Izvestiya Math.* **84**, (2020), no. 2, 348–360.

9. Mukhammedov F., Khakimov O. Phase transition and Chaos: p-adic Potts model on a Cayley tree, *Chaos, Solitons and Fractals* **87** (2016) 190–196.

10. Potts R.B. Some generalized order-disorder transformations. *Proc.Camb.Phil.Soc.* **48**, (1952), no. 1, 106–109.

11. Rahmatullaev M. M.; Samijonova N. D. Translation-invariant $p-$adic quasi gibbs measures for the Potts model with an external field on the Cayley tree, *Nanosystems: Phys. Chem. Math.*, **16**, (2025), no. 2, 164–175.

12. Rozikov U.A. Gibbs measures in biology and physics: The Potts model. World. Sci. publ., Singapore 2023.

13. Rozikov U.A.; Rahmatullaev M.M.; Khakimov R.M. Periodic Gibbs measures for the Potts model in translation-invariant and periodic external fields on the Cayley tree. *Theor. Math. Phys.* **210**, (2022), no. 1, 135–153.

14. Wu F.Y. The Potts model, *Rev. Mod. Phys.* **54** (1982), DOI: https://doi.org/10.1103/RevModPhys.54.235.

15. Vladimirov V.S.; Volovich I.V. and Zelenov E.I. p-Adic analysis and Mathematical physics. World Scientific, Singapoure 1994.

16. Volovich I.V. p-Adic String, *Classical Quantum Gravity* **4**, (1987), 83–87.

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Verification of the priority and controllability of analytical solutions determined for active parts in the gravitational field of a spheroidal planet.

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Abstract. This paper presents analytical solutions to the problem of optimal trajectory design for a point mass representing the center of mass of a spacecraft during intermediate-thrust motion in the gravitational field of an axisymmetric spheroidal planet. The mathematical model incorporates the perturbative influence of the second zonal harmonic J_2, accounting for the oblateness of the attracting body. The variational problem is formulated within the framework of optimal control theory, and a class of particular analytical solutions is obtained using the Levi-Civita regularization method. Special attention is devoted to the qualitative analysis of the derived analytical solutions. The obtained program motions are examined with respect to stability and controllability. It is shown that certain solutions exhibit regions of dynamic instability in the neighborhood of the nominal trajectory. The controllability properties of the system are analyzed in the linear approximation, and conditions under which stabilization is achievable are identified. A linear feedback regulator is constructed to ensure asymptotic stability of the investigated program motion. The results provide a theoretical basis for assessing the practical applicability of the derived analytical solutions in spacecraft guidance problems under non-central gravitational perturbations.

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Key words: Spacecraft trajectory optimization, Levi-Civita method, Second zonal harmonic, Circular trajectories, Stability, Controllability.

Introduction

Currently, there is a problem of optimizing the motion of a point (the center of mass of a spacecraft) in a gravitational field. This involves selecting, predicting, optimizing, and calculating the trajectories of controlled objects. The variational problem involves determining the controls (magnitude and direction of the reactive force) and optimal trajectories for a point moving with a limited mass flow per second. The availability of reference analytical solutions is necessary for the development of maneuvering systems. Analytical solutions, unlike numerically constructed ones, are not associated with convergence issues, allow for the pre-determination of initial parameter values, ensure the continuity of trajectory parameters when the thrust mode changes, and contain important functional dependencies between spacecraft and trajectory parameters. Knowledge of analytical solutions allows for the analysis of spacecraft parameter behavior and a qualitative assessment of the accuracy of the control algorithm [1]. Thus, analytical solutions are of particular interest due to their clarity and the possibility of comprehensive analysis. This work is devoted to the determination of particular analytical solutions for intermediate thrust sections in the case of axisymmetric gravitational fields, the determination of the magnitude and direction of the reactive force of the spacecraft, solving the variational problem, as well as the stabilization of the found program movements.

Analytical solutions

It is known that the differential equations of the variational problem of the motion of a point (the center of mass of a spacecraft) have the following form(1).

$$\dot{\vec{v}} = \frac{cm}{M}\frac{\vec{\lambda}}{\lambda} + \vec{g}(\vec{r}), \dot{\vec{r}} = \vec{v}, \quad \dot{M} = -m, \tag{1}$$

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$$\dot{\vec{\lambda}} = -\vec{\lambda}_r, \quad \dot{\vec{\lambda}}_r = -\frac{\partial \vec{g}}{\partial \vec{r}}\vec{\lambda}, \quad \dot{\lambda}_M = \frac{cm}{M^2}\lambda,$$

where $\vec{r}$ - is the radius vector of the point (the spacecraft's center of mass), $\vec{v}$ - is its velocity, $M(t)$ - is the mass of the point, c - is the relative exhaust velocity of the combustion products, m - is the mass flow rate per second ($0 \leq m \leq \bar{m}$), $\vec{g}$ - is the gravitational acceleration, and $\vec{\lambda}, \vec{\lambda}_r, \dot{\lambda}_M$ - are the Lagrange multipliers conjugate to the velocity, radius vector, and mass, respectively (1).

The gravitational field has the following form [2]:

$$U(r, \theta) = f(r) + \frac{\Phi(\theta)}{r^3}, \tag{2}$$

Here we consider the functions $f(r) = \frac{\mu}{r} - \frac{k}{r^3}$ and $\Phi(\theta) = 3ksin^2(\theta)$. where $k = \frac{\mu}{2}J_2R^2$, J_2-is the second zonal harmonic, R is the mean equatorial radius of the planet; μ is the gravitational parameter of the planet.

For (1) and (2), particular analytical solutions were determined using the Levi-Chevita method and are as follows [3].

$$v_1 = 0, v_2^2 = \frac{360k^2\cos^2\theta - 270k^2\cos^4\theta + 36k\mu r^2\cos 2\theta - 2\mu^2 r^2 - 144k^2}{45kr^3 - 105k\sin^2\theta - 3\mu r^5}, v_3 = 0,$$

$$r = const, \varphi = \varphi_0 + \frac{v_2}{rcos(\theta)}t, \theta = const, M = M_0\exp\left(\frac{N}{c\lambda_1}t\right), N = \frac{df(r)}{dr} - 2\frac{\Phi(\theta)}{r^3} + \frac{1}{r}v_2^2,$$

$$\lambda_1 = \sqrt{1-\lambda_3^2}, \lambda_2 = 0, \lambda_3 = \frac{QP + \sqrt{Q^2 + 1 - P^2}}{Q^2 + 1}, Q = \tan\theta, P = \frac{a}{2v_2\cos\theta}$$

$$\lambda_4 = 0, \lambda_5 = a, \lambda_6 = 0, \lambda_7 = \frac{c}{M} \tag{3}$$

The analytical solution (3) is defined to represent the motion of the spacecraft. This solution was determined using the Levi-Chevita method, and its detailed determination is presented in this paper [3]. The dependencies of the magnitude and direction of the reactive force on the trajectory plane's position are determined:

$$\tan\alpha = \frac{\lambda_3}{\lambda_1} = \frac{\frac{v_2^2}{r}\tan\theta - \frac{1}{r^4}\frac{d\Phi}{d\theta}}{-\frac{df(r)}{dr} + 3\frac{\Phi(\theta)}{r^4} - \frac{1}{r}v_2^2}, \tag{4}$$

where α - is the angle between the thrust force and the radial direction.

First approximation stability study.

A first-order stability study was conducted for one of the particular solutions. It was shown to be Lyapunov unstable but controllable to a first approximation, and the unperturbed motion can be stabilized by a linear controller. A linear controller was constructed, the addition of which makes the unperturbed motion asymptotically Lyapunov stable [4]. A particular solution to the variational problem for intermediate thrust sections in the case of an axisymmetric gravitational field was found. The initial state of the point can only be realized with a certain error, which may increase or decrease over time. Therefore, it is necessary to study the stability of the obtained solutions.

We write the differential equations of the unperturbed motion of a point in the form.

$$\dot{v}_1 = fe_1 - \frac{df(r)}{dr} - \frac{3\Phi(\theta)}{r^4} + \frac{1}{r}\left(v_3^2 + v_2^2\right)$$

$$\dot{v}_2 = fe_2 - \frac{v_1v_2}{r} + \frac{v_2v_3}{r}\tan\theta$$

$$\dot{v}_3 = fe_3 - \frac{v_1v_3}{r} - \frac{v_2^2}{r}\tan\theta + \frac{1}{r^4}\frac{\partial\Phi}{\partial\theta} \tag{5}$$

$$\dot{r} = v_1, \dot{\phi} = \frac{v_2}{r\cos\theta}, \dot{\theta} = \frac{v_3}{r}, \dot{M} = -m$$

Where $e_1 = \frac{\lambda_1}{\lambda}, e_2 = \frac{\lambda_2}{\lambda}, e_3 = \sqrt{1 - e_1^2 - e_2^2}, f = \frac{cm}{M}$ −jet acceleration.

Thus, the class of particular solutions has the form (3). As the unperturbed motion corresponding to the class of particular solutions under consideration, we take the following:

$$\begin{cases} v_1^* = 0 \\ v_2^* = v_{20} \\ v_3^* = 0 \\ r^* = r_0 \\ \varphi^* = \varphi_0 + \beta t, \beta = \frac{v_2}{r\cos\theta} \\ \theta^* = \theta_0 \end{cases} \tag{6}$$

Let's take as controls:

$$\begin{cases} e_3 = \lambda_{30}, \\ e_2 = 0, \\ e_1 = \sqrt{1 - \lambda_{30}^2}, \\ f^* = \frac{cm^*}{M^*} = \frac{N^*}{\lambda_{10}} > 0. \end{cases}$$

Where $f^* = f_0 = -\frac{N_0}{\lambda_{10}}, m^* = -M^* \frac{N^*}{c\lambda_1^*}$.

The disturbed motion has the form:

$$\begin{cases} v_1 = v_1^* + x_1, \\ v_2 = v_2^* + x_2, \\ v_3 = v_3^* + x_3, \\ r = r^* + x_4, \\ \varphi = \varphi^* + x_5, \\ \theta = \theta^* + x_6, \end{cases} \qquad \begin{cases} e_3 = e_3^* + u_1, \\ e_2 = e_2^* + u_2, \\ f = f^* + u_3. \end{cases}$$

This yields the following.

$$\begin{cases} v_1 = x_1, \\ v_2 = v_{20} + x_2, \\ v_3 = x_3, \\ r = r_0 + x_4, \\ \varphi = \varphi_0 + \beta t + x_5, \\ \theta = \theta_0 + x_6, \end{cases} \qquad \begin{cases} e_3 = \lambda_{30} + u_1, \\ e_2 = u_2, \\ f = f_0 + u_3. \end{cases}$$

Let us formulate the differential equations of the disturbed motion.

$$\dot{x}_1 = (f_0 + u_3)\sqrt{1 - (u_1 + \lambda_{30})^2 - u_2^2} + \frac{df(r_0 + x_4)}{d(r_0 + x_4)} - 3\frac{\Phi(\theta_0 + x_6)}{(r_0 + x_4)^4} - \frac{(x_3^2 + (v_{20} + x_2)^2)}{(r_0 + x_4)}$$

$$\dot{x}_2 = (f_0 + u_3)u_2 - \frac{x_1(v_{20} + x_2)}{r_0 + x_4} + \frac{(x_2 + v_{20})x_3}{r_0 + x_4}\tan(\theta_0 + x_6)$$

$$\dot{x}_3 = (f_0 + u_3)(\lambda_{30} + u_1) - \frac{x_1 x_3}{r_0 + x_4} - \frac{(v_{20} + x_2)^2}{(r_0 + x_4)^4}\tan(\theta_0 + x_6) + \frac{1}{(r_0 + x_4)^4}\frac{d\Phi(\theta_0 + x_6)}{d(\theta_0 + x_6)}$$

$$\dot{x}_4 = x_1$$

$$\dot{x}_5 + \beta = \frac{v_{20} + x_2}{(r_0 + x_4)(\cos(\theta_0 + x_6))},$$

$$\dot{x}_6 = \frac{x_3}{r_0 + x_4}.$$

Let's isolate the first-approximation equations. Expand the right-hand sides of the equations into a series using the formula:

$$\dot{x}_i = F(0)+(\frac{\partial F}{\partial x_1})_0 x_1+(\frac{\partial F}{\partial x_2})_0 x_2+(\frac{\partial F}{\partial x_3})_0 x_3+(\frac{\partial F}{\partial x_4})_0 x_4+(\frac{\partial F}{\partial x_5})_0 x_5+(\frac{\partial F}{\partial x_6})_0 x_6+(\frac{\partial F}{\partial u_1})_0 u_1+(\frac{\partial F}{\partial u_2})_0 u_2+(\frac{\partial F}{\partial u_3})_0 u_3+...$$

$$\dot{x}_1 = \frac{2v_{20}}{r_0}x_2 + (\left[\frac{\partial}{\partial x_4}\left(\frac{df(r)}{d(r)}\right)_{r=r_0+x_4}\right]_0 + 6\frac{\Phi(\theta_0)}{r_0^4} - \frac{v_{20}^2}{r_0^2})x_4 - \frac{2}{r_0^3}\frac{\partial(\Phi(\theta_0+x_6))}{x_6}x_6 - \frac{2f_0\lambda_{30}u_1}{\sqrt{1-\lambda_{30}^2}} + \sqrt{1-\lambda_{30}^2}u_3$$

$$\dot{x}_2 = f_0u_2 - \frac{v_{20}}{r_0}x_1 + \frac{v_{20}}{r_0}\tan(\theta_0)x_3$$

$$\dot{x}_3 = f_0u_1 + \lambda_{30}u_3 - \frac{2v_{20}}{r_0}\tan(\theta_0)x_2 + \left(\frac{v_{20}^2}{r_0^2}\tan(\theta_0) - \frac{3}{r_0^4}\left(\frac{d\Phi(\theta)}{d\theta}\right)_{\theta=\theta_0}\right)x_4+$$

$$+\left(-\frac{v_{20}^2}{r_0}\cos^2\theta_0 + \frac{1}{r_0^3}\frac{\partial}{\partial x_6}\left(\frac{\partial\Phi(\theta)}{\partial\theta}\right)_{\theta=\theta_0+x_6}\right)x_6$$

$$\dot{x}_4 = x_1,$$

$$\dot{x}_5 = \left(\frac{1}{r_0\cos\theta_0}\right)x_2 - \frac{v_{20}}{r_0^2\cos\theta_0}x_4 + \frac{v_{20}\sin\theta_0}{r_0\cos^2\theta_0}x_6,$$

$$\dot{x}_6 = \frac{1}{r_0}x_3,$$

Let's introduce the following notations:

$$R = \left[\frac{\partial}{\partial x_4}\left(\frac{df(r)}{d(r)}\right)\right]_{\theta=r_0+x_4} + 6\frac{\Phi(\theta_0)}{r_0^4} - \frac{v_{20}^2}{r_0^2},$$

$$T = \left(\frac{v_{20}^2\tan(\theta_0)}{r_0^2}\right) - \frac{3}{r_0^4}\left(\frac{d\Phi(\theta)}{d\theta}\right)_{\theta=\theta_0},$$

$$K = -\frac{v_{20}^2}{r_0\cos^2\theta_0} + \frac{1}{r_0^3}\left(\frac{\partial}{\partial x_6}\left(\frac{\partial\Phi(\theta)}{\partial\theta}\right)\right)_{\theta=\theta_0+x_6},$$

$$A = \frac{\partial\Phi(\theta_0+x_6)}{\partial x_6}.$$

We obtain the equations of the first approximation:

$$\dot{x}_1 = \frac{2v_{20}}{r_0}x_2 + Rx_4 - \frac{2}{r_0^3}Ax_6 - \frac{2f_0\lambda_{30}}{\sqrt{1-\lambda_{30}^2}}u_1 + \sqrt{1-\lambda_{30}^2}u_3,$$

$$\dot{x}_2 = f_0u_2 - \frac{v_{20}}{r_0}x_1 + \frac{v_{20}}{r_0}\tan(\theta_0)x_3$$

$$\dot{x}_3 = f_0u_1 + \lambda_{30}u_3 - \frac{2v_{20}}{r_0}\tan(\theta_0)x_2 + Tx_4 + Kx_6,$$

$$\dot{x}_4 = x_1,$$

$$\dot{x}_5 = \left(\frac{1}{r_0\cos\theta_0}\right)x_2 - \frac{v_{20}}{r_0^2\cos\theta_0}x_4 + \frac{v_{20}\sin\theta_0}{r_0\cos^2\theta_0}x_6,$$

$$\dot{x}_6 = \frac{1}{r_0} x_3,$$

We will write the system of equations of the disturbed motion in matrix form:

$$\frac{d\vec{x}}{dt} = W\vec{x} + B\vec{u} + \vec{g}(x,u)$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \vec{g}(x,u)$$

The matrix W has the form:

$$[W = \begin{pmatrix} 0 & \frac{2v_{20}}{r_0} & 0 & R & 0 & -\frac{2A}{r_0^3} \\ -\frac{v_{20}}{r_0} & 0 & \frac{v_{20}}{r_0}\operatorname{tg}\theta_0 & 0 & 0 & 0 \\ 0 & -\frac{2v_{20}}{r_0}\operatorname{tg}\theta_0 & 0 & T & 0 & K \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{r_0\cos\theta_0} & 0 & -\frac{v_{20}^2}{r_0^2\cos\theta_0} & 0 & \frac{v_{20}\sin\theta_0}{r_0\cos^2\theta_0} \\ 0 & 0 & \frac{1}{r_0} & 0 & 0 & 0 \end{pmatrix}$$

For $u = 0$, we have a problem of Lyapunov stability [4] of unperturbed motion$(x_i = 0, i = 1..6)$:

$$\dot{\vec{x}} = W\vec{x} + \vec{g}(x,u)$$

Characteristic equation of the first approximation system:

$$[W - SE] = 0$$

$$[W - SE] = \begin{pmatrix} -S & \frac{2v_{20}}{r_0} & 0 & R & 0 & -\frac{2A}{r_0^3} \\ -\frac{v_{20}}{r_0} & -S & \frac{v_{20}}{r_0}\operatorname{tg}\theta_0 & 0 & 0 & 0 \\ 0 & -\frac{2v_{20}}{r_0}\operatorname{tg}\theta_0 & -S & T & 0 & K \\ 1 & 0 & 0 & -S & 0 & 0 \\ 0 & \frac{1}{r_0\cos\theta_0} & 0 & -\frac{v_{20}^2}{r_0^2\cos\theta_0} & -S & \frac{v_{20}\sin\theta_0}{r_0\cos^2\theta_0} \\ 0 & 0 & \frac{1}{r_0} & 0 & 0 & -S \end{pmatrix} \tag{7}$$

$$S^2\left[S^4 + \left(-R - \frac{1}{r_0}K + \frac{2v_{20}^2}{r_0^2\cos^2\theta_0}\right)S^2 + \left(\frac{2TA}{r_0^4} + 4\frac{v_{20}^2 A\tan\theta_0}{r_0^6} + \frac{RK}{r_0} - \frac{2v_{20}^2 T\tan\theta_0}{r_0} - \frac{2v_{20}^2 K}{r_0} - \frac{2v_{20}^2\tan^2\theta_0}{r_0^2}\right)\right] = 0$$

The characteristic equation has two zero roots. Therefore, if at least one of the roots has a positive real part, then, according to Lyapunov's instability theorem, to a first approximation, the unperturbed motion will be unstable. For this to occur, it is sufficient that the term at S^2 the characteristic equation be negative[5].

Stabilization of programmatic movement.

We will stabilize the unperturbed motion (5), that is, we will choose a controller such that, when substituted into (7), the unperturbed motion will be asymptotically Lyapunov stable. We will verify that system (7) is controllable. The following controllability and stabilization criterion exists to a first approximation [6]:

1) System $\frac{d\vec{x}}{dt} = W\vec{x} + B\vec{u}$ is fully controllable if matrix $V = \left|B, WB, ..W^{n-1}B\right|$ has rank n, where n is the order of the system (n=6).
2) If the rank of matrix V is n, then a linear controller $\vec{u} = P\vec{x}$ exists.

Let's construct a matrix V:

$$B = \begin{pmatrix} \frac{2f_o\lambda_{30}}{\sqrt{1-\lambda_{30}^2}} & 0 & \sqrt{1-\lambda_{30}^2} \\ 0 & f_o & 0 \\ f_o & 0 & \lambda_{30} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The rank of the matrix W is equal to 6. Consequently, the unperturbed motion (5) is stabilized by the linear controller $\vec{u} = P\vec{x}$, regardless of the nonlinear terms of the $g(\vec{x,u})$, the issue of stabilization is resolved using the linear approximation [7].

The constant real matrix P must be chosen such that the unperturbed motion of the system

$$\frac{d\vec{x}}{dt} = (W + BP)\vec{x} + g(\vec{x,u})$$

is asymptotically stable, that is, so that the real parts of all eigenvalues of the matrix $(W + BP)$ are negative. Moreover, the matrix P must be as simple as possible. For example, the following matrix satisfies this requirement[8].

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & p_{22} & 0 & p_{24} & p_{25} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

that is $u_1 = 0$,$u_3 = 0$. Then the characteristic equation of the first approximation will have the form:

$$[W + BP - SE] = 0$$

or

$$\det \begin{vmatrix} -S & \frac{2v_{20}}{r_0} & 0 & R & 0 & -\frac{2A}{r_0^3} \\ -\frac{v_{20}}{r_0} & -S + f_0p_{22} & \frac{v_{20}\tan\theta_0}{r_0} & f_0p_{24} & f_0p_{25} & 0 \\ 0 & -\frac{2v_{20}\tan\theta_0}{r_0} & -S & T & 0 & K \\ 1 & 0 & 0 & -S & 0 & 0 \\ 0 & \frac{1}{r_0\cos\theta_0} & 0 & -\frac{v_{20}}{r_0^2\cos\theta_0} & -S & \frac{v_{20}\sin\theta_0}{r_0\cos^2\theta_0} \\ 0 & 0 & \frac{1}{r_0} & 0 & 0 & -S \end{vmatrix} = 0 \qquad (8)$$

We will receive:

$$S^6 - S^5 f_0 p_{22} + S^4\left[\frac{f_0 p_{25}}{r_0\cos\theta_0} - \frac{2v_{20}^2}{r_0^2} - \frac{2v_0^2\tan^2\theta_0}{r_0^2} - R + \frac{K}{r_0}\right] + S^3\left[Rf_0p_{22} - \frac{2f_0v_0p_{24}}{r_0} - \frac{Kf_0p_{22}}{r_0}\right] -$$
$$-S^2\left(\frac{2v_0^2K}{r_0^3} + \frac{2v_{20}^2A\tan\theta_0}{r_0^6} - \frac{Kf_0p_{25}}{r_0^2\cos\theta_0} - \frac{f_0p_{25}v_{20}^2\sin^2\theta_0}{r_0^3\cos^3\theta_0} + \frac{2Rv_0^2\tan^2\theta_0}{r_0^2} + \right)$$
$$\left(+\frac{2v_{20}^2T\tan\theta_0}{r_0^2} - \frac{Rf_0p_{25}}{r_0\cos\theta_0} + \frac{2f_0p_{25}v_{20}^2}{r_0^3\cos\theta_0} - \frac{RK}{r_0} + 2\frac{AT}{r_0^2}\right) + \tag{9}$$
$$+S\left[\frac{2f_0p_{25}RK}{r_0} - \frac{4Av_{20}f_0p_{24}\tan\theta_0}{r_0^3} + \frac{2ATf_0p_{22}}{r_0^2} + \frac{2Kv_{20}f_0p_{24}}{r_0^2}\right] +$$
$$+\left(-\frac{2ATf_0p_{25}}{r_0^5\cos\theta_0} - \frac{RKf_0p_{25}}{r_0^2\cos\theta_0} - \frac{2Kf_0p_{25}Av_{20}^2}{r_0^4\cos\theta_0} - \frac{4f_0p_{25}Av_{20}^2\sin\theta_0}{r_0^7\cos^2\theta_0} - \frac{2Tf_0p_{25}v_{20}^2\sin\theta_0}{r_0^3\cos^2\theta_0} - \frac{2Rf_0p_{25}v_{20}^2\sin^2\theta_0}{r_0\cos^3\theta_0}\right) = 0$$

The characteristic equation was reduced to the form:

$$b_0S^6 + b_1S^5 + b_2S^4 + b_3S^3 + b_4S^2 + b_5S^1 + b_6S^0 = 0$$

where the coefficients of the b_i $(i = 1..6)$ are as follows:

$$b_0 = 1$$
$$b_1 = -f_0p_{22}$$
$$b_2 = \frac{f_0p_{25}}{r_0\cos\theta_0} - \frac{2v_{20}^2}{r_0^2} - \frac{2v_0^2\tan^2\theta_0}{r_0^2} - R + \frac{K}{r_0}$$
$$b_3 = Rf_0p_{22} - \frac{2f_0v_0p_{24}}{r_0} - \frac{Kf_0p_{22}}{r_0}$$
$$b_4 = \left(\frac{2v_0^2K}{r_0^3} + \frac{2v_{20}^2A\tan\theta_0}{r_0^6} - \frac{Kf_0p_{25}}{r_0^2\cos\theta_0} - \frac{f_0p_{25}v_{20}^2\sin^2\theta_0}{r_0^3\cos^3\theta_0} + \frac{2Rv_0^2\tan^2\theta_0}{r_0^2} + \right) \tag{10}$$
$$\left(+\frac{2v_{20}^2T\tan\theta_0}{r_0^2} - \frac{Rf_0p_{25}}{r_0\cos\theta_0} + \frac{2f_0p_{25}v_{20}^2}{r_0^3\cos\theta_0} - \frac{RK}{r_0} + 2\frac{AT}{r_0^2}\right)$$
$$b_5 = \frac{2f_0p_{25}RK}{r_0} - \frac{4Av_{20}f_0p_{24}\tan\theta_0}{r_0^3} + \frac{2ATf_0p_{22}}{r_0^2} + \frac{2Kv_{20}f_0p_{24}}{r_0^2}$$
$$b_6 = \left(-\frac{2ATf_0p_{25}}{r_0^5\cos\theta_0} - \frac{RKf_0p_{25}}{r_0^2\cos\theta_0} - \frac{2Kf_0p_{25}Av_{20}^2}{r_0^4\cos\theta_0} - \frac{4f_0p_{25}Av_{20}^2\sin\theta_0}{r_0^7\cos^2\theta_0} - \frac{2Tf_0p_{25}v_{20}^2\sin\theta_0}{r_0^3\cos^2\theta_0} - \frac{2Rf_0p_{25}v_{20}^2\sin^2\theta_0}{r_0\cos^3\theta_0}\right)$$

The matrix P was chosen to be as simple as possible. If, for example, $u_1 = 0$, in addition to counting $u_2 = 0$ (for $u_3 \neq 0$), then the free term of the characteristic equation (8) vanishes, and one root of this equation becomes zero. Let us construct the Hurwitz matrix from the coefficients $b_i (i = 1..6)$ of equation (8).

$$\begin{pmatrix} b_1 & b_3 & b_5 & 0 & 0 & 0 \\ b_0 & b_2 & b_4 & b_6 & 0 & 0 \\ 0 & b_1 & b_3 & b_5 & 0 & 0 \\ 0 & b_0 & b_2 & b_4 & b_6 & 0 \\ 0 & 0 & b_1 & b_3 & b_5 & 0 \\ 0 & 0 & b_0 & b_2 & b_4 & b_6 \end{pmatrix}$$

Hurwitz's theorem.

In order for all roots of the algebraic equation (8) with real coefficients and a positive coefficient at the leading term to have negative real parts, it is necessary and sufficient that all main diagonal minors of the matrix (9) are positive [4].

$$b_1 > 0$$

$$\Delta_2 = \det\begin{pmatrix} b_1 & b_3 \\ b_0 & b_2 \end{pmatrix} = b_1 b_2 - b_0 b_3 > 0$$

$$\Delta_3 = \begin{vmatrix} b_1 & b_3 & b_5 \\ b_0 & b_2 & b_4 \\ 0 & b_1 & b_3 \end{vmatrix} = b_3\Delta_2 + b_0 b_1 b_5 - b_1^2 b_4 = b_3\Delta_2 + b_1 b_5 - b_1^2 b_4 > 0$$

$$\Delta_4 = \begin{vmatrix} b_1 & b_3 & b_5 & 0 \\ b_0 & b_2 & b_4 & b_6 \\ 0 & b_1 & b_3 & b_5 \\ 0 & b_0 & b_2 & b_4 \end{vmatrix} = b_4\Delta_3 - b_6(b_3\Delta_2 - b_1^2 b_4) + b_0(b_4 b_5 - b_3 b_6 - b_0 b_3 b_5 - b_0 b_6 b_3) =$$

$$= b_4\Delta_3 - b_5(b_3\Delta_2 - b_1^2 b_4) + b_1(b_4 b_5 - b_6 b_3) - b_5^2 > 0$$

$$\Delta_5 = \begin{vmatrix} b_1 & b_3 & b_5 & 0 & 0 \\ b_0 & b_2 & b_4 & b_6 & 0 \\ 0 & b_1 & b_3 & b_5 & 0 \\ 0 & b_0 & b_2 & b_4 & b_6 \\ 0 & 0 & b_1 & b_3 & b_5 \end{vmatrix} = b_5\Delta_4 - b_6\big(b_3\Delta_3 - b_1(b_3\Delta_2 - b_1^2 b_4)\big) =$$

$$= b_5\Delta_4 - b_6\Delta_3 + b_1 b_6(b_3\Delta_2 - b_1^2 b_4) + b_0 b_6(b_3 - b_1^2 b_6) > 0$$

$\Delta_6 = b_6\Delta_5 > 0$, that is $b_6 > 0$
Thus, if the parameters p_{22}, p_{24}, p_{25} satisfy the above conditions, where b_i are determined from (10), then the unperturbed motion (5) is asymptotically stable: Therefore, the stabilizing control has been found.

$$\begin{cases} u_1 = 0 \\ u_2 = p_{22}x_2 + p_{24}x_4 + p_{25}x_5 \\ u_3 = 0 \end{cases}$$

where the coefficients p_{22}, p_{24}, p_{25} satisfy the conditions found above.

Conclusion

A variational problem of the motion of a point with variable mass in intermediate momentum sections in the gravitational field of spheroidal planets is considered. The differential equations of the variational problem are written in a spherical coordinate system. The feasibility of using the Levi-Civita method to find certain solutions is demonstrated, and a specific class of solutions is found using the Levi-Civita method. One specific solution is shown to be Lyapunov unstable but can be controlled to a first approximation, and the unperturbed motion can be stabilized using a linear controller. A linear controller is constructed, the addition of which makes the unperturbed motion asymptotically Lyapunov stable. It is shown that to stabilize unstable motion to a first approximation, only perturbations in three phase coordinates need be considered.

REFERENCES

1. Natalya.A.Korshunova and Dilmurat.M.Azimov. Analytical Solutions for Thrust Arcs in a Field of Two Fixed Centers // "Journal of Guidance, Control, and Dynamics"(AIAA), USA, 2014. V.37, N5, pp. 1716-1719.

2. Demin V. G. Motion of an Artificial Satellite in an Off-Central Gravitational Field. Moscow: Nauka, 1968. 352 p.

3. R.I.Tojiyev, M.I.Ruzmatov. Analytical Solutions for Active Parts in the Gravitational Field of a Spheroidal Planet//THE BULLETIN OF YOUNG SCIENTISTS 1(4)2025,5(6-62)-C,

4. Merkin D.R. Introduction to the Theory of Stability of Motion. St. Petersburg: Lan,2003, 304 p.

5. Ruzmatov M.I. Analytical methods for optimizing trajectories in gravitational fields. Abstract of a PhD dissertation in physics and mathematics. Tashkent. 2021. 44 p.

6. Azimov D.M. and Aragon Sanabria F. Extremal Control and Guidance Solutions for Orbital Transfer with Intermediate Thrust // Journal of Astronautically Sciences. 2015, vol.62, N4, pp. 351-383.

7. Aleksandrov V.V., Lemak S.S., Parusnikov N.A. Lectures on the mechanics of controlled systems. Moscow. Moscow State University, 2020, 165p.

8. Korshunova N.A., Ruzmatov M.I. Analytical solutions to the trajectory optimization problem in the case of the Barrar gravitational field // Problems of Mechanics No. 2, 2022, pp. 3-11.

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Spectrum of the Sum of Partial Integral Operators Generated by Incomplete Orthonormal Systems

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Abstract.

In this paper, we study a class of partial integral operators acting in the Hilbert space $L_2(\Omega_1 \times \Omega_2)$, generated by incomplete orthonormal systems in the corresponding L_2 spaces. Using a direct integral decomposition, we obtain an explicit description of the spectra of these operators in terms of the essential ranges of the generating coefficient functions. It is shown that the incompleteness of the underlying orthonormal systems leads to the appearance of an additional spectral component at zero. The main result concerns the spectral analysis of the sum of two such partial integral operators. We provide a precise characterization of the spectrum of the sum operator by exploiting its fiber-wise structure. The obtained results contribute to the spectral theory of non-compact partial integral operators and can be applied to related problems in operator theory and mathematical physics.

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Key words: partial integral operator, spectrum, resolvent, essential range, direct integral, orthonormal system, non-complete system.

Introduction

In recent decades, an important direction in the development of modern operator theory has been the comprehensive study of integral operators of Fredholm and Volterra types. These classes of operators have a wide range of applications from applied mathematics and mechanics to quantum physics and the spectral analysis of Schrödinger operators.

One of the aspects that has received significant attention is the study of the influence of various perturbations on the spectral characteristics of multiplication operators acting via multiplication by a fixed function. In particular, in [1], it is investigated how non-compact partial integral operators affect the spectrum of such operators. The main focus is on identifying those components of the spectrum that change under perturbations and those that retain spectral stability. The theoretical foundation is based on classical results of spectral theory, including Weyl's theorem, which guarantees the invariance of the essential spectrum under compact perturbations. However, in the absence of compactness, new and nontrivial effects arise that influence the structure of the spectrum.

In subsequent works, for example in [2], attention is focused on operators with degenerate kernels having a specific structure-namely, in the form of a product of a positive continuous function with itself, defined on a symmetric domain. Under appropriate integrability conditions on this function, it has been possible to completely characterize the essential spectrum of the corresponding operator and rigorously prove the finiteness of the number of eigenvalues lying below its lower bound.

Of particular interest is the manifestation of the Efimov effect in the context of partial integral operators, which is studied in detail in [3]. In that work, the existence of an infinite sequence of eigenvalues is demonstrated for a certain self-adjoint operator consisting of an unperturbed part and additional integral terms under a special configuration of the kernel. It is shown that the lower bound of the essential spectrum of such a Hamiltonian is equal to zero.

More general cases of degenerate kernels have been analyzed in studies [4,5], where the kernel is represented as the product of two different functions. Under certain positivity conditions and with an appropriate

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choice of test functions, the authors obtained an analytical expression for the lower bound of the essential spectrum of the partial integral operator.

Special interest is also devoted to discrete models of Schrödinger operators describing the behavior of three quantum particles on a three-dimensional lattice. These models are formalized using partial integral operators and possess spectral properties directly related to the Efimov effect and the phenomenon of accumulation of eigenvalues in the discrete part of the spectrum, as shown in [6].

Particular attention in modern literature is paid to the analysis of spectral characteristics of partial integral operators modeling three-particle interactions. In particular, the studies presented in [4,5,10,11] are aimed at a detailed investigation of the structure of the essential and discrete spectra of such operators. These operators belong to the class of partial integral operators of Fredholm type, which naturally arise in problems of quantum mechanics and mathematical physics when describing three-particle interactions.

An analysis of the structure of the essential spectrum of a model operator describing three-particle interaction is carried out in [9]. The existence of negative eigenvalues of this operator is established, and an estimate of their number is obtained.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be infinite-dimensional Hilbert spaces, and let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ denote their tensor product [7,8]. Suppose that bounded self-adjoint operators A and B act in these spaces, respectively. Then the tensor product $A \otimes B$ is also a bounded self-adjoint operator in $\mathcal{H}$. The spectral theory of partial integral operators has been extensively studied under various assumptions, including compactness, kernel regularity, and completeness of the underlying orthonormal systems. In the classical setting, when the generating systems are complete, the corresponding operators admit relatively well-understood spectral descriptions.

In this paper, we consider a class of partial integral operators generated by incomplete orthonormal systems in $L_2(\Omega_1)$ and $L_2(\Omega_2)$. We focus on the precise description of their spectra in the space $L_2(\Omega_1 \times \Omega_2)$. Using the direct integral approach, we express these operators as measurable families of diagonal operators and describe their spectra in terms of the essential ranges of the corresponding coefficient functions.

The main emphasis is placed on the spectral analysis of the sum of two such operators. Although each operator admits a relatively simple fiber-wise representation, their sum exhibits a more intricate spectral structure. We show that this structure can still be effectively analyzed by exploiting the underlying decomposition and the orthogonality properties of the generating systems.

The results obtained in this paper extend known spectral descriptions to the case of incomplete orthonormal systems and provide a unified framework for studying non-compact partial integral operators. These findings may be useful in further investigations of operator equations and in applications where non-complete expansions naturally arise.

Formulation of the Main Problem of Spectral Analysis

Consider a linear partial integral operator of the form

$$T = T_1 + T_2, \tag{1}$$

where the operator T acts in the Hilbert space $L_2(\Omega_1 \times \Omega_2)$, with $\Omega_1 = [a, b]$ and $\Omega_2 = [c, d]$.

The operators T_1 and T_2 act in $L_2(\Omega_1 \times \Omega_2)$ as partial integral operators with respect to different variables:

$$(T_1 f)(x, y) = \int_{\Omega_1} k_1(x, s, y)\, f(s, y)\, ds,$$

$$(T_2 f)(x, y) = \int_{\Omega_2} k_2(x, t, y)\, f(x, t)\, dt.$$

Here, the kernels k_1 and k_2 are measurable functions defined on $\Omega_1^2 \times \Omega_2$ and $\Omega_1 \times \Omega_2^2$, respectively.

Assume that $\{\varphi_i\}_{i=1}^{\infty}$ is an orthonormal system in $L_2(\Omega_1)$ and $\{\psi_j\}_{j=1}^{\infty}$ is an orthonormal system in $L_2(\Omega_2)$. Let $\{h_i(y)\}_{i=1}^{\infty}$ and $\{g_j(x)\}_{j=1}^{\infty}$ be collections of essentially bounded real-valued functions defined on Ω_2 and Ω_1, respectively. Then the kernels admit the following degenerate representations:

$$k_1(x, s, y) = \sum_{i=1}^{\infty} \varphi_i(x)\, \overline{\varphi_i(s)}\, h_i(y), \tag{2}$$

$$k_2(x,t,y) = \sum_{j=1}^{\infty} \psi_j(y)\,\overline{\psi_j(t)}\,g_j(x). \tag{3}$$

Thus, both operators are partial integral operators with degenerate kernels represented as infinite sums of separable functions. Under appropriate boundedness conditions on the coefficient functions $\{h_i(y)\}$ and $\{g_j(x)\}$, these operators define bounded self-adjoint operators in $L_2(\Omega_1 \times \Omega_2)$. The operator $T_1 + T_2$ in the case where h_i and g_j are constant sequences was studied in [12].

Let $\varphi \geq 0$ be a measurable and essentially bounded function defined on a set $\Omega \subset \mathbb{R}^v$. Then

$$\operatorname{ess\,sup}_{\Omega}(\varphi) = \inf\left\{C \in \mathbb{R} : \mu\left(\{\xi \in \Omega : \varphi(\xi) > C\}\right) = 0\right\},$$

where $\mu(\cdot)$ denotes the Lebesgue measure. If there exists $\varepsilon > 0$ such that

$$\mu\left(\{\xi \in \Omega : \lambda - \varepsilon < \varphi(\xi) < \lambda + \varepsilon\}\right) > 0,$$

then the number λ is called an essential value of the function φ. The set of all such values is denoted by Essran(φ).

In this paper, we study the operator (1) with degenerate kernels of the form (2)-(3). The main goal is to investigate the spectral properties of T, in particular, to describe its essential and discrete spectra and to establish conditions for the existence of eigenvalues.

Spectral properties of the operators $T_1 + T_2$

Let the coefficients $h_i(y)$ and $g_j(x)$ satisfy the following assumptions:

- $h_i \in L_\infty(\Omega_2)$ and $h_i(y)$ is real-valued for all i, with $M_1 := \operatorname{ess\,sup}_{y\in\Omega_2} \sup_i |h_i(y)| < \infty$;
- $g_j \in L_\infty(\Omega_1)$ and $g_j(x)$ is real-valued for all j, with $M_2 := \operatorname{ess\,sup}_{x\in\Omega_1} \sup_j |g_j(x)| < \infty$.

Lemma 1. Let h_i and g_j satisfy the assumptions above. Then the operators T_1 and T_2 on $L_2(\Omega_1 \times \Omega_2)$ are bounded, with $\|T_1\| \leq M_1$ and $\|T_2\| \leq M_2$, and they are self-adjoint.

Proof. Using the orthonormality of $\{\varphi_i\}$, for $f \in L_2(\Omega_1 \times \Omega_2)$ we can write

$$(T_1 f)(x,y) = \sum_{i=1}^{\infty} h_i(y)\,\varphi_i(x) \int_{\Omega_1} f(s,y)\overline{\varphi_i(s)}\,ds.$$

Denote $c_i(y) := (f(\cdot,y), \varphi_i)_{L_2(\Omega_1)} = \int_{\Omega_1} f(s,y)\overline{\varphi_i(s)}\,ds$.

Then $(T_1 f)(x,y) = \sum_{i=1}^{\infty} h_i(y)\,\varphi_i(x)\,c_i(y)$.

If $\{\varphi_i\}$ is complete, Parseval's identity gives $\sum_{i=1}^{\infty} |c_i(y)|^2 = \|f(\cdot,y)\|^2_{L_2(\Omega_1)}$, while if $\{\varphi_i\}$ is not complete, Bessel's inequality yields $\sum_{i=1}^{\infty} |c_i(y)|^2 \leq \|f(\cdot,y)\|^2_{L_2(\Omega_1)}$.

Hence, in both cases,

$$\|T_1 f\|^2_{L_2(\Omega_1\times\Omega_2)} = \int_{\Omega_2} \sum_{i=1}^{\infty} |h_i(y)|^2 |c_i(y)|^2\,dy \leq \sup_i \|h_i\|^2_{L_\infty(\Omega_2)} \int_{\Omega_2} \sum_{i=1}^{\infty} |c_i(y)|^2\,dy \leq M^2 \|f\|^2_{L_2(\Omega_1\times\Omega_2)},$$

where $M := \sup_i \|h_i\|_{L_\infty(\Omega_2)} < \infty$.

Therefore, T_1 is bounded and $\|T_1\| \leq M$.

If $h_i(y)$ are real-valued, then for all f, g:

$$(T_1 f, g) = \int_{\Omega_2} \sum_i h_i(y) c_i(y) \overline{d_i(y)}\,dy = (f, T_1 g),$$

where $d_i(y) = (g(\cdot,y), \varphi_i)$. Therefore T_1 is self-adjoint.

Theorem 1. If the orthonormal system $\{\varphi_i\}$ is not complete, then the spectrum of T_1 in $L_2(\Omega_1\times\Omega_2)$ is given by

$$\sigma(T_1)=\overline{\bigcup_{i=1}^{\infty}\operatorname{ess\,ran}(h_i)}\ \cup\ \{0\}.$$

Proof. For each fixed $y\in\Omega_2$, define the fiber operator

$$A(y):L_2(\Omega_1)\to L_2(\Omega_1),\quad (A(y)f_y)(x)=\sum_{i=1}^{\infty}h_i(y)\langle f_y,\varphi_i\rangle\,\varphi_i(x),$$

where $f_y(x):=f(x,y)$ is the y-fiber of f.

By construction, $A(y)$ is diagonal in the orthonormal system $\{\varphi_i\}$. Its spectrum restricted to the span of $\{\varphi_i\}$ is

$$\sigma\big(A(y)\big|_{\operatorname{span}\{\varphi_i\}}\big)=\overline{\{h_i(y)\}_{i=1}^{\infty}}.$$

The operator T_1 can be expressed as a direct integral of the fiber operators:

$$T_1\cong\int_{\Omega_2}^{\oplus}A(y)\,dy.$$

According to standard results on spectra of direct integrals, we have

$$\sigma(T_1)=\overline{\bigcup_{y\in\Omega_2}\sigma(A(y)\big|_{\operatorname{span}\{\varphi_i\}})}.$$

If $\{\varphi_i\}$ is incomplete in $L_2(\Omega_1)$, then there exists a non-zero subspace $\overline{\operatorname{span}}\{\varphi_i\}_\perp\subset L_2(\Omega_1)$ orthogonal to all φ_i. For any $f_y\in H_\perp$, $(A(y)f_y)(x)=0$ for all $y\in\Omega_2$. Hence 0 is not in the spectrum of any individual $A(y)$ restricted to $\operatorname{span}\{\varphi_i\}$, but contributes to the spectrum of T_1 when taking the direct integral: $0\in\sigma(T_1)$.

Using the definition of the essential range, the union over $y\in\Omega_2$ can be rewritten as

$$\overline{\bigcup_{y\in\Omega_2}\overline{\{h_i(y)\}_{i=1}^{\infty}}}=\overline{\bigcup_{i=1}^{\infty}\operatorname{ess\,ran}(h_i)}.$$

Including the zero contribution from the orthogonal subspace, we obtain the complete spectrum:

$$\sigma(T_1)=\overline{\bigcup_{y\in\Omega_2}\overline{\{h_i(y)\}_{i=1}^{\infty}}}\ \cup\ \{0\}=\overline{\bigcup_{i=1}^{\infty}\operatorname{ess\,ran}(h_i)}\ \cup\ \{0\}.$$

Corollary 1. Zero does not belong to the spectrum of T_1 if and only if $\{\varphi_i\}_{i=1}^{\infty}$ forms a complete orthonormal system in $L_2(\Omega_1)$, and $\inf_{i\geq 1,\,y\in\Omega_2}|h_i(y)|>0$.

Theorem 2. The discrete spectrum of the operator T_1 is empty, i.e.,

$$\sigma_{\mathrm{disc}}(T_1)=\varnothing.$$

Proof. Decompose the space $L_2(\Omega_1)$ as

$$L_2(\Omega_1)=H_1\oplus H_1^\perp,\quad H_1=\overline{\operatorname{span}}\{\varphi_i\}.$$

Accordingly,

$$L_2(\Omega_1\times\Omega_2)=(H_1\otimes L_2(\Omega_2))\oplus(H_1^\perp\otimes L_2(\Omega_2)).$$

Let $\lambda\in\mathbb{R}$ be an eigenvalue of T_1 and let $f\in H_1\otimes L_2(\Omega_2)$ be a corresponding eigenfunction:

$$T_1f=\lambda f.$$

Then, for a.e. $y \in \Omega_2$, the fiber $f_y(x) = f(x, y)$ satisfies

$$\sum_{i=1}^{\infty} h_i(y)(f_y, \varphi_i)\varphi_i(x) = \lambda f_y(x).$$

By orthonormality,

$$(h_i(y) - \lambda)(f_y, \varphi_i) = 0 \quad \text{for all } i.$$

Hence

$$f(x, y) = \sum_{i:\, h_i(y)=\lambda} c_i(y)\varphi_i(x), \quad c_i \in L_2(\Omega_2).$$

Therefore, the eigenspace corresponding to λ contains functions of the form

$$\varphi_i(x) \otimes f(y), \quad f \in L_2(\Omega_2),$$

which implies that this eigenspace is infinite-dimensional.

Since $H_1^{\perp} \perp \{\varphi_i\}$, we have

$$T_1 f = 0 \quad \text{for all } f \in H_2 \otimes L_2(\Omega_2).$$

Thus, 0 is an eigenvalue and its eigenspace contains the whole subspace $H_2 \otimes L_2(\Omega_2)$, which is infinite-dimensional.

In both cases, every eigenvalue of T_1 has an infinite-dimensional eigenspace. Hence, T_1 has no eigenvalues of finite multiplicity, and therefore its discrete spectrum is empty:

$$\sigma_{\text{disc}}(T_1) = \varnothing.$$

Theorem 3. Let $\lambda \notin \sigma(T_1)$ and $\mu \notin \sigma(T_2)$. Then the resolvents of T_1 and T_2 are given by

$$\begin{aligned}
(T_1 - \lambda I)^{-1} f &= \sum_{i=1}^{\infty} \frac{\varphi_i(x)}{h_i(y) - \lambda} \int_{\Omega_1} \overline{\varphi_i(s)} f(s, y)\, ds - \frac{1}{\lambda} f_1^{\perp}(x, y), \\
(T_2 - \mu I)^{-1} f &= \sum_{j=1}^{\infty} \frac{\psi_j(y)}{g_j(x) - \mu} \int_{\Omega_2} \overline{\psi_j(t)} f(x, t)\, dt - \frac{1}{\mu} f_2^{\perp}(x, y).
\end{aligned} \tag{5}$$

where the orthogonal projections are defined by

$$f_1^{\perp}(\cdot, y) = f(\cdot, y) - \sum_{i=1}^{\infty} \varphi_i(\cdot) \int_{\Omega_1} \overline{\varphi_i(s)} f(s, y)\, ds, \quad f_2^{\perp}(x, \cdot) = f(x, \cdot) - \sum_{j=1}^{\infty} \psi_j(\cdot) \int_{\Omega_2} \overline{\psi_j(t)} f(x, t)\, dt.$$

Here $f_1^{\perp}$ and $f_2^{\perp}$ are the orthogonal projections of f onto $\mathcal{H}_1^{\perp}$ and $\mathcal{H}_2^{\perp}$, respectively.

Proof. We use the direct integral decomposition

$$L_2(\Omega_1 \times \Omega_2) \simeq \int_{\Omega_2}^{\oplus} L_2(\Omega_1)\, dy.$$

For each fixed y, define the operator

$$A(y) : L_2(\Omega_1) \to L_2(\Omega_1), \quad A(y)u = \sum_{i=1}^{\infty} h_i(y)(u, \varphi_i)\varphi_i.$$

Then

$$(T_1 f)(\cdot, y) = A(y) f(\cdot, y), \quad T_1 = \int_{\Omega_2}^{\oplus} A(y)\, dy.$$

Hence,

$$(T_1 - \lambda I)^{-1} = \int_{\Omega_2}^{\oplus} (A(y) - \lambda I)^{-1} dy,$$

provided $\lambda \notin \sigma(T_1)$.

Now fix y. Since $\{\varphi_i\}$ is orthonormal, we have the orthogonal decomposition

$$L_2(\Omega_1) = \overline{\operatorname{span}\{\varphi_i\}} \oplus \mathcal{H}_1^{\perp}.$$

On these subspaces:

$$A(y)\varphi_i = h_i(y)\varphi_i, \quad A(y)u_{\perp} = 0.$$

Therefore,

$$(A(y) - \lambda I)\varphi_i = (h_i(y) - \lambda)\varphi_i, \quad (A(y) - \lambda I)u_{\perp} = -\lambda u_{\perp}.$$

If $\lambda \notin \operatorname{ess\,ran}(h_i)$ and $\lambda \neq 0$, then

$$(A(y) - \lambda I)^{-1}\varphi_i = \frac{1}{h_i(y) - \lambda}\varphi_i, \quad (A(y) - \lambda I)^{-1}u_{\perp} = -\frac{1}{\lambda}u_{\perp}.$$

Applying this fiber-wise to $f(\cdot, y)$ yields (5).

The proof for T_2 is analogous, with the roles of x and y interchanged and φ_i replaced by ψ_j.

Let

$$\mathcal{H}_1 = \overline{\operatorname{span}}\{\varphi_i\}, \quad \mathcal{H}_2 = \overline{\operatorname{span}}\{\psi_j\}.$$

Consider the block decomposition of $L_2(\Omega_1 \times \Omega_2)$:

$$L_2(\Omega_1 \times \Omega_2) = \underbrace{\overline{\operatorname{span}\{\varphi_i \otimes \psi_j\}}}_{H_{11}} \oplus \underbrace{(\mathcal{H}_1^{\perp} \otimes \overline{\operatorname{span}\{\psi_j\}})}_{H_{21}} \oplus \underbrace{(\overline{\operatorname{span}\{\varphi_i\}} \otimes \mathcal{H}_2^{\perp})}_{H_{12}} \oplus \underbrace{(\mathcal{H}_1^{\perp} \otimes \mathcal{H}_2^{\perp})}_{H_{22}}.$$

Theorem 4. Let $\{\varphi_i\}$ and $\{\psi_j\}$ be incomplete orthonormal systems. Then

$$\sigma(T_1 + T_2) = \overline{\bigcup_{i,j}(\operatorname{ess\,ran}(h_i) + \operatorname{ess\,ran}(g_j))} \cup \overline{\bigcup_{j}\operatorname{ess\,ran}(g_j)} \cup \overline{\bigcup_{i}\operatorname{ess\,ran}(h_i)} \cup \{0\}.$$

Proof. We use the orthogonal decomposition

$$L_2(\Omega_1 \times \Omega_2) = \mathcal{H}_{11} \oplus \mathcal{H}_{12} \oplus \mathcal{H}_{21} \oplus \mathcal{H}_{22},$$

Each subspace is invariant under T_1 and T_2, hence also under $T_1 + T_2$. Therefore,

$$\sigma(T_1 + T_2) = \bigcup_{k=1}^{4} \sigma\big((T_1 + T_2)|_{\mathcal{H}_k}\big).$$

On $\mathcal{H}_{11}$, the operator admits a direct integral representation. For each (x, y)-fiber, it acts diagonally with respect to the basis $\{\varphi_i(x)\psi_j(y)\}$, and the corresponding fiber operators have spectra

$$\{h_i(y) + g_j(x) : i, j \geq 1\}.$$

Hence,

$$\sigma(\mathcal{H}_{11}) = \overline{\bigcup_{i,j}(\operatorname{ess\,ran}(h_i) + \operatorname{ess\,ran}(g_j))}.$$

On the $\mathcal{H}_{21}$ subspace, $T_1 = 0$, hence

$$(T_1 + T_2)|_{\mathcal{H}_{21}} = T_2.$$

Therefore,

$$\sigma(\mathcal{H}_{21}) = \overline{\bigcup_j \operatorname{ess\,ran}(g_j)}.$$

Similarly, $T_2 = 0$ on $\mathcal{H}_{12}$ subspace, hence

$$(T_1 + T_2)|_{\mathcal{H}_{12}} = T_1,$$

and

$$\sigma(\mathcal{H}_{12}) = \overline{\bigcup_i \operatorname{ess\,ran}(h_i)}.$$

On $\mathcal{H}_{22}$, both T_1 and T_2 vanish, so

$$(T_1 + T_2)|_{\mathcal{H}_{22}} = 0,$$

and hence

$$\sigma(\mathcal{H}_{22}) = \{0\}.$$

Combining all four blocks, we obtain

$$\sigma(T_1 + T_2) = \overline{\bigcup_{i,j} (\operatorname{ess\,ran}(h_i) + \operatorname{ess\,ran}(g_j))} \cup \overline{\bigcup_j \operatorname{ess\,ran}(g_j)} \cup \overline{\bigcup_i \operatorname{ess\,ran}(h_i)} \cup \{0\}.$$

REFERENCES

1. Kucharov, R. R.; Tuxtamurodova, T. M. Non-compact perturbation of the spectrum of multipliers given by a special form. *Nanosystems: Physics, Chemistry, Mathematics* **15** (2024), no. 1, 31–36.

2. Kucharov, R. R.; Khamraeva, R. R. Non-compact perturbations of the spectrum of multipliers given with functions. *Nanosystems: Physics, Chemistry, Mathematics* **12** (2021), no. 2, 135–141.

3. Eshkabilov, Yu. Kh.; Kucharov, R. R. Efimov's effect for partial integral operators of Fredholm type. *Nanosystems: Physics, Chemistry, Mathematics* **4** (2013), no. 4, 529–537.

4. Kucharov, R. R.; Eshkabilov, Yu. Kh. On the finiteness of negative eigenvalues of a partial integral operator. *Matematicheskie Trudy* **17** (2014), no. 1, 128–144.

5. Eshkabilov, Yu. Kh.; Kucharov, R. R. On the essential and discrete spectra of a three-particle Schrödinger operator. *Theoretical and Mathematical Physics* **170** (2012), no. 3, 409–422.

6. Rasulov, T. Kh. Asymptotics of the discrete spectrum of a model operator associated with a system of three particles on a lattice. *Theoretical and Mathematical Physics* **163** (2010), no. 1, 34–44.

7. Berezin, F. A.; Shubin, M. A. *The Schrödinger Equation.* Moscow: Moscow State University Press, 1983.

8. Reed, M.; Simon, B. *Methods of Modern Mathematical Physics. Vol. 1: Functional Analysis.* Moscow: Mir, 1977.

9. Arzikulov, G. P.; Eshkabilov, Yu. Kh. On spectral properties of a three-particle model operator. *Russian Mathematics (Iz. VUZ)* (2020), no. 5, 3–10.

10. Albeverio, S.; Lakaev, S. N.; Muminov, Z. I. On the number of eigenvalues of a model operator associated to a system of three particles on lattices. *Russian Journal of Mathematical Physics* **14** (2007), 37–387.

11. Arzikulov, G. P.; Eshkabilov, Yu. Kh. On the essential and discrete spectra of a partial integral operator of Fredholm type. *Matematicheskie Trudy* **17** (2020), no. 2, 23–40.

12. Kucharov, R. R.; Tuxtamurodova, T. M. Spectral properties and the Efimov effect for bounded self-adjoint partial integral operators. *Russian Universities Reports. Mathematics* **31** (2026), no. 153, 5–21.

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Pursuit problem for neutral-type differential-difference equations with integral constraints on the control parameters

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Abstract. In this work, the main focus is on studying a pursuit problem described by a system of linear neutral-type differential-difference equations under integral constraints on the players' control functions. In the course of this study, new sufficient conditions of Lev Semyonovich Pontryagin for the solvability of the pursuit problem are obtained. New sufficient conditions on the process parameters ensuring the completion of the game within a finite time are also derived.

MSC (2020): 49N79; 49N70; 91A24.

Key words: Differential game, pursuit problem, neutral-type differential-difference equations, terminal set, pursuer, evader, control.

Introduction

The development of the theory of differential games in such formulations is associated with the works of R. Isaacs, L. S. Pontryagin, N. Krasovskii, E. Mishchenko, B. Pshenichny, and W. Fleming.

The past 50 years have been a period of intensive development of differential game theory. A significant contribution to the development and application of this theory has been made by E. Mishchenko, Y. Osipov, R. Gamkrelidze, A. Subbotin, A. Kurzhanski, A. Kryazhimskiy, F. Chernousko, V. Ushakov, N. Lukoyanov, N. Satimov, A. Azamov, A. Chentsov, V. Tretyakov, A. Chikrii, L. Petrosyan, M. Nikolsky, V. Maksimov, N. Subbotina, P. Gusyatnikov, N. Grigorenko, V. Ukhobotov, L. Yugay, B. Riksiyev, A. Fazylov, M. Tukhtasinov, M. Mamatov, N.A. Mamadaliev, G. Ibragimov, and many other mathematicians.

The first works devoted to differential games appeared in the early 1950s. The concept of a "differential game" was first introduced in a series of classified works by the American mathematician R. Isaacs, carried out in the early 1950s under a project of the RAND Corporation, as they were considered to have potential military applications.

The studies of R. Isaacs were published in 1965 in the form of a monograph [1], in which numerous examples were considered, while theoretical issues were only briefly addressed.

The results obtained by L. S. Pontryagin and E. Mishchenko [2] led to the creation by L. S. Pontryagin of the first and second direct methods for solving the pursuit problem in linear differential games. This method provides conveniently verifiable sufficient conditions for the solvability of the pursuit problem in the class of counter-strategies.

The first method of L. S. Pontryagin served as a basis for many generalizations, in particular [3-5], and it is also closely related to the method of resolving functions [7]. In [8], a third intermediate direct method for linear differential games was developed. A. Azamov discovered the duality of the alternating integral of L. S. Pontryagin. Among the works devoted to the development and application of Pontryagin's methods, [2] should be noted.

In the general theory of differential games, pursuit-evasion problems occupy a special place due to a number of specific features. One of these is the great diversity both in the methods used and in the nature of the results obtained. This feature is already evident when considering model examples.

Thus, the strategy of parallel pursuit proposed by L. Petrosyan in a simple pursuit game with a вЂњlife lineвЂќ stimulated the development of the method of resolving functions for solving group pursuit problems

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with geometric constraints in the works of B. Pshenichny, A. Chikrii, I. Rappoport, B. Grigorenko, N. Petrov, V. Ukhobotov, A. Blagodatkikh, and others.

On the part of A. Chikrii, A. Belousov, V. Bezmagorychny, and L. Baranovskaya, attempts were made to construct an analogue of the parallel pursuit strategy for the case of integral constraints and to extend it to more general cases using the method of resolving functions.

At present, when constructing and studying mathematical models, the use of ordinary differential equations is often no longer sufficient. A more adequate approach is to employ the apparatus of differential-difference equations. In practice, there is now a need to model pursuit games by differential-difference equations that take into account the prehistory of the system's state, which allows for a more accurate description of the dynamics.

To solve such problems, the method of resolving functions has been developed, which extends the first direct method of L. S. Pontryagin. A more complex class consists of conflict-controlled processes described by neutral-type differential-difference equations containing the unknown function and its derivatives at different moments in time. In this connection, there arises a need to modify the method of resolving functions for this class of problems.

The study of differential games described by differential-difference equations has been addressed in the works of N. Krasovskii, Y. Osipov, A. Kryazhimskiy, V. Maksimov, M. Nikolsky, A. Chikrii, G. Chikrii, L. Baranovskaya, N.A. Mamadaliev, and others. In these works, sufficient conditions for the successful completion of differential-difference approach games are obtained, the structure of optimal approach strategies is clarified, a number of alternative theorems are proved, and pursuit differential-difference games with geometric and integral constraints on players' controls are studied.

In this paper, the main focus is on investigating a pursuit problem described by a system of linear neutral-type differential-difference equations under integral constraints on the players' controls. In the course of this study, new sufficient conditions of L. S. Pontryagin for the solvability of the pursuit problem are obtained. This work is closely related to publications [10] and continues the studies presented in [4-5].

Problem Statement

The dynamics of a conflict-controlled process in a finite-dimensional Euclidean space $\mathbb{R}^n$ is described by a system of linear neutral-type differential-difference equations containing the unknown function and its derivatives at different moments in time [11]:

$$\dot{z}(t) = Az(t) + \sum_{i=1}^{m} B_i \dot{z}(t-h_i) + \sum_{i=1}^{m} C_i z(t-h_i) - Du(t) + Fv(t), \quad t \geq 0, \tag{1}$$

where $z(t) \in \mathbb{R}^n, n \geq 1; A, B_i, (i = 1, 2, \ldots, m), C_i, (i = 1, 2, \ldots, m)$ are constant square matrices of order $(n \times n)$; D and F are constant matrices of orders $(n \times p)$ and $(n \times q)$, respectively; $0 = h_0 < h_1 < \cdots < h_m = h$ are real numbers; u is the control parameter of the pursuer, and v is the control parameter of the evader.

The controls $u(t)$ and $v(t)$ are chosen from the class of measurable vector functions satisfying the integral constraints:

$$\|u(\cdot)\| = \sqrt{\int_0^{+\infty} |u(t)|^2 dt} \leq \rho, \tag{2}$$

$$\|v(\cdot)\| = \sqrt{\int_0^{+\infty} |v(t)|^2 dt} \leq \sigma, \tag{3}$$

where ρ and σ are nonnegative constants.

Measurable functions $u = u(t), v = v(t), 0 \leq t < +\infty$, satisfying constraints (2), will be called *admissible controls* of the pursuer and the evader, respectively.

Moreover, in the space $\mathbb{R}^n$, a nonempty cylindrical terminal set $M = M_0 + M_1$ is specified, where M_0 is a linear subspace of $\mathbb{R}^n$, and M_1 is a compact subset of the subspace L, where L is the orthogonal complement of M_0 in $\mathbb{R}^n$ (i.e., $M_0 \oplus L = \mathbb{R}^n$).

The initial condition for the pursuit process (1) is an n-dimensional absolutely continuous function $\varphi(t)$ defined on the interval $[-h_m, 0]$, i.e.,

$$\Big\{\varphi(\cdot) : z(t) = \varphi(t), t \in [-h_m, 0], z(0) = \varphi(0) \in \mathbb{R}^n \setminus M\Big\}. \tag{4}$$

The objective of the first player (the pursuer), by choosing a control $u(t)$ (under constraint (2)), is to drive the trajectory of process (1) to the terminal set M in the shortest possible time. The objective of the second player (the evader), by means of control $v(t)$, is to prevent the trajectory of process (1) from reaching the terminal set M in finite time; if this is impossible, then to delay the termination of the game as much as possible.

Let S denote the set of points of the form

$$S = \Big\{t : t = \sum_{i=0}^{m} j_i h_i, j_i \text{ are integers}\Big\},$$

and let S^0 denote the intersection of S with the interval $(0, \infty)$, i.e.,

$$S^0 = S \cap (0, \infty).$$

Definition 1. Let $K(t), 0 < t \le \tau$, be the unique matrix function possessing the following properties [11]:

a) $K(t) = \tilde{0}, t < 0$, where $\tilde{0}$ is the zero matrix of order n;

b) $K(0) = E$, where E is the identity matrix of order n;

c) the function $\sum_{i=0}^{m} D_i K(t - h_i)$ is continuous on $[0, +\infty)$;

d) $K(t)$, for $t > 0$, satisfies the matrix differential equation

$$\dot{K}(t) = AK(t) + \sum_{i=1}^{m} A_i \dot{K}(t - h_i) + \sum_{i=1}^{m} B_i K(t - h_i), \quad t \notin S^0. \tag{5}$$

The existence and uniqueness of the matrix function $K(t), -\infty < t \le \tau$, satisfying conditions a)-b), can be proved by the standard method of successive integration of equation (5). The function $K(t)$ belongs to the class C^1 for $t > 0, t \notin S^0$, but in the general case has discontinuities of the first kind at the points of the set S^0.

Let admissible controls $u = u(t), v = v(t)$ be chosen on the interval $[0, \tau], \tau > 0$. Then, for the solution $z(t)$ of equation (1) with initial condition (4), by virtue of the Cauchy formula, the following representation holds [11]:

$$z(\tau) = \Big[K(\tau) - \sum_{i=1}^{m} K(\tau - h_i) B_i\Big]\varphi(0) + \sum_{i=1}^{m} \int_{-h_i}^{0} K(\tau - t - h_i)\Big[B_i \dot{\varphi}(t) + C_i \varphi(t)\Big] dt -$$

$$\int_{0}^{\tau} K(\tau - t)\Big[Du(t) - Fv(t)\Big] dt, \tag{6}$$

where $B_0 = -E$, and E is the identity matrix of order $n \times n$.

Definition 2. We say that in game (1) the pursuit can be completed from the initial point $\varphi(\cdot) \in \mathbb{R}^n \setminus M$ within time $T(\varphi(\cdot))$, if there exists a function $u(t, v), 0 \le t < \infty, v \in \mathbb{R}^q, u(t, v) \in \mathbb{R}^p$, such that for any square-integrable function $v(t), 0 \le t < \infty, v(t) \in \mathbb{R}^q$, satisfying the inequality $|v(\cdot)| \le \sigma$, the function $u(t) = u(t, v(t)), 0 \le t < \infty$, is also square-integrable, satisfies the inequality $|u(\cdot)| \le \rho$, and the solution $z(t), 0 \le t < \infty$, of the Cauchy problem

$$\dot{z}(t) = Az(t) + \sum_{i=1}^{m} B_i \dot{z}(t - h_i) + \sum_{i=1}^{m} C_i z(t - h_i) - Du(t) + Fv(t), \quad t \ge 0,$$

with initial condition (4), reaches the terminal set M, i.e., $z(t) \in M$ for some $t = t^* \in [0, T(\varphi(\cdot))]$.

It is assumed that the pursuer knows, at each moment of time $t \geq 0$, the solution $z(s)$ on the interval $-h_m \leq s \leq t$ and the control $v(s), 0 \leq s \leq t$.

The number $T(\varphi(\cdot))$ is called the pursuit time from the point $\varphi(\cdot)$, and the function $u(t, v), 0 \leq t \leq T(\varphi(\cdot)), v \in \mathbb{R}^q$, is called the pursuit strategy.

Let π denote the matrix of the orthogonal projection operator from $\mathbb{R}^n$ onto L [2]. Let $\tau > 0$ be an arbitrary number and $t \in [0, \tau]$.

Assumption 1.

1. For all $t \in [0, \tau]$, the inclusion holds

$$\pi K(t) D\mathbb{R}^p \supset \pi K(t) F\mathbb{R}^q.$$

2. If condition 1 of Assumption 1 is satisfied, then there exist a number $T > 0$ and a matrix function $G(t) : \mathbb{R}^q \to \mathbb{R}^p, 0 \leq t \leq T$, with square-integrable elements such that:

a) for every $t \in [0, T]$ and $v \in \mathbb{R}^q$, the equality holds

$$\pi K(t) F = \pi K(t) D G(t);$$

b) the inequality $\rho > \chi$ is satisfied, where

$$\chi = \sup\Big\{ |(Gv)(\cdot)| : |(Gv)(\cdot)| = \sqrt{\int_0^\tau |G(t)v(t)|^2 dt}, |v(\cdot)| \leq \sigma \Big\}.$$

Assume that condition 2 of Assumption 1 holds. Introduce the set $\Omega(\tau)$ consisting of vectors of the form

$$\int_0^\tau \pi K(t) D w(t) dt,$$

where $w(t)$, $0 \leq t \leq T$, is an arbitrary square-integrable function satisfying

$$|w(\cdot)|^2 = \int_0^T |w(t)|^2 dt \leq (\rho - \chi)^2. \tag{7}$$

Further, denote by $W_1(\tau)$ the set

$$W_1(\tau) = M_1 + \Omega(\tau). \tag{8}$$

Introduce the guaranteed capture time of the evader by the pursuer according to the first direct method of Lev Semyonovich Pontryagin for the conflict-controlled process (1)-(3):

$$H(\varphi(\cdot)) = \inf\{t \geq 0 : \Phi(t)\varphi(\cdot) \in W_1(\tau)\}, \tag{9}$$

where

$$\Phi(t)\varphi(\cdot) = \Big\{\pi K(t) - \sum_{i=1}^m \pi K(\tau - h_i) B_i\Big\}\varphi(0) - \sum_{i=0}^m \int_{-h_i}^0 \pi K(\tau - t - h_i)\Big[B_i \dot{\varphi}(t) + C_i \varphi(t)\Big] dt.$$

Assumption 2. There exists a number $\tau = \tau_1(\varphi(\cdot)) \in (0, T]$ such that for the initial position $\varphi(\cdot)$ the inclusion holds

$$\Phi(\tau_1)\varphi(\cdot) \in W_1(\tau_1).$$

Theorem. If Assumptions 1 and 2 stated above are satisfied, then in game (1) under constraints (2), (3), the pursuit can be completed from the initial position $\varphi(\cdot)$ within some finite time $T(\varphi(\cdot)) = \tau_1$.

Proof. By the conditions of the theorem, for $\tau = \tau_1(\varphi(\cdot))$, the inclusion (see (7), (8)) holds:

$$\Phi(\tau_1)\varphi(\cdot) \in W_1(\tau_1) = M_1 + \Omega(\tau_1). \tag{10}$$

Then, by this inclusion and the definition of the algebraic sum of sets and the integral of a multivalued mapping, there exist $m_1 \in M_1$ and $f \in \Omega(\tau_1)$ such that

$$\Phi(\tau_1)\varphi(\cdot) = m_1 + f. \tag{11}$$

According to (7), there exists a square-integrable function $\widetilde{w}(t), 0 \le t \le \tau_1$, for which

$$\int_0^{\tau_1} \pi K(t) D\widetilde{w}(t)dt = f, \quad |\widetilde{w}(\cdot)|_{L_2} \le \rho - \chi. \tag{12}$$

Then, from (11), we obtain

$$\Phi(\tau_1)\varphi(\cdot) = m_1 + \int_0^{\tau_1} \pi K(t) D\widetilde{w}(t)dt. \tag{13}$$

The control of the pursuer $u(t)$, $0 \le t \le \tau_1$, is constructed as follows:

$$u(t) = u(t,v) = \begin{cases} G(\tau_1 - t)v + \widetilde{w}(\tau_1 - t), & 0 \le t \le \tau_1, \\ 0, & \tau_1 < t < +\infty, \end{cases} \tag{14}$$

where $v \in \mathbb{R}^n$.

Now we show that if $v(t), 0 \le t < \infty$, is an arbitrary square-integrable function with $v(t) \in \mathbb{R}^q$ and $|v(\cdot)| \le \sigma$, then the function $u(t) = u(t, v(t)), 0 \le t < \infty$, is also square-integrable, $u(t) \in \mathbb{R}^p$, satisfies $|u(\cdot)| \le \rho$, and the solution $z(t)$ of equation (1) with initial condition (4) reaches the terminal set M at time $t = T(\varphi(\cdot)) = \tau_1$.

Indeed, the square integrability of $u(t)$ follows directly from its explicit form (14)[8]. Further, by the Cauchy-Schwarz inequality (see (7), (13)):

$$\left(\int_0^{\tau_1} |u(t,v(t))|^2 dt\right)^{1/2} \le \left(\int_0^{\tau_1} |G(\tau_1 - t)v(t)|^2 dt\right)^{1/2}$$

$$+\left(\int_0^{\tau_1} |\widetilde{w}(\tau_1 - t)|^2 dt\right)^{1/2} \le \chi + (\rho - \chi) = \rho, \tag{15}$$

hence $|u(\cdot)| \le \rho$.

Let the evader choose any square-integrable control $v(t)$, $0 \le t \le \tau_1$, such that

$$\int_0^{\tau_1} |v(t)|^2 dt \le \sigma^2.$$

Then $u(t) = u(t, v(t))$ is also square-integrable.

For the trajectory $z(t)$, $0 \le t \le \tau_1$, of equation

$$\dot{z}(t) = Az(t) + \sum_{i=1}^{m} B_i \dot{z}(t - h_i) + \sum_{i=1}^{m} C_i z(t - h_i) - Du(t, v(t)) + Fv(t), \tag{16}$$

with initial condition (4), we obtain:

$$\pi z(\tau_1) = \Phi(\tau_1)\varphi(\cdot) \int_0^{\tau_1} \pi K(\tau_1 - t)\big[Du(t, v(t)) - Fv(t)\big]dt$$

$$= \Phi(\tau_1)\varphi(\cdot) \int_0^{\tau_1} \pi K(\tau_1 - t)\big[D(G(\tau_1 - t)v(t) + \widetilde{w}(\tau_1 - t)) - Fv(t)\big]dt$$

$$= \Phi(\tau_1)\varphi(\cdot) \int_0^{\tau_1} \pi K(\tau_1 - t) D\widetilde{w}(\tau_1 - t)dt. \tag{17}$$

Thus, from (17), we obtain

$$\pi z(\tau_1) = \Phi(\tau_1)\varphi(\cdot)\int_0^{\tau_1} \pi K(\tau_1 - t)D\widetilde{w}(\tau_1 - t)dt = m_1 \in M_1.$$

Therefore, for the initial position $\varphi(\cdot)$, we have

$$\pi z(\tau_1) \in M_1,$$

which is equivalent to $z(\tau_1) \in M$.

The theorem is proved.

REFERENCES

1. Isaacs, R. Differential Games. *NY: John Wiley and Sons,* (1965), 385 p.
2. Pontryagin, L.S. Selected Scientific Works. *Moscow: MAKS Press,* (1988), 551 p. (in Russian)
3. Azamov, A.A.; Samatov, B.T. On a Modified Third Method in Pursuit Problems*. In: *Nonclassical Problems of Mathematical Physics. *FAN (1985), 1985, Tashkent* 174–184.
4. Mamadaliyev, N. A. A Pursuit Problem for Linear Games with Integral Constraints on the Players' Controls. *Izvestiya VUZ. Mathematics* (2020), no. 3, 12–28. (in Russian)
5. Mamadaliyev, N. A. On a Pursuit Problem with Integral Constraints on the Players' Controls. *Siberian Mathematical Journal* **56** (2015), no. 1, 129–148.
6. Chikrii, A.; Bezmagorychny V. The Method of Resolving Functions in Linear Differential Games with Integral Constraints. *Automation and Remote Control* (1993), no. 4, 26–36.
7. Chikrii, A. Conflict-Controlled Processes. *Kyiv: Naukova Dumka* (1992).
8. Satimov, N. Methods for Solving Pursuit Problems in Differential Game Theory. *Tashkent: National Library of Uzbekistan named after Alisher Navoi* (2019).
9. Satimov, N. On a Pursuit Problem in Quasilinear Differential Games of Many Persons. *In: Nonclassical Problems of Mathematical Physics. Tashkent: FAN* (1985), 154–174.
10. Nikolsky, M. Linear Differential Pursuit Games with Delay. *Differential Equations* **8** (1972), no. 2, 260–267.
11. Bellman, R.; Cooke, K. Differential-Difference Equations. *Moscow: Mir.* (1967), 254 p.
12. Varga, J. Optimal Control of Differential and Functional Equations. *Moscow: Nauka.* (1977).

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The Dirichlet problem for the wave equation in a spherical domain

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Abstract. In this paper, an ill-posed Dirichlet boundary value problem for a second-order hyperbolic equation in a spherical domain is studied. The solution is constructed using the method of separation of variables. The main focus is on determining the conditions that ensure the existence, uniqueness, and stability of the solution. In the proofs, a crucial role is played by the analysis of small denominators, based on Liouville's theorem and K. Roth's results on Diophantine approximations of algebraic numbers.

MSC (2020): 17A30; 17B30; 17B40; 17B56.

Key words: Dirichlet problem; hyperbolic equation; small denominators; algebraic numbers; Fourier method; ill-posed boundary value problems.

Introduction

In recent years, there has been a significant increase in interest in the study of ill-posed boundary value problems for equations of mathematical physics, in particular the Dirichlet boundary value problem for the wave equation in a spherical domain. As noted in the present work, the representation of solutions gives rise to the problem of small denominators, which leads to convergence difficulties of the associated series.

Main part

We consider the problem

$$u_{tt}(r,t) = a^2\Delta u(r,t), \quad 0 < r < R,\ 0 < t < T. \tag{1.1}$$

$$u(r,0) = \varphi(r), \quad u_t(r,0) = 0, \quad 0 \le r \le R. \tag{1.2}$$

$$u(R,t) = 0, \quad 0 \le t \le T. \tag{1.3}$$

The solvability of the problem is closely related to the arithmetic properties of the parameter aT/R, which arises in the analysis of the convergence of the series appearing in the solution.

We seek the solution in the form

$$u(r,t) = V(r)H(t).$$

Substituting this representation into equation (1.1) and separating the variables, we obtain

$$\frac{H''(t)}{a^2H(t)} = \frac{\Delta V(r)}{V(r)}.$$

For the function V, we arrive at the following equations:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \lambda^2 V = 0,$$

$$V'' + \frac{2}{r}V' + \lambda^2 V = 0.$$

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Introducing the substitution

$$y(r) = rV(r),$$

we obtain

$$y'' + \lambda^2 y = 0,$$

whose general solution is

$$y(r) = C_1 \cos(\lambda r) + C_2 \sin(\lambda r),$$

or equivalently,

$$V(r) = \frac{C_1 \cos(\lambda r) + C_2 \sin(\lambda r)}{r}.$$

The regularity condition at the origin ($V(0) < \infty$) implies that $C_1 = 0$. The boundary condition (1.3) gives

$$\frac{C_2 \sin(\lambda R)}{R} = 0,$$

which implies

$$\lambda R = \pi k, \quad k = 1, 2, \ldots$$

hence

$$\lambda_k = \frac{\pi k}{R}.$$

Therefore, the eigenfunctions are given by

$$R_k(r) = \frac{1}{r} \sin\left(\frac{\pi k}{R} r\right), \quad k = 1, 2, \ldots.$$

For the H, we have

$$H_k'' + a^2 \lambda_k^2 H_k = 0,$$

$$H_k(t) = A_k \cos(\lambda_k a t) + B_k \sin(\lambda_k a t).$$

Thus, the solution can be represented as

$$u(r,t) = \sum_{k=1}^{\infty} \left(A_k \cos(\lambda_k a t) + B_k \sin(\lambda_k a t)\right) \frac{1}{r} \sin\left(\frac{\pi k}{R} r\right).$$

Applying the initial conditions (1.2) yields

$$\sum_{k=1}^{\infty} A_k \frac{1}{r} \sin\left(\frac{\pi k}{R} r\right) = \varphi(r),$$

hence

$$A_k = \frac{2}{R} \int_0^R r\varphi(r) \sin\left(\frac{\pi k}{R} r\right) dr = \varphi_k, \quad k = 1, 2, \ldots.$$

From the (1.2) condition, we obtain

$$\sum_{k=1}^{\infty} \left(A_k \cos(\lambda_k a T) + B_k \sin(\lambda_k a T)\right) \frac{1}{r} \sin\left(\frac{\pi k}{R} r\right) = 0,$$

hence

$$B_k = -\frac{A_k \cos(\lambda_k a T)}{\sin(\lambda_k a T)} = -\frac{\varphi_k \cos(\lambda_k a T)}{\sin(\lambda_k a T)}.$$

Consequently, the formal solution of problem (1.1)вЂ“(1.3) is

$$u(r,t) = \sum_{k=1}^{\infty} \left(\varphi_k \cos(\lambda_k a t) - \frac{\varphi_k \cos(\lambda_k a T)}{\sin(\lambda_k a T)} \sin(\lambda_k a t)\right) \frac{1}{r} \sin\left(\frac{\pi k}{R} r\right),$$

or equivalently,

$$u(r,t)=\sum_{k=1}^{\infty}\frac{\varphi_k\sin(\lambda_k a(T-t))}{\sin(\lambda_k aT)}\frac{1}{r}\sin\left(\frac{\pi k}{R}r\right). \tag{1.4}$$

It remains to prove that the series (1.4), as well as the derivatives of this solution involved in equation (1.1), converge uniformly. In particular, we consider

$$u_{tt}(r,t)=-\left(\frac{\pi a}{R}\right)^2\sum_{k=1}^{\infty}\frac{k^2\varphi_k\sin(\lambda_k a(T-t))}{\sin(\lambda_k aT)}\frac{1}{r}\sin\left(\frac{\pi k}{R}r\right). \tag{1.5}$$

It suffices to establish the convergence of the series (1.5). A corresponding majorant series is given by

$$|u_{tt}(r,t)|\leq C\sum_{k=1}^{\infty}\frac{k^3|\varphi_k|}{|\sin(\lambda_k aT)|}. \tag{1.6}$$

We proceed by estimating the denominator of series (1.6). Note that

$$\sin\left(\frac{\pi k}{R}aT\right)=0,$$

$$\frac{\pi k}{R}aT=\pi m.$$

Introduce the notation

$$\delta_{mk}=\frac{\pi k}{R}aT-\pi m,$$

or

$$\frac{\pi k}{R}aT=\delta_{mk}+\pi m. \tag{1.7}$$

Without loss of generality, one may assume that $|\delta_{mk}|<\frac{\pi}{2}$. From (1.7) we obtain

$$\frac{aT}{R}=\frac{\delta_{mk}}{\pi k}+\frac{m}{k}. \tag{1.8}$$

As noted by D. Borzhin and R. Duffin [1] (see also [2]), when this parameter is an algebraic number, it is possible to obtain estimates that prevent the denominator from turning to zero.

In [3]-[4], Sh. A. Alimov studied the Dirichlet problem for the wave equation in a rectangular domain and demonstrated that the existence and uniqueness of the solution depend on the arithmetic properties of the parameter $\theta=T/a$. We shall similarly examine the conditions for the existence and uniqueness of the solution in a spherical domain.

Lemma. *Let the number aT/R be an algebraic number of degree 2. Then the following estimate holds*

$$|\delta_{mk}|\geq\frac{C}{k}. \tag{1.9}$$

Proof. By assumption, aT/R – is an algebraic number of degree 2. According to Liouville's theorem [6], there exists a constant $C>0$ such that for any natural numbers m and k, the inequality

$$\left|\frac{aT}{R}-\frac{m}{k}\right|\geq\frac{C}{k^2}.$$

holds. From this, we immediately obtain

$$|\delta_{mk}|=\pi k\left|\frac{aT}{R}-\frac{m}{k}\right|\geq\frac{C}{k}.$$

The lemma is thus proved.

Next, we consider the convergence of series (1.6). Taking into account (1.8), we have

$$\sin\left(\frac{\pi k}{R}aT\right) = \sin(\delta_{mk} + \pi m) = (-1)^m \sin(\delta_{mk}).$$

Since $|\delta_{mk}| < \pi/2$ and using estimate (1.9), it follows that

$$\left|\sin\left(\frac{\pi k}{R}aT\right)\right| = |\sin(\delta_{mk})| \geq \frac{2C}{\pi k}.$$

Therefore, estimate (1.6) can be rewritten as:

$$|u_{tt}(r,t)| \leq C\sum_{k=1}^{\infty}\frac{k^3|\varphi_k|}{|\sin(\lambda_k aT)|} \leq C_1\sum_{k=1}^{\infty}k^4|\varphi_k|. \tag{1.10}$$

Theorem 1. *Let aT/R be an algebraic number of degree 2. Then, for any function $\varphi \in W_2^4(|r| < R)$, the problem (1.1)-(1.3) possesses a unique solution.*

Proof. As discussed above, it remains to prove the convergence of series (1.5). Using estimate (1.10), we obtain

$$|u_{tt}(r,t)|^2 \leq C_1^2\sum_{k=1}^{\infty}k^8|\varphi_k|^2 \leq C_1^2\sum_{k=1}^{\infty}(1+k^2)^4|\varphi_k|^2.$$

The last series converges due to the assumptions on the function φ. The theorem is proved.

We remark that the solvability of the problem (1.1)-(1.3) also holds when aT/R is an arbitrary algebraic number, not necessarily of degree 2.

Theorem 2. *Let aT/R be an algebraic number of degree greater than 2. Then, for any function $\varphi \in W_2^{5+\varepsilon}(|r| < R)$, $\varepsilon > 0$, problem (1.1)-(1.3) possesses a unique solution.*

Proof. The proof is analogous to that of Theorem 1. Instead of estimate (1.9), one uses the bound obtained by K. Roth [5]:

$$\left|\frac{aT}{R} - \frac{m}{k}\right| \geq \frac{C(\varepsilon)}{k^{2+\varepsilon}}, \quad \varepsilon > 0.$$

which yields the following estimate for δ_{mk}:

$$|\delta_{mk}| = \pi k\left|\frac{aT}{R} - \frac{m}{k}\right| \geq \frac{C}{k^{1+\varepsilon}}, \quad \varepsilon > 0.$$

Consequently, estimate (1.10) takes the form

$$|u_{tt}(r,t)| \leq C\sum_{k=1}^{\infty}\frac{k^5|\varphi_k|}{|\sin(\lambda_k aT)|} \leq C_1\sum_{k=1}^{\infty}k^{6+\varepsilon}|\varphi_k|.$$

The uniform convergence of the latter series follows from the assumptions of Theorem 2.

Remark.

The result of Theorem 2 is valid not only for algebraic numbers aT/R, but, according to A. Ya. Khinchin [7], for almost all real values of aT/R in the sense of Lebesgue measure.

REFERENCES

1. David Gordon Bourgin, Richard James Duffin, The Dirichlet problem for the vibrating string equation. Bull. Amer. Math. Soc., 1939, 45(12), 851-858.

2. Bohdan Iosypovych Ptashnik, Ill-posed boundary value problems for partial differential equations. Naukova Dumka, 1984.

3. Shavkat Arifdjanovich Alimov, On Lp-solutions of one boundary value problem. Uz. Math. J., 1999, №1, 3-9.

4. Shavkat Arifdjanovich Alimov, On the solvability of an ill-posed problem. Uz. Math. J., 1999, №3, 19-29.

5. Klaus Friedrich Roth, Rational approximation to algebraic numbers, Mathematika, 1955, v.2, pp.1-20

6. Andrei Borisovich Shidlovsky, Transcendental Numbers. Moscow: Nauka, 1987.

7. Alexander Khinchin, Zur metrischen Theorie der diophantischen Approximationen, Math.Zeit. Volume 24, pages 706-714, (1926).

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THE PATH OF A LIFE DEDICATED TO SCIENCE

Science can manifest itself in various ways in human life: for some it is a profession, for others a social status or sphere of personal interest. However, there are individuals for whom science is the very essence of life, a state as natural as breathing, a destiny transformed into the true meaning of existence. For such people, scholarly activity is not confined by time, nor measured by position or material gain. They accept science not as a job, but as a way of life.

Doctor of Physical and Mathematical Sciences, Professor, Honored Public Education Worker of the Republic of Uzbekistan Gulmirza Khudayberganov was one such possessor of a remarkable scientific destiny. The 80-year life journey of this scholar is not merely a personal biography; it is the history of an entire scientific school, a spiritual stance, and an intellectual culture. When speaking of the Master, it is easy to list his research achievements: hundreds of scientific articles, monographs, textbooks, publications in international journals, dozens of students, worldwide recognition of his scientific ideas. But to fully comprehend him is far more difficult. For here it is not about an individual, but a culture of thinking; not about a single scholar, but a scientific school; not about an era, but an ongoing scientific process. His life path was not a chain of the research results, but a journey of spiritual and intellectual maturation. The land of Khorezm is a sacred ground with deep historical memory, irrigated by a spirit of respect for knowledge and enlightenment. Nourished by the legacy of great luminaries such as Abu Rayhan al-Biruni and Muhammad al-Khwarizmi, this place has for centuries been a school of thought. Gulmirza Khudayberganov, born on this very land on February 1, 1946, matured from childhood in an environment of labor, discipline, and responsibility. Rural life taught him patience and resilience, while nature taught him order, harmony, and lawfulness. These virtues later became clearly evident in his scholarly methodology as well. At the M.V. Lomonosov Secondary School in Mangit city, he accepted mathematics not as a mere subject, but as a unique instrument for shaping thought, for cultivating human intellectual culture. The striving not just to solve a problem, but to understand its internal logic and causal relationships, later shaped the profound culture of seeking root causes characteristic of his entire scientific career. In 1963, young Gulmirza Khudayberganov entered the Faculty of Mechanics and Mathematics at Tashkent State University. This step was not just an ordinary stage of student life, but a conscious choice directed towards a future as a scholar. The university environment instilled in him not only fundamental mathematical knowledge but also the benchmarks inherent to a scientist: responsibility, discipline of thought, and scholarly integrity and principledness.

His studies from 1966 to 1969 at Moscow State University named after M.V. Lomonosov further solidified his scientific destiny. The high traditions of the world mathematical school, the demanding atmosphere, and the scholarly competition fully revealed the young researcherвЃЄs talent. His diploma work, completed under the supervision of Professor B.V. Shabat, clearly demonstrated his ability for independent thought and became a solid foundation for his future scientific inquiries.

From 1969 onwards, Gulmirza Khudayberganov's life became inextricably linked with Tashkent State University (now the National University of Uzbekistan). This institution became for him not just a workplace, but a school of loyalty to science, of mentor-student traditions, and of academic integrity. For over half a century, he transformed imparting knowledge to students, conducting research, and serving the advancement of science into the very essence of his life.

The path from research intern to professor was not just a chain of positions, but a journey of inner maturation, of scientific and spiritual maturity. His early completion of his dissertation claiming for the academic degree of the candidate of sciences at the Siberian Branch of the Russian Academy of Sciences confirmed his scholarly caliber on an international scale as well. The defense of his doctoral dissertation in 1992 firmly established him as the head of an independent scientific school.

He also successfully served as head of a department and dean of a faculty. During his leadership, he placed great emphasis on preserving knowledge, safeguarding a healthy academic environment, and ensuring continuity between generations.

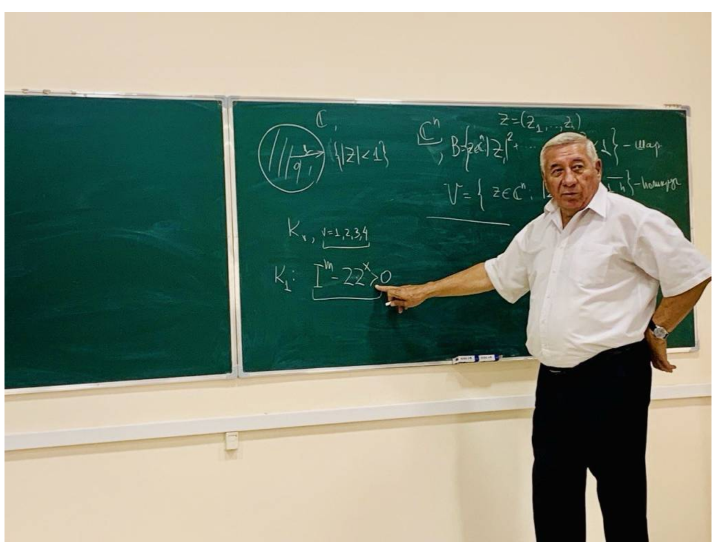
Professor G. Khudayberganov's research results opened new horizons in the field of multidimensional complex analysis. The results obtained in the theory of polynomial convexity and the analogues of Runge theorem are examples of a profound methodological approach that lies behind mathematical formulas.

The famous example constructed in 1984 in co-authorship with A.M. Kytmanov concerning three non-intersecting ellipsoids sparked new scientific discussions in world mathematics. This problem, which later became known as "Khudayberganov's shadow problem", retains its relevance to this day. This clearly demonstrates the timelessness of the scholar's scientific thought. Starting from the 1980s, he pursued systematic research in another complex and promising direction - the theory of holomorphic functions of matrix argument. Analogues of the Carleman formula, matrix forms of the Bergman and Cauchy-Szego kernels became a solid scientific foundation for contemporary researchers.

From 2001 to 2005, as rector of Karshi State University, he elevated the quality of the educational institution to a new level. For him, leadership was not merely administrative management; it was also about protecting human capital and the interests of science and enlightenment.

Most importantly, Professor Gulmirza Khudayberganov continued to manifest himself as an active scholar up to the end of his activity. He participated in seminars, worked on articles, reviewed dissertations, and advised young researchers. Under his scientific supervision, nearly 20 of his students have defended the dissertations claiming for the degree of the Doctor of Science (DSc) and Doctor of Philosophy (PhD).

In 1995, he was awarded the title "Honored Public Education Worker of the Republic of Uzbekistan", in 2001 the medal "10th Anniversary of Uzbekistan's Independence", and in 2021 the order "Sog'lom avlod uchun" ("For a Healthy Generation". However, for the Master, the greatest reward is his students – the talented people those who can think independently, with integrity and a sense of responsibility.



The phenomenon of Gulmirza Khudayberganov is not a memory of the past. It is a scientific destiny that lives today and will continue into tomorrow.

We believe that the Master's rich scientific experience, high spiritual virtues, and dedication to mentoring will remain a source of inspiration for future generations. A life devoted to science, a life path traversed with integrity and responsibility, is worthy of respect and esteem.

The memory of our Teacher will live forever in our hearts.

Bakhodir Shoimkulov
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Deputy Director for Research Activities and Innovations, Branch of Russian State University of Oil and Gas named after I.M. Gubkin in Tashkent, Tashkent, Uzbekistan