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# A NONLOCAL PROBLEM FOR MIXED-TYPE EQUATIONS OF BARENBLATT-ZHELTOV-KOCHINA AND BOUSSINESQ TYPE

ABDIMURODOVA SHOXSANAM

KARSHI STATE UNIVERSITY, KARSHI, UZBEKISTAN

shoxsanamabdimurodova@gmail.com

**ABSTRACT.** In this paper, the existence and uniqueness of a solution to a nonlocal mixed problem for abstract operator equations of Barenblatt-Zhel'tov-Kochina and Boussinesq type are investigated. The spectral method based on the method of separation of variables is applied to solve the problem. Conditions for the existence and uniqueness of the solution are established.

**MSC (2020):** 17A30; 17B30; 17B40; 17B56.

**Key words:** mixed-type equation, Fourier method, nonlocal condition, uniqueness condition.

## Introduction

Many scientific and applied studies conducted worldwide demonstrate the importance of investigating boundary value problems for partial differential equations. In recent years, considerable attention has been paid in our country to mathematics, physics, geology, and biology, as well as to fundamental scientific fields with significant potential for both theoretical research and practical applications. The main aim of this paper is to solve a nonlocal problem for mixed-type equations with an arbitrary elliptic differential operator. Nowadays, boundary value problems for mixed-type equations are considered one of the actively developing areas in the theory of partial differential equations. A large number of scientific studies and monographs have been published in this field (for example, see ([1]-[8])). Classical works on mixed-type equations can be found in the studies of A.V. Bitsadze [9], M.M. Smirnov [10], and V.N. Vragov [11].

Let  $H$  be a separable Hilbert space endowed with the scalar product  $(\cdot, \cdot)$  and the norm  $\|\cdot\|$ . Let  $A : D(A) \rightarrow H$  be an arbitrary positive, self-adjoint operator defined in  $H$  and possessing a compact inverse  $A^{-1}$ , where  $D(A) \subset H$  denotes the domain of definition of the operator  $A$ . Assume that the operator  $A$  has a complete orthonormal system of eigenfunctions  $\{v_k\}$  in  $H$  and a countable set of positive eigenvalues  $\{\lambda_k\}$ . By reordering the eigenvalues, they can be arranged in non-decreasing order, i.e.,  $0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow +\infty$ .

Let  $C([a, b]; H)$  denote the space of continuous functions  $u(t)$  defined on the interval  $t \in [a, b]$  with values in  $H$ .

## Statement of the problem

In this paper, we consider the following mixed problem:  
Let us consider the problem of finding a solution to equation

$$0 = \begin{cases} u_t + Au_t + Au + \nu^2 u, & 0 < t < \beta, \\ u_{tt} + Au_{tt} + Au + \nu^2 u, & -\alpha < t < 0, \end{cases} \quad (1)$$

satisfying the two coupling conditions

$$u(0+0) = u(0-0), \quad u_t(0+0) = u_t(0-0) \quad (2)$$

and the integral condition

$$\int_{-\alpha}^0 u(t) dt = \psi, \quad (3)$$

here,  $\alpha > 0$ ,  $\beta > 0$ ,  $\nu > 0$  are given positive real numbers,  $\psi \in H$ .

**Definition 1.** If  $u(t) \in C([- \alpha, \beta]; H)$  possesses the properties  $Au(t) \in C((-\alpha, 0) \cup (0, \beta); H)$ ;  $u_t(t)$ ,  $Au_t(t) \in C((0, \beta); H)$ ;  $u_{tt}(t)$ ,  $Au_{tt}(t) \in C((-\alpha, 0); H)$  and satisfies all conditions of problem (1)-(3), then the element  $u(t)$  is called a classical solution of problem (1)-(3).

**Construction of the solution to problem (1)-(3).**

We seek the element  $u(t)$ , which is a solution of problem (1)-(3), in the following form:

$$u(t) = \sum_{k=1}^{\infty} T_k(t) v_k. \quad (4)$$

Substituting (4) into equation (1), we obtain the following equalities:

$$0 = \begin{cases} T_k'(t)(1 + \lambda_k) + (\lambda_k + \nu^2)T_k(t), & 0 < t < \beta, \\ T_k''(t)(1 + \lambda_k) + (\lambda_k + \nu^2)T_k(t), & -\alpha < t < 0, \end{cases}$$

here,  $\mu_k^2 = \frac{\lambda_k + \nu^2}{\lambda_k + 1}$ . The general solution of these equations has the following form:

$$T_k(t) = \begin{cases} c_k e^{-\mu_k^2 t}, & t > 0, \\ a_k \cos \mu_k t + b_k \sin \mu_k t, & t < 0, \end{cases} \quad (5)$$

here,  $a_k$ ,  $b_k$  and  $c_k$  are arbitrary constants.

To determine the unknown constants  $a_k$ ,  $b_k$  and  $c_k$ , using the conditions arising from (2)

$$T_k(0+0) = T_k(0-0), \quad T_k'(0+0) = T_k'(0-0)$$

we have  $a_k = c_k$ ,  $b_k = -\mu_k c_k$ . From this, we obtain a solution of problem (1)-(2) in the form:

$$u(t) = \begin{cases} \sum_{k=1}^{\infty} c_k e^{-\mu_k^2 t} v_k, & t > 0, \\ \sum_{k=1}^{\infty} (c_k \cos \mu_k t - \mu_k c_k \sin \mu_k t) v_k, & t < 0. \end{cases} \quad (6)$$

To determine the coefficients  $c_k$ , we use the integral condition (3) and obtain the relation

$$c_k \Delta_k(\alpha) = \psi_k, \quad (7)$$

here,

$$\Delta_k(\alpha) = (1 + \frac{1}{\mu_k} \sin \mu_k \alpha - \cos \mu_k \alpha). \quad (8)$$

If  $\Delta_k(\alpha) \neq 0$ , then the coefficient  $c_k$  is determined as follows:

$$c_k = \frac{\psi_k}{\Delta_k(\alpha)}. \quad (9)$$

Then, according to (6), we obtain the following formal solution:

$$u(t) = \begin{cases} \sum_{k=1}^{\infty} \frac{\psi_k}{\Delta_k(\alpha)} e^{-\mu_k^2 t} v_k, & t > 0, \\ \sum_{k=1}^{\infty} \frac{\psi_k}{\Delta_k(\alpha)} (\cos \mu_k t - \mu_k \sin \mu_k t) v_k, & t < 0. \end{cases} \quad (10)$$

Now let us consider the case  $\Delta_k(\alpha) = 0$ . Equality  $\Delta_k(\alpha) = 0$  is equivalent to equality  $\mu_k = \mu_k \cos \mu_k \alpha - \sin \mu_k \alpha$ .

From equality  $\mu_k = \mu_k \cos \mu_k \alpha - \sin \mu_k \alpha$ , we obtain equality  $\cos(\mu_k \alpha + \varphi_k) = \frac{\mu_k}{\sqrt{1 + \mu_k^2}}$ , where  $\varphi_k = \arccos \frac{\mu_k}{\sqrt{1 + \mu_k^2}}$ ,  $0 < \varphi_k < \frac{\pi}{2}$ . For this equation to have a solution, we obtain two families of solutions:

$$1) \mu_k \alpha = 2\pi n, \quad n \in N;$$

$$2) \mu_k \alpha = -2\varphi_k + 2\pi n, \quad n \in N.$$

Since  $\mu_k \rightarrow 1$  as  $k \rightarrow \infty$  and  $\Delta_k(\alpha) \rightarrow \Delta(\alpha) = 1 + \sin \alpha - \cos \alpha$ , in this paper we assume that, in order to ensure  $\Delta(\alpha) \neq 0$ , the conditions  $\alpha \neq 2\pi n$  va  $\alpha \neq -\frac{\pi}{2} + 2\pi n$  ( $n \in Z$ ) are satisfied. Let us introduce the following notation:

$$\delta = \max \left\{ 1, 1 + \frac{\nu^2 - 1}{1 + \lambda_1} \right\}.$$

Taking into account that  $\frac{\lambda_1}{1 + \lambda_1} < \mu_k < \delta$ , it follows that  $\alpha \mu_k \in \left( \frac{\alpha \lambda_1}{1 + \lambda_1}, \alpha \delta \right)$  therefore,  $\Delta_k(\alpha) = 0$  can occur only for finitely many values of  $n$ . Let us denote by  $K_0 = \{n_1, n_2, \dots, n_s\}$  the set of indices  $n$  for which  $\Delta_k(\alpha) = 0$ .

It follows from (7) that if  $k \in K_0$ , then for problem (1)-(3) to have a solution, the following orthogonality conditions must be satisfied:

$$(\psi, v_k) = 0, \quad k \in K_0, \quad K_0 = \{n_1, n_2, \dots, n_s\}. \quad (11)$$

In this case, the formal solution of problem (1)-(3) is represented in the following form:

$$u(t) = \begin{cases} \sum_{k \notin K_0} \frac{\psi_k}{\Delta_k(\alpha)} e^{-\mu_k^2 t} v_k + \sum_{k \in K_0} c_k e^{-\mu_k^2 t} v_k, & t > 0, \\ \sum_{k \notin K_0} \frac{\psi_k}{\Delta_k(\alpha)} (\cos \mu_k t - \mu_k \sin \mu_k t) v_k \\ + \sum_{k \in K_0} c_k (\cos \mu_k t - \mu_k \sin \mu_k t) v_k, & t < 0. \end{cases} \quad (12)$$

Since the coefficients  $c_k$  are arbitrary, this solution is not unique.

In the following, we assume that the orthogonality condition (11) is satisfied in the case  $\Delta_k(\alpha) = 0$ .

If  $\Delta_k(\alpha) \neq 0$ , then there exists a constant  $C_0$  such that the following estimate holds:

$$|\Delta_k(\alpha)| \geq C_0 > 0. \quad (13)$$

Now we need to show that the solutions of the form (10) and (12) satisfy all the conditions of Definition 1.

First, we show the uniform convergence of the series (10). For this purpose, we denote the partial sums of the series (10) as follows: for the case  $t > 0$

$$S_n^+(t) = \sum_{k=1}^n \left[ \frac{\psi_k}{\Delta_k(\alpha)} e^{-\mu_k^2 t} \right] v_k$$

and for the case  $t < 0$

$$S_n^-(t) = \sum_{k=1}^n \left[ \frac{\psi_k}{\Delta_k(\alpha)} (\cos \mu_k t - \mu_k \sin \mu_k t) \right] v_k.$$

First, we study the action of the operator  $A$  on the partial sums of the series (10). For this purpose, by the definition of the operator  $A$ , we obtain the following:

$$\|AS_n^+(t)\|^2 = \left\| \sum_{k=1}^n \frac{\psi_k}{\Delta_k(\alpha)} e^{-\mu_k^2 t} \lambda_k v_k \right\|^2$$

and

$$\|AS_n^-(t)\|^2 = \left\| \sum_{k=1}^n \frac{\psi_k}{\Delta_k(\alpha)} (\cos \mu_k t - \mu_k \sin \mu_k t) \lambda_k v_k \right\|^2.$$

By Parseval's equality and using (13), we obtain the following estimates:

$$\|AS_n^+(t)\|^2 = \sum_{k=1}^n \lambda_k^2 \left| \frac{\psi_k}{\Delta_k(\alpha)} e^{-\mu_k^2 t} \right|^2 \leq C \sum_{k=1}^n \lambda_k^2 |\psi_k|^2$$

and

$$\begin{aligned} \|AS_n^-(t)\|^2 &= \sum_{k=1}^n \lambda_k^2 \left| \frac{\psi_k}{\Delta_k(\alpha)} (\cos \mu_k t - \mu_k \sin \mu_k t) \right|^2 \leq \\ &\leq C \sum_{k=1}^n \lambda_k^2 |\psi_k|^2 (1 + \mu_k)^2 \leq C_1 \sum_{k=1}^n \lambda_k^2 |\psi_k|^2, \end{aligned}$$

here,  $C$  and  $C_1$  are positive constants.

From the above estimates, it follows that if  $\psi \in D(A)$ , then  $Au(t) \in C((-\alpha, 0) \cup (0, \beta); H)$ .

Similarly, it can be shown that if  $\psi \in H$ , then  $u(t) \in C((-\alpha, 0) \cup (0, \beta); H)$ .

Now we estimate  $Au_t(t)$ . For this purpose, it is sufficient to consider the case  $t > 0$ . By the definition of the operator  $A$ , we obtain the following equality:

$$\left\| A \frac{\partial S_n^+(t)}{\partial t} \right\|^2 = \left\| \sum_{k=1}^n \left[ -\mu_k^2 \frac{\psi_k}{\Delta_k(\alpha)} e^{-\mu_k^2 t} \right] \lambda_k v_k \right\|^2.$$

By Parseval's identity and using (13), we obtain the following:

$$\begin{aligned} \left\| A \frac{\partial S_n^+(t)}{\partial t} \right\|^2 &= \sum_{k=1}^n \lambda_k^2 \left| -\mu_k^2 \frac{\psi_k}{\Delta_k(\alpha)} e^{-\mu_k^2 t} \right|^2 \leq \\ &\leq C_2 \sum_{k=1}^n \lambda_k^2 |\mu_k^2 \psi_k|^2 \leq C_3 \sum_{k=1}^n \lambda_k^2 |\psi_k|^2, \end{aligned}$$

here,  $C_2$  and  $C_3$  are positive constants. Hence, if  $\psi \in D(A)$ , then  $Au_t(t) \in C((0, \beta); H)$ .

Similarly, it can be shown that if  $\psi \in H$ , then  $u_t(t) \in C((0, \beta); H)$ .

Now we estimate  $Au_{tt}(t)$  in the case  $t < 0$ . Applying the operator  $A$  to  $S_n^-(t)$ , we obtain the following:

$$\left\| A \frac{\partial^2 S_n^-(t)}{\partial t^2} \right\|^2 = \left\| \sum_{k=1}^n \mu_k^2 \frac{\psi_k}{\Delta_k(\alpha)} (-\cos \mu_k t + \mu_k \sin \mu_k t) \lambda_k v_k \right\|^2.$$

Applying Parseval's identity, we obtain the following:

$$\begin{aligned} \left\| A \frac{\partial^2 S_n^-(t)}{\partial t^2} \right\|^2 &= \sum_{k=1}^n \lambda_k^2 \left| \mu_k^2 \frac{\psi_k}{\Delta_k(\alpha)} (-\cos \mu_k t + \mu_k \sin \mu_k t) \right|^2 \leq \\ &\leq C_4 \sum_{k=1}^n \lambda_k^2 |\psi_k|^2 |\mu_k|^4 (1 + \mu_k^2) \leq C_5 \sum_{k=1}^n \lambda_k^2 |\psi_k|^2, \end{aligned}$$

here,  $C_4$  and  $C_5$  are positive constants. From this estimate, it follows that if  $\psi \in D(A)$ , then  $Au_{tt}(t) \in C((-\alpha, 0); H)$ .

Similarly, it can be shown that if  $\psi \in H$ , then  $u_{tt}(t) \in C((-\alpha, 0); H)$ .

The uniform convergence of series (12) can be shown in the same way as above. Unlike (10), there are a finite number of additional terms, which do not affect the convergence of the series in any way.

If the set  $K_0$  is empty, i.e.,  $\Delta_k(\alpha) \neq 0$  for all  $k$ , then the coefficients  $c_k$  are uniquely determined, and problem (1)-(3) has a unique solution.

If the set  $K_0$  is nonempty, then in order for a solution to exist, the orthogonality conditions (11) must be satisfied. In this case, the solution is not unique.

Now we show that the solution is unique when  $\Delta_k(\alpha) \neq 0$ .

**Uniqueness of the solution.** To prove uniqueness, we use the standard method, namely we assume the contrary. Assume that we have two solutions  $u_1(t)$  and  $u_2(t)$ . Let  $w(t) = u_1(t) - u_2(t)$ . Since the system is linear, for  $w(t)$  we obtain the following problem:

$$0 = \begin{cases} w_t + Aw_t + Aw + \nu^2 w, & 0 < t < \beta, \\ w_{tt} + Aw_{tt} + Aw + \nu^2 w, & -\alpha < t < 0 \end{cases} \quad (14)$$

with the coupling conditions

$$w(0+0) = w(0-0), w_t(0+0) = w_t(0-0) \quad (15)$$

and the nonlocal condition

$$\int_{-\alpha}^0 w(t) dt = 0. \quad (16)$$

Thus, we are reduced to the problem of finding a function  $w(t)$  satisfying these conditions.

The solution of this equation is obtained in the same way as above and, according to expression (6), is given by:

$$w(t) = \begin{cases} \sum_{k=1}^{\infty} c_k e^{-\mu_k^2 t} v_k, & t > 0, \\ \sum_{k=1}^{\infty} c_k (\cos \mu_k t - \mu_k \sin \mu_k t) v_k, & t < 0. \end{cases} \quad (17)$$

To determine the coefficient  $c_k$ , we use the nonlocal condition (16) and obtain  $\sum_{k=1}^{\infty} c_k \Delta_k(\alpha) v_k = 0$ . From this, we obtain the relations  $c_k \Delta_k(\alpha) = 0$  for all  $k \geq 1$ . By the condition  $\Delta_k(\alpha) \neq 0$ , we conclude that  $w(t) \equiv 0$ . From condition  $\Delta_k(\alpha) \neq 0$ , we obtain  $c_k = 0$ . Accordingly, the solution of the given problem is  $w(t) \equiv 0$ . Hence, we conclude that  $u_1(t) = u_2(t)$ .

Thus, we obtain the following statement on the existence and uniqueness of a solution to problem (1)-(3):

**Theorem 1.** Let  $\psi \in D(A)$ .

- 1) If  $\Delta_k(\alpha) \neq 0$ , then problem (1)-(3) has a unique solution, which is given by the series (10).
- 2) If  $\Delta_k(\alpha) = 0$  and the orthogonality conditions (11) are satisfied, then the solution of problem (1)-(3) is given by the series (12), where  $c_k$ ,  $k \in K_0$  are arbitrary constants.

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ON THE UNIQUENESS OF THE SOLUTION TO FRANKL'S PROBLEM WITH A NON-SYMMETRIC  
NONLOCAL CONDITION FOR THE GELLERSTEDT EQUATION WITH DIFFERENT ORDERS OF  
DEGENERACY.

AMONOV BOBUR

TERMEZ STATE UNIVERSITY, TERMEZ, UZBEKISTAN;  
amonovbobur91@mail.ru

**ABSTRACT.** In this paper, in a mixed domain  $D$ , using calculations in the half-plane  $x \geq 0$ , we demonstrate examples of the uniqueness of Frankl's solution with asymmetric nonlocal conditions defined on a line segment  $x = 0$  that is asymmetric with respect to the origin for the Gellerstedt equation with various orders of degeneracy. In the Frankl problem, the nonlocal condition is defined on a line segment  $x = 0$  that is symmetric (with respect to the origin). The nonstandard formulation of the nonlocal condition and the form of the equation generate a number of new approaches to solving problems. The uniqueness of the solution to the problem is proven using the method of auxiliary functions.

**MSC (2020):** 35M10; 35M12; 35A24.

**Key words:** mixed type equation, different order of degeneration, asymmetric Frankl condition on the segment of the axis  $x = 0$ , auxiliary function, uniqueness of the solution.

## Introduction

In 1956, F.I. Frankl [1], while studying the flow around profiles by a subsonic flow with a supersonic zone ending in a normal shock wave, arrived at a mathematical model [1], currently called the Frankl problem. The first results on the Frankl problem, considered for the Lavrentyev equation, were obtained by A.V. Bitsadze [2]. Under certain restrictions on the geometry of the  $\Gamma$ -elliptic boundary curve, the most significant results were obtained in the works of Yu. V. Devingtal [3], Lin Jian-bin [4], A. P. Soldatov [5], and a detailed bibliography on the Frankl problem is given in the monograph by K. B. Sabitov [6]. In all the above-mentioned works, a characteristic feature is that the Frankl condition is specified on a segment of the line  $x = 0$  that is symmetric with respect to the origin.

This work is devoted to proving the correctness of the problem with the asymmetric Frankl condition [7] on an asymmetric, relative to the origin, segment of the  $x = 0$  axis for an equation of mixed elliptic-hyperbolic type.

## Statement of the problem $F_K$

Let  $D$  be the domain bounded by the arc  $AB$ , the normal curve  $\sigma_0 : x^2 + 4(m+2)^{-2}y^{m+2} = a^2$ ,  $x \geq 0$ , lying in the first quadrant and having ends at the points  $A(0, y_2)$ ,  $B(a, 0)$ , where  $a > 0$ ,  $y_2 = (a(m+2)/2)^{2/(m+2)}$ ; segment  $A'A$  of the axis  $x = 0$ , where  $A' = A'(0, y_1)$ ,  $y_1 < 0$ ,  $-y_1 < y_2$ ; segment  $CB$  of the axis  $y = 0$ ,  $a_1 \leq x \leq a$ ,  $0 < a_1 < a$ ,  $C = C(a_1, 0)$  and characteristic  $CA'$  of the equation

$$(\text{sign } y)|y|^m u_{xx} + u_{yy} = 0, \quad (1)$$

where  $m = m_1$  for  $y > 0$ ,  $m = m_2$  for  $y < 0$ .

Let  $D_1 = D \cap \{y > 0\}$ ,  $OP$  be the part of the characteristic of equation (1) emanating from the point  $O = O(0, 0)$  and intersecting the segment  $A'C$  at the point  $P$ ;  $D_2$  be the region bounded by the curves  $OP$ ,  $PC$ , and  $OC$ ;  $D_3$  be the region bounded by the curves  $OA'$ ,  $A'P$ , and  $PO$ .

**Problem  $F_K$ .** Find a function  $u(x, y) \in C(\bar{D}) \cap C^1(D \cup OA \cup OA' \setminus (OP \cup OC)) \cap C^2(D_1 \cup D_2 \cup D_3)$ , where  $\lim_{y \rightarrow -0} u_y = \lim_{y \rightarrow +0} u_y$  on  $OC$ , satisfying equation (1) in the domain  $D_1 \cup D_2 \cup D_3$  and the conditions:

$$u(x, y) = \varphi_1(x, y), \quad (x, y) \in \sigma_0; \quad (2)$$

$$u(x, 0) = \varphi_2(x), \quad a_1 \leq x \leq a; \quad (3)$$

$$\frac{\partial u(x, y)}{\partial x} \Big|_{x=0} = 0, \quad y_1 < y < y_2; \quad (4)$$

$$u(0, y) - \mu u(0, -ky) = f(y), \quad 0 \leq y \leq y_2; \quad (5)$$

where  $\mu = \text{const} > 0$ ,  $k = -y_1/y_2 < 1$ ,  $\varphi_1, \varphi_2, f$  are given smooth functions, and  $\varphi_2(a_1) = 0$ ,  $\varphi_2(a) = \varphi_1(a, 0) = 0$ ,  $f(0) = 0$ . (5) is the asymmetric nonlocal Frankl condition.

Note that if in the conditions of the problem  $F_K$   $y_1 = -y_2$  ( $k = 1$ ),  $\mu = 1$ , then the Frankl problem [1] follows from the problem  $F_K$ . Problems with analogues of the Frankl condition are studied in the works [8]-[12].

### Uniqueness of the solution of the problem $F_K$

**Theorem 1.** Problem  $F_K$ , subject to the conditions  $\varphi_1(x, y) \equiv 0$ ,  $\varphi_2(x) \equiv 0$ ,  $f(y) = 0$ ,

$$k^{1+m_2} \leq \mu^2, \quad (6)$$

has only a trivial solution.

Proof of Theorem 1. Let  $u(x, y)$  be a solution to the homogeneous problem  $F_K$  (i.e.  $\varphi_1(x, y) \equiv 0$ ,  $\varphi_2(x) \equiv 0$ ,  $f(y) = 0$ ), from the class of regular in  $D$  solutions of equation (1). Denote by  $v(x, y)$  a function that, together with the solution  $u(x, y)$  of problem  $F_K$ , satisfies the system of first-order equations of mixed type

$$K(y)u_x - v_y = 0, \quad u_y + v_x = 0, \quad (x, y) \in D_1 \cup D_2 \cup D_3, \quad (7)$$

where  $K(y) = (\text{sign} y)|y|^m$ . Under the condition  $v(0, 0) = 0$ , the function  $v(x, y)$  is uniquely determined from system (7):

$$v(x, y) = \int_{(0,0)}^{(x,y)} [u_\eta d\xi - K(\eta)u_\xi d\eta]. \quad (8)$$

Since  $u(x, y)$  is a solution to the homogeneous problem  $F_K$  belonging to the class of regular solutions of equation (1) in the domain  $D$ , the function  $v(x, y)$  defined by formula (8) is continuous in the closed domain  $\bar{D}$ . Taking into account condition (4), it follows from formula (8) that

$$v(0, y) = 0, \quad y_1 \leq y \leq y_2. \quad (9)$$

On the set  $D_1 \cup D_2 \cup D_3$  we consider the curvilinear integral

$$F(x, y) = \int_{(0,0)}^{(x,y)} [2uv d\xi + (K(\eta)u^2 - v^2)d\eta]. \quad (10)$$

It is easy to verify that, by (7), the integrand in (10) is a total differential of some  $w(\xi, \eta)$ , i.e.  $F(x, y) = \int_{(0,0)}^{(x,y)} dw(\xi, \eta)$ . Consequently, the value of the integral (10) does not depend on the integration path inside the domain  $D$ . Then the integral (10) defines a function continuous in  $\bar{D}$ . Our goal is to prove the equality  $F(x, y) = 0$ ,  $\forall (x, y) \in \bar{D}$ .

From formula (10), taking into account condition (9), in the region  $D$  we have

$$F(0, y) = \int_0^y K(\eta)u^2(0, \eta)d\eta \geq 0, \quad (11)$$

From (11) it follows that the function  $F(x, y)$  attains its largest non-negative value in  $\bar{D}$ . Let  $\max_{(x,y) \in \bar{D}} F(x, y) = F(Q) > 0$ . We show that the point  $Q \in A \cup \sigma_0$ .

From (10) it is easy to calculate that

$$F_x = w_x(x, y) = 2uv, \quad F_y = w_y(x, y) = K(y)u^2 - v^2. \quad (12)$$

1. Let  $y > 0$ . Let's calculate:

$$F_{xx} = 2u_xv + 2uv_x, \quad (13)$$

$$F_{yy} = m_1y^{m_1-1}u^2 + 2y^{m_1}uu_y - 2vv_y. \quad (14)$$

By virtue of (12), (13) and (14) we have

$$y^{m_1}F_{xx} + F_{yy} = m_1y^{m_1-1}u^2. \quad (15)$$

There are equalities

$$K(y)F_x^2 + F_y^2 = (y^{m_1}u^2 + v^2), \quad (16)$$

$$\frac{F_y + \sqrt{K(y)F_x^2 + F_y^2}}{2K(y)} = u^2. \quad (17)$$

By virtue of (16) and (17), from equality (15) we obtain an elliptic equation

$$y^{m_1}F_{xx} + F_{yy} = \bar{\alpha}F_x + \bar{\beta}F_y, \quad (18)$$

Where

$$\bar{\alpha} = \left(\frac{m_1}{y}\right) \frac{y^{m_1}F_x}{\sqrt{y^{m_1}F_x^2 + F_y^2}}, \quad \bar{\beta} = \left(\frac{m_1}{2y}\right) \left(1 - \frac{F_y}{\sqrt{y^{m_1}F_x^2 + F_y^2}}\right).$$

The coefficients  $\bar{\alpha}$  and  $\bar{\beta}$  are bounded in any closed subdomain of  $D_1$ . For  $y \geq h > 0$ , where  $h$  is any sufficiently small number, equation (18) is uniformly elliptic. Consequently, the internal extremum principle holds for this equation in  $D_1$ . It follows that  $Q \notin D_1$ . Consequently, the function  $F(x, y)$  attains its greatest value on the boundary of  $D_1$ , and at the point where this greatest value is attained, the inequality  $\frac{\partial F}{\partial n} < 0$  holds, where  $n$  is the internal normal to the boundary of  $D_1$ .

On the segment  $CB$  of the boundary of the region  $D_1$  the function  $F(x, 0) = \text{const}$ , since by virtue of the boundary condition (3), by virtue of (12) we have  $F_x(x, 0) = 2u(x, 0)v(x, 0) = 0$ , and by virtue of  $F(0, 0) = 0$  we obtain that  $F(x, 0) = 0$ .

2. Let  $y < 0$ , on the set  $D_2 \cup D_3 \cup OP \cup OC \cup OA'$  the function  $F(x, y)$  does not increase in  $y$ , since

$$F_y(x, y) = -(-y)^{m_2}u^2 - v^2 \leq 0, \quad (19)$$

and, consequently, the function  $F(x, y)$  does not reach its greatest positive value at the interior points of the hyperbolic part of the mixed domain and on its boundaries  $OC \cup OA'$ , i.e., the function  $F(x, y)$  can reach this value only on the characteristic  $A'C$ .

For the characteristics  $A'C$ :  $x + \frac{2}{m_2+2}(-y)^{\frac{m_2+2}{2}} = a_1$  the equality holds,

$$\frac{dx}{ds} = (-y)^{\frac{m_2}{2}} \frac{dy}{ds}. \quad (20)$$

Here  $x = x(s)$ ,  $y = y(s)$  is the parametric equation of the characteristic  $A'C$  where  $s$  is the arc length (curvilinear abscissa [8, p. 565]) of  $A'C$  measured from the point  $A'(0, y_1)$  to the point  $M(x(s), y(s)) \in A'C$  and  $x'(s) > 0$ ,  $y'(s) > 0$ .

By virtue of equality (20), on the characteristic  $A'C$  we have

$$\frac{dF(x(s), y(s))}{ds} = \frac{\partial F}{\partial x} \frac{dx}{ds} + \frac{\partial F}{\partial y} \frac{dy}{ds} =$$

$$= \left( \frac{\partial F}{\partial x} (-y)^{\frac{m_2}{2}} + \frac{\partial F}{\partial y} \right) \frac{dy}{ds} = \left[ (-y)^{\frac{m_2}{2}} u - v \right]^2 \frac{dy}{ds}. \quad (21)$$

Taking into account the inequality  $\frac{dy}{ds} > 0$ , for the points of characteristic  $A'C$  from (21) we obtain

$$\frac{dF(x(s), y(s))}{ds} \leq 0, \quad (22)$$

i.e. the function  $F(x, y)$  on the characteristic  $A'C$  does not increase, and on  $OA'$ , taking into account (9):  $v(0, y) = 0$ , from (12) we have  $F_y(0, y) = -(-y)^{m_2} u^2 \leq 0$ , therefore the point of greatest positive value of the function  $F(x, y)$  on the characteristic  $A'C$  can only be at the point  $A'$ . Thus, by virtue of (19) and (22) it follows that the function  $F(x, y)$  can reach its positive maximum in the hyperbolic part of the domain of the mixed domain  $D$  only at the point  $A'(0, y_1)$ .

Taking into account condition (6) of Theorem 1, it is easy to show that  $F(A') \leq F(A)$ . Indeed, in (11), setting  $y = y_1$ , we have

$$F(A') = - \int_0^{y_1} |t|^{m_2} u^2(0, \eta) d\eta. \quad (23)$$

In (23), having made the change of variable integration  $t = k\eta$ , where  $k = -y_1/y_2$ , we have

$$F(A') = k^{1+m_2} \int_0^{y_2} \eta^{m_2} u^2(0, -k\eta) d\eta, \quad (24)$$

here, taking into account the homogeneous asymmetric Frankl condition:  $u(0, y) - \mu u(0, -ky) = 0$  from (24) we have

$$F(A') = k^{1+m_2} \int_0^{y_2} \frac{\eta^{m_2} u^2(0, \eta)}{\mu^2} d\eta = \frac{k^{1+m_2}}{\mu^2} F(A) \leq F(A). \quad (25)$$

It follows that the function  $F(x, y)$  reaches its greatest positive value in the region  $\overline{D}$  on the curve  $A \cup \sigma_0$ .

Let  $Q \in \sigma_0$  and it has coordinates  $(x_0, y_0) = (x_0(s), y_0(s))$ .

Due to the corresponding homogeneous condition (2) (with  $\varphi_1(x, y) \equiv 0$ ) on  $\sigma_0$   $u = 0$ , then for  $(x(s), y(s)) \in \sigma_0$  from (12) we obtain

$$\frac{dF(x(s), y(s))}{ds} = \frac{\partial F}{\partial x} \frac{dx}{ds} + \frac{\partial F}{\partial y} \frac{dy}{ds} = -v^2 \frac{dy}{ds} \leq 0, \quad (26)$$

since on the curve  $\sigma_0$ ,  $\frac{dy}{ds} \geq 0$ . Then it follows from (26) that in the neighborhood of the point  $A(0, y_2)$  the function  $F(x, y)$  on  $\sigma_0$  does not decrease with decreasing arc  $s$ , therefore the function  $F(x, y)$  must take its positive maximum at an interior point of the curve  $\sigma_0$ . Consequently, the function  $F(x, y)$  does not reach its positive maximum at the point  $A = A(0, y_2)$ , then by virtue of (25) there is no such maximum at the point  $A' = A'(0, y_1)$ .

Let the function  $F(x, y)$  reach its positive maximum at some interior point  $M_0$  of the curve  $\sigma_0$ , then at this point

$$\frac{dF}{ds} = -v^2 \frac{dy}{ds} = 0,$$

it follows that  $v = 0$ , since at the interior points of the normal curve  $\sigma_0$   $\frac{dy}{ds} > 0$ . Consequently, by virtue of (12): where  $u|_{\sigma_0} = 0$  at the point of positive maximum of the function  $F(x, y)$  we have

$$\frac{\partial F}{\partial n} = \left( \frac{\partial F}{\partial x} \cos(n, x) + \frac{\partial F}{\partial y} \cos(n, y) \right) = -v^2 \cos(n, y) = 0. \quad (27)$$

Equality (27) contradicts the Zaremba-Giraud inequality [9, p. 26]:

$$\frac{\partial F}{\partial n} < 0.$$

Therefore, there is no positive maximum of the function  $F(x, y)$  on  $\sigma_0$ .

Given the equalities  $F(x, 0) = 0$  on  $CB$ , we conclude that the function  $F(x, y)$  in the domain  $\overline{D}$  does not have a positive maximum. Similarly, it can also be shown that the function in the domain  $\overline{D}$  does not have a negative minimum.

Therefore,  $F(x, y)$  cannot be non-constant, i.e.,  $F(x, y) = \text{const}$ , but since  $F(0, 0) = 0$ , then  $F(x, y) \equiv 0$ ,  $\forall (x, y) \in \overline{D}$ .

Since the integrand is the total differential of some function  $w(x, y)$ , then (10) can be written as

$$F(x, y) = \int_{(0,0)}^{(x,y)} dw(\xi, \eta) = w(x, y) - w(0, 0) \equiv 0. \quad (29)$$

Thus, from the last identity, by virtue of (19) we have  $u(x, y) \equiv 0$ ,  $v(x, y) \equiv 0$  in  $\overline{D}$  and from this follows the uniqueness of the solution to problem  $F_k$ .

Theorem 1 is proven.

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# CAPACITY OF THE SET OF LIOUVILLE NUMBERS

ASHIROV ABDUVAHHOB\*

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M.ULUGBEK, TASHKENT, UZBEKISTAN  
ashurovabduvahob007@gmail.com

BAZARBAEV SARDOR

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M.ULUGBEK, TASHKENT, UZBEKISTAN  
uzedu.bazarbaev@gmail.com

**ABSTRACT.** In this paper, we prove that the set of Liouville numbers  $\mathbb{L}$  has zero  $C_\sigma$ -capacity for every  $\sigma > 0$ , where the capacity is defined with respect to the kernel

$$k_\sigma(z, \xi) = \frac{1}{|z - \xi|^\sigma}.$$

As a consequence, the Hausdorff dimension of the set of Liouville numbers is zero. More precisely,  $H^s(\mathbb{L}) = 0$  for every  $s > 0$ .

**Key words:** Liouville numbers, capacity, Hausdorff dimension.

## Introduction

An irrational number  $\alpha \in \mathbb{R}$  is called a *Liouville number* if, for every  $n \in \mathbb{N}$ , there exist infinitely many rational numbers  $p/q$ , with  $p, q \in \mathbb{Z}$  and  $q > 1$ , such that

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^n}.$$

We denote by  $\mathbb{L}$  the set of all Liouville numbers.

It is a classical fact that the Hausdorff dimension of the set of Liouville numbers is zero. In this paper, we give a short potential-theoretic proof of this statement: for every  $\sigma > 0$ , the set  $\mathbb{L}$  has zero  $C_\sigma$ -capacity with respect to the kernel  $|z - \xi|^{-\sigma}$ .

**Theorem 1.** For every  $\sigma > 0$ , the set of Liouville numbers  $\mathbb{L}$  has zero  $C_\sigma$ -capacity with respect to the kernel

$$k_\sigma(z, \xi) = \frac{1}{|z - \xi|^\sigma}.$$

Consequently, we have  $\dim_H \mathbb{L} = 0$ . More precisely, for every  $s > 0$ ,

$$H^s(\mathbb{L}) = 0.$$

## Preliminaries

### Hausdorff measure

Let  $E \subset \mathbb{R}^n$  be a bounded set, and let  $h : [0, r_0] \rightarrow [0, +\infty)$  be a strictly increasing continuous function, where  $r_0 > 0$  and  $h(0) = 0$ . For a fixed  $\varepsilon > 0$ , consider countable covers of  $E$  by open balls  $\{B_j(x_j, r_j)\}_{j \geq 1}$  such that  $r_j < \varepsilon$ . Define

$$H^h(E, \varepsilon) = \inf \left\{ \sum_{j=1}^{\infty} h(r_j) : E \subset \bigcup_{j=1}^{\infty} B_j(x_j, r_j), r_j < \varepsilon \right\}.$$

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\*Corresponding autor

Since  $H^h(E, \varepsilon)$  is monotone as  $\varepsilon \rightarrow 0+$ , the limit

$$H^h(E) = \lim_{\varepsilon \rightarrow 0+} H^h(E, \varepsilon)$$

exists. This limit is called the *Hausdorff  $H^h$ -measure* of  $E$ . If  $h_s(t) = t^s$  for some  $s > 0$ , then  $H^{h_s}$  is the usual  $s$ -dimensional Hausdorff measure and is denoted by  $H^s$ .

### $C_\sigma$ -capacity

Let  $K \subset \mathbb{C}$  be a compact set and let  $\sigma > 0$ . We consider the kernel

$$k_\sigma(z, \xi) = \frac{1}{|z - \xi|^\sigma}.$$

For a positive Borel probability measure  $\mu$  supported on  $K$ , its  $\sigma$ -energy is defined by

$$I_\sigma(\mu) = \iint_{K \times K} \frac{d\mu(z)d\mu(\xi)}{|z - \xi|^\sigma}.$$

Set

$$W_\sigma(K) = \inf_{\mu} I_\sigma(\mu),$$

where the infimum is taken over all positive Borel probability measures supported on  $K$ . The  $C_\sigma$ -capacity of  $K$  is defined by

$$C_\sigma(K) = \frac{1}{W_\sigma(K)},$$

with the convention  $1/\infty = 0$ .

We also define the inner and outer  $C_\sigma$ -capacities with compact subsets and open supersets of any Borel set  $E$

$$\underline{C}_\sigma(E) = \sup\{C_\sigma(K) : K \subset E, K \text{ compact}\},$$

and

$$\overline{C}_\sigma(E) = \inf\{C_\sigma(U) : E \subset U, U \text{ open}\}.$$

For Borel sets, these two capacities coincide; we denote the common value by  $C_\sigma(E)$ .

We shall use the following standard properties of  $C_\sigma$ -capacity.

1. If  $E$  is a Borel set, then

$$\underline{C}_\sigma(E) = \overline{C}_\sigma(E) = C_\sigma(E).$$

2. The capacity  $C_\sigma$  is countably subadditive on Borel sets. Namely, if  $E = \bigcup_{j=1}^{\infty} E_j$ , then

$$C_\sigma(E) \leq \sum_{j=1}^{\infty} C_\sigma(E_j).$$

3. If there exists a finite positive Borel measure  $\nu$  such that

$$U_\sigma^\nu(z) := \int \frac{d\nu(\xi)}{|z - \xi|^\sigma} = +\infty$$

for every  $z \in E$ , then

$$C_\sigma(E) = 0.$$

**Remark.** Throughout the paper, we regard subsets of  $\mathbb{R}$  as subsets of  $\mathbb{C}$ . The kernel  $|z - \xi|^{-\sigma}$  is invariant under translations. Hence the corresponding capacity is also invariant under translations.

### Main part

We first prove the result on the interval  $[0, 1]$ .

**Lemma 1.** Let  $\alpha \in \mathbb{L} \cap [0, 1]$ . Then, for every  $\varepsilon > 0$  and every  $\sigma > 0$ ,

$$\sum_{q=2}^{\infty} \sum_{p=1}^{q-1} \frac{1}{\left| \alpha - \frac{p}{q} \right|^{\sigma} q^{2+\varepsilon}} = +\infty.$$

**Proof.** Since  $\alpha \in \mathbb{L} \cap [0, 1]$  and  $\alpha$  is irrational, we have  $\alpha \in (0, 1)$ . Fix  $\varepsilon > 0$  and  $\sigma > 0$ .

Choose  $n \in \mathbb{N}$  so large that

$$n\sigma > 2 + \varepsilon$$

and

$$2^{-n} < \min\{\alpha, 1 - \alpha\}.$$

By the definition of a Liouville number, there exist infinitely many rational numbers  $p_k/q_k$ , with  $q_k > 1$ , such that

$$\left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_k^n}.$$

Since  $q_k \geq 2$ , we have

$$\frac{1}{q_k^n} \leq \frac{1}{2^n} < \min\{\alpha, 1 - \alpha\}.$$

Therefore each such rational number lies in the interval  $(0, 1)$ . Indeed, if  $p_k/q_k \leq 0$ , then

$$\left| \alpha - \frac{p_k}{q_k} \right| \geq \alpha,$$

which contradicts

$$\left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_k^n} < \alpha.$$

Similarly, if  $p_k/q_k \geq 1$ , then

$$\left| \alpha - \frac{p_k}{q_k} \right| \geq 1 - \alpha,$$

which contradicts

$$\left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_k^n} < 1 - \alpha.$$

Thus  $0 < p_k/q_k < 1$ .

Since all terms in the double series are positive and the rational numbers  $p_k/q_k$  belong to  $(0, 1)$ , we obtain

$$\sum_{q=2}^{\infty} \sum_{p=1}^{q-1} \frac{1}{\left| \alpha - \frac{p}{q} \right|^{\sigma} q^{2+\varepsilon}} \geq \sum_{k=1}^{\infty} \frac{1}{\left| \alpha - \frac{p_k}{q_k} \right|^{\sigma} q_k^{2+\varepsilon}}.$$

Using

$$\left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_k^n},$$

we get

$$\frac{1}{\left| \alpha - \frac{p_k}{q_k} \right|^{\sigma} q_k^{2+\varepsilon}} > \frac{1}{q_k^{-n\sigma} q_k^{2+\varepsilon}} = q_k^{n\sigma - (2+\varepsilon)}.$$

Since  $n\sigma > 2 + \varepsilon$  and  $q_k > 1$ , each term on the right-hand side is larger than 1. Hence the series diverges. This proves the lemma.

**Proof.**[Proof of Theorem 1] Fix  $\sigma > 0$  and set

$$E_0 := \mathbb{L} \cap [0, 1].$$

We first prove that

$$C_\sigma(E_0) = 0.$$

Let  $\varepsilon > 0$  and define a positive Borel measure  $\nu$  by

$$\nu = \sum_{q=2}^{\infty} \sum_{p=1}^{q-1} \frac{\delta_{p/q}}{q^{2+\varepsilon}},$$

where  $\delta_{p/q}$  denotes the Dirac measure at  $p/q$ . This measure is finite, because

$$\nu([0, 1]) = \sum_{q=2}^{\infty} \sum_{p=1}^{q-1} \frac{1}{q^{2+\varepsilon}} \leq \sum_{q=2}^{\infty} \frac{q}{q^{2+\varepsilon}} = \sum_{q=2}^{\infty} \frac{1}{q^{1+\varepsilon}} < +\infty.$$

Moreover,  $\text{supp } \nu \subset [0, 1]$ .

For every  $\alpha \in E_0$ , Lemma 1 gives

$$U_\sigma^\nu(\alpha) = \int \frac{d\nu(\xi)}{|\alpha - \xi|^\sigma} = \sum_{q=2}^{\infty} \sum_{p=1}^{q-1} \frac{1}{\left|\alpha - \frac{p}{q}\right|^\sigma q^{2+\varepsilon}} = +\infty.$$

By the capacity criterion 3) stated above, it follows that

$$C_\sigma(E_0) = 0.$$

Now let  $m \in \mathbb{Z}$  and set

$$E_m := \mathbb{L} \cap [m, m + 1].$$

Since the class of Liouville numbers is invariant under integer translations, we have

$$E_m - m = \mathbb{L} \cap [0, 1] = E_0.$$

Since the kernel  $|z - \xi|^{-\sigma}$  is translation invariant, the capacity is translation invariant. Hence

$$C_\sigma(E_m) = C_\sigma(E_0) = 0.$$

Finally,

$$\mathbb{L} = \bigcup_{m \in \mathbb{Z}} E_m.$$

By countable subadditivity of  $C_\sigma$ -capacity,

$$C_\sigma(\mathbb{L}) \leq \sum_{m \in \mathbb{Z}} C_\sigma(E_m) = 0.$$

Therefore

$$C_\sigma(\mathbb{L}) = 0$$

for every  $\sigma > 0$ .

It remains to derive the Hausdorff-dimensional conclusion. By Frostman's lemma, if  $H^s(E) > 0$  for some Borel set  $E \subset \mathbb{R}$  and some  $s > 0$ , then for every  $0 < t < s$  the set  $E$  has positive  $C_t$ -capacity. Since we have proved that  $C_t(\mathbb{L}) = 0$  for every  $t > 0$ , it follows that  $H^s(\mathbb{L}) = 0$  for every  $s > 0$ . Consequently,

$$\dim_H \mathbb{L} = 0.$$

The theorem is proved.

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# LOCAL AND 2-LOCAL ANTI-DERIVATIONS ON SOME WITT-TYPE INFINITE-DIMENSIONAL LIE ALGEBRAS

ATAJONOV KHUSAINBOY\*

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M. ULUGBEK, TASHKENT, UZBEKISTAN

atajonovxusainboy@gmail.com

AKHMEDJANOVA ROZIYAJON

ASIA INTERNATIONAL UNIVERSITY, URGENCH, UZBEKISTAN

**ABSTRACT.** In this paper, we study anti-derivations, local anti-derivations and 2-local anti-derivations on some Witt-type infinite-dimensional Lie algebras. More precisely, we consider the Witt algebra, the Virasoro algebra and the Lie algebra  $W(a, b)$ . We prove that every anti-derivation on the Witt algebra and on the Virasoro algebra is trivial. For the Lie algebra  $W(a, b)$ , we obtain a complete description of anti-derivations. Namely, if  $b \neq 2$ , then every anti-derivation is trivial, while in the exceptional case  $b = 2$ , every anti-derivation maps the Witt part into the abelian ideal and vanishes on this ideal. Using these descriptions, we prove that every local anti-derivation and every 2-local anti-derivation on the considered algebras is an anti-derivation.

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**Key words:** infinite dimensional Lie algebras, anti-derivations, local anti-derivations, 2-local anti-derivations.

## Introduction

The theory of derivations and their generalizations plays an important role in the study of algebraic structures. In particular, derivations are one of the main tools for investigating the structure of Lie algebras and their representations. Various modifications of derivations, such as local derivations, 2-local derivations and  $\delta$ -derivations, have been actively studied in recent years.

The notion of a local derivation was introduced independently by Kadison [15] and by Larson and Sourour [16]. Later, Šemrl introduced the notions of 2-local derivations and 2-local automorphisms [18]. These mappings are generally not assumed to be global derivations at the beginning. However, for many important classes of algebras it turns out that every local or 2-local derivation is in fact a derivation.

The study of local derivations on Lie algebras was initiated by Ayupov and Kudaybergenov in [4]. They proved that every local derivation on a finite-dimensional semisimple Lie algebra is a derivation and gave examples of nilpotent Lie algebras admitting local derivations which are not derivations. Later, local derivations on solvable Lie algebras were investigated in [2]. Further results on local derivations of Lie algebras were obtained in [6, 8, 20]. In particular, in [21], local derivations were studied for Witt algebras and some other classes of infinite-dimensional Lie algebras.

Similar questions have also been considered for 2-local derivations. In [3], it was proved that every 2-local derivation on a finite-dimensional semisimple Lie algebra is a derivation. On the other hand, it was shown that finite-dimensional nilpotent Lie algebras of dimension greater than two admit 2-local derivations which are not derivations. Further investigations of 2-local derivations were carried out for generalized Witt algebras, locally simple Lie algebras, Virasoro-like algebras and Schrödinger–Virasoro algebras [5, 14, 19].

Another important generalization of derivations is the notion of  $\delta$ -derivations. This notion was introduced by Filippov in [9, 10]. A linear map  $D$  on a Lie algebra  $\mathfrak{L}$  is called a  $\delta$ -derivation if

$$D([x, y]) = \delta([D(x), y] + [x, D(y)])$$

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\*Corresponding autor

for all  $x, y \in \mathfrak{L}$ . Ordinary derivations correspond to the case  $\delta = 1$ , while anti-derivations correspond to the case  $\delta = -1$ . In [10],  $\delta$ -derivations of prime Lie algebras were studied. In particular, it was shown that prime Lie algebras usually do not admit nonzero  $\delta$ -derivations for  $\delta \neq 1, -1, 0, \frac{1}{2}$ .

Recently, local and 2-local anti-derivations were studied for several finite-dimensional algebras. In particular, local and 2-local anti-derivations on solvable Lie algebras were considered in [1], while analogous questions for filiform associative algebras, were investigated in [7].

The present paper is devoted to local and 2-local anti-derivations on some Witt-type infinite-dimensional Lie algebras. More precisely, we consider the Witt algebra, the Virasoro algebra and the Lie algebra  $W(a, b)$ . These algebras are fundamental examples of infinite-dimensional graded Lie algebras. The Witt algebra can be realized as the Lie algebra of derivations of the algebra of Laurent polynomials. The Virasoro algebra is the universal central extension of the Witt algebra. The Lie algebra  $W(a, b)$  contains the Witt algebra as a natural component and gives an important family of Witt-type infinite-dimensional Lie algebras.

The main aim of this paper is to describe anti-derivations of the above algebras and to study their local and 2-local analogues. We prove that every anti-derivation on the Witt algebra is trivial. We also show that every anti-derivation on the Virasoro algebra is trivial. For the Lie algebra  $W(a, b)$ , we obtain a complete description of anti-derivations. Namely, if  $b \neq 2$ , then every anti-derivation is trivial. In the exceptional case  $b = 2$ , every anti-derivation vanishes on the abelian ideal generated by  $\{f_n \mid n \in \mathbb{Z}\}$  and maps the Witt part into this ideal.

Using these descriptions, we prove that every local anti-derivation on the Witt algebra, the Virasoro algebra and  $W(a, b)$  is an anti-derivation. We also prove that every 2-local anti-derivation on these algebras is an anti-derivation. Thus, for the Witt-type infinite-dimensional Lie algebras considered in this paper, local and 2-local anti-derivations do not give new mappings beyond ordinary anti-derivations.

The paper is organized as follows. In the preliminary part, we recall the definitions of  $\delta$ -derivations, local  $\delta$ -derivations and 2-local  $\delta$ -derivations. Then we introduce the Witt algebra, the Virasoro algebra and the Lie algebra  $W(a, b)$ . In the main part, we describe anti-derivations of these algebras and prove the corresponding results for local and 2-local anti-derivations.

## Preliminaries

Throughout the paper, all vector spaces and Lie algebras are considered over the field of complex numbers  $\mathbb{C}$ . We denote by  $\mathbb{Z}$  the set of all integers and by  $\delta_{i,j}$  the Kronecker delta. All omitted brackets are either zero or are obtained by skew-symmetry.

**Definition 1.** Let  $(\mathfrak{L}, [-, -])$  be a Lie algebra over a field  $\mathbb{F}$ . A linear map  $D : \mathfrak{L} \rightarrow \mathfrak{L}$  is called a  $\delta$ -derivation if it satisfies

$$D([x, y]) = \delta([D(x), y] + [x, D(y)]), \quad (1)$$

for all  $x, y \in \mathfrak{L}$ , where  $\delta \in \mathbb{F}$  is fixed.

The cases  $\delta = 1$ ,  $\delta = -1$ , and  $\delta = \frac{1}{2}$  are of particular importance. A 1-derivation is an ordinary derivation, while a  $(-1)$ -derivation is called an anti-derivation. In the case of Lie algebras, anti-derivations coincide with reverse derivations, which were studied by Herstein in [12].

For a fixed  $\delta$ , we denote by  $\text{Der}_\delta(\mathfrak{L})$  the set of all  $\delta$ -derivations of  $\mathfrak{L}$ .

**Definition 2.** A linear map  $\Delta$  is called a local  $\delta$ -derivation, if for any  $x \in \mathfrak{L}$ , there exists a  $\delta$ -derivation  $D_x : \mathfrak{L} \rightarrow \mathfrak{L}$  (depending on  $x$ ) such that  $\Delta(x) = D_x(x)$ . The set of all local  $\delta$ -derivations on  $\mathfrak{L}$  we denote by  $\text{LocDer}_\delta(\mathfrak{L})$ .

**Definition 3.** A map  $\nabla : \mathfrak{L} \rightarrow \mathfrak{L}$  (not necessarily linear) is called a 2-local  $\delta$ -derivation, if for any  $x, y \in \mathfrak{L}$ , there exists a  $\delta$ -derivation  $D_{x,y} \in \text{Der}_\delta(\mathfrak{L})$  such that

$$\nabla(x) = D_{x,y}(x), \quad \nabla(y) = D_{x,y}(y).$$

It should be noted that a 2-local  $\delta$ -derivation is not necessarily linear. However, for any  $x \in \mathfrak{L}$  and any scalar  $\lambda$ , we have

$$\nabla(\lambda x) = D_{x,\lambda x}(\lambda x) = \lambda D_{x,\lambda x}(x) = \lambda \nabla(x).$$

Let  $A = \mathbb{C}[t, t^{-1}]$  be the algebra of Laurent polynomials. It has a basis  $\{t^n \mid n \in \mathbb{Z}\}$  and the multiplication  $t^m t^n = t^{m+n}$ . The Witt algebra  $W$  is the Lie algebra of all derivations of  $A$ , that is,  $W = \text{Der}(A)$ . For each

$n \in \mathbb{Z}$ , put  $e_n = -t^{n+1} \frac{d}{dt}$ . Then  $\{e_n \mid n \in \mathbb{Z}\}$  is a basis of  $W$ , and the Lie bracket is given by

$$[e_m, e_n] = (m - n)e_{m+n}, \quad m, n \in \mathbb{Z}.$$

The Virasoro algebra (denoted by  $\text{Vir}$ ) is closely related to the Witt algebra. It is a nontrivial one-dimensional central extension of the Witt algebra [13]. More precisely, the Virasoro algebra has a basis  $\{e_n, f \mid n \in \mathbb{Z}\}$ , where  $f$  is a central element, and the multiplication is given by

$$[e_m, e_n] = (m - n)e_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}f, \quad [e_m, f] = 0.$$

If the central element  $f$  is put equal to zero, then one obtains the centerless Virasoro algebra, which is usually called the Witt algebra [13].

The Lie algebras  $W(a, b)$  form an important class of Witt-type infinite-dimensional Lie algebras. They naturally arise as semidirect products of the Witt algebra and modules of the intermediate series. This class contains many important Lie algebras as special cases and has been studied in connection with extensions of the Virasoro algebra by modules of tensor densities. We now recall the definition of  $W(a, b)$ .

For  $a, b \in \mathbb{C}$ , the Lie algebra  $W(a, b) = \text{span}_{\mathbb{C}}\{e_i, f_i \mid i \in \mathbb{Z}\}$  has the following brackets:

$$\begin{aligned} [e_i, e_j] &= (i - j)e_{i+j}, \\ [e_i, f_j] &= -(j + a + bi)f_{i+j}, \quad i, j \in \mathbb{Z}. \\ [f_i, f_j] &= 0, \end{aligned}$$

The subspace  $W = \text{span}_{\mathbb{C}}\{e_m \mid m \in \mathbb{Z}\}$  is a subalgebra of  $W(a, b)$  isomorphic to the Witt algebra. Moreover,  $I = \text{span}_{\mathbb{C}}\{f_m \mid m \in \mathbb{Z}\}$  is an abelian ideal of  $W(a, b)$ , and it is a module of the intermediate series over the Witt algebra. Hence  $W(a, b) = W \ltimes I$ . The algebras  $W(a, b)$  were considered in mathematical physics [19].

### Main part

In this section, we describe anti-derivations of the Witt algebra, the Virasoro algebra and the Lie algebra  $W(a, b)$ . We then apply these descriptions to local and 2-local anti-derivations.

**Proposition 1.** Any anti-derivation on the Witt algebra  $W$  is trivial, i.e.  $\varphi \equiv 0$ .

**Proof.** Let  $\varphi$  be an anti-derivation of the Witt algebra. Then we set  $\varphi(e_i) = \sum_{j \in \mathbb{Z}} \alpha_{i,j} e_j$ . Now we apply  $\varphi$  to the elements  $e_i, e_j$ :

$$\begin{aligned} \varphi([e_i, e_j]) &= -([\varphi(e_i), e_j] + [e_i, \varphi(e_j)]) \\ &= -\left(\sum_{k \in \mathbb{Z}} \alpha_{i,k} [e_k, e_j] + \sum_{k \in \mathbb{Z}} \alpha_{j,k} [e_i, e_k]\right) \\ &= -\left(\sum_{k \in \mathbb{Z}} ((k - 2j)\alpha_{i,k-j} + (2i - k)\alpha_{j,k-i})e_k\right). \end{aligned}$$

On the other hand  $\varphi([e_i, e_j]) = (i - j)\varphi(e_{i+j})$ . Comparing coefficients of basis elements, we obtain the following relation

$$(i - j)\alpha_{i+j,k} + (k - 2j)\alpha_{i,k-j} + (2i - k)\alpha_{j,k-i} = 0. \quad (2)$$

If we put  $j = 0$  in (1), then we obtain

$$\alpha_{i,k} = \frac{k - 2i}{i + k} \alpha_{0,k-i}, \quad k \neq -i, \quad \text{and} \quad \alpha_{0,-2i} = 0, \quad i \neq 0. \quad (3)$$

If we put  $k = i + j$  in (1), then we have

$$(i - j)\alpha_{i+j,i+j} + (i - j)\alpha_{i,i} + (i - j)\alpha_{j,j} = 0 \quad (4)$$

Next, we put  $k = i$  in (2). Then we have  $\alpha_{i,i} = -\frac{1}{2}\alpha_{0,0}$ ,  $\alpha_{j,j} = -\frac{1}{2}\alpha_{0,0}$ ,  $i, j \neq 0$ . By using this and (3) we have the following:  $\alpha_{i+j,i+j} = \alpha_{0,0}$ . Thus we have  $\alpha_{i,i} = 0$ ,  $\alpha_{0,0} = 0$ . In the next step, we put  $k = -i - j$ ,  $i, j \neq 0$  in (1):

$$(i - j)\alpha_{i+j,-(i+j)} + (-i - 3j)\alpha_{i,-i-2j} + (3i + j)\alpha_{j,-2i-j} = 0.$$

We can simplify this by using (2),

$$(i - j)\alpha_{i+j,-(i+j)} + (-i - 3j)\frac{-i - 2j - 2i}{i + (-i - 2j)}\alpha_{0,-2i-2j} + (3i + j)\frac{-2i - j - 2j}{j - 2i - j}\alpha_{0,-2i-2j} = 0.$$

Due to  $\alpha_{0,-2i-2j} = 0$  and  $i \neq j$ , we have  $\alpha_{i+j,-(i+j)} = 0$ , or  $\alpha_{i,-i} = 0$ . Now we show that  $\alpha_{0,2s+1} = 0$  for all  $s \in \mathbb{Z}$ . Put  $i = 1$ ,  $j = 2$  in (1) and take  $k = i + j + (2s + 1) = 2s + 4$ . Then (1) gives

$$-\alpha_{3,2s+4} + 2s\alpha_{1,2s+2} - 2(s + 1)\alpha_{2,2s+3} = 0. \quad (5)$$

Using (2), we obtain

$$\alpha_{3,2s+4} = \frac{2s - 2}{2s + 7}\alpha_{0,2s+1}, \quad \alpha_{1,2s+2} = \frac{2s}{2s + 3}\alpha_{0,2s+1}, \quad \text{and} \quad \alpha_{2,2s+3} = \frac{2s - 1}{2s + 5}\alpha_{0,2s+1}.$$

Substituting these expressions into (4), we get

$$\left(-\frac{2s - 2}{2s + 7} + \frac{2s \cdot 2s}{2s + 3} - \frac{(2s + 2)(2s - 1)}{2s + 5}\right)\alpha_{0,2s+1} = 0.$$

A direct simplification gives

$$\frac{72}{(2s + 3)(2s + 5)(2s + 7)}\alpha_{0,2s+1} = 0.$$

Hence  $\alpha_{0,2s+1} = 0$  for all  $s \in \mathbb{Z}$ . Thus  $\varphi \equiv 0$ .

**Corollary 1.** Any local and 2-local anti-derivation on the Witt algebra  $W$  is an anti-derivation.

**Proposition 2.** Any anti-derivation on the Virasoro algebra  $\text{Vir}$  is trivial, i.e.  $\varphi \equiv 0$ .

**Proof.** The proof of the proposition is omitted, since it is technical and follows by applying the same arguments as the proof of Proposition 1.

**Corollary 2.** Any local and 2-local anti-derivation on the Virasoro algebra  $\text{Vir}$  is an anti-derivation.

**Proposition 3.** Any anti-derivation  $\varphi$  on  $W(a, b)$  has the following form:

- (i) If  $b \neq 2$ , then  $\varphi$  is trivial, i.e.  $\varphi \equiv 0$ .
- (ii) If  $b = 2$ , then  $\varphi(f_n) = 0$  for all  $n \in \mathbb{Z}$ , and there exist scalars  $(c_t)_{t \in \mathbb{Z}}$  (only finitely many  $c_t \neq 0$ ) such that

$$\varphi(e_n) = \sum_{t \in \mathbb{Z}} c_t f_{n+t}, \quad n \in \mathbb{Z}.$$

**Proof.** The proof of the proposition is omitted, since it is technical and follows by applying the same arguments as the proof of Proposition 1.

**Theorem 1.** Any local anti-derivation on  $W(a, b)$  is an anti-derivation.

**Proof.** The case  $b \neq 2$  is clear. We prove the theorem for  $b = 2$ . First, we prove several lemmas.

**Lemma 1.** Let  $\Delta$  be a local anti-derivation on  $W(a, 2)$ . If  $\Delta(e_m) = 0$ , then  $\Delta(e_{m+1}) = 0$  where  $m \in \mathbb{Z}$ .

**Proof.** By Proposition 3, any anti-derivation  $\varphi$  on  $W(a, 2)$  satisfies

$$\varphi(e_n) = \sum_{t \in \mathbb{Z}} c_t f_{n+t}, \quad n \in \mathbb{Z}.$$

Hence, there exist integers  $p \leq q$  such that

$$\varphi(e_n) = \sum_{t=p}^q c_t f_{n+t}.$$

Consider the equality

$$\varphi_1(e_{m+1}) - x\varphi_1(e_m) = \Delta(e_{m+1} - xe_m) = \Delta(e_{m+1}) = \varphi_2(e_{m+1}),$$

where  $x \in \mathbb{C}^*$  and  $\varphi_1, \varphi_2$  are anti-derivations associated with  $\Delta$ . Then

$$\sum_{t=p}^q c_t f_{m+1+t} - x \sum_{t=p}^q c_t f_{m+t} = \sum_{t=p'}^{q'} b_t f_{m+1+t},$$

where  $p' = p - 1$  and  $q' = q$ . Writing this equality with respect to the basis  $\{f_k\}$ , we obtain

$$(-xc_p - b_{p-1})f_{m+p} + \sum_{t=p}^{q-1} (c_t - xc_{t+1} - b_t)f_{m+1+t} + (c_q - b_q)f_{m+1+q} = 0.$$

Hence,

$$-xc_p - b_{p-1} = 0, \quad c_t - xc_{t+1} - b_t = 0, \quad p \leq t \leq q-1, \quad c_q - b_q = 0.$$

Consequently,

$$b_q + x^{-1}b_{q-1} + x^{-2}b_{q-2} + \cdots + x^{-(q-p)}b_p + x^{-(q-p+1)}b_{p-1} = 0.$$

Since this equality holds for all  $x \in \mathbb{C}^*$  and the coefficients  $b_t$  ( $p-1 \leq t \leq q$ ) are independent of  $x$ , it follows that

$$b_t = 0, \quad p-1 \leq t \leq q.$$

Therefore,  $\Delta(e_{m+1}) = 0$ .

**Lemma 2.** Let  $\Delta$  be a local anti-derivation on  $W(a, 2)$ . If  $\Delta(e_m) = 0$ , then  $\Delta(e_{m-1}) = 0$  where  $m \in \mathbb{Z}$ .

**Proof.** The proof is similar to that of Lemma 1.

**Lemma 3.** Let  $\Delta$  be a local anti-derivation on  $W(a, 2)$ . If  $\Delta(e_0) = 0$ , then

$$\Delta(e_m) = 0 \quad \text{for all } m \in \mathbb{Z}.$$

**Proof.** Assume  $\Delta(e_0) = 0$ . By Lemma 1, using induction, we obtain

$$\Delta(e_m) = 0 \quad \text{for all } m > 0.$$

Similarly, applying Lemma 2 yields

$$\Delta(e_m) = 0 \quad \text{for all } m < 0.$$

Hence,  $\Delta(e_m) = 0$  for all  $m \in \mathbb{Z}$ .

*Proof of Theorem 1.* Let  $\Delta$  be any local anti-derivation on  $W(a, 2)$ . Then there exists an anti-derivation  $\varphi$  on  $W(a, 2)$  such that

$$\Delta(e_0) = \varphi(e_0).$$

Set  $\Delta_1 = \Delta - \varphi$ . Then  $\Delta_1$  is a local anti-derivation such that  $\Delta_1(e_0) = 0$ . By Lemma 3,  $\Delta_1(e_m) = 0$  for all  $m \in \mathbb{Z}$ .

Moreover, by Proposition 3, every anti-derivation on  $W(a, 2)$  vanishes on the ideal generated by  $\{f_m \mid m \in \mathbb{Z}\}$ . Hence, for every  $m \in \mathbb{Z}$ , the locality of  $\Delta_1$  gives  $\Delta_1(f_m) = 0$ .

Therefore  $\Delta_1$  vanishes on both basis components  $\{e_m \mid m \in \mathbb{Z}\}$  and  $\{f_m \mid m \in \mathbb{Z}\}$ . Then  $\Delta_1 \equiv 0$ . Thus  $\Delta = \varphi$  is an anti-derivation. The proof is complete.

**Theorem 2.** Any 2-local anti-derivation on  $W(a, b)$  is an anti-derivation.

**Proof.** In the case  $b \neq 2$ , the proof is clear. We prove the theorem for  $b = 2$ .

Let  $\nabla$  be a 2-local anti-derivation on  $W(a, 2)$ , such that  $\nabla(e_0) = 0$ . Then for any element  $x = \sum_{n \in \mathbb{Z}} x_n e_n + \sum_{m \in \mathbb{Z}} y_m f_m \in W(a, 2)$  there exists an anti-derivation  $\varphi_{e_0, x}$ , such that

$$\nabla(e_0) = \varphi_{e_0, x}(e_0), \quad \nabla(x) = \varphi_{e_0, x}(x).$$

Hence,

$$0 = \nabla(e_0) = \varphi_{e_0, x}(e_0) = \sum_{t \in \mathbb{Z}} \alpha_t e_t + \sum_{t \in \mathbb{Z}} \beta_t f_t,$$

which implies,  $\alpha_t = \beta_t = 0$ ,  $t \in \mathbb{Z}$ .

Consequently, from the description of the anti-derivation  $W(a, 2)$ , we conclude that  $\varphi_{e_0, x} = 0$ . Thus, we obtain that if  $\nabla(e_0) = 0$ , then  $\nabla \equiv 0$ .

Let now  $\nabla$  be an arbitrary 2-local anti-derivation of  $W(a, 2)$ . Take an anti-derivation  $\varphi_{e_0, x}$ , such that

$$\nabla(e_0) = \varphi_{e_0, x}(e_0) \quad \text{and} \quad \nabla(x) = \varphi_{e_0, x}(x).$$

Set  $\nabla_1 = \nabla - \varphi_{e_0, x}$ . Then  $\nabla_1$  is a 2-local anti-derivation, such that  $\nabla_1(e_0) = 0$ . Hence  $\nabla_1(x) = 0$  for all  $x \in W(a, 2)$ , which implies  $\nabla = \varphi_{e_0, x}$ . Therefore,  $\nabla$  is an anti-derivation.

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WEIGHTED ERGODIC AVERAGES FOR  $d$ -DIMENSIONAL FLOWS IN IDEALS OF COMPACT OPERATORS

AZIZOV AZIZKHON

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M. ULUGBEK, TASHKENT, UZBEKISTAN  
BRANCH OF LOMONOSOV MOSCOW STATE UNIVERSITY IN TASHKENT, TASHKENT, UZBEKISTAN  
azizov.07@mail.ru

**ABSTRACT.** Let  $\mathcal{H}$  be an infinite-dimensional complex Hilbert space, and let  $(\mathcal{B}(\mathcal{H}), \|\cdot\|_\infty)$  be the  $C^*$ -algebra of all bounded linear operators acting on  $\mathcal{H}$ . Let  $\mathcal{C}_E$  be the symmetric ideal of compact operators in  $\mathcal{H}$  generated by a fully symmetric sequence space  $E \subset c_0$ . Consider a  $d$ -dimensional semigroup  $\{T_{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{R}_+^d}$  of positive Dunford-Schwartz operators on  $\mathcal{B}(\mathcal{H})$  that is strongly continuous on the trace-class ideal  $\mathcal{C}_1$ . Let  $w : \mathbb{R}_+^d \rightarrow [0, \infty)$  be a bounded measurable weight function.

In this paper, we first show that for all  $t > 0$ , the corresponding weighted ergodic averages  $A_{t,w}(x) = \frac{1}{W(t)} \int_{[0,t]^d} w(\mathbf{u}) T_{\mathbf{u}}(x) d\mathbf{u}$ , where  $W(t) = \int_{[0,t]^d} w(\mathbf{u}) d\mathbf{u}$ , are well-defined and naturally extend to positive Dunford-Schwartz operators. Using this property and the Hardy-Littlewood-Pólya majorization, we establish a non-commutative individual ergodic theorem for weighted  $d$ -dimensional flows. Specifically, we prove that if the weight  $w(\mathbf{u})$  ensures mean convergence in the Hilbert-Schmidt ideal  $\mathcal{C}_2$ , then for any  $x \in \mathcal{C}_E$ , the weighted averages  $A_{t,w}(x)$  converge to a limit  $\hat{x} \in \mathcal{C}_E$  with respect to the uniform norm  $\|\cdot\|_\infty$  as  $t \rightarrow \infty$ .

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**Key words:** symmetric sequence space, Banach ideal of compact operators, Dunford-Schwartz operator, weighted ergodic averages.

## Introduction

Let  $(\mathcal{B}(\mathcal{H}), \|\cdot\|_\infty)$  be the  $C^*$ -algebra of all bounded linear operators in a complex infinite-dimensional Hilbert space  $\mathcal{H}$ . The study of noncommutative individual ergodic theorems in the space of measurable operators affiliated with a semifinite von Neumann algebra  $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$  equipped with a faithful normal semifinite trace  $\tau$  was initiated by F. Yeadon [13], who established that for any positive  $L^1(\mathcal{M})$ -contraction  $T : L^1 \rightarrow L^1$  and every  $x \in L^1$  there exists  $\hat{x} \in L^1$  such that the averages  $A_n(T)(x) = \frac{1}{n+1} \sum_{k=0}^n T^k(x)$  converge to  $\hat{x}$  bilaterally almost uniformly. That is, given  $\varepsilon > 0$ , there exists a projection  $e \in \mathcal{M}$  such that  $\tau(\mathbf{1} - e) < \varepsilon$  and  $\|e(A_n(T)(x) - \hat{x})e\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ , where  $\mathbf{1}$  is the unit of  $\mathcal{M}$ .

The study of individual ergodic theorems beyond  $L^1(\mathcal{M}, \tau)$  started much later with another fundamental paper by M. Junge and Q. Xu [9], where, among other results, the individual ergodic theorem was extended to the case of a positive Dunford-Schwartz operator acting in the space  $L^p(\mathcal{M}, \tau)$ ,  $1 < p < \infty$ . In [2] ([3]), an individual ergodic theorem was proved for a positive Dunford-Schwartz operator in a noncommutative Lorentz (respectively, Orlicz) space. Building upon Lance's extension of the pointwise ergodic theorem for actions of the group of integers on von Neumann algebras, Conze and Dang-Ngoc [4] and Watanabe [12] studied continuous generalizations of Lance's results. In particular, noncommutative individual ergodic theorems were established for actions of the semigroups  $\mathbb{R}_+^d$ . The corresponding ergodic theorem for actions of  $\mathbb{R}_+^d$  with respect to bilaterally almost uniform convergence was initially considered by Junge and Xu [9]. In particular, they derived that these averages converge bilaterally almost uniformly in any noncommutative  $L^p$ -space for  $1 \leq p < \infty$  and almost uniformly if  $2 < p < \infty$ .

A natural and significant development of these results involves the incorporation of weight functions into the continuous flows. To this end, the fundamental properties of weighted ergodic averages must be examined. In the present work, we carry out this investigation.

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Let  $\mathcal{H}$  be a complex infinite-dimensional Hilbert space, and let  $c_0$  be the Banach space of converging to zero sequences of complex numbers. Every symmetric sequence space  $(E, \|\cdot\|_E) \subset c_0$  generates a symmetric ideal of compact operators  $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ , acting in  $\mathcal{H}$ , by the following rule (see, for example, [10, Chapter 3, Section 3.5]):

$$\mathcal{C}_E = \{x \in \mathcal{K}(\mathcal{H}) : \{s_n(x)\} \in E\}, \quad \|x\|_{\mathcal{C}_E} = \|\{s_n(x)\}\|_E,$$

where  $\mathcal{K}(\mathcal{H})$  is the two-sided ideal of compact linear operators in  $\mathcal{B}(\mathcal{H})$  and  $\{s_n(x)\}_{n=1}^{m(x)}$  is the set of eigenvalues of the compact operator  $|x|$  in the decreasing order.

Let  $\mathbb{N}$  (respectively,  $\mathbb{R}$ ) be the set of natural (respectively, real) numbers. Fix  $d \in \mathbb{N}$  and denote  $\mathbb{R}_+^d = \{\mathbf{u} = (u_1, \dots, u_d) : 0 \leq u_i \in \mathbb{R}, i = 1, \dots, d\}$ . Let  $\{T_{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{R}_+^d}$  be a semigroup of positive Dunford-Schwartz operators, such that  $T_{\mathbf{0}}(x) = x$  for all  $x \in \mathcal{B}(\mathcal{H})$  (definition of Dunford-schwartz operator see below in Preliminaries section). A semigroup  $\{T_{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{R}_+^d}$  is said to be *strongly continuous* on  $\mathcal{C}_{l_1}$  if

$$\lim_{\mathbf{u} \rightarrow \mathbf{v}} \|T_{\mathbf{u}}(x) - T_{\mathbf{v}}(x)\|_{\mathcal{C}_{l_1}} = 0.$$

for each  $x \in \mathcal{C}_{l_1}$ .

Let  $w : \mathbb{R}_+^d \rightarrow [0, \infty)$  be a bounded measurable function, and let  $\{T_{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{R}_+^d}$  be a strongly continuous on  $\mathcal{C}_{l_1}$  semigroup of Dunford-Schwartz operators and let

$$A_{t,w}(x) = \frac{1}{W(t)} \int_{[0,t]^d} w(\mathbf{u}) T_{\mathbf{u}}(x) d\mathbf{u}, \quad t > 0, \quad (1)$$

where  $W(t) = \int_{[0,t]^d} w(\mathbf{u}) d\mathbf{u}$ , be the corresponding weighted ergodic averages. We show that the operators  $A_{t,w}$  extend to the Dunford-Schwartz operators, in particular,  $A_{t,w}(\mathcal{C}_E) \subset \mathcal{C}_E$  for any fully symmetric sequence space  $E \subset c_0$ .

## Preliminaries

Let  $l^\infty$  (respectively,  $c_0$ ) be the Banach space of bounded (respectively, converging to zero) sequences  $\{\xi_n\}_{n=1}^\infty$  of complex numbers equipped with the uniform norm  $\|\{\xi_n\}\|_\infty = \sup_{n \in \mathbb{N}} |\xi_n|$ . If  $\xi = \{\xi_n\}_{n=1}^\infty \in l^\infty$ , then the non-increasing rearrangement  $\xi^* = \{\xi_n^*\}_{n=1}^\infty$  of  $\xi$  is defined by

$$\xi_n^* = \inf \left\{ \sup_{n \notin F} |\xi_n| : F \subset \mathbb{N}, |F| < n \right\}.$$

The Hardy-Littlewood-Polya partial order in the space  $l^\infty$  is defined as follows:

$$\xi = \{\xi_n\} \prec \prec \eta = \{\eta_n\} \iff \sum_{n=1}^m \xi_n^* \leq \sum_{n=1}^m \eta_n^* \quad \text{for all } m \in \mathbb{N}.$$

A non-zero linear subspace  $E \subset l^\infty$  with a Banach norm  $\|\cdot\|_E$  is called a *symmetric (fully symmetric) sequence space* if

$$\eta \in E, \xi \in l^\infty, \xi^* \leq \eta^* \text{ (respectively, } \xi^* \prec \prec \eta^*) \implies \xi \in E \text{ and } \|\xi\|_E \leq \|\eta\|_E.$$

Let  $(\mathcal{H}, (\cdot, \cdot))$  be an infinite-dimensional separable Hilbert space over the field  $\mathbb{C}$  of complex numbers, and let  $(\mathcal{B}(\mathcal{H}), \|\cdot\|_\infty)$  be the  $C^*$ -algebra of bounded linear operators in  $\mathcal{H}$ . Denote by  $\mathcal{K}(\mathcal{H})$  the two-sided ideal of compact linear operators in  $\mathcal{B}(\mathcal{H})$ .

Let  $\mathcal{B}_h(\mathcal{H}) = \{x \in \mathcal{B}(\mathcal{H}) : x = x^*\}$ ,  $\mathcal{B}_+(\mathcal{H}) = \{x \in \mathcal{B}_h(\mathcal{H}) : x \geq 0\}$ , and let  $\tau : \mathcal{B}_+(\mathcal{H}) \rightarrow [0, \infty]$  be the canonical trace on  $\mathcal{B}(\mathcal{H})$ , that is,  $\tau(x) = \sum_{i=1}^\infty (x\varphi_i, \varphi_i)$  for all  $x \in \mathcal{B}_+(\mathcal{H})$ , where  $\{\varphi_i\}_{i=1}^\infty$  is an orthonormal basis in  $\mathcal{H}$ .

Let  $\mathcal{P}(\mathcal{H})$  be the lattice of projections in  $\mathcal{H}$ . If  $\mathbf{1}$  is the identity of  $\mathcal{B}(\mathcal{H})$  and  $e \in \mathcal{P}(\mathcal{H})$ , we will write  $e^\perp = \mathbf{1} - e$ .

Let  $x \in \mathcal{B}(\mathcal{H})$ , and let  $\{e_\lambda\}_{\lambda \geq 0}$  be the spectral family of projections for the absolute value  $|x| = (x^*x)^{1/2}$  of  $x$ , that is,  $e_\lambda = \{|x| \leq \lambda\}$ . If  $t > 0$ , then the  $t$ -th generalized singular number of  $x$ , or the non-increasing rearrangement of  $x$ , is defined as (see [7])

$$\mu_t(x) = \inf\{\lambda > 0 : \tau(e_\lambda^\perp) \leq t\}.$$

A non-zero linear subspace  $X \subset \mathcal{B}(\mathcal{H})$  with a Banach norm  $\|\cdot\|_X$  is called noncommutative symmetric (fully symmetric) space if the conditions

$$x \in X, y \in \mathcal{B}(\mathcal{H}), \mu_t(y) \leq \mu_t(x) \forall t > 0$$

$$\text{(respectively, } \int_0^s \mu_t(y) dt \leq \int_0^s \mu_t(x) dt \forall s > 0 \text{ (writing } y \prec\prec x \text{))}$$

imply that  $y \in X$  and  $\|y\|_X \leq \|x\|_X$ .

The spaces  $(\mathcal{B}(\mathcal{H}), \|\cdot\|_\infty)$  and  $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty)$ , as well as the classical Banach two-sided ideals

$$\mathcal{C}_p = \{x \in \mathcal{K}(\mathcal{H}) : \|x\|_p = \tau(|x|^p)^{1/p} < \infty\}, \quad 1 \leq p < \infty,$$

are examples of noncommutative fully symmetric spaces.

If  $x \in \mathcal{K}(\mathcal{H})$ , then  $|x| = \sum_{n=1}^{m(x)} s_n(x) p_n$  (if  $m(x) = \infty$ , the series converges with respect to the uniform norm  $\|\cdot\|_\infty$ ), where  $\{s_n(x)\}_{n=1}^{m(x)}$  is the set of eigenvalues of the compact operator  $|x|$  in the decreasing order, and  $p_n$  is the projection onto the eigenspace corresponding to  $s_n(x)$ . Consequently, the non-increasing rearrangement  $\mu_t(x)$  of  $x \in \mathcal{K}(\mathcal{H})$  can be identified with the sequence  $\{s_n(x)\}_{n=1}^\infty$ ,  $s_n(x) \downarrow 0$  (if  $m(x) < \infty$ , we set  $s_n(x) = 0$  for all  $n > m(x)$ ).

Fix an orthonormal basis  $\{\varphi_n\}_{n=1}^\infty$  in  $\mathcal{H}$ . Let  $p_n$  be the one-dimensional projection on the subspace  $\mathbb{C} \cdot \varphi_n \subset \mathcal{H}$ . If  $(X, \|\cdot\|_X) \subset \mathcal{K}(\mathcal{H})$  is a symmetric space then the set

$$E(X) = \left\{ \xi = \{\xi_n\}_{n=1}^\infty \in c_0 : x_\xi = \sum_{n=1}^\infty \xi_n p_n \in X \right\}$$

(the series converges uniformly) is a symmetric sequence space with respect to the norm  $\|\xi\|_{E(X)} = \|x_\xi\|_X$ . Consequently, each symmetric space  $(X, \|\cdot\|_X) \subset \mathcal{K}(\mathcal{H})$  generates a symmetric sequence space  $(E(X), \|\cdot\|_{E(X)}) \subset c_0$ . The converse is also true: every symmetric sequence space  $(E, \|\cdot\|_E) \subset c_0$  generates a symmetric space  $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E}) \subset \mathcal{K}(\mathcal{H})$  by the following rule (see, for example, [11, Chapter 3, Section 3.5]):

$$\mathcal{C}_E = \{x \in \mathcal{K}(\mathcal{H}) : \{s_n(x)\} \in E\}, \quad \|x\|_{\mathcal{C}_E} = \|\{s_n(x)\}\|_E.$$

The pair  $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$  is called a Banach ideal of compact operators (cf. [8, Chapter III]). It is known that  $(\mathcal{C}_p, \|\cdot\|_p) = (\mathcal{C}_{l^p}, \|\cdot\|_{\mathcal{C}_{l^p}})$  for all  $1 \leq p < \infty$  and  $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty) = (\mathcal{C}_{c_0}, \|\cdot\|_{\mathcal{C}_{c_0}})$ .

Hardy-Littlewood-Polya partial order in the Banach ideal  $\mathcal{K}(\mathcal{H})$  is defined by

$$x \prec\prec y, x, y \in \mathcal{K}(\mathcal{H}) \iff \{s_n(x)\} \prec\prec \{s_n(y)\}.$$

Using Lemma 2.5 (ii) [7] and Theorem 4.4 (iii) [7] we get

$$x \prec\prec y, z \prec\prec y, x, y, z \in \mathcal{K}(\mathcal{H}) \implies x + z \prec\prec 2y. \quad (1)$$

We say that a Banach ideal  $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$  is fully symmetric, if conditions  $y \in \mathcal{C}_E$ ,  $x \in \mathcal{K}(\mathcal{H})$ ,  $x \prec\prec y$  entail that  $x \in \mathcal{C}_E$  and  $\|x\|_{\mathcal{C}_E} \leq \|y\|_{\mathcal{C}_E}$ . It is clear that  $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$  is a fully symmetric ideal if and only if  $(E, \|\cdot\|_E)$  is a fully symmetric sequence space.

Note that, along with any Schatten ideals  $\mathcal{C}_p$ ,  $1 \leq p < \infty$ , of compact operators, the family of such fully symmetric ideals  $\mathcal{C}_E$  contains many noncommutative counterparts of classical symmetric sequence spaces, examples of which are given in the last section of this note.

A linear contraction  $T : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$  is called a Dunford-Schwartz operator (writing  $T \in DS$ ), if  $T(\mathcal{C}_1) \subset \mathcal{C}_1$  and  $\|T(x)\|_1 \leq \|x\|_1$  for all  $x \in \mathcal{C}_1$ . We will write  $T \in DS^+$  if  $T$  is a positive Dunford-Schwartz operator, that is,  $T \in DS$  and  $T(\mathcal{B}_+(\mathcal{H})) \subset \mathcal{B}_+(\mathcal{H})$ .

Any fully symmetric ideal  $\mathcal{C}_E$  is an exact interpolation space in the Banach pair  $(\mathcal{C}_1, \mathcal{B}(\mathcal{H}))$  (see [5, Theorem 2.4]). It then follows that  $T(\mathcal{C}_E) \subset \mathcal{C}_E$  and  $\|T\|_{\mathcal{C}_E \rightarrow \mathcal{C}_E} \leq 1$  for all  $T \in DS$ . In particular,  $T(\mathcal{K}(\mathcal{H})) \subset \mathcal{K}(\mathcal{H})$  and the restriction of  $T$  on  $\mathcal{K}(\mathcal{H})$  is a linear contraction (also denoted by  $T$ ).

In [1], the following result was established regarding the extension of any positive linear contraction  $T : \mathcal{C}_1 \rightarrow \mathcal{C}_1$  with the property  $\|T(x)\|_\infty \leq \|x\|_\infty$  for all  $x \in \mathcal{C}_1$  up to the Dunford-Schwartz operator  $\tilde{T} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ .

**Theorem 1.** Let  $T : \mathcal{C}_1 \rightarrow \mathcal{C}_1$  be a positive linear contraction such that  $\|T(x)\|_\infty \leq \|x\|_\infty$  for all  $x \in \mathcal{C}_1$ . Then there exists a unique operator  $\tilde{T} \in DS$  such that  $\tilde{T}(x) = T(x)$  for all  $x \in \mathcal{C}_1$ , and  $\tilde{T}$  is  $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{C}_1)$ -continuous.

### Weighted ergodic averages of $d$ -dimensional flows in Banach ideals of compact operators

Let  $d \in \mathbb{N}$  and let  $\mathbb{R}_+^d = \{\mathbf{u} = (u_1, \dots, u_d) : 0 \leq u_i \in \mathbb{R}, i = 1, \dots, d\}$ . In what follows,  $\{T_{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{R}_+^d} \subset DS^+$  is a semigroup such that  $T_0(x) = x$  for all  $x \in \mathcal{B}(\mathcal{H})$ . A semigroup  $\{T_{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{R}_+^d}$  is said to be strongly continuous on fully symmetric ideal  $\mathcal{C}_1$ , if  $\lim_{\mathbf{u} \rightarrow \mathbf{v}} \|T_{\mathbf{u}}(x) - T_{\mathbf{v}}(x)\|_{\mathcal{C}_1} = 0$  for each  $x \in \mathcal{C}_1$ . Note that for  $\mathbf{u}, \mathbf{v} \in \mathbb{R}_+^d$ ,  $\mathbf{u} = (u_1, \dots, u_d)$ ,  $\mathbf{v} = (v_1, \dots, v_d)$  the convergence  $\mathbf{u} \rightarrow \mathbf{v}$  means that  $u_i \rightarrow v_i$  for each  $i = 1, \dots, d$ .

If  $\{T_{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{R}_+^d} \subset DS^+$  is a strongly continuous semigroup on  $\mathcal{C}_1$  then for any  $x \in \mathcal{C}_1$  and  $y \in \mathcal{B}(\mathcal{H})$  the function  $\varphi_{x,y}(\mathbf{u}) = \tau(T_{\mathbf{u}}(x)y)$  is continuous on  $\mathbb{R}_+^d$ . Therefore, for the Lebesgue measure  $\mu$  on  $\mathbb{R}_+^d$  we have that the map  $U_x : \mathbb{R}_+^d \rightarrow \mathcal{C}_1$  defined as  $U_x(\mathbf{u}) = T_{\mathbf{u}}(x)$  is weakly  $\mu$ -measurable, that is, the complex function  $\tau(U_x(\mathbf{u})y)$  is a measurable function on  $(\mathbb{R}_+^d, \mu)$  for all  $y \in \mathcal{B}(\mathcal{H})$  (recall that  $(\mathcal{C}_1)^* = \mathcal{B}(\mathcal{H})$  and every  $f \in (\mathcal{C}_1)^*$  has the following form  $f(x) = \tau(xy)$  for some  $y \in \mathcal{B}(\mathcal{H})$ ). Since, in addition,  $U_x(\mathbb{R}_+^d)$  is a separable subset in  $\mathcal{C}_1$ , Pettis theorem [14, Chapter V, §4] entails that the real function  $\|U_x(\mathbf{u})\|_1 = \|T_{\mathbf{u}}(x)\|_1$  is  $\mu$ -measurable on  $\mathbb{R}_+^d$ . Using the inequality  $\|T_{\mathbf{u}}(x)\|_1 \leq \|x\|_1$ , we obtain that  $\|T_{\mathbf{v}}(x)\|_1$  is an integrable function on  $[0, u_1] \times [0, u_2] \times \dots \times [0, u_d]$  for any  $\mathbf{u} = (u_1, \dots, u_d) \in \mathbb{R}_+^d$ . By [14, Chapter V, §5, Theorem 1], the function  $T_{\mathbf{v}}(x)$  is Bochner  $\mu$ -integrable on  $[0, t]^d$  for every  $t > 0$ .

Let  $w : \mathbb{R}_+^d \rightarrow [0, \infty)$  be a non-negative bounded Lebesgue measurable function such that  $W(t) = \int_{[0,t]^d} w(\mathbf{u}) d\mathbf{u} > 0$  for all  $t > 0$ . Since the semigroup  $\{T_{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{R}_+^d}$  is strongly continuous on  $\mathcal{C}_1$ , the map  $\mathbf{u} \mapsto T_{\mathbf{u}}(x)$  is strongly measurable for any  $x \in \mathcal{C}_1$ . The multiplication by a scalar measurable function preserves this property, so the map  $\mathbf{u} \mapsto w(\mathbf{u})T_{\mathbf{u}}(x)$  is also strongly measurable on  $\mathbb{R}_+^d$ . Since  $w(\mathbf{u})$  is bounded, there exists  $M > 0$  such that  $w(\mathbf{u}) \leq M$  almost everywhere. Using the fact that  $T_{\mathbf{u}}$  is a contraction in  $\mathcal{C}_1$ , we can estimate the norm of the integrand:

$$\|w(\mathbf{u})T_{\mathbf{u}}(x)\|_1 = w(\mathbf{u})\|T_{\mathbf{u}}(x)\|_1 \leq M\|x\|_1.$$

Since the function is bounded by a constant on the set  $[0, t]^d$  of finite Lebesgue measure, it is Bochner integrable. Thus, for any  $x \in \mathcal{C}_1$  and  $t > 0$  there exists the weighted Bochner integral:

$$A_{t,w}(x) = \frac{1}{W(t)} \int_{[0,t]^d} w(\mathbf{u})T_{\mathbf{u}}(x) d\mathbf{u} \in \mathcal{C}_1.$$

Since  $T_{\mathbf{u}}$  is a positive operator and  $w(\mathbf{u}) \geq 0$ , it is clear that  $A_{t,w}$  is also a positive operator. Furthermore, by the properties of the Bochner integral, we have the following norm estimates for all  $x \in \mathcal{C}_1$ :

$$\|A_{t,w}(x)\|_1 \leq \frac{1}{W(t)} \int_{[0,t]^d} w(\mathbf{u})\|T_{\mathbf{u}}(x)\|_1 d\mathbf{u} \leq \frac{1}{W(t)} \int_{[0,t]^d} w(\mathbf{u})\|x\|_1 d\mathbf{u} = \|x\|_1,$$

and similarly, since  $\|T_{\mathbf{u}}(x)\|_\infty \leq \|x\|_\infty$ ,

$$\|A_{t,w}(x)\|_\infty \leq \frac{1}{W(t)} \int_{[0,t]^d} w(\mathbf{u})\|T_{\mathbf{u}}(x)\|_\infty d\mathbf{u} \leq \|x\|_\infty.$$

Consequently, by Theorem 1, there exists a unique operator  $\tilde{A}_{t,w} \in DS^+$  such that  $\tilde{A}_{t,w}(x) = A_{t,w}(x)$  for all  $x \in \mathcal{C}_1$ , and  $\tilde{A}_{t,w}$  is  $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{C}_1)$ -continuous. Below, the operator  $\tilde{A}_{t,w}$  is simply denoted by  $A_{t,w}$ .

We summarize the preceding construction and the fundamental properties of the weighted ergodic averages in the following theorem.

**Theorem 2.** Let  $\{T_{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{R}_+^d} \subset DS^+$  be a strongly continuous semigroup on  $\mathcal{C}_1$ , and let  $w : \mathbb{R}_+^d \rightarrow [0, \infty)$  be a bounded Lebesgue measurable function such that  $W(t) = \int_{[0,t]^d} w(\mathbf{u}) d\mathbf{u} > 0$  for all  $t > 0$ . Then, for any  $t > 0$  and  $x \in \mathcal{C}_1$ , the weighted Bochner integral

$$A_{t,w}(x) = \frac{1}{W(t)} \int_{[0,t]^d} w(\mathbf{u}) T_{\mathbf{u}}(x) d\mathbf{u}$$

is well-defined and belongs to  $\mathcal{C}_1$ . Furthermore,  $A_{t,w}$  uniquely extends to a  $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{C}_1)$ -continuous positive Dunford-Schwartz operator on  $\mathcal{B}(\mathcal{H})$  (i.e.,  $A_{t,w} \in DS^+$ ).

While Theorem 2 ensures the existence and basic operator properties of the weighted averages, the mere boundedness of the weight  $w(\mathbf{u})$  is generally insufficient to guarantee the asymptotic convergence of  $A_{t,w}(x)$  as  $t \rightarrow \infty$ . To establish individual ergodic theorems in fully symmetric ideals, we must restrict our attention to a class of weights that ensure mean convergence in the space  $\mathcal{C}_2$ . This naturally motivates the following definition.

**Definition 1.** A weight function  $w \in L^\infty(\mathbb{R}_+^d)$  is called an *ergodic weight for the ideal  $\mathcal{C}_2$*  if, for any  $x \in \mathcal{C}_2$  and any strongly continuous semigroup  $\{T_{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{R}_+^d} \subset DS^+$ , the weighted averages  $A_{t,w}(x)$  converge in the norm of  $\mathcal{C}_2$  as  $t \rightarrow \infty$ .

Furthermore, the fact that  $A_{t,w}$  acts as a positive Dunford-Schwartz operator (as established in Theorem 2) immediately provides a fundamental spectral inequality for compact operators. This majorization property will serve as the crucial ingredient in transferring convergence from  $\mathcal{C}_2$  to arbitrary fully symmetric ideals.

**Corollary 1.** Let  $w : \mathbb{R}_+^d \rightarrow [0, \infty)$  be a bounded measurable weight and let  $A_{t,w}$  be the corresponding weighted ergodic average. For any compact operator  $x \in \mathcal{K}(\mathcal{H})$ , the Hardy-Littlewood-Pólya majorization  $A_{t,w}(x) \prec\prec x$  holds, that is:

$$\sum_{n=1}^m s_n(A_{t,w}(x)) \leq \sum_{n=1}^m s_n(x)$$

for all  $m \in \mathbb{N}$  and  $t > 0$ .

**Proof.** By the previous construction, the operator  $A_{t,w}$  belongs to the class  $DS^+$ . The stated majorization directly follows from the fundamental properties of positive Dunford-Schwartz operators acting on the ideal of compact operators [6, Theorem 4.7]. The proof is complete.

With the definition of an ergodic weight and the crucial majorization property established, we are now ready to state and prove the main result of this section – the non-commutative individual ergodic theorem for weighted averages in fully symmetric ideals of compact operators.

**Theorem 3.** Let  $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$  be a fully symmetric Banach ideal, and let  $\{T_{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{R}_+^d} \subset DS^+$  be a strongly continuous semigroup on  $\mathcal{C}_1$ . If  $w \in L^\infty(\mathbb{R}_+^d)$  is an ergodic weight for the ideal  $\mathcal{C}_2$ , then for any  $x \in \mathcal{C}_E$ , the weighted ergodic averages  $A_{t,w}(x)$  converge to some  $\hat{x} \in \mathcal{C}_E$  with respect to the uniform norm  $\|\cdot\|_\infty$  as  $t \rightarrow \infty$ .

**Proof.** Let  $x \in \mathcal{C}_2$ . By Definition 1, the assumption that  $w(\mathbf{u})$  is an ergodic weight for  $\mathcal{C}_2$  implies that there exists an operator  $\hat{x} \in \mathcal{C}_2$  such that  $\|A_{t,w}(x) - \hat{x}\|_2 \rightarrow 0$  as  $t \rightarrow \infty$ .

Recall that  $\mathcal{C}_2 \subset \mathcal{K}(\mathcal{H})$  and the uniform norm is majorized by the Hilbert-Schmidt norm, meaning that  $\|y\|_\infty \leq \|y\|_2$  for all  $y \in \mathcal{C}_2$ . Consequently, we immediately obtain that  $\|A_{t,w}(x) - \hat{x}\|_\infty \rightarrow 0$  as  $t \rightarrow \infty$  for any  $x \in \mathcal{C}_2$ . This implies that the net of operators  $\{A_{t,w}(x)\}_{t>0}$  is Cauchy in the Banach space  $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty)$ , which yields:

$$\|A_{t,w}(x) - A_{s,w}(x)\|_\infty \rightarrow 0 \quad \text{as } t, s \rightarrow \infty, \quad \forall x \in \mathcal{C}_2. \quad (2)$$

Since the ideal  $\mathcal{C}_2$  contains the linear subspace of all finite-dimensional operators, it is everywhere dense in  $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty)$ . Given that  $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$  is a symmetric ideal generated by the sequence space  $E \subset c_0$ , we have the continuous embedding  $\mathcal{C}_E \subset \mathcal{K}(\mathcal{H})$ . Thus, for an arbitrary element  $x \in \mathcal{C}_E$  and any  $\varepsilon > 0$ , we can choose an approximation  $x_\varepsilon \in \mathcal{C}_2$  such that  $\|x - x_\varepsilon\|_\infty < \varepsilon$ .

Using the fact that  $A_{t,w}$  is a Dunford-Schwartz operator (Theorem 2), its restriction to  $\mathcal{K}(\mathcal{H})$  is a linear contraction with respect to the uniform norm, i.e.,  $\|A_{t,w}(y)\|_\infty \leq \|y\|_\infty$  for all  $y \in \mathcal{K}(\mathcal{H})$ . This allows us to establish the following estimate for any  $t > 0$ :

$$\|A_{t,w}(x) - A_{t,w}(x_\varepsilon)\|_\infty = \|A_{t,w}(x - x_\varepsilon)\|_\infty \leq \|x - x_\varepsilon\|_\infty < \varepsilon.$$

Applying the triangle inequality along with the property (2), we obtain:

$$\begin{aligned} \|A_{t,w}(x) - A_{s,w}(x)\|_\infty &\leq \|A_{t,w}(x) - A_{t,w}(x_\varepsilon)\|_\infty + \|A_{t,w}(x_\varepsilon) - A_{s,w}(x_\varepsilon)\|_\infty + \|A_{s,w}(x_\varepsilon) - A_{s,w}(x)\|_\infty \leq \\ &\leq 2\varepsilon + \|A_{t,w}(x_\varepsilon) - A_{s,w}(x_\varepsilon)\|_\infty. \end{aligned}$$

Taking the limit superior as  $t, s \rightarrow \infty$ , we find that  $\limsup_{t,s \rightarrow \infty} \|A_{t,w}(x) - A_{s,w}(x)\|_\infty \leq 2\varepsilon$ . Since  $\varepsilon > 0$  was chosen arbitrarily, it follows that:

$$\|A_{t,w}(x) - A_{s,w}(x)\|_\infty \rightarrow 0 \quad \text{as } t, s \rightarrow \infty$$

for all  $x \in \mathcal{C}_E$ . By the completeness of the Banach space  $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty)$ , there exists a unique compact operator  $\hat{x} \in \mathcal{K}(\mathcal{H})$  such that  $\|A_{t,w}(x) - \hat{x}\|_\infty \rightarrow 0$  as  $t \rightarrow \infty$ .

Next, we show that the limit operator  $\hat{x}$  belongs to the ideal  $\mathcal{C}_E$ . According to the eigenvalue variation inequality [8, Chapter II, §2, Corollary 2.3], we have:

$$|s_n(A_{t,w}(x)) - s_n(\hat{x})| \leq \|A_{t,w}(x) - \hat{x}\|_\infty$$

for all  $n \in \mathbb{N}$ . As a consequence of the uniform convergence of the averages, it follows that  $s_n(A_{t,w}(x)) \rightarrow s_n(\hat{x})$  pointwise as  $t \rightarrow \infty$  for each  $n \in \mathbb{N}$ .

By Corollary 1, the weighted average  $A_{t,w}$  satisfies the Hardy-Littlewood-Pólya majorization  $A_{t,w}(x) \prec\prec x$  for all  $t > 0$ , which means:

$$\sum_{n=1}^m s_n(A_{t,w}(x)) \leq \sum_{n=1}^m s_n(x)$$

for all  $m \in \mathbb{N}$  and  $t > 0$ . Taking the limit as  $t \rightarrow \infty$  on the left side of the inequality, the pointwise convergence of the singular numbers yields:

$$\sum_{n=1}^m s_n(\hat{x}) \leq \sum_{n=1}^m s_n(x)$$

for all  $m \in \mathbb{N}$ , which means that  $\hat{x} \prec\prec x$ . Since  $x \in \mathcal{C}_E$  and  $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$  is a fully symmetric Banach ideal, the majorization  $\hat{x} \prec\prec x$  guarantees that  $\hat{x} \in \mathcal{C}_E$ , completing the proof.

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# CONTROL PROBLEM FOR A PROCESS DESCRIBED BY A FRACTIONAL PARTIAL DIFFERENTIAL EQUATION IN A ROD

BOBOXONOVA MARJONA

KARSHI STATE UNIVERSITY, KARSHI, UZBEKISTAN

boboxonovamarjona1012@gmail.com

**ABSTRACT.** Throughout this paper, a control problem for a process described by a fractional-order partial differential equation in a rod is investigated. The problem of choosing the right-hand side function of the equation is studied in such a way that the difference between the values of the process at different times over a given time interval follows a prescribed law.

**MSC (2020):** 17A30; 17B30; 17B40; 17B56.

**Key words:** fractional-order derivative, control problem, Fourier series, Mittag-Leffler function.

## 1. Introduction

In recent years, fractional differential equations have been widely used in physics, biology, medicine and control theory. This work is devoted to solving the problem of determining the right-hand side function of a fractional partial differential equation such that the difference between the values of the process at different times follows a given law.

Control problems for processes described by partial differential equations can be found in [1], [2]. In the works of V.A. Il'in and E.I. Moiseev (see [3]-[5]), boundary control problems related to wave propagation processes are studied in detail. Results on control problems for parabolic-type equations, particularly heat transfer processes, are presented in [6], [7]. Control problems for processes described by homogeneous heat equations were investigated in the works of academician Sh.O. Alimov (see [8]-[12]). Inverse problems for determining the right-hand side (heat source density) in various subdiffusion equations have been studied by many researchers. More detailed information can be found in [13].

Some control problems for heat equations are studied in [14]-[20], and for fractional equations in [21], [22].

## 2. Preliminaries

In this section, we present fractional-order derivative in the sense of Caputo and several pieces of data about the Mittag-Leffler function, which we will use below.

Let  $0 < \rho < 1$ . Then the fractional derivative of order  $\rho$  in the sense of Caputo is defined as follows:

$$D_t^\rho h(t) = \frac{1}{\Gamma(1-\rho)} \int_0^t \frac{h'(\tau)}{(t-\tau)^\rho} d\tau, \quad t > 0, \quad (1)$$

where  $\Gamma(\rho)$  - gamma function.

Let  $\rho$  and  $\mu$  an arbitrary complex numbers. We denote by  $E_{\rho,\mu}(z)$  the two-parameter Mittag-Leffler function:

$$E_{\rho,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + \mu)}. \quad (2)$$

## 3. Statement of the problem

In this section, we consider the problem of finding a solution to the following fractional differential equation in the sense of Caputo

$$D_t^\rho u(x, t) - a^2 u(x, t) = f(x), \quad 0 < t < T, \quad (3)$$

subject to the initial condition

$$u(x, +0) = \phi(x), \quad 0 \leq x \leq l \quad (4)$$

and the boundary conditions

$$u(0, t) = 0, \quad 0 \leq t \leq T, \quad (5)$$

$$u(l, t) = 0, \quad 0 \leq t \leq T, \quad (6)$$

where  $\phi(x)$ ,  $f(x)$  is given functions,  $a$  and  $T$  are positive constant numbers. The problem of finding a solution to (3) - (6) is called the forward problem.

**Problem.** Let  $\psi(x)$  be a given function. It is required to determine the source function  $f(x)$  such that the solution of problem (3) - (6) satisfies the following condition:

$$u(x, t_2) - u(x, t_1) = \psi(x), \quad t_1, t_2 \in (0, T). \quad (7)$$

In other words, the function  $f(x)$  must be chosen so that the difference between the states of the process at times  $t_2$  and  $t_1$  evolves according to a given law.

**Definition 3.1.** If  $u(x, t) \in C([0, l] \times [0, T])$  and  $f(x) \in C[0, l]$  the function having the properties  $D_t^\rho u(x, t)$ ,  $u_{xx}(x, t) \in C((0, l) \times (0, T))$ , satisfy conditions (3) - (7), then the pair of functions  $\{u(x, t), f(x)\}$  is called a solution of the control problem (3) - (7).

We will prove the following theorem to solve this problem.

**Theorem 3.2.** Let  $\phi(x) \in C^2[0, l]$  and have a third-order piecewise continuous derivative,  $\psi(x) \in C^4[0, l]$  have a fifth-order piecewise continuous derivative. Assume that these functions satisfy the conditions  $\phi(0) = \phi(l) = 0$ ,  $\phi''(0) = \phi''(l) = 0$ ,  $\psi(0) = \psi(l) = 0$ ,  $\psi''(0) = \psi''(l) = 0$  and  $\psi^{(IV)}(0) = \psi^{(IV)}(l) = 0$ . Then the control problem (3) - (7) has a unique solution and this solution is given by:

$$u(x, t) = \sum_{n=1}^{\infty} \left[ \phi_n E_{\rho, 1} \left( - \left( \frac{\pi n a}{l} \right)^2 t^\rho \right) + f_n t^\rho E_{\rho, \rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t^\rho \right) \right] \sin \frac{\pi n x}{l}, \quad (8)$$

where

$$f_n = \frac{\psi_n}{t_2^\rho E_{\rho, \rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_2^\rho \right) - t_1^\rho E_{\rho, \rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_1^\rho \right)} - \frac{\phi_n \left( E_{\rho, 1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_2^\rho \right) - E_{\rho, 1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_1^\rho \right) \right)}{t_2^\rho E_{\rho, \rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_2^\rho \right) - t_1^\rho E_{\rho, \rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_1^\rho \right)} \quad (9)$$

and

$$f(x) = \sum_{n=1}^{\infty} f_n \sin \frac{\pi n x}{l}. \quad (10)$$

To prove this theorem, we carry out the following steps:

1. We solve the forward problem.
2. We solve the control problem.

**Forward problem 3.3.** In this section, we present the solution of the forward problem (3) - (6). We use the Fourier method to solve this problem (see, for example [23], p. 82):

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) X_n(x)$$

and we have the following Sturm-Liouville problem:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = 0, \\ X(l) = 0. \end{cases} \quad (11)$$

Solving problem (11), we have the set of eigenfunctions  $X_n(x) = \sin \frac{\pi n x}{l}$  and eigenvalues  $\lambda_n = \left(\frac{\pi n}{l}\right)^2$ .

Thus, we have

$$u(x, t) = \sum_{n=1}^{\infty} T_n(t) \sin \frac{\pi n x}{l}. \quad (12)$$

Substituting (12) into equation (3), we have the following equation:

$$\sum_{n=1}^{\infty} \left[ D_t^\rho T_n(t) + a^2 \left(\frac{\pi n}{l}\right)^2 T_n(t) \right] = f(x).$$

Thus, using condition (4) and equality (10), we arrive at the following problem:

$$\begin{cases} D_t^\rho T_n(t) + \left(\frac{\pi n a}{l}\right)^2 T_n(t) = f_n, \\ T_n(+0) = \phi_n. \end{cases} \quad (13)$$

Based on the solution of the Cauchy problem for a linear fractional-order differential equation in the Caputo sense, the function  $T_n(t)$  is given as follows (see [24], p. 230):

$$T_n(t) = \phi_n E_{\rho,1} \left( - \left(\frac{\pi n a}{l}\right)^2 t^\rho \right) + f_n t^\rho E_{\rho,\rho+1} \left( - \left(\frac{\pi n a}{l}\right)^2 t^\rho \right). \quad (14)$$

As a result, we have that the formal solution of the forward problem (3) - (6) has the following form:

$$u(x, t) = \sum_{n=1}^{\infty} \left[ \phi_n E_{\rho,1} \left( - \left(\frac{\pi n a}{l}\right)^2 t^\rho \right) + f_n t^\rho E_{\rho,\rho+1} \left( - \left(\frac{\pi n a}{l}\right)^2 t^\rho \right) \right] \sin \frac{\pi n x}{l}.$$

**Control Problem 3.4. Existence of a solution.** In this section, we study the inverse problem (3) - (7). Assuming that the function  $f(x)$  is known, the problem (3) - (6) has a solution of the form (8). Using condition (7), we have the following equality:

$$\begin{aligned} u(x, t_2) - u(x, t_1) &= \sum_{n=1}^{\infty} \left[ \phi_n \left( E_{\rho,1} \left( - \left(\frac{\pi n a}{l}\right)^2 t_2^\rho \right) - E_{\rho,1} \left( - \left(\frac{\pi n a}{l}\right)^2 t_1^\rho \right) \right) + \right. \\ &\quad \left. + f_n \left( t_2^\rho E_{\rho,\rho+1} \left( - \left(\frac{\pi n a}{l}\right)^2 t_2^\rho \right) - t_1^\rho E_{\rho,\rho+1} \left( - \left(\frac{\pi n a}{l}\right)^2 t_1^\rho \right) \right) \right] \sin \frac{\pi n x}{l} = \psi(x). \end{aligned}$$

Expanding the function  $\psi(x)$  into a Fourier series with respect to the system  $\{\sin \frac{\pi n x}{l}\}$  and equating the corresponding coefficients, we have the following equality:

$$\begin{aligned} &\phi_n \left( E_{\rho,1} \left( - \left(\frac{\pi n a}{l}\right)^2 t_2^\rho \right) - E_{\rho,1} \left( - \left(\frac{\pi n a}{l}\right)^2 t_1^\rho \right) \right) + \\ &+ f_n \left( t_2^\rho E_{\rho,\rho+1} \left( - \left(\frac{\pi n a}{l}\right)^2 t_2^\rho \right) - t_1^\rho E_{\rho,\rho+1} \left( - \left(\frac{\pi n a}{l}\right)^2 t_1^\rho \right) \right) = \psi_n. \end{aligned}$$

From this equality, we have

$$f_n \left( t_2^\rho E_{\rho,\rho+1} \left( - \left(\frac{\pi n a}{l}\right)^2 t_2^\rho \right) - t_1^\rho E_{\rho,\rho+1} \left( - \left(\frac{\pi n a}{l}\right)^2 t_1^\rho \right) \right) =$$

$$= \psi_n - \phi_n \left( E_{\rho,1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_2^\rho \right) - E_{\rho,1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_1^\rho \right) \right).$$

From this, it follows that

$$f_n = \frac{\psi_n}{t_2^\rho E_{\rho,\rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_2^\rho \right) - t_1^\rho E_{\rho,\rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_1^\rho \right)} - \frac{\phi_n \left( E_{\rho,1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_2^\rho \right) - E_{\rho,1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_1^\rho \right) \right)}{t_2^\rho E_{\rho,\rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_2^\rho \right) - t_1^\rho E_{\rho,\rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_1^\rho \right)}.$$

Now, we prove the uniform convergence of the Fourier series of the function  $f(x)$  that is

$$f(x) = \sum_{n=1}^{\infty} f_n \sin \frac{\pi n x}{l},$$

where the Fourier coefficient  $f_n$  is defined by equality (9). For this purpose, we introduce the partial sum of the series as follows:

$$F_j(x) = \sum_{n=1}^j f_n \sin \frac{\pi n x}{l}.$$

According to the above equality, we have

$$F_j(x) = \sum_{n=1}^j \left[ \frac{\psi_n}{t_2^\rho E_{\rho,\rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_2^\rho \right) - t_1^\rho E_{\rho,\rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_1^\rho \right)} - \frac{\phi_n \left( E_{\rho,1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_2^\rho \right) - E_{\rho,1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_1^\rho \right) \right)}{t_2^\rho E_{\rho,\rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_2^\rho \right) - t_1^\rho E_{\rho,\rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_1^\rho \right)} \right] \sin \frac{\pi n x}{l}.$$

If we introduce the following notations for

$$F_j^1(x) = - \sum_{n=1}^j \frac{\phi_n \left( E_{\rho,1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_2^\rho \right) - E_{\rho,1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_1^\rho \right) \right)}{t_2^\rho E_{\rho,\rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_2^\rho \right) - t_1^\rho E_{\rho,\rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_1^\rho \right)} \sin \frac{\pi n x}{l}$$

and

$$F_j^2(x) = \sum_{n=1}^j \frac{\psi_n}{t_2^\rho E_{\rho,\rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_2^\rho \right) - t_1^\rho E_{\rho,\rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_1^\rho \right)} \sin \frac{\pi n x}{l}$$

respectively, then we have equality

$$F_j(x) = F_j^1(x) + F_j^2(x).$$

Now, we study each of these terms separately, in other words, we investigate their convergence. For this purpose, we use the asymptotic estimates of the Mittag-Leffler function:

$$E_{\rho,1}(-t) = \frac{t^{-1}}{\Gamma(1-\rho)} + O(t^{-2})$$

and

$$E_{\rho,\rho+1}(-t) = - \sum_{k=1}^2 \frac{(-t)^{-k}}{\Gamma(\rho+1-\rho k)} + O(t^{-3}) = t^{-1} - \frac{t^{-2}}{\Gamma(1-\rho)} + O(t^{-3}).$$

First, using the above asymptotic estimates, we simplify the denominators in the series  $F_j^1(x)$  and  $F_j^2(x)$ .

$$t_2^\rho E_{\rho,\rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_2^\rho \right) - t_1^\rho E_{\rho,\rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_1^\rho \right) =$$

$$\begin{aligned}
 &= t_2^\rho \left( \frac{1}{t_2^\rho} - \frac{\left(\frac{\pi na}{l}\right)^{-4}}{\Gamma(1-\rho)t_2^{2\rho}} + O\left(\frac{\left(\frac{\pi na}{l}\right)^{-6}}{t_2^{2\rho}}\right) \right) - t_1^\rho \left( \frac{1}{t_1^\rho} - \frac{\left(\frac{\pi na}{l}\right)^{-4}}{\Gamma(1-\rho)t_1^{2\rho}} + O\left(\frac{\left(\frac{\pi na}{l}\right)^{-6}}{t_1^{2\rho}}\right) \right) = \\
 &= \left(\frac{\pi na}{l}\right)^{-4} \left( -\frac{1}{\Gamma(1-\rho)t_2^\rho} + O\left(\frac{\left(\frac{\pi na}{l}\right)^{-2}}{t_2^{2\rho}}\right) + \frac{1}{\Gamma(1-\rho)t_1^\rho} - O\left(\frac{\left(\frac{\pi na}{l}\right)^{-2}}{t_1^{2\rho}}\right) \right).
 \end{aligned}$$

Using the asymptotic estimates of the Mittag-Leffler function and the last equality, we obtain the following equality:

$$\begin{aligned}
 F_j^1(x) &= - \sum_{n=1}^j \frac{\phi_n \left( \frac{\left(\frac{\pi na}{l}\right)^{-2}}{\Gamma(1-\rho)t_2^\rho} + O\left(\frac{\left(\frac{\pi na}{l}\right)^{-4}}{t_2^{2\rho}}\right) - \frac{\left(\frac{\pi na}{l}\right)^{-2}}{\Gamma(1-\rho)t_1^\rho} - O\left(\frac{\left(\frac{\pi na}{l}\right)^{-4}}{t_1^{2\rho}}\right) \right)}{\left(\frac{\pi na}{l}\right)^{-4} \left( -\frac{1}{\Gamma(1-\rho)t_2^\rho} + O\left(\frac{\left(\frac{\pi na}{l}\right)^{-2}}{t_2^{2\rho}}\right) + \frac{1}{\Gamma(1-\rho)t_1^\rho} - O\left(\frac{\left(\frac{\pi na}{l}\right)^{-2}}{t_1^{2\rho}}\right) \right)} \sin \frac{\pi nx}{l} = \\
 &= - \sum_{n=1}^j \frac{\phi_n \left(\frac{\pi na}{l}\right)^{-2} \left( \frac{1}{\Gamma(1-\rho)t_2^\rho} + O\left(\frac{\left(\frac{\pi na}{l}\right)^{-2}}{t_2^{2\rho}}\right) - \frac{1}{\Gamma(1-\rho)t_1^\rho} - O\left(\frac{\left(\frac{\pi na}{l}\right)^{-2}}{t_1^{2\rho}}\right) \right)}{\left(\frac{\pi na}{l}\right)^{-4} \left( -\frac{1}{\Gamma(1-\rho)t_2^\rho} + O\left(\frac{\left(\frac{\pi na}{l}\right)^{-2}}{t_2^{2\rho}}\right) + \frac{1}{\Gamma(1-\rho)t_1^\rho} - O\left(\frac{\left(\frac{\pi na}{l}\right)^{-2}}{t_1^{2\rho}}\right) \right)} \sin \frac{\pi nx}{l} = \\
 &= - \sum_{n=1}^j \frac{\phi_n \left(\frac{\pi na}{l}\right)^2 \left( \frac{1}{\Gamma(1-\rho)t_2^\rho} + O\left(\frac{\left(\frac{\pi na}{l}\right)^{-2}}{t_2^{2\rho}}\right) - \frac{1}{\Gamma(1-\rho)t_1^\rho} - O\left(\frac{\left(\frac{\pi na}{l}\right)^{-2}}{t_1^{2\rho}}\right) \right)}{\left( -\frac{1}{\Gamma(1-\rho)t_2^\rho} + O\left(\frac{\left(\frac{\pi na}{l}\right)^{-2}}{t_2^{2\rho}}\right) + \frac{1}{\Gamma(1-\rho)t_1^\rho} - O\left(\frac{\left(\frac{\pi na}{l}\right)^{-2}}{t_1^{2\rho}}\right) \right)} \sin \frac{\pi nx}{l} = \\
 &= C \sum_{n=1}^j n^2 \phi_n \sin \frac{\pi nx}{l}.
 \end{aligned}$$

If  $\phi(x) \in C^2[0, l]$  has a third-order piecewise continuous derivative and satisfies the conditions  $\phi(0) = \phi(l) = 0$ ,  $\phi''(0) = \phi''(l) = 0$ , then the series  $F_j^1(x)$  is uniformly convergent (see [25], p. 241).

Now, we investigate the uniform convergence of the second term  $F_j^2(x)$ . For this purpose, we use the above asymptotic estimate of the Mittag-Leffler function and obtain the following equality:

$$\begin{aligned}
 F_j^2 &= \sum_{n=1}^j \frac{\psi_n}{\left(\frac{\pi na}{l}\right)^{-4} \left( -\frac{1}{\Gamma(1-\rho)t_2^\rho} + O\left(\frac{\left(\frac{\pi na}{l}\right)^{-2}}{t_2^{2\rho}}\right) + \frac{1}{\Gamma(1-\rho)t_1^\rho} - O\left(\frac{\left(\frac{\pi na}{l}\right)^{-2}}{t_1^{2\rho}}\right) \right)} \sin \frac{\pi nx}{l} = \\
 &= \sum_{n=1}^j \frac{\psi_n \left(\frac{\pi na}{l}\right)^4}{-\frac{1}{\Gamma(1-\rho)t_2^\rho} + O\left(\frac{\left(\frac{\pi na}{l}\right)^{-2}}{t_2^{2\rho}}\right) + \frac{1}{\Gamma(1-\rho)t_1^\rho} - O\left(\frac{\left(\frac{\pi na}{l}\right)^{-2}}{t_1^{2\rho}}\right)} \sin \frac{\pi nx}{l} = \\
 &= L \sum_{n=1}^j n^4 \psi_n \sin \frac{\pi nx}{l}, \quad L > 0.
 \end{aligned}$$

It is clear that if  $\psi(x) \in C^4[0, l]$ , has a fifth-order piecewise continuous derivative and satisfies the conditions  $\psi(0) = \psi(l) = 0$ ,  $\psi''(0) = \psi''(l) = 0$  and  $\psi^{(IV)}(0) = \psi^{(IV)}(l) = 0$ , then the above series is uniformly convergent (see [25], p. 241).

Thus, the series (10), whose Fourier coefficients are defined by formula (9) is uniformly convergent.

**Uniqueness of the solution.** Assume that problem (3) - (7) has two different solutions  $\{u_1(x, t), f_1(x)\}$  and  $\{u_2(x, t), f_2(x)\}$ . If we set  $u(x, t) = u_1(x, t) - u_2(x, t)$  and  $f(x) = f_1(x) - f_2(x)$ , then problem (3) - (7)

reduces to the following problem:

$$\begin{cases} D_t^\rho u(x, t) - a^2 u_{xx}(x, t) = f(x), \\ u(x, 0) = 0, \\ u(0, t) = 0, \\ u(l, t) = 0, \\ u(x, t_1) - u(x, t_2) = 0. \end{cases} \quad (15)$$

In that case, we have following Cauchy problem:

$$\begin{cases} D_t^\rho u(x, t) - a^2 u_{xx}(x, t) = f(x), \\ u(x, 0) = 0, \\ u(0, t) = 0, \\ u(l, t) = 0. \end{cases} \quad (16)$$

According to the solution of the forward problem (3) - (6), the solution of this Cauchy problem is given by the following expression:

$$u(x, t) = \sum_{n=1}^{\infty} f_n t^\rho E_{\rho, \rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t^\rho \right) \sin \frac{\pi n x}{l}. \quad (17)$$

If we use the additional condition  $u(x, t_2) - u(x, t_1) = 0$ , we have the following equality:

$$f_n \left[ t_2^\rho E_{\rho, \rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_2^\rho \right) - t_1^\rho E_{\rho, \rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_1^\rho \right) \right] \sin \frac{\pi n x}{l} = 0,$$

since  $\rho \in (0, 1)$  and  $t_1, t_2 \in (0, T)$  satisfy  $t_2^\rho E_{\rho, \rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_2^\rho \right) - t_1^\rho E_{\rho, \rho+1} \left( - \left( \frac{\pi n a}{l} \right)^2 t_1^\rho \right) \neq 0$ , we conclude that the equality  $f_n = 0$  holds for all  $n \geq 1$ . From this, it follows that  $f(x) = 0$  and consequently, by (17), we obtain  $u(x, t) = 0$ . From these equalities, we obtain  $u_1(x, t) = u_2(x, t)$  and  $f_1(x) = f_2(x)$ , hence, the solution of problem (3) - (7) is unique.

The theorem is proved.

## 4. Conculusion

In this paper, a control problem for a process described by a fractional-order partial differential equation was studied. Using the Fourier method, an explicit representation of the solution was obtained. The results obtained can be applied in the study of fractional-order control problems and inverse problems.

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## CARLEMAN FORMULA IN THE MATRIX POLYDISC OF THE SECOND TYPE

BOZOROV JORABEK\*

TERMEZ STATE UNIVERSITY, TERMEZ, UZBEKISTAN  
jorabek.bozorov.89@mail.ru

TURAYEV TURA

TERMEZ STATE UNIVERSITY, TERMEZ, UZBEKISTAN  
turayev.tura.96@mail.ru

**ABSTRACT.** In the present paper, a Carleman-type formula is derived for holomorphic functions defined in a matrix polydisc of the second type. The domain under consideration is represented as the Cartesian product of classical matrix domains of the second type. Particular attention is devoted to the investigation of holomorphic and meromorphic functions in matrix-valued domains and to the construction of an integral representation of Carleman type for this class of matrix domains.

**Key words:** Holomorphic mappings; matrix polydisc set; matrix polydisc domain; meromorphic function; Carleman formula; classical matrix domains.

## Introduction. Statement of the Problem

Let  $D \subset \mathbb{C}$  be a bounded domain with piecewise smooth boundary  $\partial D$ . If the function  $f(z)$  is holomorphic in the domain  $D \subset \mathbb{C}$  and continuous on the closure  $\overline{D}$ , then, as is well known, the values of this function inside the domain  $D \subset \mathbb{C}$  can be reconstructed from its boundary values on  $\partial D$  by means of the Cauchy integral formula.

Naturally, the following question arises: if the function  $f(z)$  is holomorphic in the domain  $D$  and continuous on the boundary  $\partial D$ , while its values are prescribed only on a certain part of the boundary of the domain, then in what manner can the values of the function  $f(z)$  inside the domain  $D$  be reconstructed?

A positive answer to this problem was first obtained by T. Carleman in 1926 (see [1], [3]). The corresponding integral representations subsequently became known as the *Carleman formulas*.

Later, in 1933, G. M. Goluzin and V. I. Krylov (see [2]) developed a method based on the introduction of an extinguishing function, which made it possible to solve this problem for an arbitrary simply connected domain of the complex plane.

They proved the following statement: if  $f \in H^1(D)$  and the set  $M \subset \partial D$  has positive Lebesgue measure, then for every point  $z \in D$  the following Carleman formula holds

$$f(z) = \lim_{m \rightarrow \infty} \frac{1}{2\pi i} \int_M f(\xi) \left[ \frac{\varphi(\xi)}{\varphi(z)} \right]^m \frac{d\xi}{\xi - z},$$

where  $\varphi(z) \in H^\infty(D)$  satisfies the following two conditions:

1.  $|\varphi(z)| = 1$  almost everywhere on  $\partial D \setminus M$ ,
2.  $|\varphi(z)| > 1$  in  $D$ .

Subsequently, A. Kytmanov and N. Nikitina developed a new approach to deriving the Carleman formula for arbitrary homogeneous domains. By means of this method, Carleman-type formulas were obtained for a number of classical domains. Moreover, this problem has also been the subject of scientific investigations by several researchers from Uzbekistan, among whom are G. Khudoyberganov [6], B. Shoimkulov [8], U. Rakhmonov [7], R. Kosbergenov [9], J. Bozorov [10], and J. Abdullaev [12].

In the present work, the function under consideration is holomorphic in the matrix polydisc of the second type, while its values are prescribed only on a part of the skeleton. The problem consists in reconstructing the values of the function inside the domain from its known values on a part of the skeleton.

### Main part

Let

$$D_2 = \{Z \in \mathbb{C}[m \times m] : Z\bar{Z} < I\}$$

be the classical domain of the second type [3], where  $Z$  is a symmetric matrix of order  $n \times n$ ,  $\bar{Z}$  denotes the complex conjugate matrix of  $Z$  and  $I$  is the identity matrix of order  $n \times n$ . The skeleton of the domain is defined by

$$S_2 = \{\xi \in \mathbb{C}[m \times m] : \xi\bar{\xi} = I\}.$$

The Cartesian product of  $n$  copies of the domain  $D_2$  will be called the matrix polydisc of the second type

$$\underbrace{D_2 \times D_2 \times \cdots \times D_2}_n$$

and will be denoted by  $T_2^n$ , that is

$$T_2^n = \left\{ z = (z^{(1)}, z^{(2)}, \dots, z^{(n)}) \in \mathbb{C}^n[m \times m] : z^{(j)}\bar{z}^{(j)} < I, \quad j = 1, \dots, n \right\}.$$

Furthermore, we define the skeleton of the matrix polydisc of the second type as the Cartesian product of  $n$  copies of  $S_2$  and denote it by  $S(T_2^n)$ , namely,

$$S(T_2^n) = \left\{ \xi = (\xi^{(1)}, \xi^{(2)}, \dots, \xi^{(n)}) \in \mathbb{C}^n[m \times m] : \xi^{(j)}\bar{\xi}^{(j)} = I, \quad j = 1, 2, \dots, n \right\}.$$

Let the function  $f(Z)$  be defined in the domain  $T_2^n$ .

**Definition 1** ([3]). Let the function  $f(Z)$  be holomorphic in the domain  $T_2^n$  and satisfy the condition

$$\sup_{0 \leq r < 1} \int_{S(T_2^n)} |f(r\xi)| d\mu(\xi) < \infty,$$

where  $d\mu(\xi)$  denotes the normalized Lebesgue measure on  $S(T_2^n)$ . Then the function  $f(Z)$  is said to belong to the Hardy class in the domain  $T_2^n$ , and this is denoted by

$$f(Z) \in H^1(T_2^n).$$

Since the matrix polydisc of the second type is the Cartesian product of  $n$  classical domains of the second type, the automorphism of the matrix polydisc possesses  $n$  coordinates, and in each coordinate there exists an automorphism of the classical domain of the second type, namely.

$$\phi_A(Z) = (\phi_A^1(z^1), \phi_A^2(z^2), \dots, \phi_A^n(z^n)),$$

where

$$\phi_A^j(z^j) = R^j (z^j - A^j) (I - \bar{A}^j z^j)^{-1} (\bar{R}^j)^{-1}. \quad (1)$$

Here the matrices  $R^j, A^j, z^j$  satisfy the following conditions:

$$\bar{R}^j (I - A^j (A^j)^T) (R^j)^T = I$$

$$A^j \bar{R}^j + R^j \bar{A}^j = 0.$$

Mapping (1) is an automorphism of the classical domain of the second type.

The automorphism  $\phi_A(Z)$  maps the point  $A \in T_2^n$  to the origin, and the point 0 to the point  $A$ .

Let  $E \subset S(T_2^n)$  be a set of positive measure, that is,

$$(\mu E > 0).$$

Denote by  $S(T_2^{n-1})$  the projection of the set  $E$  onto the space  $\mathbb{C}^{n-1}[m \times m]$ . Let the points of the set  $S(T_2^{n-1})$  be denoted by

$$\xi = (\xi^2, \dots, \xi^n).$$

Introduce the following notation:

$$E_{0,\xi} = \left\{ Z : z \in E, \phi_0^1(z^1) = \lambda, \phi_0^j(z^j) = \lambda \phi_0^j(\xi^j), j = \overline{2, n}, \lambda \in S_2 \right\}$$

$$E_0 = \{z \in E, \mu_1 E_{0,\xi} > 0\}.$$

By Fubini's theorem, the following relations hold:

$$\mu_{n-1} E > 0, \quad E_{0,\xi} \times E_0 = E.$$

Consider the auxiliary function defined by

$$\psi_0(\xi) = \frac{1}{2\pi i} \int_{E_{0,\xi}^1} \frac{\eta + \lambda^*}{\eta - \lambda^*} \cdot \frac{d\eta}{\eta}, \quad \varphi_0 = \exp \psi_0,$$

where

$$E_{0,\xi}^1 = \{\xi^1 \in E_{0,\xi}, \xi^1 = \alpha U, \alpha = e^{i\varphi}, 0 \leq \varphi \leq 2\pi\}.$$

Denote the complementary set of  $E_{0,\xi}^1$  with respect to  $\tilde{E}_{0,\xi}$  by  $\overline{E}_{0,\xi}$ , that is,

$$E_{0,\xi}^1 \times \overline{E}_{0,\xi} = E_{0,\xi}.$$

Thus, we have defined a section of the above-mentioned set  $E \subset S(T_2^n)$  by a hyperline passing through the origin via  $E_{0,\xi}$ . Since this set is contained in  $S_2$ , for the function  $f$  belonging to the Hardy class in the classical domain of the second type, the following formula holds.

$$f(0) = \frac{m}{\int_{\overline{E}_{0,\xi}} d\mu_0} \lim_{l \rightarrow \infty} \int_{E_{0,\xi}} f(\xi) \left[ \frac{\varphi_0(\xi)}{\varphi_0(0)} \right]^l d\mu_1$$

Integrating both sides of this equality over the set  $E_0$ , we obtain

$$\int_{E_0} f(0) d\mu_{n-1} = \int_{E_0} \left[ \frac{m}{\int_{\overline{E}_{0,\xi}} d\mu_0} \lim_{l \rightarrow \infty} \int_{E_{0,\xi}} f(\xi) \left[ \frac{\varphi_0(\xi)}{\varphi_0(0)} \right]^l d\mu_1 \right] d\mu_{n-1}$$

As a result, we arrive at the equality

$$f(0) = \frac{m}{\int_{E_0} d\mu_{n-1} \int_{\overline{E}_{0,\xi}} d\mu_0} \lim_{l \rightarrow \infty} \int_E f(\xi) \left[ \frac{\varphi_0(\xi)}{\varphi_0(0)} \right]^l d\mu(\xi)$$

By Lebesgue's theorem, it follows that both the integral and the limit exist. Therefore, the formula proved above can be formulated in the form of the following lemma.

**Lemma 1.**

If

$$f(z) \in H^1(T_2^n), \quad E \subset S(T_2^n), \quad \mu E > 0,$$

then the following formula holds:

$$f(0) = \frac{m}{\int_{E_0} d\mu_{n-1} \int_{\bar{E}_{0,\xi}} d\mu_0} \lim_{l \rightarrow \infty} \int_E f(\xi) \left[ \frac{\varphi_0(\xi)}{\varphi_0(0)} \right]^l d\mu(\xi). \quad (2)$$

It follows from the above lemma that the value of the function  $f(z)$  at the origin of the domain  $T_2^n$  has been reconstructed from the values of the function  $f(z)$  on the set  $E$ .

Now, by means of this lemma, we reconstruct the values of the function  $f(z)$  at an arbitrary point of the domain  $T_2^n$ .

For this purpose, define

$$E_{A,\xi} = \left\{ Z \in E, \phi_A^1(z^1) = \lambda, \phi_A^j(z^j) = \lambda \phi_A^j(\xi^j), j = \overline{2, n}, \lambda \in S_2 \right\},$$

$$E_A = \{\xi' : \xi' \in S(T_{n-1}), \mu_1 E_{A,\xi} > 0\},$$

where

$$E_{A,\xi} \times E_A = E,$$

and

$$\varphi_A = \exp \psi_A,$$

$$\psi_A(\xi) = \frac{1}{2\pi i} \int_{E_{A,w}^1} \frac{\eta + \lambda}{\eta - \lambda} \cdot \frac{d\eta}{\eta},$$

$$E_{A,w}^1 = \{\xi^1 \in E_{A,\xi} : \xi^1 = (\phi_A^1)^{-1}(\lambda(\phi_A^1)^{-1}(w)), |\lambda| = 1\},$$

$$w \in \phi_A^1(SU(m)).$$

**Theorem 1.**

If

$$f(z) \in H^1(T_2^n), \quad E \subset S(T_2^n), \quad \mu E > 0,$$

then for every point  $A \in T_2^n$  the following formula holds:

$$f(A) = \frac{m}{\int_{E_A} d\mu_{n-1} \int_{E_{A,\xi}} d\mu_0} \lim_{l \rightarrow \infty} \int_E f(\xi) \left[ \frac{\varphi_A(\xi)}{\varphi_A(A)} \right]^l \prod_{j=1}^n H(A^j; \xi^j) d\mu(\xi). \quad (3)$$

where

$$H(A^j; \xi^j) = \frac{1}{\det^{\frac{m+1}{2}}(I - A^j \bar{\xi}^j)}$$

is the Cauchy–SzegE kernel for the classical domain of the second type. **Proof.** In the case  $A = 0$ , Theorem 1 was proved in Lemma 1. Now consider the case  $A \neq 0$ . By virtue of the lemma, the value of the function  $f(z)$  at the point 0 of the domain  $T_2^n$  was reconstructed in the following form:

$$f(0) = \frac{m}{\int_{E_0} d\mu_{n-1} \int_{\bar{E}_{0,\xi}} d\mu_0} \lim_{l \rightarrow \infty} \int_E f(\xi) \left[ \frac{\varphi_0(\xi)}{\varphi_0(0)} \right]^l d\mu(\xi).$$

Replacing  $f(z)$  in this formula by the function  $f(\phi_A(0))$ , we obtain the equality

$$f(\phi_A(0)) = \frac{m}{\int_{E_0} d\mu_{n-1} \int_{\overline{E}_{0,\xi}} d\mu_0} \lim_{l \rightarrow \infty} \int_E f(\xi) \left[ \frac{\varphi_0(\xi)}{\varphi_0(\phi_A(0))} \right]^l d\mu(\phi_A(\xi)).$$

Here  $\phi_A(0) = A$ . Taking into account the relation

$$d\mu(\phi_A(\xi)) = |\det J(\xi, A)| d\mu(\xi)$$

and

$$|\det J(\xi, A)| = P(A, \xi) = \prod_{j=1}^n \frac{H(A^j, \bar{\xi}^j) H(\xi^j, \bar{A}^j)}{H(A^j, \bar{A}^j)},$$

we obtain

$$f(A) = \frac{m}{\int_{E_A} d\mu_{n-1} \int_{E_{A,\xi}} d\mu_0} \lim_{l \rightarrow \infty} \int_E f(\xi) \left[ \frac{\varphi_A(\xi)}{\varphi_A(A)} \right]^l \prod_{j=1}^n \frac{H(A^j, \bar{\xi}^j) H(\xi^j, \bar{A}^j)}{H(A^j, \bar{A}^j)} d\mu(\xi).$$

Further, using the fact that

$$f(\xi) \left( \prod_{j=1}^n H(\xi^j, \bar{A}^j) \right)^{-1} \in H^1(T_2^n)$$

for the function under consideration, we obtain

$$\frac{f(A)}{\prod_{j=1}^n H(A^j, \bar{A}^j)} = \frac{m}{\int_{E_A} d\mu_{n-1} \int_{E_{A,\xi}} d\mu_0} \lim_{l \rightarrow \infty} \int_E f(\xi) \left[ \frac{\varphi_A(\xi)}{\varphi_A(A)} \right]^l \prod_{j=1}^n \frac{H(A^j, \bar{\xi}^j) H(\xi^j, \bar{A}^j)}{H(A^j, \bar{A}^j)} d\mu(\xi).$$

Further, we obtain

$$f(A) = \frac{m}{\int_{E_A} d\mu_{n-1} \int_{E_{A,\xi}} d\mu_0} \lim_{l \rightarrow \infty} \int_E f(\xi) \left[ \frac{\varphi_A(\xi)}{\varphi_A(A)} \right]^l \prod_{j=1}^n H(A^j, \bar{\xi}^j) d\mu(\xi). \quad (4)$$

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$C^n$  - LUZINE PROPERTIES FOR  $\alpha$  - SUBHARMONIC FUNCTIONS

GADAEV SOKHIB

SAMARKAND STATE UNIVERSITY NAMED AFTER SH.RASHIDOV, SAMARKAND, UZBEKISTAN  
sokhibgadayevev@gmail.com

**ABSTRACT.** In this paper, the smoothness properties of  $\alpha$ -subharmonic functions in the sense of Luzin are studied. For any  $\varepsilon > 0$ , it is proved that there exists an open set  $G_\varepsilon$  with small  $\alpha$ -capacity such that the function  $u$  belongs to the class  $C^k$  on compact subsets of  $D \setminus G_\varepsilon$ . The results are based on the Riesz representation theorem and estimates of Riesz potentials, generalizing the classical theorems of N. Luzin, A. Cartan, A. Sadullaev, R. Madrahimov, and S. Imomkulov to the case of  $\alpha$ -subharmonic functions.

**MSC (2020):** 31B05; 31C15; 32U05; 46E35; 26B05.

**Key words:**  $\alpha$ -subharmonic functions, Luzin property, differential smoothness, Riesz potential, Borel measure, capacity, Hausdorff measure, Whitney extension, elliptic operator, convex functions.

## 1. Introduction

N.N. Luzin's remarkable theorem states that every measurable function  $f(x)$  in the domain  $D \subset \mathbb{R}^n$  is continuous outside some thin set ( $C$ -Luzin property): For any  $\varepsilon > 0$ , there exists a subset of  $U_\varepsilon \subset D$  with Lebesgue measure  $m(U_\varepsilon) < \varepsilon$  and the constriction of the function  $f|_{D \setminus U_\varepsilon}$  is continuous (see, for example [1]).

For subharmonic functions in terms of Newtonian (logarithmic for  $n = 2$ ) capacitance, a similar theorem was proven by A. Cartan [2] (see also [3], [4]). In this case, the discarded set is even thinner: let  $u(x)$  be a subharmonic function in the domain  $D \subset \mathbb{R}^n$ . Then for any  $\varepsilon > 0$  there exists an open set  $U_\varepsilon \subset D$  such that the capacitance  $C_{n-2}(U_\varepsilon) < \varepsilon$  and the constriction  $u|_{D \setminus U_\varepsilon}$  are continuous, i.e.  $u \in C(D \setminus U_\varepsilon)$ .

In the work [6] (see also [7]), the smoothness of subharmonic functions is established outside some thicker sets than small-capacity sets. Specifically, the following theorem is proven: let  $u(x)$  be a subharmonic function in the domain  $D \subset \mathbb{R}^n$ ,  $p$  be a fixed number,  $0 < p \leq 2$ . Then for any  $\varepsilon > 0$  there exists an open set  $G_{p,\varepsilon} \subset D$  such that the Hausdorff cover  $H_{n-2+p}(G_{p,\varepsilon}) < \varepsilon$  and the function  $u(x)$  belongs to the Lipschitz class  $Lip_p$ , for  $0 < p < 2, p \neq 1$ , and also belongs to the class  $Lip_{p-0}$ , for  $p = 1, 2$ , on the compact subsets of the equality  $D \setminus G_{p,\varepsilon}$ . Here

$$Lip_{p-0} = \bigcap_{k < p} Lip_k.$$

In the works [8,9] the differentiability of subharmonic functions is studied in the cases  $p = 1, 2$ . Relatively, final results were obtained.

**Theorem 1.1 ([8]).** Let  $u(x)$  be a subharmonic function in the domain  $D \subset \mathbb{R}^n$ . Then for any  $\varepsilon > 0$  there exists an open set  $G_\varepsilon \subset D$  with Lebesgue measure  $m(G_\varepsilon) < \varepsilon$  such that  $u(x)$  belongs to the class  $C^2$  on the compact subset of the difference  $D \setminus G_\varepsilon$ .

**Theorem 1.2 ([9]).** Let  $u(x)$  be a subharmonic function in a domain  $D \subset \mathbb{R}^n$ . Then for any  $\varepsilon > 0$  there exists an open set  $G_\varepsilon \subset D$  with capacity  $C_{n-1}(G_\varepsilon) < \varepsilon$  such that  $u(x)$  belongs to the class  $C^1$  on compact subsets of the difference  $D \setminus G_\varepsilon$ .

Combining the above theorems, the following general theorem was formulated in the review paper [12].

**Theorem 1.3 ([11]).** Let  $u(x)$  be a subharmonic function in the domain  $D \subset \mathbb{R}^n$ . Then for any number  $p : 0 \leq p \leq 2$  and  $\varepsilon > 0$  there exists an open set  $G_{p,\varepsilon} \subset D$  with capacitance  $C_{n-2+p}(G_{p,\varepsilon}) < \varepsilon$  (in the case  $p = 2$  the Lebesgue measure  $m(G_{2,\varepsilon}) < \varepsilon$ ) such that  $u(x)$  belongs to the class  $C^p$  on compact subsets of the difference  $D \setminus G_{p,\varepsilon}$ .

Let us note that the proof of the above formulated theorems is based on the theorem on the Riesz representation for subharmonic functions (see, for example, [4], [5], [7], i.e., the differentiability of potentials outside a set of small capacity (Hausdorff measure) is proved).

An important subclass of the set of subharmonic functions is the class of convex functions. The properties of convex functions, in particular the property of differentiability, are well studied and are one of the main subjects of study in the theory of convex surfaces and convex analysis. You can read about them, for example, in the monographs of A.D. Alexandrov [22] and R. Rockafeller [21]. Note that a convex function has a one-sided directional derivative at any point [21]. Moreover, in the works of A.D. Alexandrov [23] and Yu.G. Reshetnyak [24] the almost everywhere double differentiability of convex functions was proved (see also [25], [26]), i.e. if  $u(x)$  is a convex function in the domain  $D \subset \mathbb{R}^n$ , then

$$u(x+h) - u(x) - \sum_{i=1}^n \frac{\partial u(x)}{\partial x_i} h_i - \frac{1}{2} \sum_{i,k=1}^n \frac{\partial^2 u(x)}{\partial x_i \partial x_k} h_i h_k = O(|h|^2)$$

in almost all points  $x \in D$ .

Obviously, such a statement does not hold in the case of subharmonic functions, generally speaking. For example, for a subharmonic function with values  $-\infty$  everywhere on a dense set of points, such a statement is not true.

In the recent work [28] (see also [27]), the following theorem is proven: let  $D \subset \mathbb{R}^n$  be a-convex domain and  $u(x) : D \rightarrow \mathbb{R}$  be a locally strictly convex function. Then for any continuous function  $\varepsilon(x) : D \rightarrow (0, 1)$  and for any number  $\varepsilon_0 > 0$  there exists a locally strictly convex function  $g(x)$  from the class  $C^2(D)$  such that

- 1)  $m\{x \in D : u(x) \neq g(x)\} < \varepsilon_0$ ;
- 2)  $|u(x) - g(x)| < \varepsilon(x), x \in D$ ;
- 3) Hausdorff measure  $H_n(\Gamma_u \Delta \Gamma_g) < \varepsilon_0$ , where  $\Gamma_u = \{x, u(x) \in \mathbb{R}^{n+1} : x \in D\}$ .

Note that in the work [10] a similar theorem is investigated for monotonic and convex functions in a one-dimensional case.

In the work of Daujanov A. Sh. [12], the differential properties of the generalized solutions of elliptic equations with constant coefficients were proven, i.e., the following theorem of the type  $C$  – Luzin’s property: Let  $f \in \mathcal{D}'(G)$  be a generalized function of the order of singularity  $s : 0 \leq s \leq m$ . Then for any  $\varepsilon > 0$  there exists an open set  $U_\varepsilon$  with Lebesgue measure  $m(U_\varepsilon) < \varepsilon$  such that the solution  $u(x)$  of the equation  $P(D)u = f$  belongs to the class  $C^{m-s}$  on compact subsets of the difference  $G \setminus U_\varepsilon$ .

Here  $P(D)u(x) = \sum_{|j| \leq m} \alpha_j D^j$  is an elliptic differential operator of the  $m$ -order with constant coefficients.

Recently, a more definitive result for functions from the Sobolev class was obtained in the works of B. Boyarsky, P. Hajlasz, and P. Strzeloski [15] (see also [13],[14]): let  $G \subset \mathbb{R}^n$  be a domain.  $u(x) \in W_{\text{loc}}^{k,p}(G)$  there exist a closed subset  $E \subset G$  and a function  $g(x) \in C^m(G) \cap W_{\text{loc}}^{m+1,p}(G)$  such that

- (i) Bessel capacity  $B_{k-m,p}(G \setminus E) < \varepsilon$ ;
- (ii)  $D^\alpha u(x) = D^\alpha g(x)$  for all  $x \in E$  and for each  $\alpha : |\alpha| \leq m$ ;
- (iii)  $u(x) - g(x) \in W^{m+1,p}(G)$ ;
- (iv)  $\|u(x) - g(x)\|_{W^{m+1,p}(G)} \leq C(n, k, p) \|u\|_{W^{m+1,p}(G \setminus E)} < \varepsilon$ .

(See also the following works [29],[30] related to this topic)

Let us give the definition of the class  $C^k(E)$  ( $k$ -non-negative number) for an arbitrary closed set  $E \subset \mathbb{R}^n$ : we will say that the function  $u(x)$  defined on  $E$  belongs to the class  $C^k(E)$  if there exist certain functions  $E$  defined on  $u^{(j)}(x)$ ,  $j \leq k$  such that  $u^{(0)} = u$ ,

$$\left| u^{(j)}(x+h) - \sum_{|j+l| \leq k} \frac{u^{(j+l)}(x)}{l!} h^l \right| \leq \alpha(h) |h|^{k-|l|} \quad (*)$$

$x, x + h \in E$ ,  $0 \leq |j| \leq k$ ,  $\alpha(h) \rightarrow 0$  at  $h \rightarrow 0$ . Here,  $j = (j_1, j_2, \dots, j_n)$  and  $l = (l_1, l_2, \dots, l_n)$  are multi-indices,  $|j| = j_1 + j_2 + \dots + j_n$ ,  $l! = l_1! l_2! \dots l_n!$ ,  $h^l = h_1^{l_1} h_2^{l_2} \dots h_n^{l_n}$ .

Note that in the case  $E = \mathbb{R}^n$ , the space  $C^k(\mathbb{R}^n)$  has a simple differential property: it consists of functions  $u$  having continuous partial derivatives up to the  $k$ -th order. The following Whitney theorem [19] (see also [20]) on extension establishes a connection between  $C^k(E)$  and  $C^k(\mathbb{R}^n)$ .

**Theorem 1.4 (Whitney).** Let  $E \subset \mathbb{R}^n$  be a closed set and  $u(x) \in C^k(E)$ . Then  $u(x)$  extends to the entire space  $\mathbb{R}^n$  as a function of class  $C^k(\mathbb{R}^n)$ , i.e., there exists a function  $F(x) \in C^k(\mathbb{R}^n)$  such that  $F(x) \equiv u(x)$  on  $E$ .

## 2. Definition of $\alpha$ -subharmonic functions and their properties

**Definition of  $\alpha$ -subharmonic functions.** (see [16]) Let  $\alpha$  be an arbitrary  $d$ -closed, strictly positive differential form of **bidegree**  $(n-1, n-1)$  in the domain  $D \subset \mathbb{C}^n$ ,  $n \geq 2$ :

$$\alpha = \left(\frac{i}{2}\right)^{n-1} \sum_{j,k=1}^n \alpha_{jk}(z) dz[j] \wedge d\bar{z}[k], \quad \alpha_{jk}(z) \in C^1(D), \quad d\alpha = 0. \quad (2.1)$$

Strictly positive  $\alpha$  means that for any compact domain  $G \subset D$  there exists a number  $\varepsilon > 0$  such that the differential form  $\alpha - \varepsilon \beta^{n-1} \geq 0$ , where  $\beta = dd^c|z|^2$ .

**Definition 2.1.** A twice smooth  $u(z) \in C^2(D)$  function is called  $\alpha$ -subharmonic in the  $D \subset \mathbb{C}^n$  domain if the differential form  $dd^c u \wedge \alpha \geq 0$  is in  $D \subset \mathbb{C}^n$ . A  $u(z) \in L^1_{\text{loc}}(D)$  function is called  $\alpha$ -subharmonic in the  $D \subset \mathbb{C}^n$  domain if

- 1) it is upper semicontinuous in  $D$ , i.e.  $u(z^0) \geq \limsup_{z \rightarrow z^0} u(z)$ ,  $\forall z^0 \in D$ ;
- 2) operator  $dd^c u \wedge \alpha \geq 0$  in a generalized sense, i.e.  $\int u(z) \alpha(z) \wedge dd^c \omega(z) \geq 0$ ,  $\forall \omega \in F(D)$ ,  $\omega \geq 0$ .

The class  $\alpha$ -of subharmonic functions is denoted by  $\alpha - \text{sh}(D)$ , including in this class, for convenience, the function  $u(z) \equiv -\infty$ .

**Definition 2.2.** A function  $D \subset \mathbb{C}^n$  that is twice smooth in a domain  $u(z) \in C^2(D)$  is called  $\alpha$ -harmonic if  $dd^c u \wedge \alpha = 0$  in  $D$ .

In work [16], it is shown that for the strictly positive closed differential form  $\alpha$  **biomogeneous**  $(n-1, n-1)$ , the operator  $dd^c u \wedge \alpha$  is a self-adjoint elliptic operator.

Since  $dd^c u \wedge \alpha$  is a second-order elliptic operator, the theory of elliptic equations of mathematical physics is applicable here. In particular, if the coefficients  $\alpha_{jk} \in C^{l+\lambda}$ , where  $C^{l+\lambda}$  is a class of  $l$ -times differentiable functions, and the  $l$ -th partial derivatives belong to the Holder class  $L_\lambda$ ,  $0 < \lambda \leq 1$ , then a solution to the equation  $dd^c u \wedge \alpha = 0$  exists and belongs to the class  $C^{l+2+\lambda}$  (see [31]).

In this work, we assume everywhere that all coefficients of the differential form belong to the Holder class  $\text{Lip}_\lambda$ ,  $0 < \lambda \leq 1$ .

- 1) According to the theory of elliptic operators, the operator  $dd^c u \wedge \alpha$  has a symmetric fundamental solution  $K_\alpha(z, w) \in C^{2+\lambda}(\mathbb{C}^n \setminus \{w\})$ . Let's list some properties of the fundamental solution:

- a)  $K_\alpha(z, w) \in C^{2+\lambda}(\mathbb{C}^n \setminus \{w\})$ ,  $K_\alpha(w, w) = -\infty$ .
- b)  $K_\alpha(z, w) < 0$ ,  $\forall z, w \in \mathbb{C}^n$  and for fixed  $w \in \mathbb{C}^n$  (remember that  $n \geq 2$ )

$$\lim_{z \rightarrow w} K_\alpha(z, w) = 0.$$

- c)  $K_\alpha(z, w)$  is  $\alpha$ -subharmonic function in  $\mathbb{C}^n$ , and it is  $\alpha$ -harmonic at  $z \neq w$ .
- d) There are constants  $c_1 > 0$  and  $c_2 > 0$  such that

$$\frac{c_1}{|z - w|^{n-2}} \leq |K_\alpha(z, w)| \leq \frac{c_2}{|z - w|^{n-2}}; \quad (2.2)$$

$$\left| \frac{\partial K_\alpha(z, w)}{\partial z_j} \right| = \left| \frac{\partial K_\alpha(z, w)}{\partial \bar{z}_j} \right| \leq \frac{c_3}{|z - w|^{n-1}}, \quad j = 1, 2, \dots, n; \quad (2.3)$$

$$\max \left\{ \left| \frac{\partial^2 K_\alpha(z, w)}{\partial z_j \partial z_k} \right|, \left| \frac{\partial^2 K_\alpha(z, w)}{\partial \bar{z}_j \partial \bar{z}_k} \right|, \left| \frac{\partial^2 K_\alpha(z, w)}{\partial \bar{z}_j \partial z_k} \right|, j, k = 1, 2, \dots, n. \right\} \leq \frac{c_4}{|z - w|^{2n}}, \quad (2.4)$$

$\forall z, w \in \mathbb{R}^n, n \geq 2$ . (see. [17], [31]).

- 2) Riesz representation (see [16]). If  $u(z)$  is  $\alpha$ -subharmonic function, then for any subdomain  $G \Subset D$  the following holds

$$u(z) = \int_G K_\alpha(z, w) d\mu(w) + g(z), \quad (2.5)$$

where  $\mu = dd^c u \wedge \alpha$  and  $g(z)$  is  $\alpha$ -harmonic functions in  $G$ .

### 3. Hausdorff measures and $C_{n-\alpha}$ -capacity

**3.1. Hausdorff measures.** Let  $E$  be an arbitrary set from  $\mathbb{R}^n$ . Let us consider coverings of a set  $E$  by finite or countable sets of balls  $B_i$  of diameters  $d_i$ :

$$E \subset \bigcup_i B_i$$

and let's put

$$\hat{H}_s(E) = \inf \sum d_i^s, \quad s > 0,$$

where **infimum** is taken over all such covers.  $\hat{H}_s(E)$  is called the Hausdorff scope of order  $s > 0$ . We impose an additional condition on  $d_i$  such that  $d_i < \varepsilon$ ,  $\varepsilon > 0$ . We denote the corresponding exact lower limit by  $H_s^\varepsilon(E)$ . Then  $H_s^\varepsilon$  will be a non-increasing function of the variable  $\varepsilon$ , i.e.

$$H_s^{\varepsilon_2}(E) \leq H_s^{\varepsilon_1}(E) \quad \text{PřCřPřPř } \varepsilon_2 > \varepsilon_1 > 0.$$

The limit  $H_s(E) = \lim_{\varepsilon \rightarrow 0+} H_s^\varepsilon(E)$  is called the outer  $\alpha$ -Hausdorff measure of the set  $E$ .

Let us note some properties of the quantities  $\hat{H}_s$  and  $H_s$ :

- 1)  $\hat{H}_s(E) \leq \sum_i \hat{H}_s(E_i)$  and  $H_s(E) \leq \sum_i H_s(E_i)$  for any finite and countable family  $\{E_i\}$  such that  $E \subset \bigcup_i E_i$ ;
- 2)  $\hat{H}_s(E) \leq H_s(E)$ ;
- 3)  $\hat{H}_s(E)$  and  $H_s(E)$  are equal to zero simultaneously.

A set  $E \subset \mathbb{R}^n$  is called  $H_s$  measurable if

$$H_s(A) = H_s(E \cap A) + H_s(A \setminus E)$$

for any  $A \subset \mathbb{R}^n$  (see [31]). If a set  $E$  is  $H_s$ -measurable, then  $H_s(E)$  is called the Hausdorff measure of the set  $E$ .

- 4) Let  $E_1, E_2, \dots, E_n, \dots$  be pairwise disjoint  $H_s$ -measurable sets, then

$$H_s \left( \bigcup_i E_i \right) = \sum_i H_s(E_i)$$

i.e.  $H_s$  is a countably additive measure;

- 5) every Borel set  $H_s$  is measurable.

Detailed information about the Hausdorff measure and its properties can be found in the monographs [4], [18], [31].

**3.2.  $C_{n,s}$  is the capacity ([4],[5]).** Let  $\mathcal{M}(\mathbb{R}^n)$  ( $0 < s < n$ ) be the set of all Borel measures in  $\mathbb{R}^n$ ,  $n \geq 2$ , and let

$$u_s^\mu(x) = \int \frac{d\mu(y)}{|x-y|^{n-s}}, \quad x \in \mathbb{R}^n,$$

potential of the measure  $\mu$ .  $C_{n-s}$  - the capacity of a compact set  $K \subset \mathbb{R}^n$  is called the quantity

$$C_{n-s}(K) = \sup\{\mu(\mathbb{R}^n) : \mu \in \mathcal{M}(\mathbb{R}^n), \text{supp}(\mu) \subset K, u_s^\mu(x) \leq 1, x \in \text{supp}(\mu)\}.$$

For an arbitrary set  $E \subset \mathbb{R}^n$ , the quantities

$$\underline{C}_{n-s}(E) = \sup\{C_{n-s}(K) : K \Subset E\} \quad \text{and} \quad \overline{C}_{n-s}(E) = \inf\{C_{n-s}(G) : G = G^0 \supset E\}$$

are called internal and external capacities, respectively.

If  $\overline{C}_{n-s}(E) = \underline{C}_{n-s}(E)$ , then the set  $E$  is called  $C$ -measurable (measurable in capacity) and we will write  $C_{n-s}(E)$  instead of  $\underline{C}_{n-s}(E)$  or  $\overline{C}_{n-s}(E)$ . It is known that all Borel sets  $C$ -are measurable (see [4]).

Let us note some properties of  $C_{n-s}$ -capacity:

- 1)  $C_{n-s}(E) \leq \sum_i C_{n-s}(E_i)$  for any finite or countable family of  $C$ -measurable sets  $\{E_i\}$  such that  $E = \bigcup_i E_i$ ;
- 2)  $C_{n-s}(B(x, r)) = A(n, s)r^{n-s}$ , where  $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ ,  $A(n, s)$  is a constant depending only on  $n$  and  $s$ .

In what follows we will need the following lemmas.

**Lemma 3.1. ([9]).** Let  $\mu$  be a finite Borel measure with compact support  $\text{supp}(\mu) \subset \mathbb{R}^n$ . Then for any  $\varepsilon > 0$  there exists an open set  $U_\varepsilon \subset \mathbb{R}^n$  with capacity  $C_{n-s}(U_\varepsilon) < \varepsilon$  such that

- a) the family of functions

$$\int_{|x-y|<\delta} \frac{d\mu(y)}{|x-y|^{n-s}}$$

converges uniformly to zero as  $\delta \rightarrow 0$  on the set  $\mathbb{R}^n \setminus U_\varepsilon$ ;

- b) the family of functions

$$\frac{\mu(B(x, t))}{t^{n-s}}$$

converges uniformly to zero as  $t \rightarrow 0$  on the set  $\mathbb{R}^n \setminus U_\varepsilon$ .

**Lemma 3.2. ([6]).** Let  $\mu$  - be a finite Borel measure in  $\mathbb{R}^n$  and  $\mu(t) = \mu(B(0, t))$  - the measure of an open ball  $B(0, t) \subset \mathbb{R}^n$ . Then for any smooth function  $h(t)$  in the interval  $(0, +\infty)$  and for any numbers  $a, b$ ,  $0 < a < b < +\infty$ , the following relation holds, connecting the Lebesgue integral with the Stiltjes integral:

$$\int_{a \leq |x| \leq b} h(|x|) d\mu(x) = \int_{(a,b]} h(t) d\mu(t).$$

## 4. Main result

**Theorem 4.1.** Let  $u(x)$ - be a  $\alpha$ - subharmonic function in a domain  $D \subset C^n \cong \mathbb{R}^{2n}$ . Then for any number  $p : 0 \leq p < 2$  and  $\varepsilon > 0$  there exists an open set  $U_{p,\varepsilon} \subset D$  with capacity  $C_{2n-2+p}(U_{p,\varepsilon}) < \varepsilon$  such that  $u(x)$  belongs to the class  $C^p$  on compact subsets of the difference  $D \setminus U_{p,\varepsilon}$ .

**Proof.** Let  $u(x) - \alpha$ - be a subharmonic function in the domain  $D \subset C^n \cong \mathbb{R}^{2n}$ . For any subdomain  $G \Subset D$ , the representation holds

$$u(z) = \int_G K_\alpha(z, w) d\mu(w) + g(z),$$

where  $\mu = dd^c u \wedge \alpha$ ,  $\text{supp } \mu \subset G$ , and  $g(z)$   $\alpha$ - are harmonic functions from the class  $C^{2+\lambda}(G)$  [see (2.5)]. According to this representation, it is sufficient for us to study the smoothness of the potential

$$v(z) = \int K_\alpha(z, w) d\mu(w)$$

of the Borel measure  $\mu$ , where  $\text{supp } \mu \subset B$ ,  $B = \{z \in \mathbb{C}^n : |z| < 1\}$ .

According to Lemma 3.1, for any  $\varepsilon > 0$  and  $0 < p < 2$ , we can take some open set  $U_{p,\varepsilon} \subset B$  such that  $C_{2n-2+p}(U_{p,\varepsilon}) < \varepsilon$  and for some non-decreasing function  $\rho(t) > 0$ ,  $t \in (0, +\infty)$ ,  $\rho(t) \rightarrow 0$  for  $t \rightarrow 0$ , the following estimates hold:

- a)  $\mu(B(z, t)) \leq \rho(t)t^{2n-2+p}$ ,  $z \in \bar{B} \setminus U_{p,\varepsilon}$ ;
- b)  $\int_{B(z,t)} \frac{d\mu(w)}{|z-w|^{2n-2+p}} \leq \rho(t)$ ,  $z \in \bar{B} \setminus U_{p,\varepsilon}$ , where  $B(z, t) = \{z \in \mathbb{C}^n : |z - w| < t\}$ .

The existence of such a set  $U_{p,\varepsilon}$  follows from Lemma 3.1.

**Case**  $0 < p < 1$ . On the compact  $\bar{B} \setminus U_{p,\varepsilon}$ , we estimate the following difference.

$$R(z, h) = v(z + h) - v(z), \quad z, z + h \in \bar{B} \setminus U_{p,\varepsilon},$$

i.e.

$$\begin{aligned} R(z, h) &= \int_{|z-w| \leq 2|h|} [K_\alpha(z + h, w) - K_\alpha(z, w)] d\mu(w) + \int_{|z-w| \geq 2|h|} [K_\alpha(z + h, w) - K_\alpha(z, w)] d\mu(w) = \\ &= J_1(z, h) + J_2(z, h). \end{aligned}$$

According to the property (2.2) of the kernel  $K_\alpha(z, w)$  we obtain the following estimate for  $J_1(z, h)$ :

$$\begin{aligned} |J_1(z, h)| &\leq \int_{|z-w| \leq 2|h|} |K_\alpha(z + h, w)| d\mu(w) + \int_{|z-w| \leq 2|h|} |K_\alpha(z, w)| d\mu(w) \leq \\ &\leq c_2 \int_{|z-w+h| \leq 3|h|} \frac{d\mu(w)}{|z-w+h|^{2n-2}} + c_2 \int_{|z-w| \leq 2|h|} \frac{d\mu(w)}{|z-w|^{2n-2}}. \end{aligned}$$

Now using Lemma 3.2 we obtain the following.

$$\begin{aligned} |J_1(z, h)| &\leq c_2 \int_0^{3|h|} \frac{d\mu(B(z + h, w))}{t^{2n-2}} + c_2 \int_0^{2|h|} \frac{d\mu(B(z, t))}{t^{2n-2}} = \\ &= c_2 \frac{\mu(B(z, t))}{t^{2n-1}} \Big|_0^{3|h|} + (2n-2)c_2 \int_0^{3|h|} \frac{\mu(B(z + h, t))}{t^{2n-1}} dt \\ &\quad + c_2 \frac{\mu(B(z, t))}{t^{2n-2}} \Big|_0^{2|h|} + (2n-2)c_2 \int_0^{2|h|} \frac{\mu(B(z, t))}{t^{2n-1}} dt = \\ &= c_2 \frac{\mu(B(z + h, t))}{t^{2n-2+p}} \Big|_0^{3|h|} + (2n-2)c_2 \int_0^{3|h|} \frac{\mu(B(z + h, t))}{t^{2n-2+p}} t^{p-1} dt + \end{aligned}$$

$$+c_2 \frac{\mu(B(z, t))}{t^{2n-2+p}} \Big|_0^{2|h|} + (2n-2)c_2 \int_0^{2|h|} \frac{\mu(B(z, t))}{t^{2n-2+p}} t^{p-1} dt$$

Since  $z, z+h \in \overline{B} \setminus U_{\rho, \varepsilon}$ , from this, according to the definition of the set  $U_{\rho, \varepsilon}$ , we obtain the estimate

$$|J_1(z, h)| \leq \left( 3^n c_2 \rho(3|h|) + (2n-2) \frac{3^p}{p} c_2 \rho(3|h|) + 2^p \rho c_2 \rho(2|h|) + (2n-2) \frac{2^p}{p} c_2 \rho(2|h|) \right) |h|^p = \gamma_1(|h|)|h|^p,$$

where

$$\gamma_1(|h|) = 3^p c_2 \rho(3|h|) + (2n-2) \frac{3^p}{p} c_2 \rho(3|h|) + 2^p c_2 \rho(2|h|) + (2n-2) \frac{2^p}{p} c_2 \rho(2|h|)$$

and  $\gamma_1(|h|) \rightarrow 0, \quad h \rightarrow 0$ .

Now we will estimate the value of  $J_2(z, h)$ . For this we will use the Taylor expansion

$$K_\alpha(z+h, w) - K_\alpha(z, w) = \sum_{j=1}^n \frac{\partial K_\alpha(z, w)}{\partial h_j} h_j + \sum_{j=1}^n \frac{\partial K_\alpha(z, w)}{\partial \bar{z}_j} \bar{h}_j + o(|h|),$$

$z, w \in \overline{B}, |z-w| \geq 2|h|$ , and equation (2.3) for sufficiently small  $h$ , we obtain the following equation:

$$|K_\alpha(z+h, w) - K_\alpha(z, w)| \leq C \frac{|h|}{|z-w|^{2n-1}},$$

$z, w \in \overline{B}, |z-w| \geq 2|h|$ ,  $C$  – constant. From here we get

$$\begin{aligned} |J_2(z, h)| &\leq C \int_{|z-w| \geq 2|h|} \frac{|h|}{|z-w|^{2n-1}} d\mu(w) = C|h| \int_{2|h|}^{+\infty} \frac{|h|}{t^{2n-1}} d(\mu(B(z, t))) = \\ &= C|h| \frac{\mu(B(z, t))}{t^{2n-1}} \Big|_{2|h|}^{+\infty} + (2n-1)C|h| \int_{2|h|}^{+\infty} \frac{\mu(B(z, t))}{t^{2n}} dt \leq \\ &\leq C2^p \rho(2|h|)|h|^p + (2n-1)C|h|^p \int_1^{+\infty} \frac{\mu(B(z, t))}{t^{2n}} dt + \\ &\quad + (2n-1)C|h|^p |h|^{1-p} \int_{2\sqrt{|h|}}^1 \frac{\mu(B(z, t))}{t^{2n}} dt + (2n-1)C|h|^p |h|^{1-p} \int_{2|h|}^{2\sqrt{|h|}} \frac{\mu(B(z, t))}{t^{2n}} dt \leq \\ &\leq C2^p \rho(2|h|)|h|^p + (2n-1)C|h|^p |h|^{1-p} \int_1^{+\infty} \frac{\mu(B(z, t))}{t^{2n}} dt + (2n-1)C|h|^p |h|^{1-p} \int_{2\sqrt{|h|}}^1 \frac{\rho(t)}{t^{2-p}} dt + \\ &\quad + (2n-1)C|h|^p |h|^{1-p} \int_{|h|}^{2\sqrt{|h|}} \frac{\rho(t)}{t^{2-p}} dt \leq \\ &\leq C(2^p \rho(2|h|) + \mu(B(0, 2))) |h|^p + \frac{2n-1}{2^{1-p}(1-p)} \rho(1)(\sqrt{|h|})^{1-p} + \\ &\quad + \frac{(2n-1)}{2^{1-p}(1-p)} \left(1 - \sqrt{|h|}^{1-p}\right) \rho\left(2, \sqrt{|h|}\right) |h|^p = \gamma_2(|h|)|h|^p. \end{aligned} \quad (4.2)$$

where  $\gamma_2(t) \rightarrow 0$  as  $t \rightarrow 0$ . Combining estimates (4.1) and (4.2) we obtain

$$|R(z, h)| \leq \gamma(|h|)|h|^p, \quad z, z+h \in \overline{B} \setminus U_{\rho, \varepsilon}, \quad \text{where } \gamma(t) \rightarrow 0 \text{ for } t \rightarrow 0.$$

**Case**  $1 \leq p < 2$ . Now, in the compact  $\bar{B} \setminus U_{p,\varepsilon}$  we evaluate the following differences

$$R_0(z, h) = v(z + h) - v(z) - \sum_{j=1}^n \frac{\partial v(z)}{\partial z_j} h_j - \sum_{j=1}^n \frac{\partial v(z)}{\partial \bar{z}_j} \bar{h}_j, \quad (4.3)$$

$$R_j(z, h) = \frac{\partial v(z + h)}{\partial z_j} - \frac{\partial g(z)}{\partial z_j}, \quad j = 1, 2, \dots, n. \quad (4.4)$$

$$\bar{R}_j(z, h) = \frac{\partial v(z + h)}{\partial \bar{z}_j} - \frac{\partial v(z)}{\partial \bar{z}_j}, \quad j = 1, 2, \dots, n. \quad (4.5)$$

First, we will estimate the value  $R_0(z, h)$ .

$$\begin{aligned} R_0(z, h) = & \int_{|z-w| < |h|} [K_\alpha(z + h, w) - K_\alpha(z, w)] d\mu(w) - \sum_{j=1}^n \int_{|z-w| < 2|h|} \frac{\partial K_\alpha(z, w)}{\partial z_j} h_j d\mu(w) - \\ & - \sum_{j=1}^n \int_{|z-w| < 2|h|} \frac{\partial K_\alpha(z, w)}{\partial \bar{z}_j} \bar{h}_j d\mu(w) - \int_{|z-u| \geq |h|} [K_\alpha(z + h, u) - K_\alpha(z, u) - \\ & - \sum_{j=1}^n \frac{\partial K_\alpha(z, w)}{\partial z_j} h_j - \sum_{j=1}^n \frac{\partial K_\alpha(z, w)}{\partial \bar{z}_j} \bar{h}_j] d\mu(w) = J_1(z, h) - J_2(z, h) + J_3(z, h). \end{aligned}$$

The value  $J_1(z, h)$  is evaluated in the same way as in the case of  $p < 1$ .

According to inequality (2.3), the value  $J_2(z, h)$  is estimated as follows

$$\begin{aligned} |J_2(z, h)| & \leq 2nC|h| \int_{|z-w| < |h|} \frac{d\mu(w)}{|z-w|^{2n-1}} = 2nC|h|^p \int_{|z-w| < 2|h|} \frac{d\mu(w)}{|z-w|^{2n-1}|h|^{p-1}} \leq \\ & \leq 2nC|h|^p \int_{|z-w| < 2|h|} \frac{d\mu(w)}{|z-w|^{2n-2+p}}. \end{aligned}$$

Due to the condition b), the last integral converges to zero on the  $\bar{B} \setminus U_{p,\varepsilon}$  compact at  $h \rightarrow 0$ , i.e.,

$$|J_2(z, h)| \leq \gamma_2(h)|h|^p, \quad z, z + h \in \bar{B} \setminus U_{p,\varepsilon},$$

and  $\gamma_2(h) \rightarrow 0$  at  $h \rightarrow 0$ .

It remains to estimate the value  $J_3(z, h)$ .

From the Taylor formula of the second degree for the kernel and according to inequality (2.4) we obtain the following estimates for  $K_\alpha(z, w)$ :

$$\left| K_\alpha(z + h, w) - K_\alpha(z, w) - \sum_{j=1}^n \frac{\partial K_\alpha(z, w)}{\partial z_j} h_j - \sum_{j=1}^n \frac{\partial K_\alpha(z, w)}{\partial \bar{z}_j} \bar{h}_j \right| \leq C \frac{|h|^2}{|z-w|^{2n}},$$

$$z, w \in \bar{B}, \quad |z-w| \geq 2|h|, \quad C - \text{constant}.$$

We have

$$|J_3(z, h)| h^2 \int_{|z-w| \geq 2|h|} \frac{d\mu(w)}{|z-w|^{2n}} \leq |h|^2 \int_{2|h|}^{\infty} \frac{d\mu(B(z, t))}{t^{2n}}.$$

Here, applying the estimation method as in (4.2), we obtain the following result.

$$|J_3(z, h)| \leq \gamma_3(h)|h|^p,$$

$\gamma_3(h) \rightarrow 0$  for  $h \rightarrow 0$ . As a result, we obtain an estimate of

$$|R_0(z, h)| \leq \gamma(h)|h|^p, \quad z, z+h \in \overline{B} \setminus U_{p,\varepsilon}.$$

$\gamma(h) \rightarrow 0$  for  $h \rightarrow 0$ . The differences are estimated exactly the same way  $R_j(z, h)$ ,  $\bar{R}_j(z, h)$ ,  $j = 1, 2, \dots, n$ . Thus, the potential  $v(z)$  belongs to the class  $C^p(\overline{B} \setminus U_{p,\varepsilon})$ .

*The theorem is proven.*

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ON  $\frac{1}{2}$ -DERIVATIONS OF THE EUCLIDEAN LIE ALGEBRAS

GAYBULLAEV RUSTAMJON\*

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M. ULUGBEK, TASHKENT, UZBEKISTAN  
r.gaybullaev@nuu.uz

NURATDINOV KAZBEK

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M. ULUGBEK, TASHKENT, UZBEKISTAN  
kazbeknur11@gmail.com

**ABSTRACT.** The purpose of this paper is to describe all  $\frac{1}{2}$ -derivations of the Euclidean Lie algebra  $\mathfrak{e}(d)$ . We determine their explicit form and show how the semidirect product structure of  $\mathfrak{e}(d)$  influences the non-existence of nontrivial  $\frac{1}{2}$ -derivations for  $d > 3$ . All  $\frac{1}{2}$ -derivations and transposed Poisson structures for the algebra  $\mathfrak{e}(3)$  are obtained. It is shown that any transposed Poisson structure on  $\mathfrak{e}(d)$ ,  $d \geq 3$  is trivial. These results contribute to the broader program of classification of generalized derivations for non-semisimple Lie algebras.

**MSC (2020):** 17A30; 17B30; 17B40; 17B56.

**Key words:** Lie algebras, Euclidean Lie algebra,  $\delta$ -derivations,  $\frac{1}{2}$ -derivations, transposed Poisson structure.

## Introduction

The idea of  $\delta$ -derivations was introduced in the theory of non-associative algebras (like Jordan algebras, Lie algebras, alternative algebras) by Filippov in [3]. They extend the concept of usual derivations to a wider class of operators. Studying inner  $\delta$ -derivations leads to understanding automorphisms and deformations. Moreover,  $\delta$ -derivations are linked with generalized derivations of Lie algebras. In [4] is shown that, in simple Lie algebras, non-trivial  $\delta$ -derivations exist only for special values of  $\delta$  (often  $\delta = \frac{1}{2}$  or  $-1$ ). The case  $\delta = \frac{1}{2}$  remains especially important because of its close relation to centroid operators and its strong appearance in semisimple, perfect, and solvable Lie algebras. In many concrete classes—such as Euclidean algebras, conformal Galilei algebras, and Heisenberg-type Lie algebras—the explicit computation of  $\frac{1}{2}$ -derivations provides detailed information about ideals, radicals, and representation behavior. This makes  $\frac{1}{2}$ -derivations particularly useful for classification problems and for understanding rigidity properties of non-semisimple Lie algebras.

The notion of  $\delta$ -derivations of Lie algebras and superalgebras, of which  $\frac{1}{2}$ -derivations are a particular case, was systematically studied by Zusmanovich in [7], where general properties of such maps and their role in the structure theory of Lie algebras were established. The study of transposed Poisson structures on Galilean and solvable Lie algebras was carried out by Kaygorodov, Lopatkin, and Zhang in [6], where in particular it was shown that any finite-dimensional solvable Lie algebra admits a non-trivial transposed Poisson structure. Transposed Poisson structures on solvable and perfect Lie algebras were described by Kaygorodov and Khudoyberdiyev in [5], where it was shown, among other results, that there exists a perfect Lie algebra admitting non-trivial  $\frac{1}{2}$ -derivations but no non-trivial transposed Poisson algebra structures.

Thus, modern investigations of  $\delta$ -derivations combine classical structural Lie theory with contemporary studies of local derivations, generalized derivations, and infinite-dimensional constructions. The subject continues to develop through applications to Lie superalgebras, Leibniz algebras, current algebras, and conformal symmetry algebras, making  $\delta$ -derivations an active and relevant topic in present-day algebra research. In recent years,  $\delta$ -derivations have been investigated for various classes of Lie algebras, including simple, semisimple, solvable, nilpotent, and infinite-dimensional algebras. For many finite-dimensional simple and semisimple Lie algebras over fields of characteristic zero, nontrivial  $\delta$ -derivations exist only for specific values of  $\delta$ , and in many cases they reduce to scalar operators or vanish identically. These results demonstrate strong rigidity properties in perfect and centerless Lie algebras.

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\*Corresponding autor

### Preliminaries

In this section, we recall some definitions and known results for studying Euclidean algebras. Throughout this paper, we denote by  $\mathbb{C}$  the set of all complex numbers. More precisely, we have

**Definition 1.** A Lie algebra over a field  $\mathbb{F}$  is a vector space  $\mathfrak{L}$  equipped with a bilinear operation  $[-, -] : \mathfrak{L} \times \mathfrak{L} \rightarrow \mathfrak{L}$ , called the Lie bracket, such that for any  $x, y, z \in \mathfrak{L}$  the following properties hold

$$\begin{aligned} [x, x] &= 0, \\ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= 0, \end{aligned}$$

for all  $x, y, z \in \mathfrak{L}$ .

**Definition 2.** Let  $\mathfrak{L}$  be a vector space equipped with two nonzero bilinear operations  $\cdot$  and  $[-, -]$ . The triple  $(\mathfrak{L}, \cdot, [-, -])$  is called a transposed Poisson algebra if  $(\mathfrak{L}, \cdot)$  is a commutative associative algebra and  $(\mathfrak{L}, [-, -])$  is a Lie algebra that satisfies the following compatibility condition

$$2z \cdot [x, y] = [z \cdot x, y] + [x, z \cdot y].$$

Transposed Poisson algebras were introduced first in a paper by Bi, Bai, Guo and Wu in [1].

**Definition 3.** Let  $(\mathfrak{L}, [-, -])$  be a Lie algebra,  $\varphi : \mathfrak{L} \rightarrow \mathfrak{L}$  be a linear map. Then  $\varphi$  is a  $\frac{1}{2}$ -derivation if it satisfies

$$\varphi([x, y]) = \frac{1}{2}([\varphi(x), y] + [x, \varphi(y)]). \quad (*)$$

A relation between  $\frac{1}{2}$ -derivations of Lie algebras and transposed Poisson algebras were established in [8] by Ferreira B.L.M., Kaygorodov I. and Lopatkin V.

**Lemma 1.**[8] Let  $(\mathfrak{L}, \cdot, [-, -])$  be a transposed Poisson algebra and  $z$  an arbitrary element from  $\mathfrak{L}$ . Then the left multiplication  $l_z$  in the associative commutative algebra  $(\mathfrak{L}, \cdot)$  gives a  $\frac{1}{2}$ -derivation of the Lie algebra  $(\mathfrak{L}, [-, -])$ .

**Lemma 2.**[8] Let  $\mathfrak{L}$  be a Lie algebra without non-trivial  $\frac{1}{2}$ -derivations. Then every transposed Poisson structure defined on  $\mathfrak{L}$  is trivial.

### Main part

The Euclidean Lie algebra  $\mathfrak{e}(d)$  is the Lie algebra of the group of rigid motions of the  $d$ -dimensional Euclidean space. It describes infinitesimal rotations and translations and is given by the semidirect sum  $\mathfrak{e}(d) = \mathfrak{so}(d) \ltimes \mathbb{R}^d$  where  $\mathfrak{so}(d)$  is the Lie algebra of rotations and  $\mathbb{R}^d$  is the abelian ideal of translations. A standard basis consists of the elements  $s_{ij}$ , corresponding to rotations, and  $P_k$ , corresponding to translations. The commutation relations are determined by the natural action of  $\mathfrak{so}(d)$  on  $\mathbb{R}^d$ . That is

$$\begin{cases} [s_{ij}, s_{kl}] = -\delta_{ik}s_{jl} - \delta_{jl}s_{ik} + \delta_{il}s_{jk} + \delta_{jk}s_{il}, \\ [s_{ij}, P_k] = -\delta_{ik}P_j + \delta_{jk}P_i, \end{cases} \quad (**)$$

where  $\delta$  is the Kronecker Delta defined as 1 for  $i = j$  and as 0 otherwise and  $i, j, k, l = 1, \dots, d$ .

Now, we present the main result which describes the space of  $\frac{1}{2}$ -derivations of  $\mathfrak{e}(d, \mathbb{C})$ .

**Theorem 1.** Every  $\frac{1}{2}$ -derivation  $\varphi$  of the Euclidean algebra  $\mathfrak{e}(d)$  is of the form

$$\begin{aligned} i) \quad \varphi &= \lambda \text{Id}_{\mathfrak{e}(3)} + \psi, & \text{if } d = 3, \\ ii) \quad \varphi &= \lambda \text{Id}_{\mathfrak{e}(d)}, & \text{if } d > 3, \end{aligned}$$

where  $\psi(s_{12}) = \beta P_3$ ,  $\psi(s_{13}) = -\beta P_2$ ,  $\psi(s_{23}) = \beta P_1$  and  $\lambda, \beta \in \mathbb{C}$ .

**Proof.**

Initially, we show that  $\varphi$  acts on  $\mathfrak{so}(d)$  diagonally. Fix the triple  $i, j, k \in \{1, 2, \dots, d\}$ . Without loss of generality, we may assume that  $i < j < k$ . Then, one can directly verify that the linear span of the elements  $s_{ij}, s_{ik}, s_{jk}$  is isomorphic to  $\mathfrak{so}(3, \mathbb{C})$ , that is, we have the following commutation relations below

$$[s_{ij}, s_{jk}] = s_{ik}, [s_{ij}, s_{ik}] = -s_{jk}, [s_{ik}, s_{jk}] = -s_{ij}, \quad (1.1)$$

Set

$$\begin{aligned} \varphi(s_{ij}) &= \sum_{1 \leq m < n \leq d} \lambda_{mn}^{ij} s_{mn} + \sum_{1 \leq t \leq d} \beta_t^{ij} P_t, \\ \varphi(s_{ik}) &= \sum_{1 \leq u < v \leq d} \lambda_{uv}^{ik} s_{uv} + \sum_{1 \leq l \leq d} \beta_l^{ik} P_l, \\ \varphi(s_{jk}) &= \sum_{1 \leq p < q \leq d} \lambda_{pq}^{jk} s_{pq} + \sum_{1 \leq w \leq d} \beta_w^{jk} P_w. \end{aligned} \quad (1.2)$$

By applying (\*) to (1.1), by bilinearity and together with (1.2) we get

$$\begin{aligned} -2\varphi(s_{ij}) &= [\varphi(s_{ik}), s_{jk}] + [s_{ik}, \varphi(s_{jk})] \\ &= \left[ \sum_{1 \leq u < v \leq d} \lambda_{uv}^{ik} s_{uv}, s_{jk} \right] + \left[ \sum_{1 \leq l \leq d} \beta_l^{ik} P_l, s_{jk} \right] + [s_{ik}, \sum_{1 \leq p < q \leq d} \lambda_{pq}^{jk} s_{pq}] + [s_{ik}, \sum_{1 \leq w \leq d} \beta_w^{jk} P_w], \\ -2\varphi(s_{jk}) &= [\varphi(s_{ij}), s_{ik}] + [s_{ij}, \varphi(s_{ik})] \\ &= \left[ \sum_{1 \leq m < n \leq d} \lambda_{mn}^{ij} s_{mn}, s_{ik} \right] + \left[ \sum_{1 \leq t \leq d} \beta_t^{ij} P_t, s_{ik} \right] + [s_{ij}, \sum_{1 \leq u < v \leq d} \lambda_{uv}^{ik} s_{uv}] + [s_{ij}, \sum_{1 \leq l \leq d} \beta_l^{ik} P_l], \\ 2\varphi(s_{ik}) &= [\varphi(s_{ij}), s_{jk}] + [s_{ij}, \varphi(s_{jk})] \\ &= \left[ \sum_{1 \leq m < n \leq d} \lambda_{mn}^{ij} s_{mn}, s_{jk} \right] + \left[ \sum_{1 \leq t \leq d} \beta_t^{ij} P_t, s_{jk} \right] + [s_{ij}, \sum_{1 \leq p < q \leq d} \lambda_{pq}^{jk} s_{pq}] + [s_{ij}, \sum_{1 \leq w \leq d} \beta_w^{jk} P_w]. \end{aligned} \quad (1.3)$$

By analyzing the coefficients at basis elements of  $\mathfrak{so}(d)$  and by (1.2) we have the following

$$\begin{aligned} -2\varphi(s_{ij}) &= -2 \sum_{1 \leq m < n \leq d} \lambda_{mn}^{ij} s_{mn} = \left[ \sum_{1 \leq u < v \leq d} \lambda_{uv}^{ik} s_{uv}, s_{jk} \right] + [s_{ik}, \sum_{1 \leq p < q \leq d} \lambda_{pq}^{jk} s_{pq}] \\ &= -\lambda_{ik}^{ij} s_{ij} - \lambda_{jk}^{ik} s_{ij} + \lambda_{ij}^{ik} s_{ik} + \lambda_{ij}^{jk} s_{jk} + x_1 + y_1, \end{aligned} \quad (1.4)$$

$$\begin{aligned} -2 \sum_{1 \leq p < q \leq d} \lambda_{pq}^{jk} s_{pq} &= \left[ \sum_{1 \leq m < n \leq d} \lambda_{mn}^{ij} s_{mn}, s_{ik} \right] + [s_{ij}, \sum_{1 \leq u < v \leq d} \lambda_{uv}^{ik} s_{uv}] \\ &= -\lambda_{ij}^{jk} s_{jk} - \lambda_{ik}^{jk} s_{jk} + \lambda_{jk}^{ij} s_{ij} + \lambda_{jk}^{ik} s_{ik} + x_2 + y_2, \end{aligned} \quad (1.5)$$

$$\begin{aligned} 2 \sum_{1 \leq u < v \leq d} \lambda_{uv}^{ik} s_{uv} &= \left[ \sum_{1 \leq m < n \leq d} \lambda_{mn}^{ij} s_{mn}, s_{jk} \right] + [s_{ij}, \sum_{1 \leq p < q \leq d} \lambda_{pq}^{jk} s_{pq}] \\ &= \lambda_{ij}^{ik} s_{ik} + \lambda_{jk}^{ik} s_{ik} - \lambda_{ik}^{ij} s_{ij} - \lambda_{ik}^{jk} s_{jk} + x_3 + y_3, \end{aligned} \quad (1.6)$$

where  $x_1, x_2, x_3 \in \langle s_{im}, s_{jm}, s_{km} \mid m \neq i, j, k \rangle$ ,  $y_1, y_2, y_3 \in \mathbb{R}^d$ . Moreover,  $\lambda_{mn}^{ij} = 0$ ,  $m, n \notin \{i, j, k\}$ , i.e.  $\varphi(s_{ij}) \in \langle s_{mn} \mid m, n \rangle$ .

If  $(m, n) = \{(i, j), (i, k), (j, k)\}$ ,  $(p, q) = \{(i, j), (i, k), (j, k)\}$ ,  $(u, v) = \{(i, j), (i, k), (j, k)\}$  in (1.4), (1.5) and (1.6), that is, the sums runs only over three pairs  $\{(i, j), (i, k), (j, k)\}$  we obtain

$$\begin{aligned} -2\lambda_{ij}^{ij} s_{ij} &= -\lambda_{ik}^{ik} s_{ij} - \lambda_{jk}^{jk} s_{ij}, & -2\lambda_{ij}^{ij} s_{ik} &= \lambda_{ij}^{ik} s_{ik}, & -2\lambda_{ij}^{ij} s_{jk} &= \lambda_{ij}^{jk} s_{jk}, \\ -2\lambda_{jk}^{jk} s_{jk} &= -\lambda_{ij}^{ij} s_{jk} - \lambda_{ik}^{ik} s_{jk}, & -2\lambda_{ik}^{ik} s_{ik} &= \lambda_{jk}^{jk} s_{ik}, & -2\lambda_{ij}^{jk} s_{ij} &= \lambda_{jk}^{ij} s_{ij}, \\ 2\lambda_{ik}^{ik} s_{ik} &= \lambda_{ij}^{ij} s_{ik} + \lambda_{jk}^{jk} s_{ik}, & 2\lambda_{jk}^{jk} s_{jk} &= -\lambda_{ik}^{ik} s_{jk}, & 2\lambda_{ij}^{ik} s_{ij} &= -\lambda_{ik}^{ij} s_{ij}. \end{aligned}$$

By comparing the corresponding coefficients at basis elements  $s_{ij}, s_{ik}, s_{jk}$  from above we obtain

$$\begin{cases} \lambda_{ij}^{ij} = \lambda_{ik}^{ik} = \lambda_{jk}^{jk}, \\ \lambda_{ik}^{ij} = \lambda_{ij}^{ik} = \lambda_{ij}^{jk} = \lambda_{jk}^{ik} = \lambda_{ik}^{jk} = \lambda_{jk}^{ij} = 0. \end{cases}$$

Thus, we have

$$\varphi(s_{ij}) = \lambda_{ij}^{ij} s_{ij} + x + y, \quad (1.7)$$

where  $x \in \langle s_{im}, s_{jm}, s_{km} \mid m \neq i, j, k \rangle$  and  $y \in \mathbb{R}^d$ .

Having established the formula (1.7) for an arbitrary triple  $i, j, k$ , we now exploit the structure of  $\mathfrak{so}(d, \mathbb{C})$ . Recall that the linear span of the elements  $\{s_{ij}\}$  is isomorphic to  $\mathfrak{so}(3, \mathbb{C})$ , and since  $\mathfrak{so}(d, \mathbb{C})$  decomposes, as a vector space, into a direct sum of such subalgebras, we may conclude that  $x = 0$  in (1.7). Consequently, the map  $\varphi$  acts on each generator  $s_{ij}$  by

$$\varphi(s_{ij}) = \lambda s_{ij} + y, \quad y \in \mathbb{R}^d,$$

that is,  $\varphi$  scales each  $s_{ij}$  by a uniform scalar  $\lambda$  and adds an element of  $\mathbb{R}^d$ .

Now, we show that  $\varphi|_{\mathfrak{so}(d, \mathbb{C})} = \lambda \text{Id}|_{\mathfrak{so}(d, \mathbb{C})}$  for  $d \geq 4$  and  $\varphi = \lambda \text{Id}_{e(3)} + \psi$ , for  $d = 3$ . To prove this, we compare the corresponding coefficients in (1.3). Thus, we have

$$\begin{aligned} -2 \sum_{1 \leq t \leq d} \beta_t^{ij} P_t &= [ \sum_{1 \leq l \leq d} \beta_l^{ik} P_l, s_{jk} ] + [ s_{ik}, \sum_{1 \leq w \leq d} \beta_w^{jk} P_w ], \\ 2 \sum_{1 \leq l \leq d} \beta_l^{ik} P_l &= [ \sum_{1 \leq t \leq d} \beta_t^{ij} P_t, s_{jk} ] + [ s_{ij}, \sum_{1 \leq w \leq d} \beta_w^{jk} P_w ], \\ -2 \sum_{1 \leq w \leq d} \beta_w^{jk} P_w &= [ \sum_{1 \leq t \leq d} \beta_t^{ij} P_t, s_{ik} ] + [ s_{ij}, \sum_{1 \leq w \leq d} \beta_w^{ik} P_w ], \end{aligned}$$

which, together with (\*\*), give  $\beta_t^{ij} = 0$ ,  $t \notin \{i, j, k\}$  and the following equalities:

$$\begin{aligned} -2\beta_i^{ij} P_i - 2\beta_j^{ij} P_j - 2\beta_k^{ij} P_k &= \beta_j^{ik} P_k - \beta_k^{ik} P_j - \beta_i^{jk} P_k + \beta_k^{jk} P_i, \\ 2\beta_i^{ik} P_i + 2\beta_j^{ik} P_j + 2\beta_k^{ik} P_k &= \beta_j^{ij} P_k - \beta_k^{ij} P_j - \beta_i^{jk} P_j + \beta_j^{jk} P_i, \\ -2\beta_i^{jk} P_i - 2\beta_j^{jk} P_j - 2\beta_k^{jk} P_k &= \beta_i^{ij} P_k - \beta_k^{ij} P_i - \beta_i^{ik} P_j + \beta_j^{ik} P_i. \end{aligned}$$

By comparing the corresponding coefficients at the basis elements, we get

$$\begin{cases} \beta_k^{ij} = -\beta_j^{ik} = \beta_i^{jk}, \\ \beta_i^{ij} = \beta_j^{ij} = \beta_i^{ik} = \beta_k^{ik} = \beta_j^{jk} = \beta_k^{jk} = 0. \end{cases}$$

Now, we show that  $\beta_k^{ij} = -\beta_j^{ik} = \beta_i^{jk} = 0$  for  $d > 3$ . Since  $d > 3$ , for any  $s_{ij}$  there exists  $s_{kl}$  such that  $[s_{ij}, s_{kl}] = 0$  ( $\{i, j\} \cap \{k, l\} = \emptyset$ ). For such a pair we apply (\*) and write the following relations:

$$\begin{aligned} 0 &= \varphi([s_{ij}, s_{kl}]) \\ &= [\varphi(s_{ij}), s_{kl}] + [s_{ij}, \varphi(s_{kl})] \\ &= [\lambda_{ij}^{ij} s_{ij} + \beta_t^{ij} P_t, s_{kl}] + [s_{ij}, \lambda_{kl}^{kl} s_{kl} + \beta_v^{kl} P_v] \\ &= \beta_k^{ij} P_l - \beta_l^{ij} P_k - \beta_i^{kl} P_j + \beta_j^{kl} P_i, \end{aligned}$$

which implies  $\beta_k^{ij} = 0$  as needed.

Now, we consider how  $\varphi$  acts on  $\mathbb{R}^d$ . Let

$$[s_{ij}, P_k] = 0, \quad (k \neq i, j), \quad (1.8)$$

$$\varphi(P_k) = \sum_{1 \leq p < q \leq d} \gamma_{pq}^k s_{pq} + \sum_{1 \leq t \leq d} \mu_t^k P_t. \quad (1.9)$$

By applying (\*) to (1.8) and by (1.9), we obtain

$$\begin{aligned} [\varphi(s_{ij}), P_k] &= [\varphi(P_k), s_{ij}], \\ [\lambda s_{ij}, P_k] &= [ \sum_{1 \leq p < q \leq d} \gamma_{pq}^k s_{pq} + \sum_{1 \leq t \leq d} \mu_t^k P_t, s_{ij} ], \\ 0 &= [ \sum_{1 \leq p < q \leq d} \gamma_{pq}^k s_{pq}, s_{ij} ] + [ \sum_{1 \leq t \leq d} \mu_t^k P_t, s_{ij} ]. \end{aligned}$$

First term on the right-hand side provides us that  $\gamma_{pq}^k = 0$ .

Finally, we show that  $\varphi$  acts on  $\mathbb{R}^d$  trivially. Let

$$[s_{ij}, P_i] = -P_j, [s_{ij}, P_j] = P_i, \quad (1.10)$$

$$\varphi(P_i) = \sum_{1 \leq t \leq d} \mu_t^i P_t, \quad \varphi(P_j) = \sum_{1 \leq l \leq d} \mu_l^j P_l. \quad (1.11)$$

By applying (\*) to (1.10) we obtain

$$\begin{aligned} 2\varphi(P_i) &= \lambda P_i + [s_{ij}, \sum_{1 \leq l \leq d} \mu_l^j P_l] \\ &= \lambda P_i - \mu_i^j P_j + \mu_j^j P_i, \\ -2\varphi(P_j) &= -\lambda P_j + [s_{ij}, \sum_{1 \leq t \leq d} \mu_t^i P_t] \\ &= -\lambda P_j - \mu_i^i P_j + \mu_j^i P_i, \end{aligned} \quad (1.12)$$

which together with (1.11) provide

$$\begin{aligned} 2\mu_i^i P_i + 2\mu_j^j P_j &= \lambda P_i + \mu_i^j P_j + \mu_j^j P_i, \\ -2\mu_i^j P_i - 2\mu_j^j P_j &= -\lambda P_j - \mu_i^i P_j + \mu_j^i P_i. \end{aligned}$$

By comparing the corresponding coefficients at basis elements, we get  $\mu_i^i = \mu_j^j = \lambda$  and  $\mu_i^j = \mu_j^i = 0$ . By substituting  $\mu_i^j$  and  $\mu_j^i$  to (1.12) we get

$$\varphi(P_i) = \lambda P_i, \quad \varphi(P_j) = \lambda P_j.$$

This completes the proof.

**Remark 1.** Note that the proof is restricted to the case  $d \geq 4$ . If  $d = 3$ , one can directly verify that the first part of theorem follows immediately.

### Description of transposed Poisson algebras structure defined on $\mathfrak{e}(3)$ .

We aim to describe the multiplication  $\cdot$  on  $\mathfrak{e}(3)$ . By lemma (1) for every basis element there is an associated  $\frac{1}{2}$ -derivation  $\varphi$  of  $(\mathfrak{L}, [-, -])$  such that  $\varphi_k(s_{ij}) = s_{ij} \cdot P_k = P_k \cdot s_{ij} = \varphi_{ij}(P_k)$ ,  $\varphi_{kl}(s_{ij}) = s_{ij} \cdot s_{kl} = s_{kl} \cdot s_{ij} = \varphi_{ij}(s_{kl})$ ,  $\varphi_i(P_j) = P_i \cdot P_j = P_j \cdot P_i = \varphi_j(P_i)$ .

**Corollary.** Any transposed Poisson structure on the Euclidean algebra  $\mathfrak{e}(d)$  ( $d \geq 3$ ) is trivial.

**Proof.** We prove the claim for  $d = 3$ , because any  $\frac{1}{2}$ -derivation of  $\mathfrak{e}(d)$  is trivial for  $d \geq 4$ . Hence, we have only trivial transposed Poisson structure for  $d \geq 4$  (see [8]).

Let  $d = 3$  and  $\varphi$  be a  $\frac{1}{2}$ -derivation of  $\mathfrak{e}(3)$ . Then by Theorem 1 we have

$$\begin{aligned} \varphi(s_{12}) &= \lambda s_{12} + \beta P_3, & \varphi(s_{13}) &= \lambda s_{13} - \beta P_2, & \varphi(s_{23}) &= \lambda s_{23} + \beta P_1, \\ \varphi(P_1) &= \lambda P_1, & \varphi(P_2) &= \lambda P_2, & \varphi(P_3) &= \lambda P_3. \end{aligned}$$

Moreover, by Lemma 1 the linear operators  $l_{s_{12}}, l_{s_{13}}, l_{s_{23}}, l_{P_1}, l_{P_2}, l_{P_3}$  are  $\frac{1}{2}$ -derivations of associative commutative algebra  $(\mathfrak{e}(3), \cdot)$ . Thus, we have

$$\begin{aligned} s_{12} \cdot s_{13} &= l_{s_{12}}(s_{13}) = \alpha s_{13} + \beta P_2, \\ s_{13} \cdot s_{12} &= l_{s_{13}}(s_{12}) = \gamma s_{12} + \delta P_3, \end{aligned}$$

where  $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ . Since  $(\mathfrak{e}(3), \cdot)$  is commutative, we derive  $\alpha = \beta = \gamma = \delta = 0$ , that is,  $s_{12} \cdot s_{13} = 0$ . By a similar way one can obtain  $s_{12} \cdot s_{23} = 0$  and  $s_{13} \cdot s_{23} = 0$ . Applying the same arguments on the operators  $l_{s_{12}}$  and  $l_{P_1}$  we get that  $s_{12} \cdot P_1 = 0$ . Similarly, one can verify that  $s_{ij} \cdot P_k = 0$  for  $i, j, k \in \{1, 2, 3\}$ .

**Remark 2.** It should be emphasized that the algebra  $\mathfrak{e}(3)$  provides a counterexample to the naive expectation that non-trivial  $\frac{1}{2}$ -derivations force the existence of non-trivial transposed Poisson structures. Indeed,  $\mathfrak{e}(3)$  possesses non-trivial  $\frac{1}{2}$ -derivations yet admits no non-trivial transposed Poisson algebra structures. Furthermore, since  $[\mathfrak{e}(3), \mathfrak{e}(3)] = \mathfrak{e}(3)$  the algebra  $\mathfrak{e}(3)$  is perfect, making this a particularly notable example and providing a concrete answer to Questions 2 and 5 raised in [2].

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# INITIAL-BOUNDARY VALUE PROBLEM FOR A THIRD-ORDER COMPOSITE TYPE DIFFERENTIAL EQUATION

KHAJIEV IKROMBEK

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M. ULUGBEK,  
TURIN POLYTECHNIC UNIVERSITY IN TASHKENT, TASHKENT, UZBEKISTAN.

RUZIMATOV JAHONGIR\*

TURIN POLYTECHNIC UNIVERSITY IN TASHKENT,  
NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M. ULUGBEK, TASHKENT, UZBEKISTAN.

**ABSTRACT.** This paper investigates an initial-boundary value problem for a third-order differential equation of composite type. The problem under consideration belongs to the class of ill-posed problems in mathematical physics, which necessitates the development of special approaches for its analysis. Using the energy integral and logarithmic convexity methods, an a priori estimate is obtained, and theorems of uniqueness and conditional stability of the solution to the problem are proved on the well-correctness set. To obtain a numerical solution, a regularization method is employed, on the basis of which an approximate solution is constructed.

**MSC (2020):** 35G15, 35R25, 65M30.

**Key words:** equation of composite type, third order, ill-posed problem, a priori estimate, uniqueness, conditional stability, well-posedness set, regularization.

## Introduction

Let  $u(x, y, t)$  be a function defined in the domain  $Q_T = \{(x, y, t) : (x, y) \in \Omega, 0 < t < T\}$ , where  $\Omega = \{0 < x < l, 0 < y < d\}$ ,  $Q_t = \{(x, y, \tau) : (x, y) \in \Omega, 0 < \tau < t\}$ , satisfying the equation

$$\left(\frac{\partial}{\partial t} + \alpha\right) \left(\frac{\partial^2}{\partial t^2} + L\right) u(x, y, t) = f(x, y, t). \quad (1)$$

Here  $\alpha \in \mathbb{R} \setminus \{0\}$ , and

$$Lu(x, y, t) = (a_1(x, y)u_x(x, y, t))_x + (a_2(x, y)u_y(x, y, t))_y - c(x, y)u(x, y, t),$$

with  $a_1(x, y) > 0$ ,  $a_2(x, y) > 0$ ,  $c(x, y) \geq 0$ , where  $a_1(x, y)$ ,  $a_2(x, y) \in C^1(\Omega)$ ,  $c(x, y) \in C(\Omega)$  and  $f(x, y, t)$  is a given source function.

**Statement of the problem.** Find a solution of the equation (1) in the domain  $Q_T$  satisfying the following conditions: initial

$$u|_{t=0} = \varphi(x, y), \quad u_t|_{t=0} = \psi(x, y), \quad u_{tt}|_{t=0} = h(x, y), \quad (x, y) \in \bar{\Omega} \quad (2)$$

boundary

$$u|_{\partial\Omega} = 0, \quad 0 \leq t \leq T. \quad (3)$$

The problem (1)-(3) falls into the category of ill-posed problems in mathematical physics. Specifically, this problem does not exhibit continuous dependence of the solution on the initial data, which indicates that it is unstable.

The theory of boundary value problems for equations of composite type and third-order equations has a rich history and continues to develop actively. The fundamental foundations of the theory of composite-type equations were laid in the works of A. V. Bitsadze [1] and V. N. Vragov [2]. A significant contribution to

the development of the solvability theory of boundary value problems for such equations was made by S. A. Tersenov [3], A. M. Nakhushiev [4], T. D. Djuraev [5-6], I. E. Egorov [7-8], and O. S. Zikirov [9-10]. Among contemporary studies, the works of A. I. Kozhanov deserve special mention [11-12]. The study of ill-posed boundary value problems for differential equations is addressed in the works of M. M. Lavrentiev, S. G. Krein, and their students [13-17]. In particular, results for differential-operator equations and mixed-type equations of the second and high orders were obtained by K. S. Fayazov and I. O. Khajiev [18-20]. Internationally, fundamental contributions have been made by J. L. Lions [22-23], who developed the theory of non-homogeneous boundary value problems and regularization methods for ill-posed problems. The general theory of linear partial differential equations, including ill-posed Cauchy problems, was significantly advanced by L. Hormander [24-25]. Classical results on ill-posed problems and their numerical treatment were obtained by H.-O. Kreiss [26] and R. Dautray [27]. The Italian school, represented by E. Magenes [28] and G. Croce [29], made important contributions to variational methods for mixed-type equations. From the Japanese mathematical school, S. Mizohata [30] studied the asymptotic properties and well-posedness of composite-type equations.

In this paper, we investigate an initial-boundary value problem for a third-order equation of composite type with three initial conditions. In many cases, such problems for similar equations have been studied in well-posed formulations. It is shown that the problem under consideration is ill-posed, and theorems of uniqueness and conditional stability of the solution are proved on the set of correctness.

Let us introduce the notation

$$u_{tt}(x, y, t) + Lu(x, y, t) = v(x, y, t), \quad (4)$$

then the problem (1)-(3) can be represented as the following subproblems.

**Problem A.** Find a function  $u(x, y, t)$  satisfying the equation

$$u_{tt} + Lu = v$$

and the conditions

$$\begin{aligned} u(x, y, 0) &= \varphi(x, y), \quad u_t(x, y, 0) = \psi(x, y), \quad (x, y) \in \bar{\Omega} \\ u(0, y, t) &= u(l, y, t) = 0, \quad u(x, 0, t) = u(x, d, t) = 0. \end{aligned}$$

**Problem B.** Find a function  $v(x, y, t)$  satisfying the equation

$$v_t + \alpha v = f$$

and the conditions

$$\begin{aligned} v(x, y, 0) &= h + L\varphi, \quad (x, y) \in \bar{\Omega}, \\ v(0, y, t) &= v(l, y, t) = 0, \quad v(x, 0, t) = v(x, d, t) = 0. \end{aligned}$$

Solving Problem B, we find  $v(x, y, t)$

$$v(x, y, t) = e^{-\alpha t}(h + L\varphi) + \int_0^t f(x, y, \tau)e^{\alpha(\tau-t)}d\tau. \quad (5)$$

Let us represent the scalar product as

$$(\varphi(x, y), \psi(x, y)) = \int_{\Omega} \varphi(x, y)\psi(x, y)d\Omega$$

and define the norm as

$$\|\varphi(x, y)\|^2 = \int_{\Omega} \varphi^2(x, y)d\Omega.$$

From equality (5) we obtain the inequality

$$\|v\| \leq e^{-\alpha t}(\|h\| + \|L\varphi\|) + \left( \frac{1 - e^{-2\alpha t}}{\alpha} \int_0^t \|f\|^2 d\tau \right)^{1/2}. \quad (6)$$

To solve Problem A, we use the method of separation of variables, i.e., the Fourier method, and proceed to the following spectral problem.

**Eigenvalue problem.** Find the values of  $\lambda$  for which the problem

$$\begin{cases} L\omega(x, y) = -\lambda\omega \\ \omega(0, y) = \omega(l, y) = 0 \\ \omega(x, 0) = \omega(x, d) = 0 \end{cases}$$

has nontrivial solutions.

The eigenvalue problem has eigenvalues  $\lambda_n$  and eigenfunctions  $\omega_n(x, y)$ . Here  $\lambda_n$  are real numbers forming a monotonically increasing sequence:

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \cdots \rightarrow \infty,$$

and  $\omega_n(x, y)$  are orthogonal functions satisfying the condition for  $k, n \in \mathbb{N}$

$$(\omega_k, \omega_n) = \frac{1}{\|\omega_k(x, y)\|^2} \int_{\Omega} \omega_k(x, y) \omega_n(x, y) d\Omega = \begin{cases} 0, & k \neq n \\ 1, & k = n \end{cases}.$$

The system of eigenfunctions is complete, which allows arbitrary continuous functions to be expanded into Fourier series with respect to this basis [21].

Suppose that a solution to the problem exists. Then problem (1)-(3) can be represented as

$$u(x, y, t) = \sum_{n=1}^{\infty} \left[ \varphi_n ch(\lambda_n t) + \frac{\psi_n sh(\lambda_n t)}{\lambda_n} + \frac{1}{\lambda_n} \int_0^t v_n(\tau) sh(t - \tau) \lambda_n d\tau \right] \omega_n(x, y) \quad (7)$$

where

$$v_n(t) = e^{-\alpha t} (h_n - \lambda_n \varphi_n) + \int_0^t f_n(\tau) e^{\alpha(\tau-t)} d\tau,$$

$$\begin{aligned} \varphi_n &= \int_{\Omega} \varphi(x, y) \omega_n(x, y) d\Omega, & \psi_n &= \int_{\Omega} \psi(x, y) \omega_n(x, y) d\Omega, \\ h_n &= \int_{\Omega} h(x, y) \omega_n(x, y) d\Omega, & f_n(t) &= \int_{\Omega} f(x, y, t) \omega_n(x, y) d\Omega. \end{aligned}$$

### Main part

**Lemma 1.** For the solution of the Problem A satisfying the conditions

$$u(0, y, t) = u(l, y, t) = 0, \quad u(x, 0, t) = u(x, d, t) = 0, \quad \forall t \in (0, T),$$

the following estimate holds:

$$\int_{Q_t} u^2 dQ_t \leq d(t) \left( T \int_{\Omega} u^2(x, y, 0) d\Omega + \gamma \right)^{1-r(t)} \left( \int_{Q_T} u^2 dQ_T + \gamma \right)^{r(t)}, \quad (8)$$

where

$$\begin{aligned} d(t) &= e^{(T+1) \frac{(1-e^{-t})T - (1-e^{-T})t}{1-e^{-T}}}, \quad \gamma = (2T^2 + 3) \int_{Q_T} v^2(x, y, \tau) dQ_T + \rho, \\ r(t) &= \frac{1 - e^{-t}}{1 - e^{-T}}, \quad \rho = 2 \left| \int_{\Omega} u(x, y, 0) u_t(x, y, 0) d\Omega \right| + T \left| \int_{\Omega} (u(x, y, 0) Lu(x, y, 0) + u_t^2(x, y, 0)) d\Omega \right|. \end{aligned}$$

**Proof.** Let

$$F(t) = \int_{Q_t} u^2(x, y, t) dQ_t + (T - t) \int_{\Omega} u^2(x, y, 0) d\Omega + \gamma.$$

then it is easy to see that

$$\begin{aligned} F'(t) &= \int_{\Omega} u^2(x, y, t) d\Omega - \int_{\Omega} u^2(x, y, 0) d\Omega, \\ F''(t) &= 2 \int_{\Omega} u(x, y, t) u_t(x, y, t) d\Omega, \end{aligned}$$

where  $F'(0) = 0$ . Then

$$\begin{aligned} F'(t) &= 2 \int_{Q_t} u(x, y, t) u_t(x, y, t) dQ_t, \\ F'''(t) &= 2 \int_{\Omega} u_t^2 d\Omega + 2 \int_{\Omega} u u_{tt} d\Omega = 2 \int_{\Omega} u_t^2 d\Omega + 2 \int_{\Omega} u(v - (a_1 u_x)_x - (a_2 u_y)_y + cu) d\Omega = \\ &= 2 \int_{\Omega} u_t^2 d\Omega + 2 \int_{\Omega} u v d\Omega - 2 \int_{\Omega} u(a_1 u_x)_x d\Omega - 2 \int_{\Omega} u(a_2 u_y)_y d\Omega + 2 \int_{\Omega} c u^2 d\Omega. \end{aligned}$$

Let us compute the derivative of the function  $\int_{\Omega} u(a_1 u_x)_x d\Omega + \int_{\Omega} u(a_2 u_y)_y d\Omega - \int_{\Omega} c u^2 d\Omega$ ,

$$\begin{aligned} \frac{d}{dt} \left( \int_{\Omega} u(a_1 u_x)_x d\Omega + \int_{\Omega} u(a_2 u_y)_y d\Omega - \int_{\Omega} c u^2 d\Omega \right) &= 2 \int_{\Omega} ((a_1 u_x)_x + (a_2 u_y)_y - cu) u_t d\Omega = \\ &= 2 \int_{\Omega} (v - u_{tt}) u_t d\Omega = 2 \int_{\Omega} v u_t d\Omega - 2 \int_{\Omega} u_{tt} u_t d\Omega = 2 \int_{\Omega} v u_t d\Omega - \frac{d}{dt} \left( \int_{\Omega} u_t^2 d\Omega \right) \end{aligned}$$

Integrating the last expression and using the initial conditions, we obtain

$$\int_{\Omega} u(a_1 u_x)_x d\Omega + \int_{\Omega} u(a_2 u_y)_y d\Omega - \int_{\Omega} c u^2 d\Omega = 2 \int_{Q_t} v u_t dQ_t - \int_{\Omega} u_t^2 d\Omega + \rho_1$$

Hence

$$\begin{aligned} F'''(t) &= 2 \int_{\Omega} u_t^2 d\Omega + 2 \int_{\Omega} u v d\Omega - 2 \left( 2 \int_{Q_t} v u_t dQ_t - \int_{\Omega} u_t^2 d\Omega + \rho_1 \right) \\ &= 4 \int_{\Omega} u_t^2 d\Omega + 2 \int_{\Omega} u v d\Omega - 4 \int_{Q_t} v u_t dQ_t - 2\rho_1, \end{aligned}$$

where  $\rho_1 = \int_{\Omega} \left( u(a_1 u_x)_x + u(a_2 u_y)_y - c u^2 + u_t^2 \right) \Big|_{t=0} d\Omega$  and integrating it, we obtain

$$\begin{aligned} F''(t) &= 4 \int_{Q_t} u_t^2(x, y, \tau) dQ_t + 2 \int_{Q_t} u(x, y, \tau) v(x, y, \tau) dQ_t - \\ &- 4 \int_0^t \int_{Q_\tau} v(x, y, \tau_1) u_t(x, y, \tau_1) dQ_\tau d\tau - 2\rho_1 t + \beta, \end{aligned}$$

where  $\beta = 2 \int_{\Omega} u(x, y, 0) u_t(x, y, 0) dx dy$ , or

$$4 \int_{Q_t} u_t^2(x, y, \tau) dQ_t = F''(t) - 2 \int_{Q_t} u(x, y, \tau) v(x, y, \tau) dQ_t +$$

$$+4 \int_0^t \int_{Q_\tau} v(x, y, \tau_1) u_t(x, y, \tau_1) dQ_\tau d\tau - (\beta - 2\rho_1 t).$$

Now, using the inequality  $2|ab| \leq sa^2 + \frac{1}{s}b^2$ , where  $s > 0$ , after elementary transformations we obtain

$$4 \int_{Q_t} u_t^2(x, y, \tau) dQ_t \leq F''(t) + \frac{2T}{s} \int_{Q_t} u_t^2(x, y, \tau) dQ_t + 2Ts \int_{Q_t} v^2(x, y, \tau) dQ_t + \int_{Q_t} u^2(x, y, \tau) dQ_t + \int_{Q_t} v^2(x, y, \tau) dQ_t + |\beta - 2\rho_1 t|,$$

If  $s = T$ , we have

$$2 \int_{Q_t} u_t^2(x, y, \tau) dQ_t \leq F''(t) + (2T^2 + 1) \int_{Q_t} v^2(x, y, \tau) dQ_t + \int_{Q_t} u^2(x, y, \tau) dQ_t + |\beta - 2\rho_1 t|,$$

from which we obtain

$$2 \int_0^t (t - \tau) \int_{Q_t} u_t^2(x, y, \tau) dQ_t dt \leq F'(t) + (2T^2 + 1)T \int_{Q_t} v^2(x, y, \tau) dQ_t + T \int_{Q_t} u^2(x, y, \tau) dQ_t + T\rho,$$

where  $\rho = |\beta| + T|\rho_1|$ , or

$$2 \int_0^t (t - \tau) \int_{Q_t} u_t^2(x, y, \tau) dQ_t + 2 \int_0^t (t - \tau) \int_{Q_t} v^2(x, y, \tau) dQ_t \leq F'(t) + TF(t).$$

Consider the function  $\mu(t) = \ln(F(t))$ , then its second derivative

$$\begin{aligned} \mu''(t) &= \frac{F''(t)F(t) - F'^2(t)}{F^2(t)} = \\ &= \frac{\left( 4 \int_{Q_t} u_t^2(x, y, \tau) dQ_t + 2 \int_{Q_t} u(x, y, \tau) v(x, y, \tau) dQ_t - 4 \int_0^t \int_{Q_\tau} v(x, y, \tau_1) u_t(x, y, \tau_1) dQ_\tau d\tau - 2\rho_1 t + \beta \right)}{F^2(t)} \times \\ &\quad \left( \int_{Q_t} u^2(x, y, \tau) dQ_t + (T - t) \int_{\Omega} u^2(x, y, 0) dx dy + \gamma \right) - \\ &\quad - \frac{\left( 2 \int_{Q_t} u(x, y, t) u_t(x, y, t) dQ_t \right)^2}{F^2(t)} = I_1 + I_2 + I_3. \end{aligned}$$

We divide the last expression into three parts and estimate each part separately.

$$\begin{aligned} I_1 &= \frac{4 \int_{Q_t} u_t^2(x, y, \tau) dQ_t \int_{Q_t} u^2(x, y, \tau) dQ_t - \left( 2 \int_{Q_t} u(x, y, t) u_t(x, y, t) dQ_t \right)^2}{F^2(t)} + \\ &+ \frac{4 \int_{Q_t} u_t^2(x, y, \tau) dQ_t \left( (T - t) \int_{\Omega} u^2(x, y, 0) d\Omega + \gamma \right)}{F^2(t)} \geq 0 \end{aligned}$$

$$\begin{aligned}
 I_2 &= \frac{\left(2 \int_{Q_t} u(x, y, \tau) v(x, y, \tau) dQ_t - 2\rho_1 t + \beta\right) \left(\int_{Q_t} u^2(x, y, \tau) dQ_t (T-t) \int_{\Omega} u^2(x, y, 0) d\Omega + \gamma\right)}{F^2(t)} = \\
 &= \frac{\left(2 \int_{Q_t} u(x, y, \tau) v(x, y, \tau) dQ_t - 2\rho_1 t + \beta\right)}{F(t)} \geq \frac{\left(-\int_{Q_t} u^2(x, y, \tau) dQ_t - \int_{Q_t} v^2(x, y, \tau) dQ_t - |\rho|\right)}{F(t)} \geq -1 \\
 I_3 &= \frac{-4 \int_0^t \int_{Q_\tau} v(x, y, \tau_1) u_t(x, y, \tau_1) dQ_\tau d\tau \left(\int_{Q_t} u^2(x, y, \tau) dQ_t + (T-t) \int_{\Omega} u^2(x, y, 0) d\Omega + \gamma\right)}{F^2(t)} = \\
 &= \frac{-4 \int_0^t \int_{Q_\tau} v(x, y, \tau_1) u_t(x, y, \tau_1) dQ_\tau d\tau}{F(t)} \geq \frac{-4 \int_0^t (t-\tau) \int_{Q_\tau} u_t(x, y, \tau_1) v(x, y, \tau_1) dQ_\tau d\tau}{F(t)} \geq \\
 &\geq \frac{-F'(t) - TF(t)}{F(t)} \geq -\mu'(t) - T.
 \end{aligned}$$

We combine the above inequalities and obtain

$$\mu''(t) + 2\mu'(t) + 2T + 1 \geq 0.$$

Then the following inequality holds:

$$\mu(t) \leq \mu(0)(1 - r(t)) + \mu(T)r(t) + c(t).$$

Hence the desired inequality (8) follows. **Lemma 1 is proved.**

Taking into account inequality (6) and based on inequality (8),

$$\int_{Q_t} u^2 dQ_t \leq d(t) \left(T \int_{\Omega} u^2(x, y, 0) d\Omega + \gamma_1\right)^{1-r(t)} \left(\int_{Q_T} u^2 dQ_T + \gamma_1\right)^{r(t)} \quad (9)$$

the inequality will be appropriate here

$$\gamma_1 = 3(2T^2 + 3) \int_0^T \left(e^{-2\alpha t} \left(\|h\|^2 + 3(\|a_1 \varphi_x\|_{W_2^{1,0}}^2 + \|a_2 \varphi_y\|_{W_2^{0,1}}^2 + \|c\varphi\|^2)\right) + \frac{1 - e^{-2\alpha t}}{\alpha} \int_0^t \|f\|^2 d\tau\right) dt + \rho$$

We define the well-posedness set  $M$  as follows:

$$M = \{u(x, y, t) : \int_0^T \|u(x, y, t)\|^2 dt \leq m^2\}$$

**Theorem 1.** Suppose a solution of the problem (1)-(3) exists and belongs to the set  $M$ . Then the solution of the problem (1)-(3) is unique.

**Proof.** Suppose there exist two solutions  $u_1, u_2$  of the problem (1)-(3). Moreover  $u_1, u_2$  satisfy equation (1) with homogeneous initial and boundary conditions.

Let us introduce the notation  $u = u_1 - u_2$ . Then the function  $u(x, y, t)$  satisfies the equation

$$\left(\frac{\partial}{\partial t} + \alpha\right)(u_{tt} + Lu) = 0$$

the initial

$$u(x, y, 0) = 0, u_t(x, y, 0) = 0, u_{tt}(x, y, 0) = 0$$

and the boundary

$$u(0, y, t) = u(l, y, t) = 0, \quad u(x, 0, t) = u(x, d, t) = 0$$

conditions. It follows that  $\gamma_1 = 0$  and based on inequality (9) we obtain

$$\int_{Q_t} u^2 dQ_t \leq 0$$

or  $u(x, y, t) \equiv 0$ . Hence  $u_1 = u_2$  i.e., the solution is unique.

$$\begin{aligned} &\text{Let us introduce the notation } m_1 = \sup_{(x,y) \in \Omega} |a_1(x, y)|, \quad m_2 = \sup_{(x,y) \in \Omega} |a_2(x, y)|, \\ m_3 = \sup_{(x,y) \in \Omega} |c(x, y)|, \quad m_4 = \sup_{(x,y) \in \Omega} |a_{1x}(x, y)|, \quad m_5 = \sup_{(x,y) \in \Omega} |a_{2y}(x, y)|. \end{aligned}$$

**Theorem 2.** Suppose a solution of the problem (1)-(3) exists and  $u, u_\varepsilon \in M$  and also

$$\|\varphi - \varphi_\varepsilon\|_{W_2^2} \leq \varepsilon, \quad \|\psi - \psi_\varepsilon\| \leq \varepsilon, \quad \|h - h_\varepsilon\| \leq \varepsilon, \quad \sup_t \|f - f_\varepsilon\| \leq \varepsilon.$$

Then for the solution of the problem (1)-(3), the following estimate holds:

$$\int_0^t \|u - u_\varepsilon\|^2 d\tau \leq d(t) (T\varepsilon^2 + \gamma_\varepsilon)^{1-r(t)} (4m^2 + \gamma_\varepsilon)^{r(t)},$$

where

$$\begin{aligned} \gamma_\varepsilon = & 3(2T^2 + 3)\varepsilon^2 \left( 1 + 6(m_1^2 + m_2^2 + \frac{m_3^2}{2} + m_4^2 + m_5^2) \right) \frac{1 - e^{-2\alpha T}}{2\alpha} \\ & + 3(2T^2 + 3)\varepsilon^2 \frac{4\alpha T + 3e^{-2\alpha T} - 1}{4\alpha^2} + 2T\varepsilon^2 (m_1^2 + m_2^2 + \frac{m_3^2}{2} + m_4^2 + m_5^2) \end{aligned}$$

**Proof.** Let us introduce the notation  $U(x, y, t) = u(x, y, t) - u_\varepsilon(x, y, t)$ . Then the function  $U(x, y, t)$  satisfies the equation

$$\left( \frac{\partial}{\partial t} + \alpha \right) (U_{tt} + LU) = f(x, y, t) - f_\varepsilon(x, y, t)$$

with the initial and boundary conditions

$$\begin{aligned} U(x, y, 0) &= \varphi(x, y) - \varphi_\varepsilon(x, y), \quad U_t(x, y, 0) = \psi(x, y) - \psi_\varepsilon(x, y), \quad U_{tt}(x, y, 0) = h(x, y) - h_\varepsilon(x, y), \\ U(0, y, t) &= U(l, y, t) = 0, \quad U(x, 0, t) = U(x, d, t) = 0. \end{aligned}$$

Let  $U_{tt} + LU = V$ . Then

$$\begin{aligned} V_t + \alpha V &= f(x, y, t) - f_\varepsilon(x, y, t), \\ V(x, y, 0) &= h(x, y) - h_\varepsilon(x, y) + L(\varphi(x, y) - \varphi_\varepsilon(x, y)), \\ U(x, y, 0) &= \varphi(x, y) - \varphi_\varepsilon(x, y), \quad U_t(x, y, 0) = \psi(x, y) - \psi_\varepsilon(x, y), \\ U(0, y, t) &= U(l, y, t) = 0, \quad U(x, 0, t) = U(x, d, t) = 0. \end{aligned}$$

From inequality (6)

$$\begin{aligned} \|V(x, y, t)\|^2 &\leq 3e^{-2\alpha t} (\|h - h_\varepsilon\|^2 + \|L(\varphi - \varphi_\varepsilon)\|^2) + \frac{1 - e^{-2\alpha t}}{\alpha} \int_0^t \|f - f_\varepsilon\|^2 d\tau \leq \\ &\leq 3e^{-2\alpha t} \varepsilon^2 \left( 1 + 6 \left( m_1^2 + m_2^2 + \frac{m_3^2}{2} + m_4^2 + m_5^2 \right) \right) + \frac{3(1 - e^{-2\alpha t}) t \varepsilon^2}{\alpha}. \end{aligned}$$

Now we estimate the function  $\rho$

$$\rho = 2 \left| \int_{\Omega} U(x, y, 0) U_t(x, y, 0) d\Omega \right| + T \left| \int_{\Omega} (U(x, y, 0) LU(x, y, 0) + U_t^2(x, y, 0)) d\Omega \right|.$$

Applying the initial conditions, we estimate

$$\begin{aligned} \rho &= 2 \left| \int_{\Omega} (\varphi(x, y) - \varphi_{\varepsilon}(x, y))(\psi(x, y) - \psi_{\varepsilon}(x, y)) d\Omega \right| + \\ &+ T \left| \int_{\Omega} ((\varphi(x, y) - \varphi_{\varepsilon}(x, y))L(\varphi(x, y) - \varphi_{\varepsilon}(x, y)) + (\psi(x, y) - \psi_{\varepsilon}(x, y))^2) d\Omega \right| \leq \\ &\leq 2T\varepsilon^2 \left( m_1^2 + m_2^2 + \frac{m_3^2}{2} + m_4^2 + m_5^2 \right) \\ \gamma &= (2T^2 + 3) \int_{\Omega} \|V(x, y, t)\|^2 d\Omega + |\rho| \leq \gamma_{\varepsilon}. \end{aligned}$$

If we denote

$$\begin{aligned} \gamma_{\varepsilon} &= 3(2T^2 + 3)\varepsilon^2 \left( 1 + 6(m_1^2 + m_2^2 + \frac{m_3^2}{2} + m_4^2 + m_5^2) \right) \frac{1 - e^{-2\alpha T}}{2\alpha} \\ &+ 3(2T^2 + 3)\varepsilon^2 \frac{4\alpha T + 3e^{-2\alpha T} - 1}{4\alpha^2} + 2T\varepsilon^2 (m_1^2 + m_2^2 + \frac{m_3^2}{2} + m_4^2 + m_5^2) \end{aligned}$$

we obtain

$$\int_0^t \|u - u_{\varepsilon}\|^2 d\tau \leq d(t)(T\varepsilon^2 + \gamma_{\varepsilon})^{1-r(t)} (4m^2 + \gamma_{\varepsilon})^{r(t)}.$$

Which was required to prove.

### Approximate Solution

**Case 1.** Without loss of generality, we assume that  $\varphi(x, y) = 0$ ,  $\psi(x, y) = 0$ ,  $f(x, y, t) = 0$ ,  $h(x, y) \neq 0$ . Suppose a solution of the problem (1)-(3) exists, then it can be represented in the form

$$u(x, y, t) = \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n^2} e^{-\alpha t} h_n \frac{\alpha sh(\lambda_n t) - \lambda_n ch(\lambda_n t) + \lambda_n e^{-\alpha t}}{\alpha^2 - \lambda_n^2} \right) \omega_n(x, y).$$

As an approximate solution, consider the function

$$u^N(x, y, t) = \sum_{n=1}^N \left( \frac{1}{\lambda_n^2} e^{-\alpha t} h_n \frac{\alpha sh(\lambda_n t) - \lambda_n ch(\lambda_n t) + \lambda_n e^{-\alpha t}}{\alpha^2 - \lambda_n^2} \right) \omega_n(x, y),$$

where  $N$  is an integer regularization parameter,

$$h_n = \int_{\Omega} h(x, y) \omega_n(x, y) dx dy$$

and as an approximate solution with approximate data, consider the function

$$u^{N\varepsilon}(x, y, t) = \sum_{n=1}^N \left( \frac{1}{\lambda_n^2} e^{-\alpha t} h_{n\varepsilon} \frac{\alpha sh(\lambda_n t) - \lambda_n ch(\lambda_n t) + \lambda_n e^{-\alpha t}}{\alpha^2 - \lambda_n^2} \right) \omega_n(x, y),$$

where  $h_{n\varepsilon} = \int_{\Omega} h_{\varepsilon}(x, y) \omega_n(x, y) d\Omega$ .

Suppose that  $\lambda_n \neq \alpha$ ,  $\exists \delta > 0$ ,  $\min |\alpha - \lambda_n| \geq \delta$ , and  $\|h - h_{\varepsilon}\| \leq \varepsilon$ ,  $u \in M$ . Then we estimate the difference

$$\|u - u^{N\varepsilon}\| \leq \|u - u^N\| + \|u^N - u^{N\varepsilon}\|. \quad (10)$$

Consider the second term on the right-hand side of inequality (10)

$$\begin{aligned} \|u^N - u^{N\varepsilon}\|^2 &= \sum_{n=1}^N \left( \frac{1}{\lambda_n^2} e^{-\alpha t} (h_n - h_{n\varepsilon}) \left( \frac{\alpha sh(\lambda_n t) - \lambda_n ch(\lambda_n t) + \lambda_n e^{-\alpha t}}{\delta^2(\alpha + \lambda_n)} \right) \omega_n(x, y) \right)^2 \leq \\ &\leq \frac{1}{\lambda_1^4} e^{2t(|\alpha| + \lambda_N)} \frac{(|\alpha| + 2\lambda_N)^2}{\delta^2(\alpha + \lambda_1)^2} \varepsilon^2. \end{aligned}$$

Now consider the first term on the right-hand side of inequality (10). To do this, we estimate the norm of the difference between the exact and approximate solutions.

We have:

$$\|u - u^N\|^2 = \sum_{n=N+1}^{\infty} \left( \frac{1}{\lambda_n^2} e^{-\alpha t} h_n \left( \frac{\alpha sh(\lambda_n t) - \lambda_n ch(\lambda_n t) + \lambda_n e^{-\alpha t}}{\alpha^2 - \lambda_n^2} \right) \omega_n(x, y) \right)^2. \quad (11)$$

Using the well-posedness set, we obtain the following inequality where

$$\|u(x, y, T)\|^2 = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^4} e^{-2\alpha T} h_n^2 \left( \frac{\alpha sh(\lambda_n T) - \lambda_n ch(\lambda_n T) + \lambda_n e^{-\alpha T}}{\alpha^2 - \lambda_n^2} \right)^2.$$

The coefficients  $h_n$  are chosen as follows:

$$h_n = \begin{cases} \frac{\lambda_{N+1}^2 e^{\alpha T} (\alpha^2 - \lambda_{N+1}^2) m}{\alpha sh(\lambda_{N+1} T) - \lambda_{N+1} ch(\lambda_{N+1} T) + \lambda_{N+1} e^{-\alpha T}}, n = N + 1 \\ 0, n \neq N + 1 \end{cases}$$

Then we obtain the following estimate for the norm of the difference of the solutions:

$$\|u - u^N\| \leq \frac{e^{\alpha(T-t)} (\alpha sh(\lambda_{N+1} t) - \lambda_{N+1} ch(\lambda_{N+1} t) + \lambda_{N+1} e^{-\alpha t}) m}{(\alpha sh(\lambda_{N+1} T) - \lambda_{N+1} ch(\lambda_{N+1} T) + \lambda_{N+1} e^{-\alpha T})}.$$

If approximate data are used instead of exact data, the corresponding estimate takes the form

$$\|u - u_\varepsilon^N\| \leq \frac{e^{\alpha(T-t)} (\alpha sh(\lambda_{N+1} t) - \lambda_{N+1} ch(\lambda_{N+1} t) + \lambda_{N+1} e^{-\alpha t}) m}{(\alpha sh(\lambda_{N+1} T) - \lambda_{N+1} ch(\lambda_{N+1} T) + \lambda_{N+1} e^{-\alpha T})} + \frac{1}{\lambda_1^2} e^{t(|\alpha| + \lambda_N)} \frac{(|\alpha| + 2\lambda_N) \varepsilon}{\delta^2 (\alpha + \lambda_1)}.$$

To find the parameter  $N$ , we compute:

$$\inf_N \left\{ \frac{e^{\alpha(T-t)} (\alpha sh(\lambda_{N+1} t) - \lambda_{N+1} ch(\lambda_{N+1} t) + \lambda_{N+1} e^{-\alpha t}) m}{(\alpha sh(\lambda_{N+1} T) - \lambda_{N+1} ch(\lambda_{N+1} T) + \lambda_{N+1} e^{-\alpha T})} + \frac{1}{\lambda_1^2} e^{t(|\alpha| + \lambda_N)} \frac{(|\alpha| + 2\lambda_N) \varepsilon}{\delta^2 (\alpha + \lambda_1)} \right\}.$$

**Case 2.** Without loss of generality, we assume that  $\varphi(x, y) = 0$ ,  $\psi(x, y) \neq 0$ ,  $f(x, y, t) = 0$ ,  $h(x, y) = 0$ . Suppose a solution of the problem (1)-(3) exists, then it can be represented in the form

$$u(x, y, t) = \sum_{n=1}^{\infty} \frac{\psi_n sh(\lambda_n t)}{\lambda_n} \omega_n(x, y).$$

where  $\psi_n = \int_{\Omega} \psi(x, y) \omega_n(x, y) d\Omega$ .

Similarly, the norm of the difference between the exact and the approximate data is obtained, which has the form:

$$\|u - u_\varepsilon^N\| \leq \frac{m \lambda_{N+1} sh \lambda_{N+1} t}{sh \lambda_{N+1} T} + \frac{e^{\lambda_N t}}{\lambda_N} \varepsilon.$$

For the purpose of determining the regularization parameter, the following minimization problem is considered:

$$\inf_N \left\{ \frac{m \lambda_{N+1} sh \lambda_{N+1} t}{sh \lambda_{N+1} T} + \frac{e^{\lambda_N t}}{\lambda_N} \varepsilon \right\}.$$

**Case 3.** Without loss of generality, we assume that  $\varphi(x, y) = 0$ ,  $\psi(x, y) = 0$ ,  $f(x, y, t) \neq 0$ ,  $h(x, y) = 0$ . Suppose a solution to problem (1)-(3) exists, then it can be represented in the form

$$u(x, y, t) = \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n^2} \int_0^t \int_0^\tau f_n(\tau_1) e^{\alpha(\tau_1 - \tau)} d\tau_1 sh((t - \tau) \lambda_n) d\tau \right) \omega_n(x, y)$$

where

$$f_n(\tau_1) = \int_{\Omega} f(x, y, \tau_1) \omega_n(x, y) d\Omega$$

As an approximate solution, consider the function

$$u^N(x, y, t) = \sum_{n=1}^N \left( \frac{1}{\lambda_n^2} \int_0^t \int_0^{\tau} f_n(\tau_1) \cdot e^{\alpha(\tau_1-\tau)} d\tau_1 sh((t-\tau)\lambda_n) d\tau \right) \omega_n(x, y)$$

where  $N$  is an integer regularization parameter, and as an approximate solution with approximate data, consider the function

$$u^{N_\varepsilon}(x, y, t) = \sum_{n=1}^N \left( \frac{1}{\lambda_n^2} \int_0^t \int_0^{\tau} f_{n_\varepsilon}(\tau_1) \cdot e^{\alpha(\tau_1-\tau)} d\tau_1 sh((t-\tau)\lambda_n) d\tau \right) \omega_n(x, y)$$

where  $f_n(\tau_1) = \int_{\Omega} f(x, y, \tau_1) \omega_n(x, y) d\Omega$ ,  $f_{n_\varepsilon}(\tau_1) = \int_{\Omega} f_\varepsilon(x, y, \tau_1) \omega_n(x, y) d\Omega$ .

Let

$$\|f(x, y, \tau_1) - f_\varepsilon(x, y, \tau_1)\| \leq \varepsilon$$

and  $u \in M$ . Then we estimate the difference

$$\|u - u^{N_\varepsilon}\| \leq \|u - u^N\| + \|u^N - u^{N_\varepsilon}\| \quad (12)$$

Consider the second term on the right-hand side of inequality (12)

$$\begin{aligned} \|u^N - u^{N_\varepsilon}\|^2 &= \sum_{n=1}^N \left( \frac{1}{\lambda_n^2} \omega_n(x, y) \int_0^t \int_0^{\tau} (f_n(\tau_1) - f_{n_\varepsilon}(\tau_1)) e^{\alpha(\tau_1-\tau)} d\tau_1 sh((t-\tau)\lambda_n) d\tau \right)^2 \leq \\ &\leq \frac{1}{\lambda_1^4} \int_0^t \int_0^{\tau} e^{2\alpha(\tau_1-\tau)} \varepsilon^2 d\tau_1 \frac{e^{2\lambda_N(t-\tau)}}{4} d\tau = \frac{1}{\lambda_1^4} \left( \frac{\lambda_N (e^{2\lambda_N t} + e^{-2\alpha t} - 1) - \alpha}{16\lambda_N \alpha (\lambda_N + \alpha)} \right) \varepsilon^2 \\ \|u^N - u^{N_\varepsilon}\| &\leq \frac{1}{4\lambda_1^2} \left( \frac{\lambda_N (e^{2\lambda_N t} + e^{-2\alpha t} - 1) - \alpha}{\lambda_N \alpha (\lambda_N + \alpha)} \right)^{1/2} \varepsilon. \end{aligned}$$

Now consider the first term on the right-hand side of inequality (12)

$$\|u - u^N\|^2 = \sum_{n=N+1}^{\infty} \left( \frac{1}{\lambda_n^2} \int_0^t \int_0^{\tau} f_n(\tau_1) \cdot e^{\alpha(\tau_1-\tau)} d\tau_1 sh((t-\tau)\lambda_n) d\tau \right)^2 \quad (13)$$

From the set of correctness, we obtain the following inequality:

$$\|u(x, y, t)\|^2 = \sum_{n=1}^{\infty} \left( \frac{e^{\lambda_n T}}{2\lambda_n^2} \left( \int_0^T \int_0^{\tau} f_n(\tau_1) \cdot e^{\alpha(\tau_1-\tau)-\lambda_n \tau} d\tau_1 d\tau - \int_0^T \int_0^{\tau} f_n(\tau_1) \cdot e^{\alpha(\tau_1-\tau)-\lambda_n(\tau-2T)} d\tau_1 d\tau \right) \right)^2 \leq m^2.$$

We introduce the notation. Then, using the Lagrange multiplier method, the coefficients are chosen as follows:

$$\begin{aligned} \int_0^T \int_0^{\tau} f_n(\tau_1) \cdot e^{\alpha(\tau_1-\tau)-\lambda_n \tau} d\tau_1 d\tau &= g_n \\ e^{-2\lambda_n T} \int_0^T \int_0^{\tau} f_n(\tau_1) \cdot e^{\alpha(\tau_1-\tau)+\lambda_n \tau} d\tau_1 d\tau &= \tilde{g}_n \end{aligned}$$

$$g_n = \begin{cases} 2m\lambda_{N+1}^2 e^{-\lambda_n T} + \tilde{g}_{N+1}, & n = N+1 \\ 0, & n \neq N+1 \end{cases}$$

$$\begin{aligned} \|u - u^N\|^2 &\leq \sum_{n=N+1}^{\infty} \frac{e^{2\lambda_n t}}{4\lambda_n^4} \left( \int_0^t \int_0^\tau f_n(\tau_1) \cdot e^{\alpha(\tau_1-\tau)-\lambda_n \tau} d\tau_1 d\tau - \int_0^t \int_0^\tau f_n(\tau_1) \cdot e^{\alpha(\tau_1-\tau)+\lambda_n(\tau-2t)} d\tau_1 d\tau \right)^2 = \\ &= \sum_{n=N+1}^{\infty} \frac{e^{2\lambda_n t}}{4\lambda_n^4} \left( \int_0^T \int_0^\tau f_n(\tau_1) e^{\alpha(\tau_1-\tau)-\lambda_n \tau} d\tau_1 d\tau - \int_t^T \int_0^\tau f_n(\tau_1) e^{\alpha(\tau_1-\tau)+\lambda_n(\tau-2t)} d\tau_1 d\tau \right)^2 \leq A_1 + A_2 + A_3, \end{aligned}$$

where

$$\begin{aligned} A_1 &= \sum_{n=N+1}^{\infty} \frac{3e^{2\lambda_n t}}{4\lambda_n^4} \left( \int_0^T \int_0^\tau f_n(\tau_1) \cdot e^{\alpha(\tau_1-\tau)-\lambda_n \tau} d\tau_1 d\tau \right)^2, \\ A_2 &= \sum_{n=N+1}^{\infty} \frac{3e^{2\lambda_n t}}{4\lambda_n^4} \left( \int_t^T \int_0^\tau f_n(\tau_1) \cdot e^{\alpha(\tau_1-\tau)-\lambda_n \tau} d\tau_1 d\tau \right)^2, \\ A_3 &= \sum_{n=N+1}^{\infty} \frac{3e^{2\lambda_n t}}{4\lambda_n^4} \left( \int_0^t \int_0^\tau f_n(\tau_1) \cdot e^{\alpha(\tau_1-\tau)+\lambda_n(\tau-2t)} d\tau_1 d\tau \right)^2. \end{aligned}$$

We divide the last expression into three parts and estimate each part separately.

$$\begin{aligned} A_1 &\leq \sum_{n=N+1}^{\infty} \frac{3e^{2\lambda_n t}}{4\lambda_n^4} (g_n^2) = \sum_{n=N+1}^{\infty} \frac{3e^{2\lambda_n t}}{4\lambda_n^4} (2m\lambda_n^2 e^{-\lambda_n T} + \tilde{g}_n)^2 \leq 6me^{2\lambda_{N+1}(t-T)} \lambda_{N+1}^2 + \frac{3e^{2\lambda_{N+1}(t-T)} T^2}{2\lambda_{N+1}^4 N^2}, \\ A_2 &\leq (T-t) \sum_{n=N+1}^{\infty} \frac{3e^{2\lambda_n t}}{4\lambda_n^4} \int_t^T \left( \int_0^\tau f_n(\tau_1) \cdot e^{\alpha(\tau_1-\tau)-\lambda_n \tau} d\tau_1 \right)^2 d\tau \leq \\ &\leq (T-t) \sum_{n=N+1}^{\infty} \frac{3}{4\lambda_n^4} \int_t^T \int_0^\tau f_n^2(\tau_1) d\tau_1 \cdot \frac{e^{2\lambda_n(t-\tau)} - e^{2\lambda_n(t-\tau)-2\tau\alpha}}{2\alpha} d\tau \leq \left( \frac{\sqrt{3}(T-t)T}{2N\lambda_{N+1}^2} \right)^2, \\ A_3 &\leq \sum_{n=N+1}^{\infty} \frac{3e^{2\lambda_n t}}{4\lambda_n^4} \int_0^t \int_0^\tau f_n^2(\tau_1) d\tau_1 \int_0^\tau e^{2\alpha(\tau_1-\tau)+2\lambda_n(\tau-2t)} d\tau_1 d\tau \leq \\ &\leq \sum_{n=N+1}^{\infty} \frac{3t}{4\lambda_n^4} \int_0^t \int_0^\tau f_n^2(\tau_1) d\tau_1 \frac{e^{2\lambda_n(\tau-t)} - e^{2\lambda_n(\tau-t)-2\alpha\tau}}{2\alpha} d\tau \leq \left( \frac{\sqrt{3}t}{2N\lambda_{N+1}^2} \right)^2. \end{aligned}$$

We combine the above inequalities and obtain

$$\|u - u^N\| \leq \left( 3me^{2\lambda_{N+1}(t-T)} \lambda_{N+1}^2 + \frac{3e^{2\lambda_{N+1}(t-T)} T^2}{2\lambda_{N+1}^4 N^2} + \left( \frac{\sqrt{3}(T-t)T}{2N\lambda_{N+1}^2} \right)^2 + \left( \frac{\sqrt{3}t}{2N\lambda_{N+1}^2} \right)^2 \right)^{1/2}.$$

Here, too, to find the parameter  $N$ , the following expression is computed:

$$\inf_N \left\{ \frac{1}{4\lambda_1^2} \left( \frac{\lambda_N (e^{2\lambda_N t} + e^{-2\alpha t} - 1) - \alpha}{\lambda_N \alpha (\lambda_N + \alpha)} \right)^{1/2} \varepsilon + \left( 3me^{2\lambda_{N+1}(t-T)} \lambda_{N+1}^2 + \frac{3e^{2\lambda_{N+1}(t-T)} T^2}{2\lambda_{N+1}^4 N^2} + \left( \frac{\sqrt{3}(T-t)T}{2N\lambda_{N+1}^2} \right)^2 + \left( \frac{\sqrt{3}t}{2N\lambda_{N+1}^2} \right)^2 \right)^{1/2} \right\}$$

### Conclusion

The paper investigates an initial-boundary value problem for a third-order composite-type differential equation. This problem belongs to the class of ill-posed problems in mathematical physics. During the study, a priori estimates were obtained using the energy integral and logarithmic convexity methods, and theorems on the uniqueness and conditional stability of the solution were proved. The problem was reduced to a spectral problem by means of the Fourier method, and an analytical representation of the solution was derived.

In addition, the set of well-posedness was determined, and the stability of the solution on this set was investigated. To construct convergent approximate solutions, the Tikhonov regularization method was applied, and the problem of optimal selection of the regularization parameter was considered. Approximate solution formulas and their error estimates were obtained for various cases of initial data.

In conclusion, theoretical and numerical methods for solving an ill-posed third-order composite-type equation were developed in the paper, and it was shown that the proposed regularization approach ensures the stability of the solution on the well-posedness set.

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## LIMITING GIBBS MEASURE ON A NON-REGULAR TREE

KHALILOV AKBAR

KARSHI STATE UNIVERSITY, KARSHI, UZBEKISTAN  
akbarx1991@gmail.com

**ABSTRACT.** In this article we consider ferromagnetic Ising model on an infinite non-regular rooted tree, specifically an alternating tree where the branching number depends on the parity of the generation. We prove that a phase transition occurs if and only if  $x = g(x)$  admits more than one solution, which is equivalent to the condition  $d_1 d_2 \cdot f'(u_0) f'(v_0) > 1$ , where  $(u_0, v_0)$  is the critical point of the composite map.

**MSC (2020):** 17A30; 17B30; 17B40; 17B56.

**Key words:** Gibbs measures, Ising model, alternating tree, Cayley tree, phase transition, limiting Gibbs states, fixed-point equations, boundary conditions, non-regular trees.

## Introduction

Gibbs measures on infinite graphs are fundamental to statistical mechanics and probability theory, with the Ising model serving as the canonical example for studying phase transitions and multiple equilibrium states [2, 5]. The geometry of the underlying graph profoundly influences the structure of Gibbs states [1, 3].

Infinite trees occupy a special position due to their recursive structure, which allows exact analytic treatment through nonlinear recurrence relations [3, 5]. The classical theory has primarily focused on regular trees (Cayley trees or Bethe lattices), where every vertex has the same number of neighbors. In this setting, Higuchi [1] established that every “Markov chain” Gibbs state can be obtained as a limiting Gibbs state with suitable boundary conditions, based on the foundational work of Preston [3] and Spitzer [5].

This article extends Higuchi’s framework to the ferromagnetic Ising model on an infinite non-regular rooted tree, specifically an alternating tree where the branching number depends on the parity of the generation. The alternating structure introduces a coupled system of fixed-point equations, replacing the single equation that governs the regular case.

Our main results provide sharp conditions for uniqueness versus multiplicity of limiting Gibbs states, revealing a critical boundary determined jointly by the branching numbers and interaction strength. Below this boundary the limiting Gibbs state is unique; above it, three fixed points emerge, signaling phase coexistence

## Preliminaries

Let  $T$  be an infinitely rooted tree with root  $0 \in T$ . Unlike the regular  $(d + 1)$ -tree, we do not assume that every vertex has the same number of neighbors. Instead, the branching structure may vary from vertex to vertex and, in particular, may alternate periodically or follow a more general pattern.

For any vertex  $t \in T$ , denote by:  $|t|$  the graph distance from the root  $0$  to  $t$ ,  $N(t)$  the set of neighbors of  $t$ ;  $b(t) = |N(t) \cap \{u : |u| > |t|\}|$  the number of *children* of  $t$  (i.e., neighbors further from the root);  $p(t) = |N(t) \cap \{u : |u| < |t|\}|$  the number of *parents* of  $t$  (for  $t \neq 0$ ,  $p(t) = 1$  since the tree has no cycles; for  $t = 0$ ,  $p(0) = 0$ ).

Thus every vertex  $t \neq 0$  has exactly one parent, and  $b(t) \geq 0$  children. The root has  $b(0)$  children.

For  $t, s \in T$ , we write:  $t \rightarrow s$  if  $t \neq s$  and the unique simple path from  $0$  to  $s$  passes through  $t$  (i.e.,  $t$  is an ancestor of  $s$ );  $t \rightarrow_1 s$  if  $t \rightarrow s$  and  $t$  and  $s$  are neighbors (i.e.,  $s$  is a child of  $t$ ).

For any vertex  $s \neq 0$ , we denote by  $t(s)$  its unique parent (the vertex such that  $t(s) \rightarrow_1 s$ ).

For any finite  $V \subset T$ , the *boundary*  $\partial V$  is defined as:

$$\partial V = \{t \in T \setminus V : \exists s \in V \text{ such that } t \text{ and } s \text{ are neighbors}\}.$$

We will primarily consider the sequence of *boxes* or *balls*:

$$V_n = \{t \in T : |t| \leq n\}, \quad n \geq 1.$$

For any vertex  $s \in T$ , define the *subtree rooted at*  $s$ :

$$T_s = \{t \in T : s \rightarrow t\} \cup \{s\},$$

and its intersection with  $V_n$ :

$$V_{n,s} = T_s \cap V_n.$$

Let  $\Phi_V = \{-1, +1\}^V$  be the space of spin configurations on a subset  $V \subset T$ . For the whole tree we write  $\Phi = \Phi_T$ . The product topology is assigned to  $\Phi$ .

For each  $t \in T$ , define the coordinate projection  $X_t : \Phi \rightarrow \{-1, +1\}$  by  $X_t(\sigma) = \sigma(t)$ . The  $\sigma$ -algebra generated by  $\{X_t : t \in V\}$  is denoted by  $\mathcal{B}_V$  that is  $\mathcal{B}_V = \sigma(\{\pi^{-1}(A) : A \subset \{-1, 1\}^V\})$ . In particular,  $\mathcal{B} = \mathcal{B}_T$ .

Fix a real number  $H$  (external magnetic field) and a positive number  $J$  (ferromagnetic nearest-neighbor coupling). For a finite subset  $V \subset T$ , an inner configuration  $\sigma \in \Phi_V$ , and a boundary condition  $\omega \in \Phi$ , the interaction energy on  $V$  is defined by:

$$E_V^\omega(\sigma) = -H \sum_{t \in V} \sigma(t) - J \sum_{\substack{t, s \in V \\ t \sim s}} \sigma(t)\sigma(s) - J \sum_{\substack{t \in V, s \in \partial V \\ t \sim s}} \sigma(t)\omega(s),$$

where  $t \sim s$  denotes that  $t$  and  $s$  are neighbors in  $T$ . The first sum runs over all vertices in  $V$ , the second over internal bonds (both endpoints in  $V$ ), and the third over bonds crossing the boundary.

The finite-volume Gibbs state on  $\Phi_V$  is:

$$P_V^\omega(\sigma) = \frac{1}{Z_V^\omega} \exp[-E_V^\omega(\sigma)], \quad \sigma \in \Phi_V,$$

where the partition function is:

$$Z_V^\omega = \sum_{\delta \in \Phi_V} \exp[-E_V^\omega(\delta)].$$

We may view  $P_V^\omega$  as a probability measure on  $(\Phi, \mathcal{B})$  by concentrating on configurations that equal  $\omega$  outside  $V$  (though this extension is not strictly necessary for limiting Gibbs states).

For fixed  $H$  and  $J$ , let  $\{V_n\}$  be an increasing sequence of finite subsets such that  $V_n \uparrow T$  (i.e.,  $\bigcup_n V_n = T$ ). If for a given boundary condition  $\omega \in \Phi$  the weak limit

$$P^\omega = \text{w-lim}_{n \rightarrow \infty} P_{V_n}^\omega$$

exists, then  $P^\omega$  is called a *limiting Gibbs state* with boundary condition  $\omega$ .

A probability measure  $P$  on  $(\Phi, \mathcal{B})$  is called a *Gibbs state* (or DLR state) for  $(H, J)$  if for every finite  $V \subset T$  and every  $A \in \mathcal{B}_V$ ,

$$P(A \mid \mathcal{B}_{V^c})(\omega) = P_V^\omega(A) \quad \text{a.s. } (P),$$

where  $V^c = T \setminus V$  and  $\mathcal{B}_{V^c}$  is the  $\sigma$ -algebra generated by spins outside  $V$ .

It is well known that:

- The set  $\mathcal{G}(H, J)$  of all Gibbs states for fixed  $(H, J)$  is non-empty, compact, and convex.
- Every limiting Gibbs state  $P^\omega$  is a Gibbs state.
- Conversely, every extremal point of  $\mathcal{G}(H, J)$  can be realized as a limiting Gibbs state for some boundary condition  $\omega$ .

In the specific case of an *alternating tree* where:

$$b(t) = \begin{cases} d_2, & \text{if } |t| \text{ is even,} \\ d_1, & \text{if } |t| \text{ is odd,} \end{cases}$$

for all  $t \in T$  (with the root at level 0), we denote:

- The children of the root:  $1, 2, \dots, d_1$ .
- For any vertex  $s \neq 0$ , let  $b_s = b(s)$  be its number of children.

The recursions simplify because  $b_s$  depends only on the parity of  $|s|$ .

### Main part

In this section we define limiting Gibbs states and derive recurrence relations for the quantities  $R_{n,s}(\omega)$  and  $Y_{n,s}(\omega)$  in the context of a non-regular tree  $T$  (specifically, the alternating tree described above). We follow the general approach of Higuchi but adapt it to the case where the number of children depends on the generation.

From now on we consider the increasing sequence of boxes:

$$V_n = \{t \in T : |t| \leq n\}, \quad n \geq 1.$$

For each vertex  $s \in T$  we define the subtree rooted at  $s$ :

$$T_s = \{s\} \cup \{t \in T : s \rightarrow t\},$$

and its intersection with  $V_n$ :

$$V_{n,s} = T_s \cap V_n, \quad n \geq |s|.$$

The vertices of  $V_1 \setminus \{0\}$  (the children of the root) are numbered  $1, 2, \dots, b(0)$ , where  $b(0) = d_1$  in our alternating model.

For any boundary condition  $\omega \in \Phi$ , any vertex  $s \in T \setminus \{0\}$ , and any  $n \geq |s|$ , define:

$$W_{n,s}^\omega(\varepsilon) = \sum_{\substack{\sigma \in \Phi_{V_{n,s}} \\ \sigma(s) = \varepsilon}} \exp[-E_{V_{n,s}}^\omega(\sigma) - J\varepsilon\omega(t)], \quad \varepsilon = \pm 1, \quad (1)$$

where  $t$  is the unique parent of  $s$  (i.e., the vertex such that  $t \rightarrow_1 s$ ). The term  $-J\varepsilon\omega(t)$  is the contribution from the bond connecting  $s$  to its parent, separated out for convenience.

Then define:

$$R_{n,s}(\omega) = \frac{W_{n,s}^\omega(+1)}{W_{n,s}^\omega(-1)}, \quad Y_{n,s}(\omega) = \frac{1}{2} \log R_{n,s}(\omega).$$

**Theorem 1.** For  $s \in T \setminus \{0\}$ , let its children be denoted by  $u_1, u_2, \dots, u_{b(s)}$ , where  $b(s)$  is the number of children of  $s$ . Then:

$$W_{n,s}^\omega(\varepsilon) = \exp(\varepsilon H) \prod_{i=1}^{b(s)} \left[ e^{\varepsilon J} W_{n,u_i}^\omega(+1) + e^{-\varepsilon J} W_{n,u_i}^\omega(-1) \right], \quad \varepsilon = \pm 1.$$

**Proof.** If we denote

$$-E_{V_{n,s}}^{\omega\uparrow}(\sigma) = H \sum_{x \in V_{n,s}} \sigma(x) + J \sum_{\substack{x \sim y \\ x, y \in V_{n,s}}} \sigma(x)\sigma(y) + J \sum_{\substack{x \rightarrow_1 y \\ x \in V_{n,s} \\ y \in \partial V_{n,s}}} \sigma(x)\omega(y) \quad (1)$$

the expression (1) can be written as

$$W_{n,s}^\omega(\varepsilon) = \sum_{\sigma \in \Phi^{V_{n,s}}, \sigma(s)=\varepsilon} \exp \left[ -E_{V_{n,s}}^{\omega\uparrow} \right], \quad \varepsilon = \pm 1,$$

Decompose the energy: Separating the root  $s$  from its subtrees gives

$$E_{V_{n,s}}^{\omega\uparrow}(\sigma) = -H\varepsilon - J \sum_{i=1}^{b(s)} \varepsilon \sigma(u_i) + \sum_{i=1}^{b(s)} E_{V_{n,u_i}}^{\omega\uparrow}(\sigma^{(i)})$$

where  $\sigma^{(i)}$  restriction of  $\sigma$  to  $V_{n,u_i}$  depending only on the configuration in that subtree.

Since the subtrees  $V_{n,u_1}, \dots, V_{n,u_{b(s)}}$  are pairwise disjoint, the sum factorises:

$$\begin{aligned} W_{n,s}^\omega(\varepsilon) &= e^{H\varepsilon} \sum_{\substack{\sigma \in \Phi^{V_{n,s}} \\ \sigma(s)=\varepsilon}} \prod_{i=1}^{b(s)} \left( e^{J\varepsilon\sigma(u_i)} \cdot e^{-E_{V_{n,u_i}}^{\omega\uparrow}(\sigma(u_i))} \right) = \\ &= e^{H\varepsilon} \prod_{i=1}^{b(s)} \left[ \sum_{\sigma^{(i)}} \left( e^{J\varepsilon\sigma^{(i)}(u_i)} \cdot e^{-E_{V_{n,u_i}}^{\omega\uparrow}(\sigma^{(i)}(u_i))} \right) \right] \end{aligned}$$

Summing  $\sigma^{(i)}(u_i)$  over  $\{-1, +1\}$  and recognising the definition of  $W_{n,u_i}$ :

$$\begin{aligned} W_{n,s}^\omega(\varepsilon) &= e^{H\varepsilon} \sum_{\substack{\sigma \in \Phi^{V_{n,s}} \\ \sigma(s)=\varepsilon}} \prod_{i=1}^{b(s)} \left( e^{J\varepsilon\sigma(u_i)} \cdot e^{-E_{V_{n,u_i}}^{\omega\uparrow}(\sigma(u_i))} \right) = \\ &= e^{\varepsilon H} \prod_{u_i: s \rightarrow_1 u_i} \left[ e^{\varepsilon J} W_{n,u_i}^\omega(+1) + e^{-\varepsilon J} W_{n,u_i}^\omega(-1) \right] \end{aligned}$$

Hence (1) holds. Then we get

$$\begin{aligned} W_{n,s}^\omega(+1) &= e^H \prod_{u_i: s \rightarrow_1 u_i} [e^J W_{n,u_i}^\omega(+1) + e^{-J} W_{n,u_i}^\omega(-1)] \\ W_{n,s}^\omega(-1) &= e^{-H} \prod_{u_i: s \rightarrow_1 u_i} [e^{-J} W_{n,u_i}^\omega(+1) + e^J W_{n,u_i}^\omega(-1)] \\ R_{n,s}(\omega) &= \frac{W_{n,s}^\omega(+1)}{W_{n,s}^\omega(-1)} = \frac{e^H \prod_{u_i: s \rightarrow_1 u_i} [e^J W_{n,u_i}^\omega(+1) + e^{-J} W_{n,u_i}^\omega(-1)]}{e^{-H} \prod_{u_i: s \rightarrow_1 u_i} [e^{-J} W_{n,u_i}^\omega(+1) + e^J W_{n,u_i}^\omega(-1)]} \\ R_{n,s}(\omega) &= e^{2H} \prod_{u_i: s \rightarrow_1 u_i} \frac{e^J R_{n,u_i}(\omega) + e^{-J}}{e^{-J} R_{n,u_i}(\omega) + e^J} \end{aligned} \tag{2}$$

where  $\{\sigma(0) = +1\} := \{\sigma \in \Phi^{V_n} : \sigma(0) = +1\}$  is a cylinder set.

$$P_{V_n}(\sigma(0) = +1) = \frac{1}{Z_{V_n}} \sum_{\substack{\sigma \in \Phi^{V_n} \\ \sigma(0)=+1}} \exp\{-E_{V_n}^\omega(\sigma_+)\} = \frac{1}{Z_{V_n}} W_{n,0}^\omega(+1)$$

similary

$$P_{V_n}(\sigma(0) = -1) = \frac{1}{Z_{V_n}} \sum_{\substack{\sigma \in \Phi^{V_n} \\ \sigma(0)=-1}} \exp\{-E_{V_n}^\omega(\sigma_-)\} = \frac{1}{Z_{V_n}} W_{n,0}^\omega(-1)$$

It is easy to see from the above formulas that the following holds

$$\frac{P_{V_n}(\sigma(0) = +1)}{P_{V_n}(\sigma(0) = -1)} = e^{2H} \prod_{i=1}^{d_1} \frac{e^J R_{n,i}(\omega) + e^{-J}}{e^{-J} R_{n,i}(\omega) + e^J}$$

From this, dividing the equations for  $\varepsilon = +1$  and  $\varepsilon = -1$ , we obtain:

$$R_{n,s}(\omega) = e^{2H} \prod_{i=1}^{b(s)} \frac{e^J R_{n,u_i}(\omega) + e^{-J}}{e^{-J} R_{n,u_i}(\omega) + e^J}.$$

In terms of  $Y_{n,s}(\omega) = \frac{1}{2} \ln R_{n,s}(\omega)$ , this becomes:

$$Y_{n,s}(\omega) = H + \sum_{i=1}^{b(s)} f(Y_{n,u_i}(\omega)), \quad (3)$$

here the function  $f$  is defined by:

$$f(x) = \frac{1}{2} \log \frac{\text{ch}(x+J)}{\text{ch}(x-J)} = \text{th}^{-1}(\text{th } J \cdot \text{th } x).$$

For the root 0 with children  $1, 2, \dots, b(0)$ , we have a similar expression without the parent term:

$$\frac{P_{V_n}^\omega(\sigma(0) = +1)}{P_{V_n}^\omega(\sigma(0) = -1)} = e^{2H} \prod_{i=1}^{b(0)} \frac{e^J R_{n,i}(\omega) + e^{-J}}{e^{-J} R_{n,i}(\omega) + e^J}.$$

Equivalently,

$$\frac{P_{V_n}^\omega(\sigma(0) = +1)}{P_{V_n}^\omega(\sigma(0) = -1)} = \exp\left(2H + 2 \sum_{i=1}^{b(0)} f(Y_{n,i}(\omega))\right).$$

For  $n > m$  and for any  $\eta \in \Phi_{V_m}$ , the finite-volume Gibbs measure satisfies:

$$P_{V_n}^\omega(\{\sigma(s) = \eta(s), s \in V_m\}) = \frac{\exp[-E_{V_{m-1}}^\omega(\eta)]}{\sum_{\xi \in \Phi_{V_m}} \exp[-E_{V_{m-1}}^\omega(\xi)] \prod_{s \in \partial V_{m-1}} W_{n,s}^\omega(\eta(s))}, \quad (4)$$

where  $E_{V_{m-1}}^\omega(\eta)$  is the energy restricted to  $V_{m-1}$  with boundary condition  $\omega$  and inner configuration  $\eta$ , and  $\partial V_{m-1}$  is the set of vertices at level  $m$ . The following lemma provides a sufficient condition for the existence of the weak limit  $P^\omega = \text{w-lim}_{n \rightarrow \infty} P_{V_n}^\omega$ .

**Lemma 2.** Let  $\omega \in \Phi$  be given. If there exists an integer  $N > 0$  such that for every vertex  $s \in T$  with  $|s| \geq N$  the limit

$$\lim_{n \rightarrow \infty} R_{n,s}(\omega) \text{ exists,}$$

then the limiting Gibbs state  $P^\omega = \text{w-lim}_{n \rightarrow \infty} P_{V_n}^\omega$  exists.

**Proof.** The existence of limits for  $R_{n,s}(\omega)$  implies, via the recurrence relations, the convergence of the finite-dimensional distributions of  $P_{V_n}^\omega$ . The latter are expressed in terms of products of ratios involving  $R_{n,s}(\omega)$ . By the martingale convergence theorem or direct verification of the Cauchy criterion for weak convergence, the limit exists.

Let  $d_1$  and  $d_2$  simultaneously odd or even. Firstly, us consider a boundary condition  $\omega$  such that  $c(\omega) = \sum_{s:t \rightarrow s} \omega(s)$  is independent of  $t \in T \setminus \{0\}$ .

We obtain

$$Y_{n,s}(\omega) = H + \sum_{i=1}^{b(s)} f(Y_{n,u_i}(\omega)),$$

If we separete parity levels by writing

$$a_k := Y_{n,s}, \quad |s| = 2k \quad b_k := Y_{n,s}, \quad |s| = 2k + 1$$

then

$$\begin{cases} a_k = H + d_1 f(b_{k-1}) \\ b_k = H + d_2 f(a_k) \end{cases} \quad (5)$$

This coupled structure is the fundamental departure from the regular tree setting, where a single scalar recurrence  $Y_n = H + df(Y_{n-1})$  suffices[1].

### The Fixed-Point Equations

Passing to the infinite-volume limit, we seek fixed points  $(x, y) \in \mathbb{R}^2$  of system (6), that is, solutions of

$$\begin{cases} x = H + d_2 f(y) \\ y = H + d_1 f(x) \end{cases} \quad (6)$$

Substituting the second equation into the first yields the following scalar fixed-point equation in the single variable  $x$ :

$$x = H + d_2 f(H + d_1 f(x)) \quad (7)$$

where  $f(x) = \frac{1}{2} \ln \frac{\cosh(x+J)}{\cosh(x-J)}$ , with parameters  $J > 0$ ,  $d_1, d_2 \in \mathbb{N}$ , and  $H \in \mathbb{R}$ .

The number and stability of solutions to (7) govern the uniqueness or multiplicity of limiting Gibbs states.

### Properties of $f$

The function  $f$  satisfies the following properties:

1.  $f \in C^\infty(\mathbb{R})$ .
2.  $f$  is odd:  $f(-x) = -f(x)$  for all  $x \in \mathbb{R}$ .
3.  $f$  is strictly monotonically increasing with derivative

$$f'(x) = \frac{\sinh(2J)}{\cosh(2x) + \cosh(2J)} > 0.$$

4.  $\sup_{x \in \mathbb{R}} f'(x) = f'(0) = \tanh J$ .
5.  $\lim_{x \rightarrow \pm\infty} f(x) = \pm J$ .

Indeed, properties (1) and (2) follow directly from the definition. For (3), we compute:

$$\begin{aligned} f'(x) &= \frac{1}{2} [\tanh(x+J) - \tanh(x-J)] \\ &= \frac{1}{2} \left( \frac{\sinh(2J)}{\cosh(x+J) \cosh(x-J)} \right) \\ &= \frac{\sinh(2J)}{\cosh(2x) + \cosh(2J)} > 0. \end{aligned}$$

For (4), note that  $f'(x)$  is an even function and the denominator  $\cosh(2x) + \cosh(2J)$  attains its minimum at  $x = 0$ , giving  $1 + \cosh(2J)$ . Therefore,

$$f'(x) \leq f'(0) = \frac{\sinh(2J)}{1 + \cosh(2J)} = \tanh J.$$

(5) follows from  $\lim_{x \rightarrow \pm\infty} \tanh(x \pm J) = \pm 1$ .

### Properties of $g$

The function  $g(x) = H + d_2 f(H + d_1 f(x))$  satisfies:

1.  $g \in C^\infty(\mathbb{R})$ ;
2.  $g'(x) = d_1 d_2 f'(H + d_1 f(x)) f'(x)$ ;
3.  $g(x)$  has a unique point  $x_0$  which  $g'(x)$  attains its maximum;
4.  $g'(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ ;
5.  $\max_{x \in \mathbb{R}} |g'(x)| = d_1 d_2 \tanh^2 J$ ;
6.  $\lim_{x \rightarrow \pm\infty} g(x) = \pm d_2 J$ .

All properties of  $g(x)$  follow from properties of  $f(x)$  function which proof above.

**Theorem 3.** [Contraction Mapping Principle][2] If  $\sup_{x \in \mathbb{R}} |g'(x)| < 1$ , then  $g$  is a contraction mapping on  $\mathbb{R}$  and possesses a unique fixed point.

**Proof.** By the Mean Value Theorem, for any  $x, y \in \mathbb{R}$ :

$$|g(x) - g(y)| = |g'(\xi)| |x - y| \leq L |x - y|,$$

where  $L = \sup_{x \in \mathbb{R}} |g'(x)| < 1$ . Since  $\mathbb{R}$  is a complete metric space, the Banach fixed-point theorem guarantees the existence of a unique fixed point.

**Theorem 4.** [Analogue of Preston's theorem on non-regular tree] For fixed parameters  $H \in \mathbb{R}$  and  $J > 0$ , the following three conditions are mutually equivalent:

- (i) There exist at least two distinct Gibbs states for the interaction  $(H, J)$ ; that is, a phase transition occurs.
- (ii) The fixed-point equation (3) admits more than one solution.
- (iii)

$$d_1 d_2 f(u_0) f(v_0) > 1, \quad |H| \leq H_c(d_1, d_2, J),$$

where  $(u_0, v_0)$  is the unique positive solution of the critical system

$$u_0 = d_1 f(v_0), \quad v_0 = d_2 f(u_0), \quad f'(u_0) f'(v_0) = \frac{1}{d_1 d_2},$$

and  $H_c(d_1, d_2, J)$  denotes the critical external field.

**Proof.** (i)  $\Leftrightarrow$  (ii). The FKG inequality [3] and the monotonicity of the ferromagnetic interaction remain valid on the alternating tree, since these properties depend only on the nearest-neighbour structure and the sign of  $J$  not on the regularity of branching. By Lemma 1, the existence of a limiting Gibbs state  $P^\omega$  is equivalent to the convergence of  $R_{n,s}(\omega)$  as  $n \rightarrow \infty$  for all  $s$  outside a finite set. The limit points of the sequence  $(a_k, b_k)$  under iteration of (5) are precisely the fixed points of (6), and hence the solutions of (7). If (7) has multiple solutions, distinct boundary conditions produce distinct limiting states, establishing non-uniqueness.

(ii)  $\Leftrightarrow$  (iii). Define the composite function  $g : \mathbb{R} \rightarrow \mathbb{R}$  by

$$g(x) = H + d_2 f(H + d_1 f(x))$$

Then (8) is equivalent to the fixed-point equation  $x = g(x)$ . Since  $f$  is smooth, concave on  $(0, \infty)$ , and satisfies  $f''(x) < 0$ , the derivative  $g''(x) < 0$  therefore graph of function  $g$  has an S-shaped, and equation  $x = g(x)$  admits more than one solution if and only if  $\max_{x \in \mathbb{R}} g'(x) > 1$ , which occurs precisely when condition (iii) holds. The critical point is accordingly

$$J_c = \frac{1}{2} \ln \frac{\sqrt{d_1 d_2} + 1}{\sqrt{d_1 d_2} - 1}.$$

When the fixed-point equation (7) admits three solutions, label them  $x_\alpha \leq x_\beta \leq x_\gamma$  with corresponding  $y$ -components  $y_\alpha \leq y_\beta \leq y_\gamma$  determined by system (7). The intermediate solution  $(x_\beta, y_\beta)$  is unstable under iteration of (5), while  $(x_\alpha, y_\alpha)$  and  $(x_\gamma, y_\gamma)$  are stable.

For a boundary condition  $\omega \in \Phi$ , define the spin sum  $c_t(\omega) = \sum_{s:t \rightarrow 1 s} \omega(s)$  for  $t \in T \setminus \{0\}$ .

**Proposition.** Under the same hypotheses as Theorem 2, the iterates  $(a_k(\omega), b_k(\omega))$  satisfy

$$\lim_{k \rightarrow \infty} (a_k(\omega), b_k(\omega)) = \begin{cases} (x_\alpha, y_\alpha), & H + Jc_t(\omega) < x_\beta \text{ for all } t \in T \setminus \{0\} \\ (x_\beta, y_\beta), & H + Jc_t(\omega) = x_\beta, \\ (x_\gamma, y_\gamma), & H + Jc_t(\omega) > x_\beta \text{ for all } t \in T \setminus \{0\} \end{cases} \quad (2)$$

**Proof.** From (5) we have  $b_k = g(b_{k-1})$  where  $g(x)$  define above. Once the congruence  $\{b_k\}$  is established, the convergence of  $\{a_k\}$  follows immediately from the continuity of  $f$ . So suffices to analyze the iteration  $b_k = g(b_{k-1})$ .

Indeed if  $\max_{x \in \mathbb{R}} g'(x) > 1$ ,  $x = g(x)$  equation exist exactly three fixed points  $x_\alpha < x_\beta < x_\gamma$  or  $\max_{x \in \mathbb{R}} g'(x) \leq 1$  than  $g(x)$  has one fixed point. Each fixed point under iteration  $b_k = g(b_{k-1})$  is stability that at  $x_\alpha$  and  $x_\gamma$ :  $g'(x_\alpha) < 1$  and  $g'(x_\gamma) < 1$ , at  $x_\beta$ :  $g'(x_\beta) > 1$ .

Let  $x^* \in \{x_\alpha, x_\gamma\}$  be stable fixed point of  $g$  then we easily obtain if  $b_0 > x_\beta$ , the sequence  $\{b_k\}$  converges monotonically to  $x_\gamma$ ; if  $b_0 < x_\beta$ , the sequence  $\{b_k\}$  converges monotonically to  $x_\alpha$ ; if  $b_0 = x_\beta$ , then  $b_k = x_\beta$  for all  $k$ , another hand  $b_0(\omega) = H + Jc_t(\omega)$ .

**Corollary.** Let  $P^{\omega+}$  and  $P^{\omega-}$  denote the limiting Gibbs states corresponding to the all-plus and all-minus boundary conditions, respectively. Then:

$$P^\omega = \begin{cases} P^{\omega+}, & H + Jc_t(\omega) > x_\beta \quad \forall t \in T \setminus \{0\}, \\ P^{\omega-}, & H + Jc_t(\omega) < x_\beta \quad \forall t \in T \setminus \{0\}. \end{cases}$$

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## A DIRECT PROBLEM AND A TIME-DEPENDENT INVERSE COEFFICIENT PROBLEM FOR A SUB-DIFFUSION EQUATION INVOLVING THE $\psi$ -PRABHAKAR-CAPUTO OPERATOR

KHASANOV SHOKHZODBEK

FERGANA STATE UNIVERSITY, FERGANA, UZBEKISTAN

shohzodbekxasanov@gmail.com

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**ABSTRACT.** In this paper, a direct problem and a time-dependent inverse coefficient problem for a sub-diffusion equation with the  $\psi$ -Prabhakar-Caputo differential operator of fractional order are considered. The direct problem is reduced to a Volterra integral equation of the second kind with respect to  $t$ , and the existence and uniqueness of the solution to the direct problem are proved based on the theory of Volterra integral equations and the completeness of the system of eigenfunctions. In the inverse problem, a time-dependent coefficient is determined, and the existence and uniqueness of the solution are proved using the contraction mapping principle and Banach's fixed-point theorem. In addition, stability estimates for both problems are established.

**MSC (2020):** 34A08; 35R11; 35R30.

**Key words:**  $\psi$ -Prabhakar-Caputo fractional derivative, sub-diffusion equation, inverse coefficient problem, contraction mapping principle, Banach's fixed-point theorem, bivariate Mittag-Leffler type function.

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### Introduction

Currently, researchers are focusing on systematically developing generalised fractional calculus and its applications to ordinary and partial differential equations, since generalised fractional operators explicitly characterise the nature of the underlying processes. For instance, in [5] and [19], the history of fractional calculus,  $\psi$ -fractional calculus, and various types of fractional and generalised fractional integral-differential operators, along with their properties and applications, are presented in detail. In [6],[16],[17],[18], the authors investigate the properties of  $\psi$ -Prabhakar fractional integral-differential operators, their generalisations, and selected applications.

In [1],[3],[4],[15], the authors consider inverse coefficient problems for time-fractional diffusion and advection-dispersion equations, employing the Fourier method, the contraction mapping principle, and Banach's fixed-point theorem to establish well-posedness. However, the number of such works remains limited, and the existing studies primarily rely on the Riemann-Liouville, Caputo, and Hilfer fractional derivatives. When studying fractional differential equations, it is also essential to understand the properties of special functions, in particular, their upper and lower bound estimates, complete monotonicity, and asymptotic behaviour, since solutions to fractional differential equations are typically expressed in terms of such functions. Analysing these properties is, in general, a non-trivial task. In [8],[9],[11], researchers employed the bivariate Mittag-Leffler function  $E_2(x, y)$  to investigate direct and inverse problems for sub-diffusion equations and mixed-type equations.

In the present paper, we consider both a direct problem and a time-dependent inverse coefficient problem for a sub-diffusion equation involving the  $\psi$ -Prabhakar-Caputo fractional differential operator of fractional order with a time-dependent coefficient. The main contributions of this work are the proof of uniqueness, the establishment of local existence, and the analysis of the stability of solutions to both problems. To the best of our knowledge, this is the first study to address such problems in the framework of the  $\psi$ -Prabhakar-Caputo operator, and the results obtained here generalise several previously known results in the literature.

### Preliminaries

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In this section, we recall some well-known definitions, lemmas, and results that will be used throughout the paper.

**Definition 1.** [7] The bivariate Mittag-Leffler function  $E_2(x, y)$  is defined by

$$E_2 \left( \begin{matrix} \gamma_1, \alpha_1, \beta_1; \gamma_2, \alpha_2 \\ \delta_1, \alpha_3, \beta_2; \delta_2, \alpha_4, \beta_3 \end{matrix} \middle| \begin{matrix} x \\ y \end{matrix} \right) = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} \frac{(\gamma_1)_{\alpha_1 m + \beta_1 n} (\gamma_2)_{\alpha_2 m}}{\Gamma(\delta_1 + \alpha_3 m + \beta_2 n) \Gamma(\delta_2 + \alpha_4 m) \Gamma(\delta_3 + \beta_3 n)} \frac{x^m}{\Gamma(\delta_1 + \alpha_3 m + \beta_2 n)} \frac{y^n}{\Gamma(\delta_2 + \alpha_4 m) \Gamma(\delta_3 + \beta_3 n)}. \quad (1)$$

Here,  $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \delta_1, \delta_2, \delta_3, x, y \in \mathbb{R}$ ,  $\min\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3\} > 0$ .

**Lemma 1.** [10] If  $\alpha_3 > -\alpha_4$ ,  $\alpha_3 > 0$ ,  $\beta_2 > -\beta_3$ ,  $\beta_2 > 0$  and  $\frac{\delta_1}{2} > \alpha_3 \theta$ ,  $\frac{\delta_1}{2} > \beta_2 \theta$ ,  $\gamma_1 > \theta(\alpha_1 + \beta_1)$ ,  $x < 0$ ,  $y < 0$ ,  $0 \leq \theta < \frac{1}{2}$  conditions are valid, then following estimates holds;

$$\left| E_2 \left( \begin{matrix} \gamma_1, \alpha_1, \beta_1; 1, 0 \\ \delta_1, \alpha_3, \beta_2; \delta_2, \alpha_4, \beta_3 \end{matrix} \middle| \begin{matrix} x \\ y \end{matrix} \right) \right| \leq \frac{\tilde{C}}{(-x)^\theta (-y)^\theta} \frac{\Gamma(\frac{\delta_1}{2} - \alpha_3 \theta) \Gamma(\frac{\delta_1}{2} - \beta_2 \theta) \Gamma(\gamma_1 - \theta(\alpha_1 + \beta_1))}{\Gamma(\delta_1 - \theta(\alpha_3 + \beta_2))}. \quad (2)$$

**Definition 2.** [13, pp.40-45] The function

$$E_{\alpha, \beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \quad (3)$$

called the Mittag-Leffler function with two parameters. Here  $\alpha, \beta, x \in \mathbb{R}$ ,  $\alpha > 0$ .

**Theorem 1.** [2] Let  $u, v$  be two integrable functions and  $g$  continuous, with domain  $[a, b]$ . Let  $\psi \in C^1[a, b]$  an increasing function such that  $\psi'(t) \neq 0 \forall t \in [a, b]$ . Assume that

1.  $u$  and  $v$  are nonnegative;
2.  $g$  is nonnegative and non-decreasing.

If

$$u(t) \leq v(t) + g(t) \int_a^t \psi'(t) (\psi(t) - \psi(\tau))^{\alpha-1} u(\tau) d\tau,$$

then

$$u(t) \leq v(t) + \int_a^t \sum_{n=1}^{+\infty} \frac{[g(t) \Gamma(\alpha)]^k}{\Gamma(\alpha k)} \psi'(t) (\psi(t) - \psi(\tau))^{\alpha k - 1} v(\tau) d\tau, \quad \forall t \in [a, b]. \quad (4)$$

**Corollary 1.** [2] Under the hypothesis of Theorem 1, let  $v$  be a non-decreasing function on  $[a, b]$ . Then, we have

$$u(t) \leq v(t) E_{\alpha, 1}(g(t) \Gamma(\alpha) (\psi(t) - \psi(a))^\alpha), \quad \forall t \in [a, b]. \quad (5)$$

In the next section, we will propose our main results.

### A direct problem

Let us consider a partial differential equation with fractional order:

$${}^{PC}D_{0t; \psi}^{\alpha, \beta, \gamma, \delta} u(t, x) - u_{xx}(t, x) + p(t) u(t, x) = f(t, x), \quad (6)$$

on the following domain:  $\Omega = \{(t, x) : 0 < t < T, 0 < x < b\}$ .

Here  $T, b, \alpha, \beta, \gamma, \delta \in \mathbb{R}$ ,  $\alpha > 0, 0 < \beta \leq 1$ . Also  $f(t, x), p(t)$  are given functions,  ${}^{PC}D_{0t; \psi}^{\alpha, \beta, \gamma, \delta}$  is  $\psi$ -Prabhakar-Caputo fractional derivative and defined as follows in [6], [17]

$$\begin{aligned} {}^{PC}D_{0t; \psi}^{\alpha, \beta, \gamma, \delta} u(t, x) &= {}^PI_{at; \psi}^{\alpha, 1-\beta, -\gamma, \delta} \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right) u(t, x) = \\ &= \int_0^t \psi'(s) (\psi(t) - \psi(s))^{-\beta} E_{\alpha, 1-\beta}^{-\gamma} (\delta(\psi(t) - \psi(s))^\alpha) \left( \frac{1}{\psi'(s)} \frac{d}{ds} \right) u(s, x) ds, \quad t > 0, \end{aligned}$$

where,  $\psi(t) \in C^1[0, T], \psi'(t) > 0$  for all  $t \in [0, T]$ .

**Problem 1.** To find a regular solution  $\{u(t, x)\}$  of the equation (6) which satisfies

- Regularity conditions:

$$u(t, x) \in C(\bar{\Omega}), {}^{PC}D_{0t; \psi}^{\alpha, \beta, \gamma, \delta} u(t, x) \in C(\Omega), u_{xx}(t, x) \in C(\Omega), u_x(t, x) \in C(\bar{\Omega}). \quad (7)$$

- Initial condition:

$$u(0, x) = \varphi(x), x \in [0, b], \quad (8)$$

- Boundary conditions:

$$u(t, 0) = 0, u(t, b) = 0, t \in [0, T]. \quad (9)$$

Here,  $\varphi(x)$  is a real, given function and  $\varphi(0) = \varphi(b) = 0$ .

**Definition 3.** A function  $u(t, x)$  is called a regular solution to Problem 1, if  $u(t, x)$  satisfies equation (6) and conditions (7)-(9).

We point out that in [3], a problem similar to Problem 1 was considered by D.K. Durdiev and J.J. Jumaev, where the Caputo fractional derivative was employed. Note that if  $\gamma = 1$  and  $\psi(t) = t$  in equation (6), then the  $\psi$ -Prabhakar–Caputo fractional derivative reduces to the Caputo fractional derivative.

### Existence and Uniqueness of the Solution to the Problem

**Theorem 2.** If  $\alpha > 0, \gamma > 0, \delta < 0, p(t) \in C[0, T], \psi(t) \in C^1[0, T], \psi'(t) > 0, \varphi^{(i)}(0) = \varphi^{(i)}(b) = 0, (i = 0, 2, 4), \varphi^{(4)}(x) \in C(\bar{\Omega}), \varphi^{(5)}(x) \in L_2(0, b), f(t, 0) = f(t, b) = f_{xx}^{(2)}(t, 0) = f_{xx}^{(2)}(t, b) = 0, f_{xx}^{(2)}(t, x) \in C(\bar{\Omega}), f_{xxx}^{(3)}(t, x) \in L_2((0, b); C[0, T])$ , then there exists a unique solution to the Problem 1, and the following estimates hold true:

$$|u(t, x)| \leq E_{\beta, 1} \left( C_5 \Gamma(\beta) \|p(t)\|_{C[0, T]} (\psi(T) - \psi(0))^\beta \right) \left( \frac{1+C_3+C_4}{12} + \frac{1}{2} \|\varphi^{(1)}(x)\|_{L_2(0, b)}^2 \right. \\ \left. + \left( \frac{C_3}{2} \|\varphi^{(3)}(x)\|_{L_2(0, b)}^2 + \frac{1}{2} \|f_x^{(1)}(t, x)\|_{L_2((0, b), C[0, T])}^2 \right) (\psi(T) - \psi(0))^\beta \right), \quad (10)$$

$$|u_{xx}(t, x)| \leq E_{\beta, 1} \left( C_5 \Gamma(\beta) \|p(t)\|_{C[0, T]} (\psi(T) - \psi(0))^\beta \right) \left( \frac{1+C_3+C_4}{12} + \frac{1}{2} \|\varphi^{(3)}(x)\|_{L_2(0, b)}^2 \right. \\ \left. + \left( C_3 \frac{1}{2} \|\varphi^{(5)}(x)\|_{L_2(0, b)}^2 + \frac{1}{2} \|f_{xxx}^{(3)}(t, x)\|_{L_2((0, b), C[0, T])}^2 \right) (\psi(T) - \psi(0))^\beta \right), \quad (11)$$

$$|{}^P D_{0t; \psi}^{\alpha, \beta, \gamma, \delta} u(t, x)| \leq \frac{1}{12} + \frac{1}{2} \|f_x^{(1)}(t, x)\|_{L_2((0, b), C[0, T])}^2 \\ + E_{\beta, 1} \left( C_5 \Gamma(\beta) \|p(t)\|_{C[0, T]} (\psi(T) - \psi(0))^\beta \right) \|p(t)\|_{C[0, T]} \left( \frac{1+C_3+C_4}{12} + \frac{1}{2} \|\varphi^{(2)}(x)\|_{L_2(0, b)}^2 \right. \\ \left. + \left( \frac{C_3}{2} \|\varphi^{(3)}(x)\|_{L_2(0, b)}^2 + \frac{1}{2} \|f_x^{(1)}(t, x)\|_{L_2((0, b), C[0, T])}^2 \right) (\psi(T) - \psi(0))^\beta \right) \\ + E_{\beta, 1} \left( C_5 \Gamma(\beta) \|p(t)\|_{C[0, T]} (\psi(T) - \psi(0))^\beta \right) \left( \frac{1+C_3+C_4}{12} + \frac{1}{2} \|\varphi^{(3)}(x)\|_{L_2(0, b)}^2 \right. \\ \left. + \left( \frac{C_3}{2} \|\varphi^{(5)}(x)\|_{L_2(0, b)}^2 + \frac{1}{2} \|f_{xxx}^{(3)}(t, x)\|_{L_2((0, b), C[0, T])}^2 \right) (\psi(T) - \psi(0))^\beta \right). \quad (12)$$

Here,  $C_3 = C_1 \Gamma(\gamma)$ ,  $C_4 = C_2 \frac{\Gamma(\gamma)}{\beta}$ ,  $C_5 = C_2 \Gamma(\gamma)$ , and

$$\|u(t, \cdot)\|_{L_2(0, b)} = \sqrt{\int_0^b u^2(t, \cdot) dx}, \quad \|u(\cdot, x)\|_{C(0, T)} = \max_{0 < t \leq T} |u(\cdot, x)|,$$

$$\|u(t, \cdot)\|_{L_2((0, b); C[0, T])} = \sqrt{\int_0^b \|u(t, \cdot)\|_{C[0, T]}^2 dx}.$$

**Proof.** To prove Theorem 2, we first establish the existence of a solution to Problem 1, and then prove the uniqueness of such a solution.

# First part of the proof. Existence.

Let us seek a solution to Problem 1, as follows:

$$u(t, x) = \sum_{n=1}^{+\infty} u_n(t) X_n(x), \quad (13)$$

where

$$X_n(x) = \sqrt{\frac{2}{b}} \sin \frac{\pi n x}{b}, \quad \lambda_n = \frac{(\pi n)^2}{b^2}, \quad u_n(t) = \int_0^b u(t, x) X_n(x) dx.$$

Taking function (13) into account, from equation (1), we have

$${}^P D_{0t; \psi}^{\alpha, \beta, \gamma, \delta} u_n(t) + \lambda_n u_n(t) = G_n(t), \quad n = 1, 2, \dots, \quad (14)$$

where

$$G_n(t) = f_n(t) - p(t) u_n(t), \quad f_n(t) = \int_0^b f(x, t) X_n(x) dx.$$

The initial condition (8) leads to the relations

$$u_n(0) = \varphi_n, \quad n = 1, 2, \dots, \quad (15)$$

where

$$\varphi_n = \int_0^b \varphi(x) X_n(x) dx.$$

The solution to the problem {(14),(15)} obtained in [12] as follows:

$$u_n(t) = \varphi_n - \lambda_n \Gamma(\gamma) \varphi_n (\psi(t) - \psi(0))^\beta E_2 \left( \begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta + 1, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda_n (\psi(t) - \psi(0))^\beta \\ \delta (\psi(t) - \psi(0))^\alpha \end{matrix} \right) + \Gamma(\gamma) \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\beta-1} E_2 \left( \begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda_n (\psi(t) - \psi(s))^\beta \\ \delta (\psi(t) - \psi(s))^\alpha \end{matrix} \right) G_n(s) ds.$$

Including, the form of the function  $G_n(t)$ , we arrive at the following integral equation:

$$u_n(t) = \varphi_n - \lambda_n \Gamma(\gamma) \varphi_n (\psi(t) - \psi(0))^\beta E_2 \left( \begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta + 1, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda_n (\psi(t) - \psi(0))^\beta \\ \delta (\psi(t) - \psi(0))^\alpha \end{matrix} \right) + \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\beta-1} \Gamma(\gamma) E_2 \left( \begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda_n (\psi(t) - \psi(s))^\beta \\ \delta (\psi(t) - \psi(s))^\alpha \end{matrix} \right) f_n(s) ds, \quad (16) - \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\beta-1} \Gamma(\gamma) E_2 \left( \begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda_n (\psi(t) - \psi(s))^\beta \\ \delta (\psi(t) - \psi(s))^\alpha \end{matrix} \right) (p(s) u_n(s)) ds.$$

Based on integral equation (16), we have the following Volterra integral equation of the second kind with weak singularity

$$u_n(t) = h_n(t) + \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\beta-1} K_n(t, s) u_n(s) ds, \quad (17)$$

where

$$h_n(t) = \varphi_n - \lambda_n \Gamma(\gamma) \varphi_n (\psi(t) - \psi(0))^\beta E_2 \left( \begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta + 1, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda_n (\psi(t) - \psi(0))^\beta \\ \delta (\psi(t) - \psi(0))^\alpha \end{matrix} \right) + \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\beta-1} \Gamma(\gamma) E_2 \left( \begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda_n (\psi(t) - \psi(s))^\beta \\ \delta (\psi(t) - \psi(s))^\alpha \end{matrix} \right) f_n(s) ds, \quad (18)$$

$$K_n(t, s) = -\Gamma(\gamma) E_2 \left( \begin{array}{c} \gamma, \gamma, 1; 1, 0 \\ \beta, \beta, \alpha; \gamma, \gamma; 1, 1 \end{array} \middle| \begin{array}{c} -\lambda_n(\psi(t) - \psi(s))^\beta \\ \delta(\psi(t) - \psi(s))^\alpha \end{array} \right) p(s). \quad (19)$$

In view of the facts that  $K_n(t, s) \in C([0, T] \times [0, T])$ ,  $0 < \beta < 1$ ,  $\psi(t) \in C^1[0, T]$ ,  $\psi'(t) > 0$  for all  $t \in [0, T]$ ,  $p(t) \in C[0, T]$ , and  $f_n(t) \in C[0, T]$ , by the theory of Volterra integral equations, equation (17) has a unique solution in  $C[0, T]$  for each  $n$ .

We proceed to derive estimates (10)–(12). For this purpose, we first establish estimates for the following functions:  $u_n(t)$  and  ${}^P D_{0t; \psi}^{\alpha, \beta, \gamma, \delta} u_n(t)$ , for each  $n \in \mathbb{N}$ .

Using estimate (2) and taking the conditions of Theorem 2, for  $\theta = 0$  into account, we obtain the following estimates:

$$\begin{aligned} \left| E_2 \left( \begin{array}{c} \gamma, \gamma, 1; 1, 0 \\ \beta + 1, \beta, \alpha; \gamma, \gamma; 1, 1 \end{array} \middle| \begin{array}{c} -\lambda_n(\psi(t) - \psi(0))^\beta \\ \delta(\psi(t) - \psi(0))^\alpha \end{array} \right) \right| &\leq C_1, \\ \left| E_2 \left( \begin{array}{c} \gamma, \gamma, 1; 1, 0 \\ \beta, \beta, \alpha; \gamma, \gamma; 1, 1 \end{array} \middle| \begin{array}{c} -\lambda_n(\psi(t) - \psi(s))^\beta \\ \delta(\psi(t) - \psi(s))^\alpha \end{array} \right) \right| &\leq C_2. \end{aligned} \quad (20)$$

Using estimates (20), function (18), considering facts that  $\psi(t) \in C^1[0, T]$ ,  $\psi'(t) > 0$  for all  $t \in [0, T]$  and  $p(t) \in C[0, T]$ ,  $f_n(t) \in C[0, T]$ , we obtain

$$|h_n(t)| \leq |\varphi_n| + \left( C_3 \left| \varphi_n^{(2)} \right| + C_4 \|f_n(t)\|_{C[0, T]} \right) (\psi(T) - \psi(0))^\beta. \quad (21)$$

In the same way, from (19) we get

$$|K_n(t, s)| \leq C_5 \|p(t)\|_{C[0, T]}. \quad (22)$$

In light of estimates (21), (22) and using inequality (5), we have

$$\begin{aligned} |u_n(t)| &\leq E_{\beta, 1} \left( C_5 \Gamma(\beta) \|p(t)\|_{C[0, T]} (\psi(T) - \psi(0))^\beta \right) \\ &\times \left( |\varphi_n| + \left( C_3 \left| \varphi_n^{(2)} \right| + C_4 \|f_n(s)\|_{C[0, T]} \right) (\psi(T) - \psi(0))^\beta \right). \end{aligned} \quad (23)$$

Applying equation (14) and estimate (23), we arrive at

$$\begin{aligned} \left| {}^P D_{0t; \psi}^{\alpha, \beta, \gamma, \delta} u_n(t) \right| &\leq \|f_n(t)\|_{C[0, T]} + \left( |\varphi_n| + \left( C_3 \left| \varphi_n^{(2)} \right| + C_4 \|f_n(t)\|_{C[0, T]} \right) (\psi(T) - \psi(0))^\beta \right) \\ &\times E_{\beta, 1} \left( C_5 \Gamma(\beta) \|p(t)\|_{C[0, T]} (\psi(T) - \psi(0))^\beta \right) \|p(t)\|_{C[0, T]} \\ &+ \left( \left| \varphi_n^{(2)} \right| + \left( C_3 \left| \varphi_n^{(4)} \right| + C_4 \|f_n^{(2)}(t)\|_{C[0, T]} \right) (\psi(T) - \psi(0))^\beta \right) \\ &\times E_{\beta, 1} \left( C_5 \Gamma(\beta) \|p(t)\|_{C[0, T]} (\psi(T) - \psi(0))^\beta \right). \end{aligned} \quad (24)$$

Here,

$$\varphi_n^{(2)} = \int_0^b \varphi^{(2)}(x) X_n(x) dx, \quad \varphi_n^{(4)} = \int_0^b \varphi^{(4)}(x) X_n(x) dx, \quad f_n^{(2)}(s) = \int_0^b f_{xx}^{(2)}(s, x) X_n(x) dx.$$

In light of estimates (23), (24), and function (13), and in view of the following well-known results

$$2ab \leq a^2 + b^2, \quad \sum_{m=1}^{+\infty} |g_m|^2 \leq \|g(x)\|_{L_2(a, b)}^2, \quad \sum_{m=1}^{+\infty} \frac{1}{(\pi m)^2} = \frac{1}{6}, \quad (25)$$

and conditions of the theorem, we derive estimates (10)–(12).

**Second part of the proof. Uniqueness.** It has been shown above that equation (14) has a unique solution  $u_n(t)$  for each  $n \in \mathbb{N}$ . Furthermore, by virtue of estimate (10), we establish the absolute and uniform convergence of the series representing the solution  $u(t, x)$  of Problem 1. These results, together with the completeness of the system of eigenfunctions  $\{X_n(x)\}_{n \in \mathbb{N}}$ , imply that Problem 1 has a unique solution.

### Stability of the Solution to the Problem

Let us suppose that the function  $\tilde{u}_n(t)$  be a solution of equation (16) with corresponding coefficient  $\tilde{\varphi}_n$  and functions  $\tilde{p}(t), \tilde{f}_n(t)$ . Namely,

$$\begin{aligned} \tilde{u}_n(t) = & \tilde{\varphi}_n - \lambda_n \Gamma(\gamma) \tilde{\varphi}_n (\psi(t) - \psi(0))^\beta E_2 \left( \begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta + 1, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda_n (\psi(t) - \psi(0))^\beta \\ \delta (\psi(t) - \psi(0))^\alpha \end{matrix} \right) \\ & + \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\beta-1} \Gamma(\gamma) E_2 \left( \begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda_n (\psi(t) - \psi(s))^\beta \\ \delta (\psi(t) - \psi(s))^\alpha \end{matrix} \right) \tilde{f}_n(s) ds, \\ & - \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\beta-1} \Gamma(\gamma) E_2 \left( \begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda_n (\psi(t) - \psi(s))^\beta \\ \delta (\psi(t) - \psi(s))^\alpha \end{matrix} \right) (\tilde{p}(s) \tilde{u}_n(s)) ds, \end{aligned} \quad (26)$$

where

$$\tilde{u}_n(t) = \int_0^b \tilde{u}(t, x) X_n(x) dx, \tilde{f}_n(t) = \int_0^b \tilde{f}(t, x) X_n(x) dx, \tilde{\varphi}_n = \int_0^b \tilde{\varphi}(x) X_n(x) dx.$$

For convenience, we introduce the following notations:

$$\bar{u}(t, x) = u(t, x) - \tilde{u}(t, x), \bar{\varphi}(x) = \varphi(x) - \tilde{\varphi}(x), \bar{f}(t, x) = f(t, x) - \tilde{f}(t, x), \bar{p}(t) = p(t) - \tilde{p}(t). \quad (27)$$

Then the following theorem holds.

**Theorem 3** If  $\alpha > 0, \gamma > 0, \delta < 0, p(t) \in C[0, T], \psi(t) \in C^1[0, T], \psi'(t) > 0, \varphi^{(i)}(0) = \varphi^{(i)}(b) = 0, (i = 0, 2), \varphi^{(2)}(x) \in C(\bar{\Omega}), \varphi^{(3)}(x) \in L_2(0, b), f(t, 0) = f(t, b) = 0, f(t, x) \in C(\bar{\Omega}), f_x^{(1)}(t, x) \in L_2((0, b); C[0, T])$ , then the following estimate holds true:

$$\begin{aligned} \|\bar{u}(t, x)\|_{C[0, T]} \leq & E_{\beta, 1} \left( C_5 \Gamma(\beta) \|p(t)\|_{C[0, T]} (\psi(T) - \psi(0))^\beta \right) \left( \frac{1}{12} + \frac{1}{2} \|\bar{\varphi}^{(2)}(x)\|_{L_2(0, b)}^2 \right. \\ & + \left( \frac{C_3 + C_4}{12} + C_3 \frac{1}{2} \|\bar{\varphi}^{(3)}(x)\|_{L_2(0, b)}^2 + \frac{1}{2} \|\bar{f}_x^{(1)}(t, x)\|_{L_2((0, b), C[0, T])}^2 \right) (\psi(T) - \psi(0))^\beta \\ & + C_4 \|\bar{p}(t)\|_{C[0, T]} \left( E_{\beta, 1} \left( C_5 \Gamma(\beta) \|p(t)\|_{C[0, T]} (\psi(T) - \psi(0))^\beta \right) \right)^2 \left( \frac{1}{12} + \frac{1}{2} \|\bar{\varphi}^{(2)}(x)\|_{L_2(0, b)}^2 \right. \\ & + \left. \left( \frac{C_3 + C_4}{12} + C_3 \frac{1}{2} \|\bar{\varphi}^{(3)}(x)\|_{L_2(0, b)}^2 + \frac{1}{2} \|\bar{f}_x^{(1)}(t, x)\|_{L_2((0, b), C[0, T])}^2 \right) (\psi(T) - \psi(0))^\beta \right). \end{aligned} \quad (28)$$

**Proof** Using functions (16), (26) and applying notations (27), we derive the following equation

$$\begin{aligned} \bar{u}_n(t) = & \bar{\varphi}_n - \lambda_n \Gamma(\gamma) \bar{\varphi}_n (\psi(t) - \psi(0))^\beta E_2 \left( \begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta + 1, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda_n (\psi(t) - \psi(0))^\beta \\ \delta (\psi(t) - \psi(0))^\alpha \end{matrix} \right) \\ & + \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\beta-1} \Gamma(\gamma) E_2 \left( \begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda_n (\psi(t) - \psi(s))^\beta \\ \delta (\psi(t) - \psi(s))^\alpha \end{matrix} \right) \bar{f}_n(s) ds, \\ & - \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\beta-1} \Gamma(\gamma) E_2 \left( \begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda_n (\psi(t) - \psi(s))^\beta \\ \delta (\psi(t) - \psi(s))^\alpha \end{matrix} \right) \bar{p}(s) \bar{u}_n(s) ds \\ & - \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\beta-1} \Gamma(\gamma) E_2 \left( \begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda_n (\psi(t) - \psi(s))^\beta \\ \delta (\psi(t) - \psi(s))^\alpha \end{matrix} \right) p(s) \bar{u}_n(s) ds. \end{aligned} \quad (29)$$

Taking into account the conditions of the Theorem and estimates (20), we obtain that

$$\begin{aligned} |\bar{u}_n(t)| \leq & \left( |\bar{\varphi}_n| + \left( C_3 |\bar{\varphi}_n^2| + C_4 \|\bar{f}_n(t)\|_{C[0, T]} \right) (\psi(T) - \psi(0))^\beta \right) \\ & \times E_{\beta, 1} \left( C_5 \Gamma(\beta) \|p(t)\|_{C[0, T]} (\psi(T) - \psi(0))^\beta \right) \\ & + C_4 \|\bar{p}(t)\|_{C[0, T]} \left( E_{\beta, 1} \left( C_5 \Gamma(\beta) \|p(t)\|_{C[0, T]} (\psi(T) - \psi(0))^\beta \right) \right)^2 \\ & \times \left( |\varphi_n| + \left( C_3 |\varphi_n^{(2)}| + C_4 \|f_n(t)\|_{C[0, T]} \right) (\psi(T) - \psi(0))^\beta \right). \end{aligned} \quad (30)$$

Then, applying the first relation in (27), conditions of the Theorem, the series representation of  $u(t, x)$  given by equality (13), and the classical results (25), together with estimate (30), we obtain estimate (28).

Thus, in this section, we have established the existence, uniqueness, and stability of Problem (1). If we assume that certain functions appearing in equation (6)  $\{u(t, x), p(t)\}$  are unknown, such a setting leads to

a more challenging and practically relevant problem. In the next section, we investigate the time-dependent inverse coefficient problem.

# INVERSE COEFFICIENT PROBLEM

**Problem 2.** Find a function  $p(t)$  in equation (6), if the solution of Problem 1 satisfies the following integral condition

$$\int_0^b \omega(x) u(x, t) dx = E(t), \quad 0 \leq t \leq T, \quad (31)$$

where, functions  $\omega(x), E(t)$  are given functions.

**Theorem 4.** Let us assume that the following conditions  $\alpha > 0, \gamma > 0, \delta < 0, \psi(t) \in C^1[0, T], \psi'(t) > 0, \varphi^{(i)}(0) = \varphi^{(i)}(b) = 0, (i = 0, 2, 4), \varphi^{(4)}(x) \in C(\bar{\Omega}), \varphi^{(5)}(x) \in L_2(0, b), f(t, 0) = f(t, b) = f_{xx}^{(2)}(t, 0) = f_{xx}^{(2)}(t, b) = 0, f_{xx}^{(2)}(t, x) \in C(\bar{\Omega}), f_{xxx}^{(3)}(t, x) \in L_2((0, b); C[0, T]),$  are valid. If  $E(t) \in C^1[0, T], |E(t)| \geq q > 0, \omega(x) \in C^2[0, L], \omega(0) = \omega(L) = 0$ , then there exists a number  $T^*$  such that Problem 2 has a unique solution belonging to the class  $C[0, T^*]$ .

Here,  $q$  is a sufficiently small number.

**Proof.** To prove the theorem, we multiply both sides of equation (6) by  $\omega(x)$  and integrate from 0 to  $b$  with respect to  $x$ , then we have

$$\int_0^b \omega(x) {}^{PC}D_{0t;\psi}^{\alpha,\beta,\gamma,\delta} u(t, x) dx - \int_0^b \omega(x) u_{xx}(t, x) dx + \int_0^b \omega(x) p(t) u(t, x) dx = \int_0^b \omega(x) f(t, x) dx. \quad (31)$$

Applying condition (31), to equation (32), we get

$${}^{PC}D_{0t;\psi}^{\alpha,\beta,\gamma,\delta} E(t) - \int_0^b \omega(x) u_{xx}(t, x) dx + p(t) E(t) = \int_0^b \omega(x) f(t, x) dx \quad (32)$$

Applying the integration by parts to equation (33) and using boundary conditions (9), equality (13), we have

$${}^{PC}D_{0t;\psi}^{\alpha,\beta,\gamma,\delta} E(t) - \sum_{n=1}^{+\infty} u_n(t) \omega_n^{(2)} + p(t) E(t) = \int_0^b \omega(x) f(t, x) dx, \quad (34)$$

where

$$\omega_n^{(2)} = \int_0^b \omega''(x) X_n(x) dx.$$

Taking equality (16) into account, from (34), we obtain the following equation

$$\begin{aligned} & {}^{PC}D_{0t;\psi}^{\alpha,\beta,\gamma,\delta} E(t) - \sum_{n=1}^{+\infty} \left( \varphi_n - \lambda_n \Gamma(\gamma) \varphi_n (\psi(t) - \psi(0))^\beta \right. \\ & \times E_2 \left( \begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta + 1, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda_n (\psi(t) - \psi(0))^\beta \\ \delta (\psi(t) - \psi(0))^\alpha \end{matrix} \right) \omega_n^{(2)} \\ & - \sum_{n=1}^{+\infty} \left( \Gamma(\gamma) \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\beta-1} \right. \\ & \times E_2 \left( \begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda_n (\psi(t) - \psi(s))^\beta \\ \delta (\psi(t) - \psi(s))^\alpha \end{matrix} \right) (f_n(s)) ds \Big) \omega_n^{(2)} \\ & + \sum_{n=1}^{+\infty} \left( \Gamma(\gamma) \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\beta-1} \right. \\ & E_2 \left( \begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta, \beta, \alpha; \gamma, \gamma; 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda_n (\psi(t) - \psi(s))^\beta \\ \delta (\psi(t) - \psi(s))^\alpha \end{matrix} \right) (p(s) u_n(s)) ds \Big) \omega_n^{(2)} \\ & + p(t) E(t) = \int_0^b \omega(x) f(t, x) dx, \end{aligned}$$

Since the last equation, we have the following operator equation with respect to the function  $p(t)$

$$p(t) = \mathbb{S}[p(t)]. \quad (35)$$

Here,

$$\begin{aligned} \mathbb{S}[p(t)] = M(t) - \frac{1}{E(t)} \sum_{n=1}^{+\infty} \left( \omega_n^{(2)} \Gamma(\gamma) \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\beta-1} \right. \\ \left. \times E_2 \left( \begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta, \beta, \alpha; \gamma, \gamma, 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda_n (\psi(t) - \psi(s))^\beta \\ \delta (\psi(t) - \psi(s))^\alpha \end{matrix} \right) (p(s) u_n(s)) \right) ds, \end{aligned} \quad (36)$$

$$\begin{aligned} M(t) = \frac{1}{E(t)} g(t) + \frac{1}{E(t)} \sum_{n=1}^{+\infty} \left( \omega_n^{(2)} \varphi_n - \lambda_n \Gamma(\gamma) \omega_n^{(2)} \varphi_n \right. \\ \left. \times (\psi(t) - \psi(0))^\beta E_2 \left( \begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta + 1, \beta, \alpha; \gamma, \gamma, 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda_n (\psi(t) - \psi(0))^\beta \\ \delta (\psi(t) - \psi(0))^\alpha \end{matrix} \right) \right. \\ \left. + \frac{1}{E(t)} \sum_{n=1}^{+\infty} \left( \omega_n^{(2)} \Gamma(\gamma) \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\beta-1} \right. \right. \\ \left. \left. \times E_2 \left( \begin{matrix} \gamma, \gamma, 1; 1, 0 \\ \beta, \beta, \alpha; \gamma, \gamma, 1, 1 \end{matrix} \middle| \begin{matrix} -\lambda_n (\psi(t) - \psi(s))^\beta \\ \delta (\psi(t) - \psi(s))^\alpha \end{matrix} \right) f_n(s) ds \right) \right), \end{aligned} \quad (37)$$

$$g(t) = \int_0^b \omega(x) f(t, x) dx.$$

Next, we apply the contraction mapping principle to show equation (35) has a unique solution. For this purpose, we find an estimate of the function  $M(t)$ . Considering estimates (20), theorem conditions, relations (25), and function (37), we have the following estimate

$$\begin{aligned} |M(t)| \leq \frac{1}{q} \|g(t)\|_{C[0,T]} + \frac{1}{q} \left( \frac{1}{2} \left( \|\omega^{(2)}(x)\|_{L_2(0,b)}^2 + \|\varphi(x)\|_{L_2(0,b)}^2 \right) \right. \\ \left. - C_3 \frac{1}{2} \left( \|\omega^{(2)}(x)\|_{L_2(0,b)}^2 + \|\varphi^{(2)}(x)\|_{L_2(0,b)}^2 \right) (\psi(T) - \psi(0))^\beta \right) \\ + \frac{C_4}{q} \left( \frac{1}{2} \left( \|\omega^{(2)}(x)\|_{L_2(0,b)}^2 + \|f(t, x)\|_{L_2((0,b), C[0,T])}^2 \right) (\psi(T) - \psi(0))^\beta \right) := R. \end{aligned}$$

Let us define the following ball of functions:

$$F^T(p(t), M(t), R) := \left\{ p(t) : p(t) \in C[0, T], \|p(t) - M(t)\|_{C[0,T]} \leq R \right\},$$

with a radius  $R$  and a center at the point  $M(t)$ . Based on the definition of the ball, we have

$$\|p(t)\|_{C[0,T]} \leq 2R. \quad (38)$$

Using expressions (36), estimates (20),(23),(25), and (38), we obtain the following estimate

$$\begin{aligned} \|\mathbb{S}[p(t)] - M(t)\|_{C[0,T]} \leq \frac{2RC_4}{q} (\psi(T) - \psi(0))^\beta E_{\beta,1} \left( 2RC_5 \Gamma(\beta) (\psi(T) - \psi(0))^\beta \right) \\ \times \left( \frac{1}{2} \left( \|\omega^{(2)}(x)\|_{L_2(0,b)}^2 + \|\varphi(x)\|_{L_2(0,b)}^2 \right) + \frac{C_3}{2} \left( \|\omega^{(2)}(x)\|_{L_2(0,b)}^2 + \|\varphi^{(2)}(x)\|_{L_2(0,b)}^2 \right) (\psi(T) - \psi(0))^\beta \right. \\ \left. + \frac{C_4}{2} \left( \|\omega^{(2)}(x)\|_{L_2(0,b)}^2 + \|f(t, x)\|_{L_2((0,b), C[0,T])}^2 \right) (\psi(T) - \psi(0))^\beta \right). \end{aligned}$$

Let  $T_1$  be a positive root of the following equation with respect to  $T$

$$\begin{aligned} \frac{2RC_4}{q} (\psi(T) - \psi(0))^\beta E_{\beta,1} \left( 2RC_5 \Gamma(\beta) (\psi(T) - \psi(0))^\beta \right) \\ \times \left( \frac{1}{2} \left( \|\omega^{(2)}(x)\|_{L_2(0,b)}^2 + \|\varphi(x)\|_{L_2(0,b)}^2 \right) + \frac{C_3}{2} \left( \|\omega^{(2)}(x)\|_{L_2(0,b)}^2 + \|\varphi^{(2)}(x)\|_{L_2(0,b)}^2 \right) (\psi(T) - \psi(0))^\beta \right. \\ \left. + \frac{C_4}{2} \left( \|\omega^{(2)}(x)\|_{L_2(0,b)}^2 + \|f(t, x)\|_{L_2((0,b), C[0,T])}^2 \right) (\psi(T) - \psi(0))^\beta \right) = R. \end{aligned}$$

Then for any  $T \in [0, T_1]$ , the  $\mathbb{S}$  maps a ball to the same ball, that is,  $\mathbb{S}[q(t)] \in F^T(p(t), M(t), R)$ .

Also, using equality (36), notations (27), estimate (23),(26), for  $f_n(t) = \tilde{f}_n(t)$ ,  $\varphi_n = \tilde{\varphi}_n$ , we have the following estimate

$$\begin{aligned} \|\mathbb{S}(p(t)) - \mathbb{S}(\tilde{p}(t))\|_{C[0,T]} &\leq \left[ \frac{2RC_4^2}{q} (\psi(T) - \psi(s))^\beta \left( E_{\beta,1} \left( 2RC_5 \Gamma(\beta) (\psi(T) - \psi(0))^\beta \right) \right)^2 \right. \\ &+ \frac{C_4}{q} E_{\beta,1} \left( 2RC_5 \Gamma(\beta) (\psi(T) - \psi(0))^\beta \right) \left. \left( \frac{1}{2} \left( \|\omega^{(2)}(x)\|_{L_2(0,b)}^2 + \|\varphi(x)\|_{L_2(0,b)}^2 \right) \right. \right. \\ &+ \frac{1}{2} \left( C_3 \left( \|\omega^{(2)}(x)\|_{L_2(0,b)}^2 + \|\varphi^{(2)}(x)\|_{L_2(0,b)}^2 \right) \right. \\ &\left. \left. + C_4 \left( \|\omega^{(2)}(x)\|_{L_2(0,b)}^2 + \|f(t,x)\|_{L_2((0,b),C[0,T])}^2 \right) \right) (\psi(T) - \psi(s))^\beta \right] \|p(t) - \tilde{p}(t)\|_{C[0,T]}. \end{aligned}$$

Let  $T_2$  be a positive solution of the following equation with respect to  $T$ :

$$\begin{aligned} &\left[ \frac{2RC_4^2}{q} (\psi(T) - \psi(s))^\beta \left( E_{\beta,1} \left( 2RC_5 \Gamma(\beta) (\psi(T) - \psi(0))^\beta \right) \right)^2 \right. \\ &+ \frac{C_4}{q} E_{\beta,1} \left( 2RC_5 \Gamma(\beta) (\psi(T) - \psi(0))^\beta \right) \left. \left( \frac{1}{2} \left( \|\omega^{(2)}(x)\|_{L_2(0,b)}^2 + \|\varphi(x)\|_{L_2(0,b)}^2 \right) \right. \right. \\ &+ \frac{1}{2} \left( C_3 \left( \|\omega^{(2)}(x)\|_{L_2(0,b)}^2 + \|\varphi^{(2)}(x)\|_{L_2(0,b)}^2 \right) \right. \\ &\left. \left. + C_4 \left( \|\omega^{(2)}(x)\|_{L_2(0,b)}^2 + \|f(t,x)\|_{L_2((0,b),C[0,T])}^2 \right) \right) (\psi(T) - \psi(s))^\beta \right] = 1. \end{aligned}$$

For  $T \in (0, T^*]$ , the operator  $\mathbb{S}$  reduces the distance between any two elements  $p(t)$  and  $\tilde{p}(t)$  in  $F^T(p(t), M(t), R)$ . Consequently, by selecting  $T^* < \min\{T_1, T_2\}$ , the operator  $\mathbb{S}$  becomes a contraction on the ball  $F^T(p(t), M(t), R)$ . By the Banach fixed-point theorem (see [13], p.67-68 and [14], p.43-51),  $\mathbb{S}$  admits a unique fixed point in  $F^T(p(t), M(t), R)$ , which implies the existence and uniqueness of a solution to Equation(35).

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## NUMERICAL SOLUTION OF THE NON-STATIONARY CONVECTION-DIFFUSION PROBLEM IN THE ONE-DIMENSIONAL CASE

KHAYITKULOV BOBOMUROD

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M.ULUGBEK, TASHKENT, UZBEKISTAN  
b.hayitkulov@mail.ru

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**ABSTRACT.** This work is devoted to the numerical solution of a non-stationary problem of optimal placement of minimum-power heat sources. The formulation of the problem requires the simultaneous fulfillment of two conditions. The first condition is to ensure that the temperature remains within the specified minimum and maximum limits by optimally placing minimum-power heat sources in a rod. The second condition is that the total power of the heat sources used for heating must be minimal. This problem has been studied under stationary conditions in the works of other researchers. However, the non-stationary case has not been considered. Since finding an analytical (continuous) solution to the boundary value problem is difficult, a numerical solution is sought. It is challenging to find the integral operator with a continuous kernel (Green's function). A numerical approximation of the Green's function in matrix form has been obtained. A new algorithm for the numerical solution of the non-stationary optimal control problem of placing minimum-power heat sources in processes described by parabolic partial differential equations is proposed. A new numerical solution method is introduced. A mathematical and numerical model of the processes described by the convection-diffusion equation, defined for the first boundary value problem, has been constructed. The boundary value problem is studied in the one-dimensional case. An implicit finite difference scheme was used for the numerical solution of the problem. Based on this scheme, a system of difference equations was constructed. The resulting system of difference equations was transformed into a linear programming problem. The linear programming problem is solved using the M-method. For each time value, the linear programming problem is solved. A new approach to the numerical solution of such problems is proposed. A general block diagram of the algorithm for solving the non-stationary optimal control problem of placing minimum-power heat sources is provided. An algorithm and software have been developed for the numerical solution of the problem. A brief description of the software is given. Specific examples demonstrate that the numerical solution of the boundary value problem stays within the specified limits, and the sum of optimally placed minimum-power heat sources minimizes the objective functional. The results of the computational experiment have been visualized.

**MSC (2020):** 65K05; 65N06.

**Key words:** non-stationary problems, optimal placement, heat sources, convection-diffusion equations, M-method.

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## INTRODUCTION

One of the most common elements in various fields of human activity is the system of heat sources and the thermal balance in heated spaces. Mathematical modeling of such systems poses the problem of optimal placement of heat sources in heated areas, which is closely related to resource-saving engineering technologies. The problem of optimal placement of heat sources in heated areas has always been relevant in design work in construction, greenhouses, and other technical and technological fields. Problems of this type have been studied in the stationary case. However, the problem of optimal placement of minimum-power heat sources in the non-stationary case has not yet been studied.

The work [1] proposes a solution to the problem of optimal placement of sources in inhomogeneous media, in which scalar stationary fields are described by elliptic equations. The algorithms for solving the problem are based on methods for estimating the values of the functional on the set of possible locations of sources, which

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makes it possible to choose the optimal variant by implementing the branch and bound method. The paper [2] considers the problem of optimal space heating based on the Pontryagin maximum principle. The paper [3] considers the problem of energy-efficient heat supply of a building in a central heating system. In [4], the differential-difference problem of controlling the diffusion process was studied, an analogue of the maximum principle was obtained, which allows one to determine the moments of switching on and off the source of maximum power. In [5], the problem of optimal control of processes described by the heat equation was studied. The control parameter is set in the boundary condition and has reached the minimum of the functional given by the integral quadratic expression. A method for finding an admissible control that gives a minimum to the functional is shown.

In [6, 7, 8], a method and an algorithm for solving the non-stationary problem of the optimal choice of the density of heat sources in simple geometric regions were developed so that the temperature inside the region under consideration was within the specified limits. At the same time, the heat sources provided the specified temperature regime of the minimum total power and the temperature in the specified corridor filled with a homogeneous or inhomogeneous medium. In [9], a boundary value problem of parabolic type was considered. The distribution of heat in the considered body is controlled by a function that is located on the boundary of the body.

In [10], the convection-diffusion equation is solved numerically in two-dimensional geometry to simulate heat transfer. Testing was carried out and results were obtained for the cases of boundary conditions of three types. We also note that in [11, 12, 13] numerical schemes with explicit convective and implicit diffusion transport are considered.

In this paper, the problem of convection-diffusion control based on optimization of the linear objective functional with restrictions is considered, which is solved based on approximation and reduction to the problem of linear programming. The paper proposes a method and algorithm for solving the non-stationary problem of maintaining the temperature inside the region within specified limits, by optimally placing heat sources in the rod. Software was developed to conduct computational experiments.

## STATEMENT OF THE PROBLEM AND ITS FINITE-DIMENSIONAL APPROXIMATION

In the rectangle  $D = \{a \leq x \leq b, 0 \leq t \leq T\}$ , it is required to find a function  $f(x, t) \geq 0$  such that for any  $t \in [0, T]$  the linear functional

$$J\{f\} = \int_a^b f(x, t) dx \rightarrow \min, \quad (1)$$

reaches a minimum and satisfies the following conditions:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \chi(x, t) \frac{\partial^2 u}{\partial x^2} - v(x, t) \frac{\partial u}{\partial x} + f(x, t), \\ a < x < b, \quad 0 < t \leq T, \\ u(x, 0) &= u_0(x), \quad a \leq x \leq b, \\ u(a, t) &= \mu_1(t), \quad u(b, t) = \mu_2(t), \quad 0 < t \leq T, \end{aligned} \quad (2)$$

$$m(x, t) \leq u(x, t) \leq M(x, t), \quad (x, t) \in D, \quad (3)$$

where  $u = u(x, t)$  – is the temperature of the rod at point  $x$  at time  $t$ ;  $\chi(x, t) > 0$  – is the diffusion coefficient (thermal diffusivity);  $v(x, t)$  – is the convection velocity;  $u_0(x)$ ,  $\mu_1(t)$ ,  $\mu_2(t)$ ,  $m(x, t)$ ,  $M(x, t)$  – are given continuous functions satisfying the conjugation conditions  $\mu_1(0) = u_0(a)$ ,  $\mu_2(0) = u_0(b)$ . The functions  $m(x, t)$  and  $M(x, t)$  – are the minimum and maximum temperatures defined in the domain  $D$ . The power of volumetric heat sources is described by a square-integrable function  $f(x, t)$  in the space  $L_2(D)$ .

Let  $Lu = \frac{\partial u}{\partial t} - \chi(x, t) \frac{\partial^2 u}{\partial x^2} + v(x, t) \frac{\partial u}{\partial x}$ . The operator  $L$  defined in  $L_2(D)$  has an inverse  $L^{-1}$ . Here  $L^{-1}$  is an integral operator with a continuous kernel (Green's function). Using it, you can write the problem (1)-(3) in the following form:

$$\begin{aligned} m(x, t) &\leq (L^{-1}f)(x, t) \leq M(x, t), \\ f(\cdot, \cdot) &\in L_2(D), \quad f(x, t) \geq 0. \end{aligned} \quad (4)$$

Since it is difficult to find a continuous solution to problem (1)-(4), we look for a numerical solution to the problem. In this case, using an implicit scheme, we replace problem (2) with a finite-difference equation.

Let us introduce into  $D$  a difference grid uniform in both variables  $\bar{\omega}_h^T = \bar{\omega}_h \times \bar{\omega}_\tau = \{(x_i, t_j): x_i = ih, t_j = j\tau, i = 0, 1, \dots, N_1, j = 0, 1, \dots, N_2, h = (b-a)/N_1, \tau = T/N_2\}$ .

The implicit difference scheme for problem (2) has the form:

$$\left\{ \begin{array}{l} \frac{u_i^{j+1} - u_i^j}{\tau} = \chi_i^{j+1} \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2} - \\ v_i^{j+1} \frac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2h} + f_i^{j+1}, \\ i = 1, 2, \dots, N_1 - 1, \quad j = 0, 1, \dots, N_2 - 1, \\ u_i^0 = u_0(x_i), \quad i = 0, 1, \dots, N_1, \\ u_0^{j+1} = \mu_1(t_{j+1}), \quad j = 0, 1, \dots, N_2 - 1, \\ u_{N_1}^{j+1} = \mu_2(t_{j+1}), \quad j = 0, 1, \dots, N_2 - 1. \end{array} \right. \quad (5)$$

Here  $\chi_i^{j+1} = \chi(x_i, t_{j+1})$ ,  $v_i^{j+1} = v(x_i, t_{j+1})$ ,  $f_i^{j+1} = f(x_i, t_{j+1})$ .

Let us introduce the notation

$$X = \left( \frac{1}{\tau} + \frac{2\chi_i^{j+1}}{h^2} \right), \quad X^+ = \left( -\frac{\chi_i^{j+1}}{h^2} + \frac{v_i^{j+1}}{2h} \right),$$

$$X^- = \left( -\frac{\chi_i^{j+1}}{h^2} - \frac{v_i^{j+1}}{2h} \right).$$

Let's consider the extended matrix of the system:

$$A = \begin{bmatrix} X & X^+ & \dots & \dots & 0 \\ X^- & X & X^+ & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & X^- & X & X^+ \\ 0 & \dots & \dots & X^- & X \end{bmatrix}.$$

We get

$$G = A^{-1}.$$

We approximate the problem (1)-(5) as a linear programming problem. We divide the domain  $D$  into equal parts by  $x$  on  $N_1$ , and by  $t$  on  $N_2$ :  $D = \bigcup_{j=1}^{N_2} \bigcup_{i=1}^{N_1} D_i^j$ , where  $D_i^j = \{(x, t), x_{i-1} \leq x \leq x_i, t_{j-1} \leq t \leq t_j\}$ ,  $i = 1, 2, \dots, N_1, j = 1, 2, \dots, N_2$ . In the space  $L_2(D)$  the functions  $f_i^j = f(x_i, t_j)$ ,  $(x, t) \in D_i^j$  ( $i = 1, 2, \dots, N_1 - 1, j = 1, 2, \dots, N_2$ ) are defined as piecewise constant functions. From this we obtain  $f(x, t) \approx \sum_{j=1}^{N_2} \sum_{i=1}^{N_1-1} f_i^j$ .

Let  $g_{ik}^j = G$ ,  $m_i^j = m(x_i, t_j)$ ,  $M_i^j = M(x_i, t_j)$ ,  $i = \overline{1, N_1 - 1}$ ,  $k = \overline{1, N_1 - 1}$ ,  $j = \overline{1, N_2}$ . Substitute the expression  $f(x, t)$  into (1) and replace inequality (4) with grid functions.

After this we get the following linear programming problem:

$$\begin{aligned} J_j\{f\} &= \sum_{i=1}^{N_1-1} (mesD_i^j) f_i^j \rightarrow \min, \quad j = 1, 2, \dots, N_2, \\ m_i^j &\leq \sum_{k=1}^{N_1-1} g_{ik}^j f_k^j \leq M_i^j, \quad i = \overline{1, N_1-1}, \quad j = \overline{1, N_2}, \\ f_k^j &\geq 0, \quad k = 1, 2, \dots, N_1-1, \quad j = 1, 2, \dots, N_2. \end{aligned} \quad (6)$$

Problem (6) is solved by the M-method [14, 15]. The numerical solution to problem (2) is found using  $u_i^j = \sum_{k=1}^{N_1-1} g_{ik}^j f_k^j$ . The found  $f_k^j$  is a function that gives a minimum to the functional (1).

### DESCRIPTION OF THE ALGORITHM AND MODELING RESULTS

For an approximate solution of problem (1)-(6), software in the *C#* language has been developed. It allows you to represent all the necessary input data: constants, coefficients, grid parameters, as well as temperature functions, initial and boundary conditions, in the form of scripts. Graphical modules have been developed to present the results.

The flowchart (Fig. 1) shows a general algorithm for the numerical solution of the non-stationary problem of controlling the optimal location of heat sources.

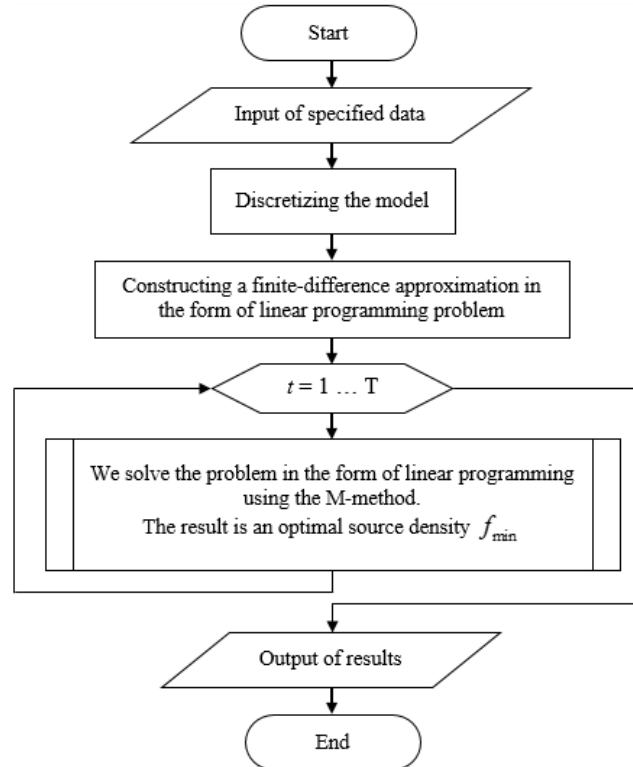


Рис. 1: Flowchart of the general algorithm for solving the problem

**Computational experiment.** It is required to find the optimal location of heat sources with minimal power in a parallelepiped. The problem was solved with the following input data: the computational domain uses a cube  $x \in [0, 1]$  with thermal diffusivity functions  $\chi(x, t) = x^2 t$  m<sup>2</sup>/s and velocity components  $v(x, t) = xt$  m/s. The initial and boundary conditions are determined by the functions:  $u_0(x) = 2 + x^2$  m/s,  $\mu_1(t) = 2 + t^2$  m/s,  $\mu_2(t) = 3 + t^2$  m/s. The minimum and maximum temperatures are given by the functions  $m(x, t) = 1 + x^2 + t^2$  K,

$M(x, t) = 4 + x^2 + t^2$  K, and the end of time  $T = 1$ . Calculation grid with the number of sources  $N_2 \times (N_1 - 1) = 50 \times 49$ . The minimum value of the functional in the numerical solution is  $J_{\min} = 2982.17$  K·m/s. In fig. 2 presents the results of a numerical solution to problem (6). The results are presented with the minimum (blue borders, below), maximum (red borders, above) and approximate (green, middle) temperature values. In fig. 3 shows the optimal placement of heat sources with minimum power in the form of a histogram.

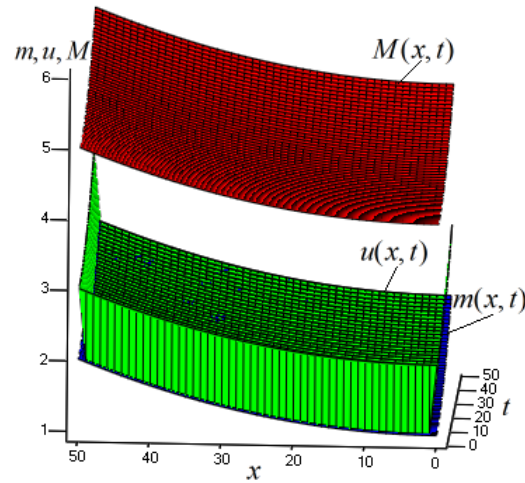


Рис. 2: Problem solution graph (6)

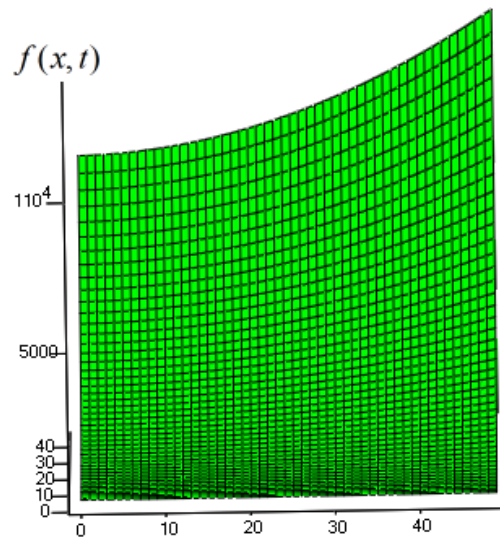


Рис. 3: Optimal placement of heat sources  $f(x, t)$

As can be seen from figure 2, the solution to the boundary value problem lies in the limit of minimum and maximum temperatures, i.e. satisfies inequality (3). It can be seen that the value of  $u(x, t)$  is almost equal to the minimum temperature. This means that the functional  $J_f\{f\}$  reaches a minimum.

In figure 3, the power of optimally placed heat sources is displayed in the form of a histogram. The height of the bars in the histogram indicates the power of the heat sources. The black lines in the figure are formed by the superposition of heat sources at each fixed point in time.

## CONCLUSION

A method and algorithm are proposed for solving the non-stationary problem of maintaining the temperature

inside a region within specified limits by optimally placing heat sources in a rectangle. The problem was solved on the basis of numerical modeling of the convection-diffusion process and sequential solution of linear programming problems. The results of the computational experiment confirm that the functional achieves the minimum and solves the main problem.

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# ON THE PURSUIT PROBLEM IN LINEAR DIFFERENTIAL-DIFFERENCE GAMES OF NEUTRAL TYPE

MAMADALIEV NUMANJON\*

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M.ULUGBEK, TASHKENT, UZBEKISTAN  
V.I. ROMANOVSKIY INSTITUTE OF MATHEMATICS,  
UZBEKISTAN ACADEMY OF SCIENCES, TASHKENT, UZBEKISTAN  
m\_numana59@mail.ru

SAYFULLAEVA RAYKHONA

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M.ULUGBEK, TASHKENT, UZBEKISTAN  
Sayfullayevarayhona0527@gmail.com

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**ABSTRACT.** This work is devoted to the method of directional pursuit, which arose in the consideration of many-player differential games [10]. This method is applied to the study of the pursuit problem in linear differential games described by a system of linear differential-difference equations of neutral type under geometric constraints on the players' controls. Modifications of the third pursuit method for differential-difference equations of neutral type have been developed. New sufficient conditions on the process parameters for the completion of the game in a guaranteed finite time have been obtained.

**MSC (2020):** 49N79; 49N70; 91A24.

**Key words:** differential game, pursuit problem, integral constraint, resolving function, differential-difference equations of neutral type, terminal set, pursuer, evader, control.

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## Introduction

The theory of conflict-controlled processes is an intensively developing branch of modern mathematics. This theory investigates the control problems of dynamic processes under conditions of conflict, which implies the presence of two or more parties capable of influencing the process with opposing or non-coincident goals and optimizing specified performance functionals of the process. Dynamic processes can be described by differential, integral, difference, hybrid, and other equations. Conflict-controlled processes described by ordinary differential equations are called differential games [7], a term introduced by the American mathematician R. Isaacs, one of the founders of the theory of differential games.

Differential games represent one of the most important branches of mathematical control theory, which studies object control in conflict situations. The theory of differential games develops the ideas and methods of optimal control theory; the latter can be considered a special case of the former when the number of players equals one. By considering differential games as a part of mathematical control theory, we thereby emphasize the managerial aspect of the theory, its origin, and the field of possible applications.

The systematic development of the theory of differential games began in the late 1950s and early 1960s. It was stimulated by practical needs and the great successes of mathematical control theory, game theory, and operations research. The progress of the theory of differential games is primarily associated with the names of L. S. Pontryagin, N. N. Krasovskii, E. F. Mishchenko, R. Isaacs, and W. Fleming. A major contribution to the development of the theory of differential games was made by Yu. S. Osipov, B. N. Pshenichnyi, L. A. Petrosyan, A. I. Subbotin, N. Yu. Satimov, F. L. Chernousko, and A. A. Chikrii. Interesting results were obtained in the studies of A. A. Azamov, A. Ya. Azimov, E. G. Al'brekht, V. D. Batukhtin, A. Bryson, N. L. Grigorenko, R. V. Gamkrelidze, P. B. Gusyatnikov, V. G. Guseinov, M. I. Zelikin, A. V. Kryazhinskii, A. B. Kurzhanskii, V. N. Lagunov, A. A. Melikyan, M. S. Nikol'skii, N. N. Petrov, E. S. Polovinkin, N. N. Subbotina, V. E. Tret'yakov, V. N. Ushakov, V. I. Ukhobotov, A. Friedman, A. Z. Fazylov, A. G. Chentsov, L. P. Yugai, and their students.

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\*Corresponding author

A fundamental contribution to the development of the theory of differential games was made by the schools of L. S. Pontryagin and N. N. Krasovskii.

In the present work, the main focus is on the investigation of the pursuit problem described by a system of linear differential-difference equations of neutral type under geometric constraints on the players' controls. Analogs of the third pursuit method and the directional pursuit method have been developed and applied to solve the pursuit problem. To solve the formulated problem, the method of resolving functions is applied. In the study of this problem, new sufficient L. S. Pontryagin conditions for the solvability of the pursuit problem have been obtained.

### Problem Formulation

Consider a pursuit problem described by a system of differential-difference equations of neutral type [1, 8]:

$$\dot{z}(t) = Az(t) + \sum_{i=1}^m B_i \dot{z}(t - h_i) + \sum_{i=0}^m C_i z(t - h_i) - Du(t) + Fv(t), \quad (1)$$

where  $z(t) \in \mathbb{R}^n, n \geq 1$ ;  $A, B_i (i = 1, 2, \dots, m)$ , and  $C_i (i = 0, 1, 2, \dots, m)$  are constant square matrices of order  $n \times n$ ;  $D$  and  $F$  are constant matrices of order  $n \times p$  and  $n \times q$ , respectively;  $0 = h_0 < h_1 < \dots < h_m$  are constants;  $u$  is the control parameter of the pursuer, and  $v$  is the control parameter of the evader. The parameters  $u$  and  $v$  are chosen in the form of measurable vector functions  $u = u(\cdot)$  and  $v = v(\cdot)$  satisfying the geometric constraints:

$$u(t) \in P, \quad v(t) \in Q, \quad 0 \leq t < +\infty, \quad (2)$$

where  $P$  and  $Q$  are non-empty compact subsets of the spaces  $\mathbb{R}^p$  and  $\mathbb{R}^q$ , respectively. Measurable functions  $u = u(t)$  and  $v = v(t)$ ,  $0 \leq t < +\infty$ , satisfying constraints (2) are called admissible controls of the pursuer and the evader, respectively. Furthermore, a non-empty cylindrical terminal set  $M = M_0 + M_1$  is specified in the space  $\mathbb{R}^n$ , where  $M_0$  is a linear subspace of  $\mathbb{R}^n$ , and  $M_1$  is a compact subset of the subspace  $L$ , where  $L$  is the orthogonal complement of the subspace  $M_0$  in  $\mathbb{R}^n$  (i.e.,  $M_0 \oplus L = \mathbb{R}^n$ ). The initial position for the pursuit problem (1) is given by an  $n$ -dimensional absolutely continuous function  $\varphi(t)$  defined on the interval  $[-h_m, 0]$ , i.e.,

$$\left\{ \varphi(\cdot) : z(t) = \varphi(t), \quad t \in [-h_m, 0], \quad z(0) = \varphi(0) \in \mathbb{R}^n \setminus M \right\}. \quad (3)$$

The goal of the first player, the pursuer  $u$ , is to bring the trajectory of equation (1) onto the terminal set  $M$  in a finite time by choosing a control  $u(t)$  (subject to constraints (2)). The goal of the second player, the evader  $v$ , is to use the control  $v(t)$  to prevent the trajectory of equation (1) from meeting the terminal set  $M$  at any finite moment of time, or, if this is impossible, to delay the end of the game as much as possible. Let us denote by  $S$  the set of points of the form  $S = \left\{ t : t = \sum_{i=0}^m j_i h_i, \quad j_i \text{ is an integer} \right\}$ , and let  $S^0$  denote the intersection of  $S$  with the interval  $(0, \infty)$ , i.e.,  $S^0 = S \cap (0, \infty)$ .

**Definition 1.** Let  $K(t)$  be the unique matrix function possessing the following properties [1]: a)  $K(t) = \tilde{0}$  for  $t < 0$ , where  $\tilde{0}$  is the  $n \times n$  zero matrix; b)  $K(0) = E$ , where  $E$  is the  $n \times n$  identity matrix; c) the function  $\sum_{i=0}^m C_i K(t - h_i)$  is continuous on  $[0, +\infty)$ ; d)  $K(t)$  satisfies the matrix differential equation:

$$\dot{K}(t) = AK(t) + \sum_{i=1}^m B_i \dot{K}(t - h_i) + \sum_{i=0}^m C_i K(t - h_i), \quad \text{for } t > 0, \quad t \notin S^0. \quad (4)$$

It should be noted that the matrix function  $K(t)$  satisfying conditions a)-d) can be obtained by the conventional method of successive integration of equation (4). The function  $K(t)$  belongs to the class  $C^1$  for  $t > 0, t \notin S^0$ , but in the general case, it has discontinuities of the first kind at the points of the set  $S^0$ . Suppose that admissible controls  $u = u(t)$  and  $v = v(t)$  are chosen on the interval  $[0, t]$ ,  $t > 0$ . Then, for the solution  $z(t)$  of equation (1) under the initial condition (3), the following formula holds true [1]:

$$z(t) = \left[ K(t) - \sum_{i=0}^m K(t - h_i) B_i \right] \varphi(0) + \sum_{i=0}^m \int_{-h_i}^0 K(t - r - h_i) \left[ B_i \dot{\varphi}(r) + C_i \varphi(r) \right] dr -$$

$$-\int_0^t K(t-r) [Du(r) - Fv(r)] dr, \quad (5)$$

where  $B_0 = -E$ , and  $E$  is the  $n \times n$  identity matrix. (Note: In the second integral of formula (5), the upper limit  $\tau$  in the original text was corrected to  $t$  to ensure mathematical consistency with the variable  $z(t)$ ).

**Definition 2.** We shall say that in the game (1), (2), from the initial position  $\varphi(\cdot)$ , the pursuit can be completed in time  $T = T(\varphi(\cdot))$ ,  $0 \leq T < +\infty$ , if there exists a function  $u(t, v)$ ,  $0 \leq t \leq T(\varphi(\cdot))$ ,  $v \in Q$ , taking values from the set  $P$ , such that for any measurable function  $v = v(t)$ ,  $0 \leq t \leq T(\varphi(\cdot))$ ,  $v(t) \in Q$ , the function  $u(t, v(t))$ ,  $0 \leq t \leq T(\varphi(\cdot))$  is measurable, and the corresponding trajectory  $z = z(t)$  of equation (1), subject to the initial condition (3), reaches the terminal set  $M$  at some moment  $t = t^* \in [0, T]$ , i.e., the inclusion  $z(t^*) \in M$  holds true. Moreover, to conduct the pursuit, the pursuer requires information about the objects. It is assumed that at each moment of time  $t \geq 0$ , the pursuer knows the solution  $z(s)$  on the interval  $-h_m \leq s \leq t$  and the control  $v(s)$ ,  $0 \leq s \leq t$ .

The number  $T(\varphi(\cdot))$  is called the guaranteed pursuit time from the initial position  $\varphi(\cdot)$ , and the function  $u(t, v)$ ,  $0 \leq t \leq T(\varphi(\cdot))$ ,  $v \in Q$ , is called the pursuit function. By the solution of the pursuit problem, we mean: a) finding the initial positions  $\varphi(\cdot)$  from which the completion of the pursuit is possible in the game (1), (2) in the sense of the definition introduced above; b) finding the pursuit function  $u(t, v)$  in an explicit form or specifying a method for its construction.

### Formulation and Proof of the Main Results

Let us denote by  $\pi$  the matrix of the orthogonal projection operator from  $\mathbb{R}^n$  onto  $L$  [4], and by

$$X \underset{*}{\setminus} Y = \{x \in \mathbb{R}^n : x + Y \subset X\} = \bigcap_{y \in Y} (X - y),$$

the geometric difference (Minkowski difference) of the sets  $X$  and  $Y$ , where  $X, Y \subset \mathbb{R}^n$  [4].

**A.** Let  $\tau$  be an arbitrary fixed positive number, and  $t \in [0, \tau]$ . Let  $M(t) \subset L$ ,  $0 \leq t \leq \tau$ , denote an arbitrary measurable closed-valued set-valued function (mapping) such that [8]:

$$\int_0^\tau M(t) dt \subset M_1. \quad (6)$$

Let us introduce the set-valued mapping

$$\widehat{W}(M(t), t, \tau) = [M(t) + \pi K(t)DP] \underset{*}{\setminus} \pi K(t)FQ, \quad 0 \leq t \leq \tau. \quad (7)$$

**Assumption 1.** For the initial position  $\varphi(\cdot)$ , there exist a positive number  $\tau$  and a set-valued mapping  $M(t)$ ,  $0 \leq t \leq \tau$ , satisfying condition (6), such that for all  $t \in [0, \tau]$ , the set  $\widehat{W}(M(t), t, \tau)$  is non-empty.

By virtue of Assumption 1 and the properties of the set-valued mapping  $\widehat{W}(M(t), t, \tau)$ , there exists at least one Borel-measurable integrable selector, which allows us to introduce the Lebesgue integral [4] of the set-valued mapping  $\widehat{W}(M(t), t, \tau)$ , i.e.,

$$W(M(\cdot), \tau) = \int_0^\tau \widehat{W}(M(t), t, \tau) dt, \quad (8)$$

$$W_3(\tau) = \bigcup_{M(\cdot)} W(M(\cdot), \tau). \quad (9)$$

Note that  $W_3(0) = M_1$ .

The set  $W_3(\tau)$  will be called the modified first integral of L.S. Pontryagin (corresponding to game (1)). Furthermore, the set  $W_3(\tau)$  is a convex closed set that depends continuously on  $\tau$ .

Let us fix some initial position  $\varphi(\cdot)$ . We introduce the following vector function:

$$\begin{aligned} \xi[\tau, t, \varphi(\cdot), \tilde{w}(\cdot)] &= \left[ \pi K(\tau) - \sum_{i=0}^m \pi K(\tau - h_i) B_i \right] \varphi(0) + \\ &+ \sum_{i=0}^m \int_{-h_i}^0 \pi K(\tau - t - h_i) \left[ B_i \dot{\varphi}(t) + C_i \varphi(t) \right] dt - w, \end{aligned}$$

where  $w \in W_3(\tau)$ . In accordance with the definition of the integral (8), there exists a Borel-measurable integrable selector  $\tilde{w}(t) \in \widehat{W}(M(t), t, \tau)$ ,  $0 \leq t \leq \tau$ , such that the equality  $w = \int_0^\tau \tilde{w}(\tau - t) dt$  holds. Let us fix it. Then, the vector function  $\xi[\tau, t, \varphi(\cdot), \tilde{w}(\cdot)]$  takes the form:

$$\begin{aligned} \xi[\tau, t, \varphi(\cdot), \tilde{w}(\cdot)] &= \left[ \pi K(\tau) - \sum_{i=0}^m \pi K(\tau - h_i) B_i \right] \varphi(0) + \\ &+ \sum_{i=0}^m \int_{-h_i}^0 \pi K(\tau - t - h_i) \left[ B_i \dot{\varphi}(t) + C_i \varphi(t) \right] dt - \int_0^\tau \tilde{w}(\tau - t) dt. \end{aligned} \quad (10)$$

Let us set

$$\eta[\tau, t, \varphi(\cdot), \tilde{w}(\cdot)] = \begin{cases} \frac{\xi[\tau, t, \varphi(\cdot), \tilde{w}(\cdot)]}{\|\xi[\tau, t, \varphi(\cdot), \tilde{w}(\cdot)]\|} & \text{if } \xi[\tau, t, \varphi(\cdot), \tilde{w}(\cdot)] \neq 0, \\ \eta^0 & \text{if } \xi[\tau, t, \varphi(\cdot), \tilde{w}(\cdot)] = 0, \end{cases} \quad (11)$$

where  $\eta^0$  is an arbitrary unit vector from  $L$ .

It is easy to show that if  $\xi[\tau, t, \varphi(\cdot), \tilde{w}(\cdot)] = 0$  for some positive  $\tau$ , then the pursuit problem is solvable by the moment  $\tau$ . Therefore, in what follows, it is assumed that  $\xi[\tau, t, \varphi(\cdot), \tilde{w}(\cdot)] \neq 0$  for all  $\tau \geq 0$ .

For an arbitrary vector  $v \in Q$ , we define the numerical function:

$$\begin{aligned} \lambda(\varphi(\cdot), \tau, t, v) &= \\ &= \begin{cases} \sup \left\{ \lambda \geq 0 : \lambda \eta[\tau, t, \varphi(\cdot), \tilde{w}(\cdot)] \in \left[ M(\tau - t) + \pi K(\tau - t) DP \right] - \right. \\ \left. - \pi K(\tau - t) FQ - \tilde{w}(\tau - t) \right\} & \text{if } \xi[\tau, t, \varphi(\cdot), \tilde{w}(\cdot)] \neq 0, \\ \tau^{-1} & \text{if } \xi[\tau, t, \varphi(\cdot), \tilde{w}(\cdot)] = 0. \end{cases} \end{aligned} \quad (12)$$

Let us introduce the notation  $\lambda(\varphi(\cdot), \tau, t) = \inf \{ \lambda(\varphi(\cdot), \tau, t, v) : v \in Q \}$ .

**Assumption 2.** For the initial position  $\varphi(\cdot)$ , there exist a positive number  $\tau = \tau_1(\varphi(\cdot)) > 0$ , a set-valued mapping  $M(t)$ ,  $0 \leq t \leq \tau_1$ , (see (6)), and an integrable function  $\tilde{w}(t)$ ,  $0 \leq t \leq \tau_1$ , such that:

a) the function  $\lambda(\varphi(\cdot), \tau_1, t)$ ,  $0 \leq t \leq \tau_1$ , as well as the superposition  $\lambda(\varphi(\cdot), \tau_1, t, v(t))$ ,  $0 \leq t \leq \tau_1$ , of the function  $\lambda(\varphi(\cdot), \tau_1, t, v)$ ,  $0 \leq t \leq \tau_1$ ,  $v \in Q$ , and an arbitrary integrable function  $v(t)$ ,  $0 \leq t \leq \tau_1$ , are integrable;

b) the following inequality holds:

$$|\xi[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)]| - \int_0^{\tau_1} \lambda(\varphi(\cdot), \tau_1, t) dt \leq 0. \quad (13)$$

**Theorem 1.** If the conditions of Assumptions 1 and 2 are satisfied, then in the game (1) under constraints (2), from the initial position  $\varphi(\cdot)$ , the pursuit can be completed in a finite time  $T = \tau_1(\varphi(\cdot))$ .

**Proof.** Suppose that for the initial position  $\varphi(\cdot)$ , all conditions of Theorem 1 are fulfilled. For an arbitrary measurable function  $v(t)$ ,  $0 \leq t \leq \tau_1$ ,  $v(t) \in Q$ , we consider the control function  $h(t; v(t))$ ,  $0 \leq t \leq \tau_1$  defined as follows [7-9]:

$$h(t; v(t)), \quad 0 \leq t \leq \tau_1 = |\xi[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)]| - \int_0^{\tau_1} \lambda(\varphi(\cdot), \tau_1, t, v(t)) dt.$$

It is claimed that there exists a moment of time  $t = t^* \in [0, \tau_1]$  such that  $h(t^*; v(t), 0 \leq t \leq t^*) = 0$  (by virtue of item b) of Assumption 2, such a moment of time exists for each initial position  $\varphi(\cdot)$ ). Clearly, if  $\xi[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)] = 0$ , then one can set  $t^* = 0$ .

Suppose that  $\xi[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)] \neq 0$  and  $h(t; v(t), 0 \leq t \leq \tau_1) > 0$  on the interval  $[0, \tau_1]$ . Indeed, in this case, we have:

$$\begin{aligned} 0 < h(t; v(t), 0 \leq t \leq \tau_1) &= |\xi[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)]| - \int_0^{\tau_1} \lambda(\varphi(\cdot), \tau_1, t, v(t)) dt \leq \\ &\leq |\xi[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)]| - \int_0^{\tau_1} \lambda(\varphi(\cdot), \tau_1, t) dt, \end{aligned}$$

which contradicts inequality (13). Thus, let  $h(t^*; v(t), 0 \leq t \leq t^*) = 0$ . Moreover,  $h(t; v(t), 0 \leq t \leq t) > 0$  for all  $t$  such that  $0 \leq t < t^*$ .

Taking this fact into account, we choose the functions  $u(t) \in P$  and  $m_1(t) \in M(t)$ , which, at the moments of time  $t \in [0, t^*]$ , are defined as solutions to the equation:

$$\begin{aligned} m_1(\tau_1 - t) + \pi K(\tau_1 - t) Du(t) - \pi K(\tau_1 - t) Fv(t) &= \\ &= \tilde{w}(\tau_1 - t) + \lambda(\varphi(\cdot), \tau_1, t, v(t)) \eta[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)], \end{aligned} \quad (14)$$

and at the moments of time  $t \in (t^*, \tau_1]$ , as solutions to the equation:

$$m_1(\tau_1 - t) + \pi K(\tau_1 - t) Du(t) - \pi K(\tau_1 - t) Fv(t) = \tilde{w}(\tau_1 - t). \quad (15)$$

In what follows, the time intervals  $[0, t^*]$  and  $(t^*, \tau_1]$  will be called active and passive, respectively [9].

By virtue of item a) of Assumption 2, there exists one or multiple solutions to equations (14) and (15). According to Proposition 2 [3], the function  $\lambda(\varphi(\cdot), \tau, t, v)$  defined by equality (12) is a measurable function with respect to  $t$  for fixed  $v$ . Consequently, by virtue of the Filippov-Castaing theorem [2], equations (14) and (15) are solvable in the class of measurable functions.

Let us show that with this method of managing the parameter  $u(t)$ , the pursuer can guarantee the completion of the pursuit within the time  $\tau_1$ . Let us verify that the trajectory  $z(t)$ ,  $0 \leq t \leq \tau_1$ , of equation (1), subject to the initial condition (3), reaches  $M$  by the moment of time  $\tau_1$ . Using the Cauchy formula (see (5)), after projecting it onto  $L$ , adding and subtracting the value  $\int_0^{\tau_1} \tilde{w}(\tau_1 - t) dt$ , and taking into account the choice of controls (14) and (15), we obtain:

$$\begin{aligned} \pi z(\tau_1) &= \left[ \pi K(\tau_1) - \sum_{i=0}^m \pi K(\tau_1 - h_i) B_i \right] \varphi(0) + \sum_{i=0}^m \int_{-h_i}^0 \pi K(\tau_1 - t - h_i) \left[ B_i \dot{\varphi}(t) + C_i \varphi(t) \right] dt - \\ &- \int_0^{\tau_1} \pi K(\tau_1 - t) \left[ Du(t) - Fv(t) \right] dt + \int_0^{\tau_1} \tilde{w}(\tau_1 - t) dt - \int_0^{\tau_1} \tilde{w}(\tau_1 - t) dt = \\ &= \xi[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)] - \int_0^{t^*} \left[ \tilde{w}(\tau_1 - t) - m_1(\tau_1 - t) + \lambda(\varphi(\cdot), \tau_1, t, v(t)) \eta[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)] \right] dt + \\ &+ \int_0^{\tau_1} \tilde{w}(\tau_1 - t) dt - \int_{t^*}^{\tau_1} \left[ \tilde{w}(\tau_1 - t) - m_1(\tau_1 - t) \right] dt = \\ &= \xi[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)] + \int_0^{\tau_1} \tilde{w}(\tau_1 - t) dt - \int_0^{t^*} \tilde{w}(\tau_1 - t) dt - \int_0^{t^*} \lambda(\varphi(\cdot), \tau_1, t, v(t)) \eta[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)] dt + \end{aligned}$$

$$\begin{aligned}
 & + \int_0^{t^*} m_1(\tau_1 - t) dt - \int_{t^*}^{\tau_1} \tilde{w}(\tau_1 - t) dt + \int_{t^*}^{\tau_1} m_1(\tau_1 - t) dt = \\
 & = \int_0^{\tau_1} m_1(\tau_1 - t) dt.
 \end{aligned} \tag{16}$$

Indeed, as established above:

$$\begin{aligned}
 & \xi[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)] - \int_0^{t^*} \lambda(\varphi(\cdot), \tau_1, t, v(t)) \eta[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)] dt = \\
 & = | \xi[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)] | \eta[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)] - \int_0^{t^*} \lambda(\varphi(\cdot), \tau_1, t, v(t)) \eta[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)] dt = \\
 & = \left[ | \xi[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)] | - \int_0^{t^*} \lambda(\varphi(\cdot), \tau_1, t, v(t)) dt \right] \eta[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)] = 0.
 \end{aligned}$$

Consequently, from (16) we have:

$$\pi z(\tau_1) = \int_0^{\tau_1} m_1(\tau_1 - t) dt \in \int_0^{\tau_1} M(\tau_1 - t) dt \subset M_1.$$

Thus, for the initial position  $\varphi(\cdot)$ , the inclusion  $\pi z(\tau_1) \in M_1$  holds, which is equivalent to  $z(\tau_1) \in M$ , meaning that the pursuit terminates within the time  $\tau_1$ . Theorem 1 is proved.

**Assumption 3.** For the initial position  $\varphi(\cdot)$ , there exists a matrix function  $G(t) : L \rightarrow L$ ,  $t \in [0, \tau]$ , which is upper semicontinuous with respect to  $t$ , such that the set

$$\hat{w}(t) = \pi K(\tau - t) DP \ast G(\tau - t) \pi K(\tau - t) FQ$$

is non-empty for all  $t \in [0, \tau]$ .

We define the numerical function  $\lambda(\varphi(\cdot), \tau, t, v)$  as follows:

$$\begin{aligned}
 & \lambda(\varphi(\cdot), \tau, t, v) = \\
 & = \begin{cases} \sup \left\{ \lambda \geq 0 : \lambda \eta[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)] \in \pi K(\tau - t) DP - \right. \\ \left. - G(\tau - t) \pi K(\tau - t) Fv - \tilde{w}(\tau - t) \right\} & \text{if } \xi[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)] \neq 0, \\ \tau^{-1} & \text{if } \xi[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)] = 0, \end{cases}
 \end{aligned}$$

where the vector functions  $\xi[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)]$  and  $\eta[\tau_1, t, \varphi(\cdot), \tilde{w}(\cdot)]$  are defined in the same way as in Theorem 1.

**Assumption 4.** a) For the initial position  $\varphi(\cdot)$ , there exists a number  $\tau = \tau_2(\varphi(\cdot)) > 0$  such that the following inequality holds:

$$\inf_{v(\cdot) \in Q} \left[ \int_0^{\tau_2} \lambda(\varphi(\cdot), \tau_2, t, v(t)) dt \right] > 1;$$

b) for any admissible control  $v = v(t)$ ,  $0 \leq t \leq \tau_2$ , of the evader, the following inclusion holds:

$$\int_0^{\tau_2} [E - G(\tau_2 - t)] \pi K(\tau_2 - t) Fv(t) dt \subset M_1.$$

**Theorem 2.** If the conditions of Assumptions 3 and 4 are satisfied, then in the game (1) under constraints (2), from the initial position  $\varphi(\cdot)$ , the pursuit can be completed in the time  $T = \tau_2(\varphi(\cdot))$ .

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## FINITE SOFT PRODUCTS AND SYMMETRIC POWERS IN SOFT TOPOLOGY

MAMADALIYEV NODIRBEK

KIMYO INTERNATIONAL UNIVERSITY IN TASHKENT, TASHKENT, UZBEKISTAN  
nodirbekm.topol25@gmail.com

JURAEV RUSTAM\*

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M.ULUGBEK, TASHKENT, UZBEKISTAN  
TASHKENT UNIVERSITY OF INFORMATION TECHNOLOGY, TASHKENT, UZBEKISTAN  
rmjurayev@mail.ru

TOSHBUVAYEV BOBURMIRZO

FERGANA STATE UNIVERSITY, FERGANA, UZBEKISTAN  
toshbuvayevboburmirzo@gmail.com

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**ABSTRACT.** We investigate finite soft product spaces and symmetric powers within the framework of soft topology. We prove that the weight of a soft topological space is preserved under finite Cartesian powers and that the canonical soft projections are soft bi-open. We then introduce the  $n$ -th symmetric power of a soft space as a soft quotient of the  $n$ -fold product and study its basic properties. In particular, we show that soft compactness is preserved under symmetric powers and that the weight of the symmetric power does not exceed the weight of the original space. An example illustrates that the canonical quotient projection need not be soft open in the soft setting.

**MSC (2020):** 54B10; 54B20; 54D30.

**Key words:** soft topology, soft product topology, soft symmetric power, soft compactness, weight, soft mappings.

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## Introduction

Since its introduction by Molodtsov [12], soft set theory has become an important mathematical tool for modeling uncertainty via parameterized families of subsets. Unlike fuzzy set theory, a soft set assigns to each parameter a crisp subset of the universe, providing a flexible framework for handling incomplete or approximate information.

The development of soft topology, initiated in [3,7,8,13,16], extends classical topological ideas to the soft setting. In this framework, notions such as soft continuity, soft openness, soft compactness, and soft separation axioms are defined in terms of parameterized families of subsets. Although these concepts parallel classical topology, their behavior may differ significantly due to the presence of parameters.

Product constructions play a fundamental role in topology. In classical topology, finite products preserve compactness and important cardinal invariants such as weight. It is therefore natural to investigate to what extent these properties extend to soft topological spaces. In particular, understanding the behavior of weight under soft product constructions is essential for the structural study of soft spaces.

The first aim of this paper is to study the finite Cartesian powers of a soft topological space. We prove that the weight of a soft space coincides with the weight of its finite powers. We also show that canonical soft projections are soft bi-open and that fundamental mapping properties-such as soft continuity, soft openness, soft injectivity, and soft surjectivity-are preserved under finite powers.

The second aim of the paper is to investigate symmetric powers in soft topology.

The  $n$ -th symmetric power of a soft space is introduced as a soft quotient of the  $n$ -fold product under the action of the symmetric group. We examine compactness and weight properties of these spaces and prove that soft compactness is preserved under symmetric powers. Moreover, we establish that the weight of the

symmetric power does not exceed the weight of the original space. An example demonstrates that, unlike in classical topology, the canonical quotient projection need not be soft open.

The paper is organized as follows. Section 2 recalls basic definitions and notation from soft topology. Section 3 studies finite soft products and the preservation of weight. Section 4 introduces symmetric powers of soft spaces and establishes their compactness and weight properties.

### Preliminaries

Throughout,  $X$  denotes a universe set and  $E$  a fixed parameter set. A mapping  $F: E \rightarrow \mathcal{P}(X)$  is called a *soft set* over  $X$  with respect to  $E$  and is denoted by  $F_E$ . The pair  $(X, E)$  is referred to as a *soft class*. For  $e \in E$ , the value  $F(e) \subseteq X$  is called the *e-approximation* of  $F_E$ .

A soft set  $F_E$  is called the *null soft set* if  $F(e) = \emptyset$  for all  $e \in E$ , and the *absolute soft set* if  $F(e) = X$  for all  $e \in E$ . These are denoted by  $\Phi_E$  and  $\tilde{X}_E$ , respectively.

Given soft sets  $F_E$  and  $G_E$  over the same universe  $X$  and parameter set  $E$ , their union and intersection are defined parameterwise:

$$(F_E \sqcup G_E)(e) = F(e) \cup G(e), \quad (F_E \sqcap G_E)(e) = F(e) \cap G(e), \quad e \in E.$$

Arbitrary unions and finite intersections of soft sets are defined similarly. The complement of  $F_E$  is the soft set  $F_E^c$  given by

$$F_E^c(e) = X \setminus F(e), \quad e \in E.$$

**Definition 2.1.** A *soft topological space* is a triple  $(X, \mathcal{T}, E)$  where  $\mathcal{T}$  is a family of soft sets over  $X$  satisfying:

1.  $\Phi_E$  and  $\tilde{X}_E$  belong to  $\mathcal{T}$ ;
2. the union of any family of members of  $\mathcal{T}$  belongs to  $\mathcal{T}$ ;
3. the intersection of finitely many members of  $\mathcal{T}$  belongs to  $\mathcal{T}$ .

Elements of  $\mathcal{T}$  are called *soft open sets*, and *soft closed sets* are complements of soft open sets [3].

**Example 2.2.** Let  $X = \{a, b, c\}$  and  $E = \{e_1, e_2, e_3\}$ . Let  $\mathcal{T} = \{\Phi_E, \tilde{X}_E, F_E, G_E, H_E\}$ , where

$$\begin{aligned} F(e_1) &= \{a, c\}, & F(e_2) &= \{a, b\}, & F(e_3) &= \{c\}, \\ G(e_1) &= \{a, b\}, & G(e_2) &= X, & G(e_3) &= \{a, b\}, \\ H(e_1) &= \{a\}, & H(e_2) &= \{a, b\}, & H(e_3) &= \emptyset. \end{aligned}$$

Then  $\mathcal{T}$  is a soft topology on  $X$ .

**Definition 2.3.** Let  $(X, \mathcal{T}, E)$  be a soft topological space and let  $Y \subseteq X$ . The family

$$\mathcal{T}_Y = \{F_E \sqcap \tilde{Y}_E : F_E \in \mathcal{T}\}$$

is a soft topology on  $Y$ , and  $(Y, \mathcal{T}_Y, E)$  is called the *soft subspace* of  $(X, \mathcal{T}, E)$ , where  $\tilde{Y}_E$  is the constant soft set defined by  $\tilde{Y}_E(e) = Y$  for all  $e \in E$  [7].

A soft subspace  $(Y, \mathcal{T}_Y, E)$  is called a **soft open** (resp. **soft closed**) **subspace** of  $(X, \mathcal{T}, E)$  if  $\tilde{Y}_E$  is **soft open** (resp. **soft closed**) over  $X$ .

**Definition 2.4.** A subfamily  $\mathcal{B} \subseteq \mathcal{T}$  is called a *soft base* for  $\mathcal{T}$  if for every  $F_E \in \mathcal{T}$  there exists a subfamily  $\mathcal{B}_0 \subseteq \mathcal{B}$  such that

$$F_E = \bigsqcup \{U_E : U_E \in \mathcal{B}_0\}.$$

The *weight* of  $(X, \mathcal{T}, E)$  is

$$w(X, \mathcal{T}, E) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a soft base for } \mathcal{T}\}.$$

**Proposition 2.5.** Let  $(X, \mathcal{T}, E)$  be a soft space and let  $(Y, \mathcal{T}_Y, E)$  be a subspace of  $X$ . Then

$$w(Y, \mathcal{T}_Y, E) \leq w(X, \mathcal{T}, E).$$

**Proof.** Let  $w(X, \mathcal{T}, E) = \kappa$ . Then there exists a soft base

$$\mathcal{B} = \{U_{\alpha E} : \alpha \in \Lambda\}$$

for  $\mathcal{T}$  such that  $|\mathcal{B}| = \kappa$ . It is enough to show that

$$\mathcal{B}_Y = \{U_{\alpha E} \cap \tilde{Y}_E : \alpha \in \Lambda\}$$

is a soft base for  $\mathcal{T}_Y$ . Let  $V_E$  be an arbitrary soft open set over  $Y$ . By the definition of a soft subspace, there exists a soft open set  $U_E$  over  $X$  such that

$$V_E = U_E \cap \tilde{Y}_E.$$

Since  $\mathcal{B}$  is a soft base for  $\mathcal{T}$ , we have

$$U_E = \bigsqcup_{\alpha \in \Lambda_0} U_{\alpha E}, \quad \text{for some } \Lambda_0 \subseteq \Lambda.$$

Therefore,

$$V_E = \left( \bigsqcup_{\alpha \in \Lambda_0} U_{\alpha E} \right) \cap \tilde{Y}_E = \bigsqcup_{\alpha \in \Lambda_0} (U_{\alpha E} \cap \tilde{Y}_E).$$

Hence  $\mathcal{B}_Y$  is a soft base for  $Y$ , and thus  $|\mathcal{B}_Y| \leq \kappa$ . This proves Proposition 2.5.  $\square$

### Soft maps

Let  $(X, E)$  and  $(Y, E')$  be soft classes, and let  $f: X \rightarrow Y$  and  $\varphi: E \rightarrow E'$  be mappings. The pair  $f_\varphi$  is called a *soft map* from  $(X, E)$  to  $(Y, E')$ .

**Definition 2.6.** Let  $F_E$  be a soft set over  $X$ . The *soft image*  $f_\varphi(F_E)$  is the soft set over  $Y$  defined by

$$(f_\varphi(F_E))(e') = \bigcup_{e \in \varphi^{-1}(e')} f(F(e)), \quad e' \in E'$$

[10].

**Definition 2.7.** Let  $G_{E'}$  be a soft set over  $Y$ . The *soft inverse image*  $(f_\varphi)^{-1}(G_{E'})$  is the soft set over  $X$  defined by

$$((f_\varphi)^{-1}(G_{E'}))(e) = f^{-1}(G(\varphi(e))), \quad e \in E$$

[10].

**Example 2.8.** Let  $X = \{a, b, c\}$ ,  $Y = \{x, y, z\}$ ,  $E = \{e_1, e_2, e_3, e_4\}$ ,  $E' = \{e'_1, e'_2, e'_3\}$ . Then  $(X, E)$  and  $(Y, E')$  are soft classes. Define  $f_\varphi: (X, E) \rightarrow (Y, E')$  as:

$$f(a) = x, \quad f(b) = f(c) = y,$$

$$\varphi(e_1) = \varphi(e_2) = \varphi(e_4) = e'_2, \quad \varphi(e_3) = e'_3.$$

Choose two soft sets over  $X$  and  $Y$  respectively as:

$$F_E : F(e_1) = \{b, c\}, \quad F(e_2) = \{b\}, \quad F(e_3) = \{a, c\}, \quad F(e_4) = \emptyset,$$

$$G_{E'} : G(e'_1) = \{x, y\}, \quad G(e'_2) = \{x, z\}, \quad G(e'_3) = \{z\}.$$

Then the mapping  $f_\varphi: (X, E) \rightarrow (Y, E')$  is given as: for a soft set  $F_E$  over  $X$ ,  $f_\varphi(F_E)$  is a soft set over  $Y$  obtained as follows:

$$f_\varphi(F_E)(e'_1) = \emptyset, \quad f_\varphi(F_E)(e'_2) = f(F(e_1) \cup F(e_2) \cup F(e_4)) = f(\{b, c\}) = \{y\},$$

$$f_\varphi(F_E)(e'_3) = f(F(e_3)) = f(\{a, c\}) = \{x, y\}$$

and

$$\begin{aligned} f_\varphi^{-1}(G_{E'})(e_1) &= f_\varphi^{-1}(G_{E'})(e_2) = f_\varphi^{-1}(G_{E'})(e_4) = f^{-1}(G(e'_2)) = \{a\}, \\ f_\varphi^{-1}(G_{E'})(e_3) &= f^{-1}(G(e'_3)) = \emptyset. \end{aligned}$$

Thus we have

$$f_\varphi(F_E) = \{(e'_1, \emptyset), (e'_2, \{y\}), (e'_3, \{x, y\})\}$$

and

$$f_\varphi^{-1}(G_{E'}) = \{(e_1, \{a\}), (e_2, \{a\}), (e_3, \emptyset), (e_4, \{a\})\}.$$

**Definition 2.9.** Let  $(X, \mathcal{T}, E)$  and  $(Y, \mathcal{T}', E')$  be soft topological spaces. A soft map  $f_\varphi$  is called

- *soft continuous* if  $(f_\varphi)^{-1}(G_{E'}) \in \mathcal{T}$  for every  $G_{E'} \in \mathcal{T}'$ ;
- *soft open* if  $f_\varphi(F_E) \in \mathcal{T}'$  for every  $F_E \in \mathcal{T}$ ;
- *soft bi-open* if it is both *soft continuous* and *soft open*.

Moreover,  $f_\varphi$  is called *soft injective* (resp. *soft surjective*) if both  $f$  and  $\varphi$  are injective (resp. surjective) [8].

**Proposition 2.10.** If  $f_\varphi: (X, E) \rightarrow (Y, E')$  and  $h_\psi: (Y, E') \rightarrow (Z, E'')$  are soft continuous maps, then

$$h_\psi \circ f_\varphi = (h \circ f)_{\psi \circ \varphi}$$

is soft continuous [7].

**Lemma 2.11.** Let  $f_\varphi: (X, \mathcal{T}_X, E) \rightarrow (Y, \mathcal{T}_Y, E')$  be a soft bi-open surjective map. Then

$$w(Y, \mathcal{T}_Y, E') \leq w(X, \mathcal{T}_X, E).$$

**Proof.** Let  $w(X, \mathcal{T}_X, E) = \kappa$ , and let

$$\mathcal{B} = \{F_{\alpha E} : \alpha \in \Lambda\}$$

be a soft base for  $(X, \mathcal{T}_X, E)$  with  $|\mathcal{B}| = \kappa$ . Put

$$\mathcal{B}' = \{f_\varphi(F_{\alpha E}) : \alpha \in \Lambda\}.$$

Since  $f_\varphi$  is soft open, we have  $\mathcal{B}' \subseteq \mathcal{T}_Y$  and  $|\mathcal{B}'| \leq \kappa$ .

Let  $G_{E'} \in \mathcal{T}_Y$ . By soft continuity,  $(f_\varphi)^{-1}(G_{E'}) \in \mathcal{T}_X$ , hence

$$(f_\varphi)^{-1}(G_{E'}) = \bigsqcup_{\alpha \in \Lambda_0} F_{\alpha E}$$

for some  $\Lambda_0 \subseteq \Lambda$ . Applying  $f_\varphi$ , we obtain

$$f_\varphi((f_\varphi)^{-1}(G_{E'})) = \bigsqcup_{\alpha \in \Lambda_0} f_\varphi(F_{\alpha E}).$$

Since  $f$  and  $\varphi$  are surjective, for every  $e' \in E'$  we have  $\varphi^{-1}(e') \neq \emptyset$ , and

$$f_\varphi((f_\varphi)^{-1}(G_{E'}))(e') = \bigcup_{e \in \varphi^{-1}(e')} f(f^{-1}(G(e'))) = G(e').$$

Hence  $f_\varphi((f_\varphi)^{-1}(G_{E'})) = G_{E'}$ . Therefore  $\mathcal{B}'$  is a soft base for  $(Y, \mathcal{T}_Y, E')$ , and so  $w(Y, \mathcal{T}_Y, E') \leq \kappa$ .  $\square$

For every soft space  $(X, \mathcal{T}, E)$ , the map  $(\text{id}_X)_{\text{id}_E}$  is called the *identity soft map*. It is soft continuous by definition.

### The weight of soft product space

Let  $\{(X_\alpha, E_\alpha) : \alpha \in \Delta\}$  be a family of soft classes. For soft sets  $F_{\alpha, E_\alpha}$  over  $X_\alpha$ , their Cartesian product is the soft set

$$\left(\prod_{\alpha \in \Delta} F_\alpha\right)_{\prod_{\alpha \in \Delta} E_\alpha} \quad \text{over} \quad \prod_{\alpha \in \Delta} X_\alpha,$$

defined by

$$\left(\prod_{\alpha \in \Delta} F_\alpha\right)((e_\alpha)_{\alpha \in \Delta}) = \prod_{\alpha \in \Delta} F_\alpha(e_\alpha), \quad (e_\alpha)_{\alpha \in \Delta} \in \prod_{\alpha \in \Delta} E_\alpha.$$

**Definition 2.12.** Let  $\{(X_\alpha, \mathcal{T}_\alpha, E_\alpha) : \alpha \in \Delta\}$  be a family of soft topological spaces. Put

$$X = \prod_{\alpha \in \Delta} X_\alpha, \quad E = \prod_{\alpha \in \Delta} E_\alpha.$$

The initial soft topology on  $X$  generated by the family of soft projections

$$(p_q)_\alpha : (X, E) \longrightarrow (X_\alpha, E_\alpha), \quad \alpha \in \Delta,$$

is called the *soft product topology* on  $X$  [3].

**Proposition 2.13.** The family of all product soft sets  $\prod_{\alpha \in \Delta} (F_\alpha, E_\alpha)$ , where  $(F_\alpha, E_\alpha)$  is a soft open set over  $X_\alpha$  and  $(F_\alpha, E_\alpha) \neq \tilde{X}_{\alpha, E_\alpha}$  for only finitely many  $\alpha \in \Delta$ , is a soft base for the Cartesian product  $\prod_{\alpha \in \Delta} X_\alpha$ . Moreover, if for each  $\alpha \in \Delta$  a soft base  $\mathcal{B}_\alpha$  for  $(X_\alpha, \mathcal{T}_\alpha, E_\alpha)$  is fixed, then the subfamily consisting of those products  $\prod_{\alpha \in \Delta} (F_\alpha, E_\alpha)$  for which  $(F_\alpha, E_\alpha) \in \mathcal{B}_\alpha$  whenever  $(F_\alpha, E_\alpha) \neq \tilde{X}_{\alpha, E_\alpha}$ , and  $(F_\alpha, E_\alpha) \neq \tilde{X}_{\alpha, E_\alpha}$  for only finitely many  $\alpha \in \Delta$ , is also a soft base [3].

We generalize the definition of soft compactness to arbitrary soft sets.

**Definition 2.14.** Let  $(X, \mathcal{T}, E)$  be a soft topological space. A soft set  $F_E$  over  $X$  is called **soft compact** if every soft open cover of  $F_E$  admits a finite subcover. That is, for every family  $\{F_{\alpha E}\}_{\alpha \in \Lambda} \subseteq \mathcal{T}$  such that

$$F_E \subseteq \bigcup_{\alpha \in \Lambda} F_{\alpha E},$$

there exist  $\alpha_1, \dots, \alpha_n \in \Lambda$  such that

$$F_E \subseteq \bigcup_{i=1}^n F_{\alpha_i E}.$$

A soft topological space  $(X, \mathcal{T}, E)$  is called **soft compact** if its absolute soft set  $\tilde{X}_E$  is soft compact.

**Proposition 2.15.** Any soft closed subspace of a soft compact space is also soft compact.

**Proof.** Let  $(X, \mathcal{T}, E)$  be a soft compact space and let  $(Y, \mathcal{T}_Y, E)$  be a soft closed subspace. Then  $(\tilde{Y}_E)^c$  is soft open over  $X$ . Let  $\mu = \{F_{\alpha E} : \alpha \in \Lambda\}$  be a soft open cover of  $\tilde{Y}_E$ . Then

$$\mu' = \mu \cup \{(\tilde{Y}_E)^c\}$$

is a soft open cover of  $\tilde{X}_E$ . Since  $(X, \mathcal{T}, E)$  is soft compact, there exists a finite subcover

$$\nu = \{F_{\alpha_1 E}, \dots, F_{\alpha_n E}, (\tilde{Y}_E)^c\}$$

such that

$$\tilde{X}_E = \left(\bigcup_{i=1}^n F_{\alpha_i E}\right) \sqcup (\tilde{Y}_E)^c.$$

Hence  $\tilde{Y}_E \subseteq \bigcup_{i=1}^n F_{\alpha_i E}$ , so  $(Y, \mathcal{T}_Y, E)$  is soft compact.  $\square$

In classical topology it is known that the Cartesian product of compact spaces is compact. An analogous statement holds in soft topology.

**Theorem 2.16 (Tychonoff-type Theorem for Soft Compact Spaces).** Let  $\{(X_\alpha, \mathcal{T}_\alpha, E_\alpha) : \alpha \in \Delta\}$  be a family of soft compact spaces. Then the soft product space  $\prod_{\alpha \in \Delta} X_\alpha$ , equipped with the *soft product topology*, is also soft compact [2].

We consider the case  $\Delta = \{1, 2, \dots, n\}$  and  $X_\alpha = X$ ,  $E_\alpha = E$  for all  $\alpha \in \Delta$ . In this case the Cartesian product is denoted by  $(X^n, E^n)$ . For soft sets  $F_{1E}, \dots, F_{nE}$  over  $X$ , define their product soft set on  $X^n$  by

$$(F_{1E} \times \dots \times F_{nE})(e) = F_1(e_1) \times F_2(e_2) \times \dots \times F_n(e_n), \quad e = (e_1, \dots, e_n) \in E^n.$$

Let  $(X, \mathcal{T}, E)$  be a soft topological space. Then the family

$$\mathcal{B} = \{F_{1E} \times \dots \times F_{nE} : F_{iE} \in \mathcal{T}, i = 1, \dots, n\}$$

is a base for a soft topology on  $X^n$ , and this topology is denoted by  $\mathcal{T}^n$ . In other words,

$$\mathcal{T}^n = \left\{ \bigcup \mathcal{B}_0 : \mathcal{B}_0 \subseteq \mathcal{B} \right\}.$$

**Proposition 2.17.** For each  $i \in \{1, \dots, n\}$ , the soft projection

$$(p_q)_i : (X^n, \mathcal{T}^n, E^n) \longrightarrow (X, \mathcal{T}, E),$$

induced by the classical projections  $p_i : X^n \rightarrow X$  and  $q_i : E^n \rightarrow E$ , is soft bi-open.

**Proof.** Fix  $i \in \{1, \dots, n\}$ .

*Soft continuity.* Let  $F_E \in \mathcal{T}$ . For  $e = (e_1, \dots, e_n) \in E^n$  we have

$$\left( ((p_q)_i)^{-1}(F_E) \right)(e) = p_i^{-1}(F(q_i(e))) = p_i^{-1}(F(e_i)).$$

Hence  $((p_q)_i)^{-1}(F_E)$  is the basic product soft set  $H_{1E} \times \dots \times H_{nE}$ , where  $H_{iE} = F_E$  and  $H_{jE} = \tilde{X}_E$  for  $j \neq i$ . Thus  $((p_q)_i)^{-1}(F_E) \in \mathcal{T}^n$ .

*Soft openness.* Take a basic open set  $F_{1E} \times \dots \times F_{nE} \in \mathcal{T}^n$ . For  $e_i \in E$ ,

$$\left( (p_q)_i(F_{1E} \times \dots \times F_{nE}) \right)(e_i) = \bigcup_{e \in q_i^{-1}(e_i)} p_i \left( (F_{1E} \times \dots \times F_{nE})(e) \right) = F_i(e_i).$$

Therefore  $(p_q)_i(F_{1E} \times \dots \times F_{nE}) = F_{iE} \in \mathcal{T}$ . So  $(p_q)_i$  is soft open, and hence soft bi-open.  $\square$

**Proposition 2.18.** For any soft topological space  $(X, \mathcal{T}, E)$  the following equality holds:

$$w(X, \mathcal{T}, E) = w(X^n, \mathcal{T}^n, E^n).$$

**Proof.** First we prove that  $w(X^n, \mathcal{T}^n, E^n) \leq w(X, \mathcal{T}, E)$ . Let  $w(X, \mathcal{T}, E) = \kappa$ . Then there exists a soft base  $\mathcal{B}$  for  $(X, \mathcal{T}, E)$  such that  $|\mathcal{B}| = \kappa$ . By Proposition 2.13 (for finite products), the family

$$\mathcal{B}^\otimes = \{F_{1E} \times \dots \times F_{nE} : F_{iE} \in \mathcal{B}, i = 1, \dots, n\}$$

is a soft base for  $(X^n, \mathcal{T}^n, E^n)$ . Clearly,  $|\mathcal{B}^\otimes| = \kappa$ , hence  $w(X^n, \mathcal{T}^n, E^n) \leq \kappa$ .

Now we prove that  $w(X, \mathcal{T}, E) \leq w(X^n, \mathcal{T}^n, E^n)$ . By Proposition 2.17, the projection  $(p_q)_i$  is soft bi-open and surjective. Therefore Lemma 2.11 yields

$$w(X, \mathcal{T}, E) \leq w(X^n, \mathcal{T}^n, E^n).$$

$\square$

Let  $(X, \mathcal{T}_1, E_1)$  and  $(Y, \mathcal{T}_2, E_2)$  be soft topological spaces, and let  $f_\varphi : (X, \mathcal{T}_1, E_1) \rightarrow (Y, \mathcal{T}_2, E_2)$  be a soft map. Consider the soft map

$$(\Pi^n f)_{\Pi^n \varphi} : (X^n, \mathcal{T}_1^n, E_1^n) \rightarrow (Y^n, \mathcal{T}_2^n, E_2^n)$$

defined by

$$\Pi^n f(x_1, \dots, x_n) = (f(x_1), \dots, f(x_n)), \quad \Pi^n \varphi(e_1, \dots, e_n) = (\varphi(e_1), \dots, \varphi(e_n)).$$

### $n$ -th symmetric power of a soft space

The symmetric group  $S_n$  acts on  $X^n$  by permuting coordinates:

$$\sigma \cdot (x_1, x_2, \dots, x_n) = (x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}).$$

This action naturally induces an equivalence relation  $\sim$  on  $X^n$ :

$$(x_1, \dots, x_n) \sim (y_1, \dots, y_n) \iff \exists \sigma \in S_n \text{ such that } (y_1, \dots, y_n) = \sigma \cdot (x_1, \dots, x_n).$$

Equivalently, two points of  $X^n$  are equivalent if one is a permutation of the other.

Let  $SP^n X := X^n / S_n$  be the set of unordered  $n$ -tuples (i.e., multisets) of elements of  $X$ , and let  $SP^n \mathcal{T}$  be the quotient soft topology induced from the soft product topology  $\mathcal{T}^n$  via the canonical soft projection map

$$\pi_\rho: (X^n, \mathcal{T}^n, E^n) \longrightarrow (SP^n X, SP^n \mathcal{T}, SP^n E),$$

where

$$\pi(x_1, \dots, x_n) := [(x_1, \dots, x_n)], \quad \rho(e_1, \dots, e_n) := [(e_1, \dots, e_n)],$$

and  $[x]$  denotes the equivalence class of  $x$  under the action of the symmetric group  $S_n$  by coordinate permutations.

Finally, we define the  $n$ -th symmetric power of a soft space  $(X, \mathcal{T}, E)$  as the soft topological space

$$(SP^n X, SP^n \mathcal{T}, SP^n E).$$

That is,

$$(H, SP^n E) \in SP^n \mathcal{T} \iff (\pi_\rho)^{-1}(H, SP^n E) \in \mathcal{T}^n.$$

More details about this space are given in [15,9,4].

**Proposition 2.19.** Let  $(X, \mathcal{T}, E)$  be a soft compact space. Then the symmetric power soft space  $(SP^n X, SP^n \mathcal{T}, SP^n E)$  is also soft compact.

**Proof.** We proceed in two steps.

**Step 1.** Let  $(X, \mathcal{T}, E)$  be a soft compact space. Then, by Theorem 2.16, the soft product space  $(X^n, \mathcal{T}^n, E^n)$  is soft compact.

**Step 2.** The canonical soft projection

$$\pi_\rho: (X^n, \mathcal{T}^n, E^n) \longrightarrow (SP^n X, SP^n \mathcal{T}, SP^n E)$$

is soft continuous and surjective. Since the image of a soft compact space under a soft continuous surjection is soft compact, it follows that  $(SP^n X, SP^n \mathcal{T}, SP^n E)$  is soft compact. This proves Proposition 2.19.  $\square$

It is well known that the canonical projection  $\pi: (X^n, \mathcal{T}^n) \rightarrow (SP^n X, SP^n \mathcal{T})$  is an open map in classical topology (see [15]). However, in the setting of soft sets, this property need not hold.

**Example 2.20.** Let  $X = \{a, b\}$ ,  $E = \{e_1, e_2\}$ , and let

$$\mathcal{T} = \{\Phi_E, \tilde{X}_E, F_E\}, \quad F(e_1) = \{a\}, \quad F(e_2) = \{b\}.$$

A basic family for the soft product topology on  $X^2$  is

$$\mathcal{B} = \{F_E \times F_E, F_E \times \tilde{X}_E, \tilde{X}_E \times F_E, \tilde{X}_E \times \tilde{X}_E\},$$

where, for  $E^2 = E \times E$ , we have

$$\begin{aligned} (F \times F)_{E^2}(e_1, e_1) &= \{(a, a)\}, & (F \times F)_{E^2}(e_1, e_2) &= \{(a, b)\}, \\ (F \times F)_{E^2}(e_2, e_1) &= \{(b, a)\}, & (F \times F)_{E^2}(e_2, e_2) &= \{(b, b)\}, \\ (F \times \tilde{X})_{E^2}(e_1, e_1) &= (F \times \tilde{X})_{E^2}(e_1, e_2) &= \{(a, a), (a, b)\}, \end{aligned}$$

$$\begin{aligned}(F \times \tilde{X})_{E^2}(e_2, e_1) &= (F \times \tilde{X})_{E^2}(e_2, e_2) = \{(b, a), (b, b)\}, \\ (\tilde{X} \times F)_{E^2}(e_1, e_1) &= (\tilde{X} \times F)_{E^2}(e_2, e_1) = \{(a, a), (b, a)\}, \\ (\tilde{X} \times F)_{E^2}(e_1, e_2) &= (\tilde{X} \times F)_{E^2}(e_2, e_2) = \{(a, b), (b, b)\}.\end{aligned}$$

Hence

$$\mathcal{T}^2 = \left\{ \bigsqcup \mathcal{B}_0 : \mathcal{B}_0 \subseteq \mathcal{B} \right\}$$

is a soft topology on  $X^2$ .

Assume that the canonical soft projection

$$\pi_\rho : (X^2, \mathcal{T}^2, E^2) \longrightarrow (SP^2X, SP^2\mathcal{T}, SP^2E)$$

is a soft open map. Then  $\pi_\rho(F_E \times F_E)$  is a soft open set over  $SP^2X$ , where

$$\pi_\rho(F_E \times F_E)([(e_1, e_1)]) = \{[(a, a)]\}, \quad \pi_\rho(F_E \times F_E)([(e_1, e_2)]) = \{[(a, b)]\},$$

and

$$\pi_\rho(F_E \times F_E)([(e_2, e_2)]) = \{[(b, b)]\}.$$

Since  $\pi_\rho$  is the soft quotient map, it follows that

$$(\pi_\rho)^{-1}(\pi_\rho(F_E \times F_E))$$

is a soft open set over  $X^2$ . However,

$$(\pi_\rho)^{-1}(\pi_\rho(F_E \times F_E))(e_1, e_1) = \{(a, a)\},$$

$$(\pi_\rho)^{-1}(\pi_\rho(F_E \times F_E))(e_1, e_2) = (\pi_\rho)^{-1}(\pi_\rho(F_E \times F_E))(e_2, e_1) = \{(a, b), (b, a)\},$$

and

$$(\pi_\rho)^{-1}(\pi_\rho(F_E \times F_E))(e_2, e_2) = \{(b, b)\}.$$

Thus  $(\pi_\rho)^{-1}(\pi_\rho(F_E \times F_E)) \notin \mathcal{T}^2$ , a contradiction. Therefore,  $\pi_\rho$  is not a soft open map.  $\square$

Let  $f_\varphi : (X, E) \rightarrow (Y, E')$  be a soft mapping. Define the mapping

$$(SP^n f)_{SP^n \varphi} : (SP^n X, SP^n E) \longrightarrow (SP^n Y, SP^n E')$$

by

$$SP^n f([x_1, \dots, x_n]) := [f(x_1), \dots, f(x_n)],$$

and

$$SP^n \varphi([e_1, \dots, e_n]) := [\varphi(e_1), \dots, \varphi(e_n)].$$

**Theorem 2.21.** Let  $f_\varphi : (X, \mathcal{T}, E) \rightarrow (Y, \mathcal{T}', E')$  be a soft continuous map. Then

$$(SP^n f)_{SP^n \varphi} : (SP^n X, SP^n \mathcal{T}, SP^n E) \longrightarrow (SP^n Y, SP^n \mathcal{T}', SP^n E')$$

is also soft continuous.

**Proof.** Consider the canonical soft projection mappings

$$(\pi_X)_{\rho_X} : (X^n, E^n) \longrightarrow (SP^n X, SP^n E)$$

and

$$(\pi_Y)_{\rho_Y} : (Y^n, (E')^n) \longrightarrow (SP^n Y, SP^n E').$$

It is known that they are soft continuous mappings. By Proposition 2.10, the map  $\Pi^n f_{\Pi^n \varphi}$  is soft continuous. The following diagram is commutative:

$$\begin{array}{ccc}(SP^n X, SP^n E) & \xrightarrow{(SP^n f)_{SP^n \varphi}} & (SP^n Y, SP^n E') \\ (\pi_X)_{\rho_X} \uparrow & & \uparrow (\pi_Y)_{\rho_Y} \\ (X^n, E^n) & \xrightarrow{(\Pi^n f)_{\Pi^n \varphi}} & (Y^n, (E')^n)\end{array}$$

That is,

$$(SP^n f)_{SP^n \varphi} \circ (\pi_X)_{\rho_X} = (\pi_Y)_{\rho_Y} \circ \Pi^n f_{\Pi^n \varphi}.$$

Hence  $(SP^n f)_{SP^n \varphi} \circ (\pi_X)_{\rho_X}$  is soft continuous. Since  $(\pi_X)_{\rho_X}$  is a soft quotient map, it follows that  $(SP^n f)_{SP^n \varphi}$  is soft continuous.  $\square$

**Proposition 2.22.** For any soft topological space  $(X, \mathcal{T}, E)$  the following inequality holds:

$$w(SP^n X, SP^n \mathcal{T}, SP^n E) \leq w(X, \mathcal{T}, E).$$

**Proof.** Let  $w(X, \mathcal{T}, E) = \kappa$ . Then by proposition, we have  $w(X^n, \mathcal{T}^n, E^n) = \kappa$ . In this case there exists a soft base  $\mathcal{B} = \{(H_\alpha, E^n) : \alpha \in \Delta\}$  for  $SP^n \mathcal{T}$  such that  $|\mathcal{B}| = \kappa$ . We prove that

$$\mathcal{N} = \{\pi_\rho(H_\alpha, E^n) : \alpha \in \Delta\} \cap SP^n \mathcal{T}$$

is a soft base for  $SP^n \mathcal{T}$ . Take an arbitrary soft open set  $(G, SP^n E)$  over  $SP^n X$ . Since  $\pi_\rho$  is soft quotient map, it follows that  $(\pi_\rho)^{-1}(G, SP^n E)$  is soft open set over  $X^n$ . Then there is a  $\Delta' \in \Delta$  such that

$$(\pi_\rho)^{-1}(G, SP^n E) = \bigcup_{\beta \in \Delta'} (H_\beta, E^n).$$

Since  $\pi_\rho$  is a soft surjective map, implies that

$$(G, SP^n E) = \bigcup_{\beta \in \Delta'} \pi_\rho(H_\beta, E^n).$$

Proposition 2.22 is proved.  $\square$

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# A PIECEWISE CONSTANT ARGUMENT APPROACH USING THE ADAMS-BASHFORTH METHOD FOR NUMERICAL SOLUTION OF SECOND-ORDER SYSTEMS OF ODES

MUMINOV MUKHIDDIN\*

FACULTY OF MATHEMATICS, SAMARKAND STATE UNIVERSITY, SAMARKAND, UZBEKISTAN;  
V.I ROMANOVSKIY INSTITUTE OF MATHEMATICS, UZBEKISTAN ACADEMY OF SCIENCES, TASHKENT,  
UZBEKISTAN  
mmuminov@mail.ru

SHERMUKHAMMEDOV BAKHTIYOR

FACULTY OF MATHEMATICS, SAMARKAND STATE UNIVERSITY, SAMARKAND, UZBEKISTAN  
shermuxammedovbaxtiyor47@gmail.com

**ABSTRACT.** The aim of this work is to develop a numerical method for solving initial value problems involving second order nonlinear system of ordinary differential equations (ODEs). This is achieved by using the Piecewise Constant Argument Method (PCAM) in combination with the Adams-Bashforth technique. To facilitate this, we construct a differential equation with piecewise constant arguments (DEPCA) that is parameterized by a positive integer  $n$ , which represents the number of subintervals. We derive the explicit form of the unique solution to the initial value problem for the resulting second order DEPCA. Furthermore, we prove that this solution converges to the solution of the original problem as  $n$  increases. To validate the proposed approach, we solve several test problems that arise from physical models with specified initial conditions. The numerical results show that the method is efficient and highly accurate.

**MSC (2020):** 34A06, 34A12, 34A34, 65A05.

**Key words:** Approximate solution, Initial value problems, Piecewise constant argument.

## Introduction

In this paper, we show a construction of an approximate solution for the second order nonlinear system of differential equation of the form

$$\mathbf{y}''(t) = \mathbf{f}(t, \mathbf{y}(t), \mathbf{y}'(t)), \quad t \in [0, 1], \quad (1)$$

with initial conditions

$$\mathbf{y}(0) = \mathbf{y}_0, \quad \mathbf{y}'(0) = \mathbf{y}'_1, \quad (2)$$

where  $\mathbf{y}(t) = (y_1(t), y_2(t), \dots, y_m(t))$ ,  $\mathbf{f} = (f_1, f_2, \dots, f_m)$  are vector functions on  $R^m$ ,  $f_i$ ,  $i = 1, \dots, m$ , are real valued differentiable, continuous functions on  $[0, 1] \times R^{2m}$ ,  $\mathbf{y}_0, \mathbf{y}'_1 \in R^m$  are given vectors.

Systems of ODEs are widely utilized in various scientific and engineering fields. For instance, mixing problems involving two tanks, electrical networks, and mass-spring systems can all be modeled using systems of ODEs [1].

Various analytical and numerical techniques have been developed and applied to solve systems of ODEs. The Adomian decomposition method (ADM) is presented in [2]. Haar wavelet collocation method (HWC) is investigated in the literatures [3,4]. In [5], an Artificial Neural Network (ANN) method is proposed for solving well-known classes of Lane-Emden type equations, a class of important nonlinear singular second-order differential equations. The functional matrix of integration via Bernoulli wavelets and generated a competent numerical scheme is developed to solve the nonlinear system of singular differential equations which is Lane-Emden form by Bernoulli wavelets collocation technique (BWCT) with different physical conditions in [6].

The piecewise constant argument method is employed to solve various nonlinear ordinary differential equations, such as the Bernoulli, Lane-Emden, and Duffing equations [7-9].

To construct an approximate solution, we first introduce second order system of differential equation with piecewise constant arguments corresponding to the considered system of ODE with initial conditions, which does depend on a positive integer number  $n$ . Next, we define a definition of the solution of second order system of differential equation with piecewise constant arguments as a piecewise-smooth function with respect to a positive integer  $n$ . It is proved that this equation has a unique piecewise-smooth solution, which will be an approximate solution of the considered initial value problem for large  $n$ . Numerical results are presented showing the effectiveness of the proposed methods, and the errors between the approximate and exact solutions are estimated.

### Differential equations with piecewise constant arguments

Let  $\mathbf{y}(t)$  be solution of the initial value problem (1)-(2). Then there exists  $A > 0$  such that  $|y_p(t)| \leq A$  for all  $t \in [0, 1]$  and  $p = 1, 2, \dots, m$ .

In what follows, we suppose that the functions  $f_p(t, x_1, \dots, x_{2m})$ ,  $p = 1, 2, \dots, m$ , have the compact support  $D = [0, 1] \times [-A, A]^{2m}$ , i.e. we assume that

$$f_p(t, x_1, \dots, x_{2m}) = 0 \quad \text{for} \quad (t, x_1, \dots, x_{2m}) \notin D. \quad (3)$$

Moreover, under this assumption on the functions  $f_1, \dots, f_m$  the solution  $\mathbf{y}(t)$  of the initial value problem (1)-(2) is preserved.

We remark that the assumption (3) ensures the boundedness of the functions  $f_p(t, x_1, \dots, x_{2m})$  in  $D$ . Thus, under these conditions, we can construct a numerical approximation algorithm for the solution.

Let  $n$  be a positive integer number and  $t_k = \frac{k}{n}$ ,  $k = 0, 1, \dots, n$ . This study focuses on using piecewise constant argument method and Adams-Bashforth technique for solving the initial-value problems. This technique is thoroughly discussed in many references, for example in [10].

Consider second order nonlinear system of differential equation with piecewise constant arguments of the form

$$\begin{cases} \mathbf{y}''(t) = \mathbf{F}_n(t), \\ \mathbf{y}(0) = \mathbf{y}_0, \quad \mathbf{y}'(0) = \mathbf{y}'_1, \end{cases} \quad (4)$$

where  $\mathbf{F}_n(t)$  is a vector function with piecewise continuous functions on  $[0, 1]$  defined as

$$\mathbf{F}_n(t) = \mathbf{f}_k \quad \text{for} \quad t \in [t_{k-1}, t_k), \quad k = 1, 2, \dots, n.$$

Here the vectors  $\mathbf{f}_k \in R^m$  are chosen by using Adams-Bashforth technique (See pages 302-307 in [10])

$$\begin{aligned} \mathbf{f}_1 &= \mathbf{f}(t_0, \mathbf{y}(t_0), \mathbf{y}'(t_0)), \quad t \in [t_0, t_1), \\ \mathbf{f}_2 &= \frac{3}{2}\mathbf{f}(t_1, \mathbf{y}(t_1), \mathbf{y}'(t_1)) - \frac{1}{2}\mathbf{f}(t_0, \mathbf{y}(t_0), \mathbf{y}'(t_0)), \quad t \in [t_1, t_2), \\ \mathbf{f}_3 &= \frac{3}{2}\mathbf{f}(t_2, \mathbf{y}(t_2), \mathbf{y}'(t_2)) - \frac{1}{2}\mathbf{f}(t_1, \mathbf{y}(t_1), \mathbf{y}'(t_1)), \quad t \in [t_2, t_3), \\ \mathbf{f}_k &= \frac{1}{24} \left( 55\mathbf{f}(t_{k-1}, \mathbf{y}(t_{k-1}), \mathbf{y}'(t_{k-1})) - 59\mathbf{f}(t_{k-2}, \mathbf{y}(t_{k-2}), \mathbf{y}'(t_{k-2})) \right. \\ &\quad \left. + 37\mathbf{f}(t_{k-3}, \mathbf{y}(t_{k-3}), \mathbf{y}'(t_{k-3})) - 9\mathbf{f}(t_{k-4}, \mathbf{y}(t_{k-4}), \mathbf{y}'(t_{k-4})) \right), \quad t \in [t_{k-1}, t_k), \quad k = 4, 5, \dots, n, \end{aligned}$$

the vectors  $\mathbf{y}(t_k)$  and  $\mathbf{y}'(t_k)$  are taken as a limit of the solution of the ODE

$$\mathbf{y}''(t) = \mathbf{f}_k, \quad t \in [t_{k-1}, t_k)$$

$$\text{as } \mathbf{y}(t_k) = \lim_{t \rightarrow t_k-0} \mathbf{y}(t), \quad \mathbf{y}'(t_k) = \lim_{t \rightarrow t_k-0} \mathbf{y}'(t), \quad k = 2, 3, \dots, n.$$

We introduce the definition of a solution to the system of equations (4).

**Definition 1.** A vector function  $\mathbf{y}(t) := \mathbf{y}_n(t)$  is called a solution of the initial value problem (4) if the following conditions are satisfied

- (i)  $\mathbf{y}(t), \mathbf{y}'(t)$  are continuous on  $[0, 1]$ ;
- (ii)  $\mathbf{y}''(t)$  exists and continuous in  $[0, 1]$  with possible exception at points  $\frac{k}{n}, k = 0, 1, \dots, n$ , where one-sided derivatives exist;
- (iii)  $\mathbf{y}(t)$  satisfies initial value problem (4) in  $[0, 1]$ , with the possible exception at the points  $\frac{k}{n}, k = 0, 1, \dots, n$ .

The system of equation (4) is called the second order system of differential equation with piecewise constant arguments corresponding to the system of equation (1).

**Theorem 1.** For any positive integer number  $n$  the initial value problem (4) has a unique solution  $\mathbf{y}_n$  defined as

$$\mathbf{y}(t) = \int_{t_{k-1}}^t \int_{t_{k-1}}^s \mathbf{F}_n(s_1) ds_1 ds + (t - t_{k-1})\mathbf{y}'(t_{k-1}) + \mathbf{y}(t_{k-1}), \quad t \in [t_{k-1}, t_k], \quad (5)$$

where  $\mathbf{y}(t_{k-1}) = \lim_{t \rightarrow t_{k-1}-0} \mathbf{y}(t), \quad \mathbf{y}'(t_{k-1}) = \lim_{t \rightarrow t_{k-1}-0} \mathbf{y}'(t), \quad k = 2, 3, \dots, n.$

**Lemma 1.** Let  $\mathbf{y}_n$  be solution of (4). Then

$$\lim_{n \rightarrow \infty} |\mathbf{f}(t, \mathbf{y}(t), \mathbf{y}'(t)) - \mathbf{F}_n(t)| \rightarrow 0, \quad (6)$$

for all  $t \in [0, 1]$ , where  $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$ .

The following theorem claims that the function  $\mathbf{y}_n$  can be approximated for the solution of the initial value problem (1)-(2).

**Theorem 2.** For any  $\varepsilon > 0$  there exists a positive integer number  $n_0 = n(\varepsilon)$  such that for any  $n > n_0$  the estimation

$$\sup_{t, t \neq \frac{k}{n}} |\mathbf{y}_n''(t) - \mathbf{f}(t, \mathbf{y}(t), \mathbf{y}'(t))| < \varepsilon \quad (7)$$

holds, where  $\mathbf{y}_n$  is a solution of the initial value problem (4).

### Proof of the Main results

**Proof of the Theorem 1.** For  $t \in [t_0, t_1)$  the equation (4) takes the following form

$$\mathbf{y}''(t) = \mathbf{F}_n(t). \quad (8)$$

Integrating both sides of equation (8) (2)-times over  $[t_0, t]$ , we have

$$\mathbf{y}'(t) = \int_{t_0}^t \mathbf{F}_n(s) ds + \mathbf{y}'(t_0), \quad (9)$$

$$\mathbf{y}(t) = \int_{t_0}^t \int_{t_0}^s \mathbf{F}_n(s_1) ds_1 ds + \mathbf{y}'(t_0)t + \mathbf{y}(t_0). \quad (10)$$

The following limits exist  $\mathbf{y}(t_1) = \lim_{t \rightarrow t_1-0} \mathbf{y}(t), \quad \mathbf{y}'(t_1) = \lim_{t \rightarrow t_1-0} \mathbf{y}'(t)$ . Further, for  $t \in [t_1, t_2)$  we rewrite equation (4) as

$$\mathbf{y}''(t) = \mathbf{F}_n(t). \quad (11)$$

Integrating (11) (2)-times over  $[t_1, t]$  gives

$$\mathbf{y}'(t) = \int_{t_1}^t \mathbf{F}_n(s) ds + \mathbf{y}'(t_1), \quad (12)$$

$$\mathbf{y}(t) = \int_{t_1}^t \int_{t_1}^s \mathbf{F}_n(s_1) ds_1 ds + (t - t_1) \mathbf{y}'(t_1) + \mathbf{y}(t_1). \quad (13)$$

One can see that the functions  $\mathbf{y}(t)$ ,  $t \in [t_0, t_2]$ , are twice differentiable in  $(t_0, t_2)$  and continuous on  $[t_0, t_2]$ . The function  $\mathbf{y}''(t)$  continuous in  $(t_0, t_1) \cup (t_1, t_2)$  and at points  $t = t_1$  and  $t = t_2$  one-sided derivatives exist.

By similar way, we can show that a solution of the system of equation

$$\mathbf{y}''(t) = \mathbf{F}_n(t), \quad t \in [t_{k-1}, t_k], \quad k = 3, \dots, n,$$

represents as

$$\mathbf{y}'(t) = \int_{t_{k-1}}^t \mathbf{F}_n(s) ds + \mathbf{y}'(t_{k-1}), \quad t \in [t_{k-1}, t_k], \quad (14)$$

$$\mathbf{y}(t) = \int_{t_{k-1}}^t \int_{t_{k-1}}^s \mathbf{F}_n(s_1) ds_1 ds + (t - t_{k-1}) \mathbf{y}'(t_{k-1}) + \mathbf{y}(t_{k-1}), \quad t \in [t_{k-1}, t_k]. \quad (15)$$

The function  $\mathbf{y}_n(t) =: \mathbf{y}(t)$  defined by (15) is twice differentiable in  $(t_0, t_1) \cup (t_1, t_2) \cup \dots \cup (t_{k-1}, 1)$  and a differentiable in  $(0, 1)$  and continuous on  $[0, 1]$ . One can see that the function  $\mathbf{y}_n(t)$  is a unique solution of the system of IVP (4). Theorem 1 is proved.

**Lemma 2.** There exists a positive number  $C$  such that for any  $n$  and  $k \in \{1, 2, \dots, n\}$  the inequalities

$$|\mathbf{y}'(t_k)| \leq C, \quad (16)$$

$$|\mathbf{y}(t_k)| \leq C \quad (17)$$

hold.

**Proof.** Let

$$M = \max_{0 \leq t \leq 1} |\mathbf{F}_n(t)|. \quad (18)$$

Then for  $t \in [0, t_1)$  from (9) and (10) we obtain

$$|\mathbf{y}'(t_1)| \leq \frac{1}{n} M + |\mathbf{y}'(0)| \quad (19)$$

and

$$|\mathbf{y}(t_1)| \leq \frac{1}{2n^2} M + \frac{1}{n} |\mathbf{y}'(0)| + |\mathbf{y}(0)|, \quad (20)$$

respectively.

For  $t \in [t_1, t_2)$  applying the equality (18), from (12) and (13), we obtain

$$|\mathbf{y}'(t_2)| \leq \frac{1}{n} M + |\mathbf{y}'(t_1)| \quad (21)$$

and

$$|\mathbf{y}(t_2)| \leq \frac{1}{2n^2} M + \frac{1}{n} |\mathbf{y}'(t_1)| + |\mathbf{y}(t_1)|. \quad (22)$$

Analysing similarly, from (14), (15) for  $t \in [t_{k-1}, t_k)$ ,  $k = 3, 4, \dots, n$ , we obtain the following inequalities

$$|\mathbf{y}'(t_k)| \leq \frac{1}{n} M + |\mathbf{y}'(t_{k-1})| \quad (23)$$

and

$$|\mathbf{y}(t_k)| \leq \frac{1}{2n^2} M + \frac{1}{n} |\mathbf{y}'(t_{k-1})| + |\mathbf{y}(t_{k-1})|. \quad (24)$$

Using the recurrence inequality (23) and taking account (19), we get

$$|\mathbf{y}'(t_k)| \leq \frac{M}{n} + |\mathbf{y}'(t_{k-1})| \leq \frac{(k-1)M}{n} + |\mathbf{y}'(t_1)| \leq \frac{kM}{n} + |\mathbf{y}'(0)|. \quad (25)$$

Hence

$$|\mathbf{y}'(t_k)| \leq \frac{kM}{n} + |\mathbf{y}'(0)| \leq M + |\mathbf{y}'(0)|. \quad (26)$$

Similarly, using (24), we obtain the following evaluations

$$\begin{aligned} |\mathbf{y}(t_k)| &\leq \frac{1}{2n^2}M + \frac{1}{n}|\mathbf{y}'(t_{k-1})| + |\mathbf{y}(t_{k-1})| \\ &\leq \frac{1}{2n^2}M + \frac{1}{n}|\mathbf{y}'(t_{k-1})| + \frac{1}{2n^2}M + \frac{1}{n}|\mathbf{y}'(t_{k-2})| + |\mathbf{y}(t_{k-2})| \\ &\leq \frac{1}{2n^2}M + \frac{1}{n}|\mathbf{y}'(t_{k-1})| + \frac{1}{2n^2}M + \frac{1}{n}|\mathbf{y}'(t_{k-2})| \\ &\quad + \dots + \frac{1}{2n^2}M + \frac{1}{n}|\mathbf{y}'(t_1)| + |\mathbf{y}(t_1)| \\ &\leq \frac{k}{2n^2}M + \frac{1}{n}|\mathbf{y}'(t_{k-1})| + \frac{1}{n}|\mathbf{y}'(t_{k-2})| + \dots + \frac{1}{n}|\mathbf{y}'(0)| + |\mathbf{y}(0)|. \end{aligned}$$

Then applying (25), we have

$$\begin{aligned} |\mathbf{y}(t_k)| &\leq \frac{k}{2n^2}M + \frac{1}{n} \left( \frac{(k-1)M}{n} + |\mathbf{y}'(0)| + \frac{(k-2)M}{n} + |\mathbf{y}'(0)| \right. \\ &\quad \left. + \dots + \frac{M}{n} + |\mathbf{y}'(0)| \right) + |\mathbf{y}(0)| \\ &\leq \frac{k}{2n^2}M + \frac{1}{n} \left( \frac{k(k-1)M}{2n} + k|\mathbf{y}'(0)| \right) + |\mathbf{y}(0)| \\ &\leq \frac{k}{2n^2}M + \frac{k(k-1)}{2n^2}M + \frac{k}{n}|\mathbf{y}'(0)| + |\mathbf{y}(0)|. \end{aligned}$$

Hence

$$|\mathbf{y}(t_k)| \leq \frac{k}{2n^2}M + \frac{k(k-1)}{2n^2}M + \frac{k}{n}|\mathbf{y}'(0)| + |\mathbf{y}(0)| \leq \frac{M}{2} + |\mathbf{y}'(0)| + |\mathbf{y}(0)|. \quad (27)$$

Since  $k \leq n$ , from (26) and (27) we obtain the inequalities (16), (17).

**Proof of the Lemma 1.** For convenience we denote  $\mathbf{f}(t) := \mathbf{f}(t, y_1(t), y_1'(t), \dots, y_m(t), y_m'(t))$  and

$$\mathbf{R}_n(t) = \mathbf{f}(t) - \mathbf{F}_n(t).$$

Since  $\mathbf{f}$  is continuous on  $[t_0, t_1]$ ,  $\mathbf{R}_n(t)$  estimated as

$$\begin{aligned} |\mathbf{R}_n(t)| &= |\mathbf{f}(t) - \mathbf{f}(t_0)| \leq L(|t - t_0| + |y_{1,n}(t) - y_1(t_0)| + |y_{1,n}'(t) - y_1'(t_0)| \\ &\quad + \dots + |y_{m,n}(t) - y_m(t_0)| + |y_{m,n}'(t) - y_m'(t_0)|). \end{aligned}$$

Similarly, for  $t \in [t_1, t_2]$  we have

$$\begin{aligned} \mathbf{R}_n(t) &= |\mathbf{f}(t) - [\frac{3}{2}\mathbf{f}(t_1) - \frac{1}{2}\mathbf{f}(t_0)]| \leq |\mathbf{f}(t) - \mathbf{f}(t_1)| + \frac{1}{2}|\mathbf{f}(t_1) - \mathbf{f}(t_0)| \leq \\ &\leq L(|t - t_1| + |y_{1,n}(t) - y_1(t_1)| + |y_{1,n}'(t) - y_1'(t_1)| + \dots + |y_{m,n}(t) - y_m(t_1)| + |y_{m,n}'(t) - y_m'(t_1)|) \\ &\quad + \frac{L}{2}(|t_1 - t_0| + |y_{1,n}(t_1) - y_1(t_0)| + |y_{1,n}'(t_1) - y_1'(t_0)| + \dots + |y_{m,n}(t_1) - y_m(t_0)| + |y_{m,n}'(t_1) - y_m'(t_0)|). \end{aligned}$$

For  $t \in [t_2, t_3]$

$$\begin{aligned} \mathbf{R}_n(t) &= |\mathbf{f}(t) - [\frac{3}{2}\mathbf{f}(t_2) - \frac{1}{2}\mathbf{f}(t_1)]| \leq |\mathbf{f}(t) - \mathbf{f}(t_2)| + \frac{1}{2}|\mathbf{f}(t_2) - \mathbf{f}(t_1)| \\ &\leq L(|t - t_2| + |y_{1,n}(t) - y_1(t_2)| + |y_{1,n}'(t) - y_1'(t_2)| + \dots + |y_{m,n}(t) - y_m(t_2)| + |y_{m,n}'(t) - y_m'(t_2)|) \\ &\quad + \frac{L}{2}(|t_2 - t_1| + |y_{1,n}(t_2) - y_1(t_1)| + |y_{1,n}'(t_2) - y_1'(t_1)| + \dots + |y_{m,n}(t_2) - y_m(t_1)| + |y_{m,n}'(t_2) - y_m'(t_1)|). \end{aligned}$$

For  $t \in [t_{k-1}, t_k]$ ,  $k = 4, 5, \dots, n$

$$\begin{aligned} \mathbf{R}_n(t) &= |\mathbf{f}(t) - [\frac{55}{24}\mathbf{f}(t_{k-1}) - \frac{59}{24}\mathbf{f}(t_{k-2}) + \frac{37}{24}\mathbf{f}(t_{k-3}) - \frac{9}{24}\mathbf{f}(t_{k-4})]| \\ &\leq |\mathbf{f}(t) - \mathbf{f}(t_{k-1})| + \frac{31}{24}|\mathbf{f}(t_{k-1}) - \mathbf{f}(t_{k-2})| + \frac{28}{24}|\mathbf{f}(t_{k-2}) - \mathbf{f}(t_{k-3})| + \frac{9}{24}|\mathbf{f}(t_{k-3}) - \mathbf{f}(t_{k-4})| \\ &\leq L(|t - t_{k-1}| + |y_{1,n}(t) - y_1(t_{k-1})| + |y'_{1,n}(t) - y'_1(t_{k-1})| \\ &\quad + \dots + |y_{m,n}(t) - y_m(t_{k-1})| + |y'_{m,n}(t) - y'_m(t_{k-1})|) \\ &\quad + \frac{31L}{24}(|t_{k-1} - t_{k-2}| + |y_{1,n}(t_{k-1}) - y_1(t_{k-2})| + |y'_{1,n}(t_{k-1}) - y'_1(t_{k-2})| \\ &\quad + \dots + |y_{m,n}(t_{k-1}) - y_m(t_{k-2})| + |y'_{m,n}(t_{k-1}) - y'_m(t_{k-2})|) \\ &\quad + \frac{28L}{24}(|t_{k-2} - t_{k-3}| + |y_{1,n}(t_{k-2}) - y_1(t_{k-3})| + |y'_{1,n}(t_{k-2}) - y'_1(t_{k-3})| \\ &\quad + \dots + |y_{m,n}(t_{k-2}) - y_m(t_{k-3})| + |y'_{m,n}(t_{k-2}) - y'_m(t_{k-3})|) \\ &\quad + \frac{9L}{24}(|t_{k-3} - t_{k-4}| + |y_{1,n}(t_{k-3}) - y_1(t_{k-4})| + |y'_{1,n}(t_{k-3}) - y'_1(t_{k-4})| \\ &\quad + \dots + |y_{m,n}(t_{k-3}) - y_m(t_{k-4})| + |y'_{m,n}(t_{k-3}) - y'_m(t_{k-4})|). \end{aligned}$$

According to the equalities (9) and (10), we obtain

$$\mathbf{y}'_n(t) - \mathbf{y}'(t_{k-1}) = \int_{t_{k-1}}^t \mathbf{F}_n(s)ds, \quad \text{for } t \in [t_{k-1}, t_k] \quad (28)$$

and

$$\mathbf{y}_n(t) - \mathbf{y}(t_{k-1}) = \int_{t_{k-1}}^t \int_{t_{k-1}}^s \mathbf{F}_n(s_1)ds_1ds + \mathbf{y}'(t_{k-1})(t - t_{k-1}), \quad \text{for } t \in [t_{k-1}, t_k], \quad (29)$$

by (28) and (29) we have

$$|\mathbf{y}'_n(t) - \mathbf{y}'(t_{k-1})| \leq M \frac{1}{n},$$

and

$$|\mathbf{y}_n(t) - \mathbf{y}(t_{k-1})| \leq (M + 2C) \frac{1}{n}.$$

Those yield

$$|\mathbf{y}_n^{(r)}(t) - \mathbf{y}^{(r)}(t_{k-1})| \leq (M + 2C) \frac{1}{n}, \quad r = 0, 1.$$

Then, we obtain

$$\mathbf{R}_n \leq \begin{cases} L(1 + 2m(M + 2C)) \frac{1}{n}, & [t_0, t_1] \\ \frac{3L}{2}(1 + 2m(M + 2C)) \frac{1}{n}, & [t_1, t_2] \\ \frac{3L}{2}(1 + 2m(M + 2C)) \frac{1}{n}, & [t_2, t_3] \\ \frac{23L}{6}(1 + 2m(M + 2C)) \frac{1}{n}, & [t_{k-1}, t_k], \quad k = 4, 5, \dots, n. \end{cases} \quad (30)$$

Since the numbers  $M$  and  $C$  do not depend of  $n$ , the last inequality yields the proof of the Lemma 1.

**Proof of the Theorem 2.** Let  $\mathbf{y}_n(t)$  be a solution of (4), i.e. the equality

$$\mathbf{y}_n''(t) = \mathbf{F}_n(t), \quad t \in [0, 1]. \quad (31)$$

Then the equality

$$\mathbf{Q}_n(t) = \mathbf{y}_n''(t) - \mathbf{f}(t, \mathbf{y}(t), \mathbf{y}'(t)) = \mathbf{F}_n(t) - \mathbf{f}(t, \mathbf{y}(t), \mathbf{y}'(t))$$

holds. According to the (30) inequality yields the proof of the Theorem 2.

### Stability analysis

We introduce Zero-stability for the proposed piecewise constant argument method mixed the Adams-Bashforth technique.

**Definition 2.** (Zero-stability) The numerical method applied in (5) for approximating problem (1)-(2) is said to be *zero-stable* if there exists a constant  $C$  such that, for every  $\varepsilon > 0$ , there exists a positive integer  $n_0 = n_0(\varepsilon)$  for which the inequality

$$|\mathbf{y}_\varepsilon(t) - \mathbf{y}(t)| \leq C\varepsilon,$$

holds for all  $t \in [0, 1]$  and  $n > n_0$ . Here,  $\mathbf{y} = \mathbf{y}_n$  and  $\mathbf{y}_\varepsilon = \mathbf{y}_{n,\varepsilon}$  denote, respectively, the solutions of the exact and the perturbed problems (4) and

$$\begin{cases} \mathbf{y}_\varepsilon''(t) = \mathbf{F}_{n,\varepsilon}(t), \\ \mathbf{y}_\varepsilon(0) = \mathbf{y}_0 + \delta, \quad \mathbf{y}_\varepsilon'(0) = \mathbf{y}_1 + \delta, \end{cases} \quad (32)$$

where  $|\delta| \leq \varepsilon$ ,  $\mathbf{F}_{n,\varepsilon}(t) = \mathbf{f}_{k,\varepsilon}$ ,  $t \in [t_{k-1}, t_k]$ ,

$$\begin{aligned} \mathbf{f}_{1,\varepsilon} &= \mathbf{f}_\varepsilon(t_0, \mathbf{y}_\varepsilon(t_0), \mathbf{y}_\varepsilon'(t_0)), \quad t \in [t_0, t_1], \\ \mathbf{f}_{2,\varepsilon} &= \frac{3}{2}\mathbf{f}_\varepsilon(t_1, \mathbf{y}_\varepsilon(t_1), \mathbf{y}_\varepsilon'(t_1)) - \frac{1}{2}\mathbf{f}_\varepsilon(t_0, \mathbf{y}_\varepsilon(t_0), \mathbf{y}_\varepsilon'(t_0)), \quad t \in [t_1, t_2], \\ \mathbf{f}_{3,\varepsilon} &= \frac{3}{2}\mathbf{f}_\varepsilon(t_2, \mathbf{y}_\varepsilon(t_2), \mathbf{y}_\varepsilon'(t_2)) - \frac{1}{2}\mathbf{f}_\varepsilon(t_1, \mathbf{y}_\varepsilon(t_1), \mathbf{y}_\varepsilon'(t_1)), \quad t \in [t_2, t_3], \\ \mathbf{f}_{k,\varepsilon} &= \frac{1}{24} \left( 55\mathbf{f}_\varepsilon(t_{k-1}, \mathbf{y}_\varepsilon(t_{k-1}), \mathbf{y}_\varepsilon'(t_{k-1})) - 59\mathbf{f}_\varepsilon(t_{k-2}, \mathbf{y}_\varepsilon(t_{k-2}), \mathbf{y}_\varepsilon'(t_{k-2})) \right. \\ &\quad \left. + 37\mathbf{f}_\varepsilon(t_{k-3}, \mathbf{y}_\varepsilon(t_{k-3}), \mathbf{y}_\varepsilon'(t_{k-3})) - 9\mathbf{f}_\varepsilon(t_{k-4}, \mathbf{y}_\varepsilon(t_{k-4}), \mathbf{y}_\varepsilon'(t_{k-4})) \right), \quad t \in [t_{k-1}, t_k], \quad k = 4, \dots, n. \end{aligned}$$

**Theorem 3.** (Zero-stability) Assume that  $\mathbf{f}$  is differentiable function. Then the numerical method applied in (5) for approximating problem (1)-(2) is zero-stable.

### Numerical experiments and Discussions

In order to derive an approximate solution to equation (1), it is necessary to compute  $\mathbf{y}(t)$ , which is explicitly defined by (5). Additionally, Theorem 2 establishes the convergence of the algorithm, with inequality (7) demonstrating the diminishment of the residual error as  $n$  becomes large.

The procedures of the PCAM are outlined as follows

**Input:**  $n$  is a positive integer number, initial conditions.

**Step 1:** Compute  $\mathbf{y}(t)$ ,  $\mathbf{y}'(t)$  for  $t \in [0, t_1]$ ;

**Step 2:** Find limits  $\mathbf{y}(t_1) = \lim_{t \rightarrow t_1-0} \mathbf{y}(t)$ ,  $\mathbf{y}'(t_1) = \lim_{t \rightarrow t_1-0} \mathbf{y}'(t)$ ;

**Step 3:** Let  $\mathbf{y}(t)$  be solution of (4) in  $[t_{k-2}, t_{k-1}]$  and

$$\mathbf{y}(t_{k-2}) = \lim_{t \rightarrow t_{k-2}-0} \mathbf{y}(t), \quad \mathbf{y}'(t_{k-2}) = \lim_{t \rightarrow t_{k-2}-0} \mathbf{y}'(t).$$

Compute  $\mathbf{y}(t)$ ,  $\mathbf{y}'(t)$ ,  $k = 2, \dots, n$  using (5), and find corresponding limits

$$\mathbf{y}(t_{k-1}) = \lim_{t \rightarrow t_{k-1}-0} \mathbf{y}(t), \quad \mathbf{y}'(t_{k-1}) = \lim_{t \rightarrow t_{k-1}-0} \mathbf{y}'(t);$$

**Step 4:** Construct an approximate solution from (5);

**Output:** Errors are found.

Let us define absolute errors as follows

$$\text{Absolute Error (AE)} = |Y_i - y_i|, \quad i = 1, 2, \dots, n,$$

where  $Y$  and  $y$  are exact and approximate solutions respectively. All of the computations have been performed using MAPLE 12.

**Problem 1.** A nonlinear system of Lane-Emden equations is solved by proposed method. Consider the nonlinear of initial value problem is expressed in [4-6], following as

$$\begin{aligned} y_1''(t) + \frac{1}{t}y_1'(t) &= y_1^2(t)y_2^3(t) + y_2^3(t), \quad y_1(0) = 1, \quad y_1'(0) = 0, \\ y_2''(t) + \frac{3}{t}y_2'(t) &= -y_1^2(t)y_2^5(t) - 3y_2^5(t), \quad y_2(0) = 1, \quad y_2'(0) = 0, \quad t \in [0, 1]. \end{aligned} \quad (33)$$

The exact solution is  $y_1(t) = \sqrt{1+t^2}$ ,  $y_2(t) = \frac{1}{\sqrt{1+t^2}}$ . Absolute errors are computed, obtained using Artificial neural networks [5], Haar wavelet collocation method [4], Bernoulli wavelets collocation technique [6] and present method evaluated at the chosen points are reported in Tables 1-2.

| t   | ANN [5]               | HWCM [4]              | BWCT [6]              | Present method<br>$n = 10^2$ | Present method<br>$n = 10^3$ | Present method<br>$n = 10^4$ |
|-----|-----------------------|-----------------------|-----------------------|------------------------------|------------------------------|------------------------------|
| 0.0 | $3.96 \times 10^{-6}$ | 0                     | 0                     | 0                            | 0                            | 0                            |
| 0.1 | $3.53 \times 10^{-6}$ | $1.19 \times 10^{-6}$ | $5.60 \times 10^{-9}$ | $3.15 \times 10^{-8}$        | $7.86 \times 10^{-10}$       | $8.13 \times 10^{-12}$       |
| 0.2 | $2.17 \times 10^{-6}$ | $3.67 \times 10^{-6}$ | $5.79 \times 10^{-9}$ | $1.47 \times 10^{-7}$        | $2.99 \times 10^{-9}$        | $3.03 \times 10^{-11}$       |
| 0.3 | $1.62 \times 10^{-6}$ | $7.28 \times 10^{-6}$ | $3.61 \times 10^{-9}$ | $4.31 \times 10^{-7}$        | $6.03 \times 10^{-9}$        | $6.07 \times 10^{-11}$       |
| 0.4 | $5.79 \times 10^{-7}$ | $1.16 \times 10^{-5}$ | $4.31 \times 10^{-9}$ | $7.36 \times 10^{-7}$        | $9.19 \times 10^{-9}$        | $9.23 \times 10^{-11}$       |
| 0.5 | $5.85 \times 10^{-7}$ | $1.18 \times 10^{-5}$ | $2.39 \times 10^{-9}$ | $9.96 \times 10^{-7}$        | $1.18 \times 10^{-8}$        | $1.18 \times 10^{-10}$       |
| 0.6 | $1.63 \times 10^{-6}$ | $2.09 \times 10^{-5}$ | $3.12 \times 10^{-9}$ | $1.17 \times 10^{-6}$        | $1.36 \times 10^{-8}$        | $1.36 \times 10^{-10}$       |
| 0.7 | $2.67 \times 10^{-6}$ | $2.69 \times 10^{-5}$ | $1.42 \times 10^{-9}$ | $1.25 \times 10^{-6}$        | $1.45 \times 10^{-8}$        | $1.45 \times 10^{-10}$       |
| 0.8 | $3.78 \times 10^{-6}$ | $3.38 \times 10^{-2}$ | $3.83 \times 10^{-2}$ | $1.25 \times 10^{-6}$        | $1.45 \times 10^{-8}$        | $1.46 \times 10^{-10}$       |
| 0.9 | $5.00 \times 10^{-6}$ | $3.21 \times 10^{-5}$ | $3.44 \times 10^{-9}$ | $1.21 \times 10^{-6}$        | $1.41 \times 10^{-8}$        | $1.41 \times 10^{-10}$       |
| 1.0 | $6.06 \times 10^{-6}$ | $3.27 \times 10^{-5}$ | $2.34 \times 10^{-9}$ | $1.15 \times 10^{-6}$        | $1.35 \times 10^{-8}$        | $1.36 \times 10^{-10}$       |

Таблица 2: Comparisons of  $y_1(t)$  of absolute errors are presented for Problem 1.

| t   | ANN [5]               | HWCM [4]              | BWCT [6]              | Present method<br>$n = 10^2$ | Present method<br>$n = 10^3$ | Present method<br>$n = 10^4$ |
|-----|-----------------------|-----------------------|-----------------------|------------------------------|------------------------------|------------------------------|
| 0.0 | $8.20 \times 10^{-6}$ | 0                     | 0                     | 0                            | 0                            | 0                            |
| 0.1 | $9.63 \times 10^{-6}$ | $2.43 \times 10^{-6}$ | $3.04 \times 10^{-9}$ | $3.24 \times 10^{-8}$        | $1.78 \times 10^{-9}$        | $1.79 \times 10^{-11}$       |
| 0.2 | $7.49 \times 10^{-6}$ | $3.63 \times 10^{-5}$ | $3.49 \times 10^{-9}$ | $4.95 \times 10^{-7}$        | $6.34 \times 10^{-9}$        | $6.35 \times 10^{-11}$       |
| 0.3 | $6.74 \times 10^{-6}$ | $6.50 \times 10^{-7}$ | $3.57 \times 10^{-9}$ | $1.03 \times 10^{-6}$        | $1.15 \times 10^{-8}$        | $1.16 \times 10^{-10}$       |
| 0.4 | $6.42 \times 10^{-6}$ | $2.96 \times 10^{-6}$ | $3.53 \times 10^{-9}$ | $1.42 \times 10^{-6}$        | $1.52 \times 10^{-8}$        | $1.53 \times 10^{-10}$       |
| 0.5 | $4.55 \times 10^{-6}$ | $3.98 \times 10^{-6}$ | $3.31 \times 10^{-9}$ | $1.49 \times 10^{-6}$        | $1.59 \times 10^{-8}$        | $1.59 \times 10^{-10}$       |
| 0.6 | $3.68 \times 10^{-6}$ | $3.78 \times 10^{-7}$ | $3.01 \times 10^{-9}$ | $1.21 \times 10^{-6}$        | $1.30 \times 10^{-8}$        | $1.30 \times 10^{-10}$       |
| 0.7 | $2.87 \times 10^{-6}$ | $5.56 \times 10^{-6}$ | $2.69 \times 10^{-9}$ | $6.30 \times 10^{-7}$        | $7.02 \times 10^{-9}$        | $7.03 \times 10^{-11}$       |
| 0.8 | $1.66 \times 10^{-6}$ | $1.48 \times 10^{-5}$ | $2.48 \times 10^{-2}$ | $1.70 \times 10^{-7}$        | $1.06 \times 10^{-9}$        | $1.05 \times 10^{-11}$       |
| 0.9 | $1.13 \times 10^{-6}$ | $1.24 \times 10^{-5}$ | $1.78 \times 10^{-9}$ | $1.07 \times 10^{-6}$        | $1.01 \times 10^{-8}$        | $1.01 \times 10^{-10}$       |
| 1.0 | $4.67 \times 10^{-7}$ | $2.35 \times 10^{-5}$ | $1.68 \times 10^{-9}$ | $1.99 \times 10^{-6}$        | $1.94 \times 10^{-8}$        | $1.94 \times 10^{-10}$       |

Таблица 3: Comparisons of  $y_2(t)$  of absolute errors are presented for Problem 1.

**Problem 2.** Consider the nonlinear initial value problem, following as

$$\begin{aligned} y_1''(t) &= y_1^2(t) + y_2'(t) - (t+1)\sin(t) - \cos(t) - \sin^2(t) - t^2, \quad y_1(0) = 0, \quad y_1'(0) = 2, \\ y_2''(t) &= y_1'(t)y_2(t) - 2\sin(t) - t\cos^2(t)2t\cos(t), \quad y_2(0) = 0, \quad y_2'(0) = 1, \quad t \in [0, 1]. \end{aligned} \quad (34)$$

The exact solution is  $y_1(t) = \sin(t) + t$ ,  $y_2(t) = t\cos(t)$ . Absolute errors are computed at the chosen points are reported in Tables 3-4.

## Conclusion

Firstly, employing the Adams-Bashforth technique is constructed second order system of differential equation (4) with the piecewise constant argument corresponding to the considered problem (1)-(2) which has the unique twice differentiable solution of the form (5). Subsequently, it is shown that the residual error of the approximation  $\mathbf{y}_n(t)$  tends to zero as the value of  $n$  becomes sufficiently large. Several numerical problems are represented to illustrate the absolute errors of the approximate solutions for nonlinear second order system of ODEs and to demonstrate the effectiveness of the proposed method.

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| t   | Present method<br>$n = 10^2$ | Present method<br>$n = 10^3$ | Present method<br>$n = 10^4$ | Present method<br>$n = 10^5$ |
|-----|------------------------------|------------------------------|------------------------------|------------------------------|
| 0.0 | 0                            | 0                            | 0                            | 0                            |
| 0.1 | $4.62 \times 10^{-6}$        | $4.89 \times 10^{-8}$        | $4.91 \times 10^{-10}$       | $4.92 \times 10^{-12}$       |
| 0.2 | $1.10 \times 10^{-5}$        | $1.13 \times 10^{-7}$        | $1.13 \times 10^{-9}$        | $1.13 \times 10^{-11}$       |
| 0.3 | $1.91 \times 10^{-5}$        | $1.95 \times 10^{-7}$        | $1.95 \times 10^{-9}$        | $1.95 \times 10^{-11}$       |
| 0.4 | $2.92 \times 10^{-5}$        | $2.95 \times 10^{-7}$        | $2.96 \times 10^{-9}$        | $2.96 \times 10^{-11}$       |
| 0.5 | $4.15 \times 10^{-5}$        | $4.19 \times 10^{-7}$        | $4.19 \times 10^{-9}$        | $4.19 \times 10^{-11}$       |
| 0.6 | $5.66 \times 10^{-5}$        | $5.71 \times 10^{-7}$        | $5.71 \times 10^{-9}$        | $5.71 \times 10^{-11}$       |
| 0.7 | $7.53 \times 10^{-5}$        | $7.58 \times 10^{-7}$        | $7.59 \times 10^{-9}$        | $7.59 \times 10^{-11}$       |
| 0.8 | $9.84 \times 10^{-5}$        | $9.91 \times 10^{-7}$        | $9.92 \times 10^{-9}$        | $9.92 \times 10^{-11}$       |
| 0.9 | $1.27 \times 10^{-4}$        | $1.28 \times 10^{-6}$        | $1.28 \times 10^{-8}$        | $1.28 \times 10^{-10}$       |
| 1.0 | $1.64 \times 10^{-4}$        | $1.65 \times 10^{-6}$        | $1.65 \times 10^{-8}$        | $1.65 \times 10^{-10}$       |

Таблица 4: Absolute errors of  $y_1(t)$  are presented for Problem 2.

| t   | Present method<br>$n = 10^2$ | Present method<br>$n = 10^3$ | Present method<br>$n = 10^4$ | Present method<br>$n = 10^5$ |
|-----|------------------------------|------------------------------|------------------------------|------------------------------|
| 0.0 | 0                            | 0                            | 0                            | 0                            |
| 0.1 | $1.17 \times 10^{-5}$        | $1.24 \times 10^{-7}$        | $1.25 \times 10^{-9}$        | $1.25 \times 10^{-11}$       |
| 0.2 | $2.46 \times 10^{-5}$        | $2.53 \times 10^{-7}$        | $2.54 \times 10^{-9}$        | $2.54 \times 10^{-11}$       |
| 0.3 | $3.83 \times 10^{-5}$        | $3.90 \times 10^{-7}$        | $3.91 \times 10^{-9}$        | $3.91 \times 10^{-11}$       |
| 0.4 | $5.30 \times 10^{-5}$        | $5.38 \times 10^{-7}$        | $5.39 \times 10^{-9}$        | $5.39 \times 10^{-11}$       |
| 0.5 | $6.94 \times 10^{-5}$        | $7.03 \times 10^{-7}$        | $7.04 \times 10^{-9}$        | $7.04 \times 10^{-11}$       |
| 0.6 | $8.80 \times 10^{-5}$        | $8.89 \times 10^{-7}$        | $8.90 \times 10^{-9}$        | $8.90 \times 10^{-11}$       |
| 0.7 | $1.09 \times 10^{-4}$        | $1.10 \times 10^{-6}$        | $1.10 \times 10^{-8}$        | $1.10 \times 10^{-10}$       |
| 0.8 | $1.33 \times 10^{-4}$        | $1.35 \times 10^{-6}$        | $1.35 \times 10^{-8}$        | $1.35 \times 10^{-10}$       |
| 0.9 | $1.62 \times 10^{-4}$        | $1.63 \times 10^{-6}$        | $1.63 \times 10^{-8}$        | $1.63 \times 10^{-10}$       |
| 1.0 | $1.95 \times 10^{-4}$        | $1.97 \times 10^{-6}$        | $1.97 \times 10^{-8}$        | $1.97 \times 10^{-10}$       |

Таблица 5: Absolute errors of  $y_2(t)$  are presented for Problem 2.

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## EXAMPLES OF ADMISSIBLE CONVEX KERNELS FOR GENERALIZED POTENTIALS

NARZILLAEV NURBEK\*

NATIONAL UNIVERSITY OF UZBEKISTAN, TASHKENT, UZBEKISTAN  
n.narzillaev@nuu.uz

SUYUNOVA MADINA

NATIONAL UNIVERSITY OF UZBEKISTAN, TASHKENT, UZBEKISTAN  
madinasuyunova7476@gmail.com

## RESUME

We study concrete examples of admissible convex kernels arising in generalized potential theory. The kernels considered in this paper are positive, non-increasing, convex near the origin, singular at the origin, locally integrable with respect to the weight  $r dr$ , lower semicontinuous, and satisfy a strengthened local doubling condition of the form

$$k(qr) \leq Ak(r), \quad qA < 1.$$

Such assumptions appear naturally in Wiener-type estimates for generalized capacities and  $k$ -thin sets. We give detailed verifications for logarithmic kernels, power kernels of order less than one, iterated logarithmic kernels, and finite positive linear combinations of such kernels.

**Key words:** generalized potential, admissible kernel, convex kernel, logarithmic kernel,  $k$ -capacity, thin set, Wiener-type condition.

**MSC 2020:** 31A15, 31C15, 28A75.

## 1. Introduction

In classical potential theory in the complex plane, the logarithmic kernel plays a central role. Many fine properties of potentials, capacities, and thin sets are formulated in terms of the logarithmic potential. In generalized potential theory, the logarithmic kernel can be replaced by a more general positive kernel  $k(|z - w|)$ , where  $k$  is a singular function defined near the origin. Such a replacement is useful when one wants to study capacities and thinness associated with kernels having different rates of blow-up.

The aim of this paper is to present explicit examples of kernels satisfying the standard assumptions needed for generalized potentials. We consider positive functions which are non-increasing, convex near zero, singular at zero, locally integrable with respect to the measure  $r dr$ , lower semicontinuous, and satisfy a strengthened local doubling condition. The additional restriction  $qA < 1$  is important in applications to Wiener-type estimates, because it guarantees useful estimates for integrals of the kernel over small intervals.

The present note is motivated by the class of generalized kernels used in the necessary Wiener-type condition for generalized potentials studied in [8]. We do not prove a new Wiener criterion here. Instead, we give detailed elementary verifications for several concrete kernels satisfying the assumptions that arise in that theory.

The paper is organized as follows. In Section 2 we define the class of admissible convex kernels. In Section 3 we recall the associated generalized potential, energy, capacity, and thinness. In Sections 4–6 we verify the assumptions for logarithmic, power, and iterated logarithmic kernels. In Section 7 we prove stability under finite positive linear combinations. Section 8 contains concluding remarks.

## 2. Admissible Convex Kernels

We introduce the class of kernels considered throughout the paper.

**Definition 1.** Let  $a > 0$ . A positive function  $k : (0, a) \rightarrow (0, \infty)$  is called an *admissible convex kernel* if it satisfies the following conditions:

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\*Corresponding autor

K1  $k$  is non-increasing on  $(0, a)$  and convex in a neighborhood of 0;

K2

$$\lim_{r \rightarrow 0^+} k(r) = +\infty;$$

K3 there exists  $b \in (0, a)$  such that

$$\int_0^b rk(r) dr < \infty;$$

K4 there exist constants  $q \in (0, 1)$  and  $A > 0$  such that

$$k(qr) \leq Ak(r)$$

for all sufficiently small  $r > 0$ , and

$$qA < 1;$$

K5  $k$  is lower semicontinuous on  $(0, a)$ .

**Remark 1.** Condition (K4) is a strengthened local doubling condition near the origin. The inequality  $qA < 1$  is stronger than ordinary doubling. It is useful, for instance, for proving estimates of the form

$$\int_0^r k(t) dt \leq Crk(r),$$

which appear naturally in the study of one-dimensional density properties of thin sets.

When such a kernel is used in the definition of potentials on the whole complex plane, it may be extended to a positive lower semicontinuous function on  $(0, \infty)$ . In the present paper only the behavior of  $k$  near the origin is essential.

### 3. Generalized Potentials and Capacities

Let  $\mu$  be a finite positive Borel measure with compact support in  $\mathbb{C}$ . The generalized  $k$ -potential of  $\mu$  is defined by

$$U_k^\mu(z) = - \int_{\mathbb{C}} k(|z - w|) d\mu(w), \quad z \in \mathbb{C}.$$

The sign convention is chosen so that larger positive kernels correspond to more negative potentials.

Let  $E \subset \mathbb{C}$  be compact and denote by  $\mathcal{M}(E)$  the set of all probability measures supported on  $E$ . The  $k$ -energy of  $E$  is

$$W_k(E) = \inf_{\mu \in \mathcal{M}(E)} \int_E \int_E k(|z - w|) d\mu(z) d\mu(w).$$

The corresponding  $k$ -capacity is

$$C_k(E) = \frac{1}{W_k(E)},$$

with the usual convention that  $C_k(E) = 0$  if  $W_k(E) = +\infty$ .

**Definition 2.** A set  $E \subset \mathbb{C}$  is called  $k$ -thin at a point  $z_0 \in \overline{E}$  if there exists a finite positive Borel measure  $\mu$  with compact support such that  $U_k^\mu(z_0) > -\infty$  and

$$\limsup_{\substack{z \rightarrow z_0 \\ z \in E \setminus \{z_0\}}} U_k^\mu(z) < U_k^\mu(z_0).$$

If  $z_0 \notin \overline{E}$ , then  $E$  is called  $k$ -thin at  $z_0$  by convention.

For an admissible kernel, the natural Wiener-type series associated with a point  $z_0$  is

$$\sum_{n=1}^{\infty} k(q^n) C_k(E_n),$$

where

$$E_n = E \cap \{z \in \mathbb{C} : q^{n+1} \leq |z - z_0| \leq q^n\}.$$

In Wiener-type problems for generalized potentials, this series appears naturally as a necessary condition for  $k$ -thinness at  $z_0$ ; see [8]. The purpose of the present paper is not to prove such a criterion, but to exhibit concrete kernels for which the underlying assumptions hold.

#### 4. Logarithmic Kernels

We first consider powers of the logarithm.

**Proposition 1.** Let  $\alpha > 0$  and define

$$k(r) = |\log r|^\alpha, \quad 0 < r < a < 1.$$

Then  $k$  is an admissible convex kernel.

**Proof.** For  $0 < r < a < 1$ , one has  $|\log r| = -\log r$ , and therefore

$$k(r) = (-\log r)^\alpha.$$

A direct computation gives

$$k'(r) = -\frac{\alpha(-\log r)^{\alpha-1}}{r} < 0.$$

Thus  $k$  is non-increasing. Moreover,

$$k''(r) = \frac{\alpha(-\log r)^{\alpha-2}}{r^2} ((-\log r) + \alpha - 1).$$

Since  $-\log r \rightarrow +\infty$  as  $r \rightarrow 0^+$ , we obtain  $k''(r) > 0$  for all sufficiently small  $r > 0$ . Hence  $k$  is convex near the origin.

The singularity condition is immediate:

$$\lim_{r \rightarrow 0^+} |\log r|^\alpha = +\infty.$$

For the weighted integrability condition, put  $t = -\log r$ . Then  $r = e^{-t}$  and  $dr = -e^{-t} dt$ . Hence

$$\int_0^b r |\log r|^\alpha dr = \int_{-\log b}^\infty t^\alpha e^{-2t} dt < \infty,$$

because exponential decay dominates polynomial growth.

The function  $k$  is continuous on  $(0, a)$ ; hence it is lower semicontinuous on  $(0, a)$ .

It remains to check the strengthened doubling condition. Let  $q \in (0, 1)$  be fixed. Since

$$\frac{k(qr)}{k(r)} = \left( \frac{|\log(qr)|}{|\log r|} \right)^\alpha = \left( 1 + \frac{|\log q|}{|\log r|} \right)^\alpha,$$

we have

$$\lim_{r \rightarrow 0^+} \frac{k(qr)}{k(r)} = 1.$$

Choose  $A > 1$  so close to 1 that  $qA < 1$ . For sufficiently small  $r > 0$ , the above limit implies

$$k(qr) \leq Ak(r).$$

Thus (K4) holds, and  $k$  is admissible.

#### 5. Power Kernels

We next consider power-type singularities. The condition  $qA < 1$  imposes the restriction that the power exponent must be less than one.

**Proposition 2.** Let  $0 < \alpha < 1$  and define

$$k(r) = r^{-\alpha}, \quad r > 0.$$

Then  $k$  is an admissible convex kernel.

**Proof.** We have

$$k'(r) = -\alpha r^{-\alpha-1} < 0$$

and

$$k''(r) = \alpha(\alpha+1)r^{-\alpha-2} > 0.$$

Thus  $k$  is decreasing and convex on  $(0, \infty)$ . Since  $\alpha > 0$ ,

$$\lim_{r \rightarrow 0^+} r^{-\alpha} = +\infty.$$

Moreover,

$$\int_0^b r k(r) dr = \int_0^b r^{1-\alpha} dr < \infty,$$

because  $1 - \alpha > -1$ . In fact, this weighted integral is finite for every  $\alpha < 2$ , but the strengthened doubling condition below requires  $\alpha < 1$ .

The function  $k$  is continuous on  $(0, \infty)$ ; hence it is lower semicontinuous.

For  $q \in (0, 1)$ ,

$$k(qr) = (qr)^{-\alpha} = q^{-\alpha} k(r).$$

Thus the doubling condition holds with  $A = q^{-\alpha}$ . Since  $0 < \alpha < 1$ , we have

$$qA = q^{1-\alpha} < 1.$$

Therefore  $k$  is admissible.

**Remark 2.** If one only assumes the weaker doubling condition  $k(qr) \leq Ak(r)$  without requiring  $qA < 1$ , then the weighted integrability condition allows  $r^{-\alpha}$  for every  $0 < \alpha < 2$ . However, for the stronger class relevant to the Wiener-type estimates considered here, one needs  $0 < \alpha < 1$ .

## 6. Iterated Logarithmic Kernels

We now consider a slower singularity involving an iterated logarithm.

**Proposition 3.** Let  $\beta > 0$  and define

$$k(r) = |\log |\log r||^\beta$$

for all sufficiently small  $r > 0$ . Then  $k$  is an admissible convex kernel.

**Proof.** Take  $0 < r < e^{-1}$ . Then  $|\log r| = -\log r > 1$ , and hence

$$k(r) = (\log(-\log r))^\beta.$$

Put  $t = -\log r$ . Then  $t \rightarrow +\infty$  as  $r \rightarrow 0^+$  and  $k(r) = (\log t)^\beta$ . Differentiating gives

$$k'(r) = -\frac{\beta(\log t)^{\beta-1}}{rt}, \quad t = -\log r.$$

Thus  $k'(r) < 0$  for all sufficiently small  $r > 0$ , so  $k$  is non-increasing near the origin.

A second differentiation yields

$$k''(r) = \frac{\beta(\log t)^{\beta-2}}{r^2 t^2} [t \log t - \log t + \beta - 1], \quad t = -\log r.$$

Since  $t \log t - \log t + \beta - 1 \rightarrow +\infty$ , we obtain  $k''(r) > 0$  for all sufficiently small  $r > 0$ . Hence  $k$  is convex near zero.

The blow-up condition follows from

$$\lim_{r \rightarrow 0^+} \log(-\log r) = +\infty.$$

For integrability, using again  $t = -\log r$ , we obtain

$$\int_0^b r |\log |\log r||^\beta dr = \int_{-\log b}^\infty e^{-2t} (\log t)^\beta dt < \infty.$$

The function  $k$  is continuous for all sufficiently small  $r > 0$ ; hence it is lower semicontinuous there.

Finally, let  $q \in (0, 1)$  and set  $c = |\log q|$ . Then

$$|\log(qr)| = -\log r + c = t + c.$$

Thus

$$\frac{k(qr)}{k(r)} = \left( \frac{\log(t+c)}{\log t} \right)^\beta \rightarrow 1$$

as  $t \rightarrow +\infty$ . Choose  $A > 1$  so close to 1 that  $qA < 1$ . Then, for all sufficiently small  $r > 0$ ,

$$k(qr) \leq Ak(r).$$

Thus the strengthened doubling condition holds, and  $k$  is admissible.

## 7. Finite Positive Linear Combinations

The above examples can be combined to produce many further kernels.

**Proposition 4.** Let  $k_1, \dots, k_m$  be admissible convex kernels, and let  $c_1, \dots, c_m > 0$ . Then

$$k(r) = \sum_{j=1}^m c_j k_j(r)$$

is also an admissible convex kernel.

**Proof.** The positivity is clear. Since each  $k_j$  is non-increasing near the origin, their positive linear combination is also non-increasing near the origin. Since convexity is preserved under positive linear combinations,  $k$  is convex near the origin.

Since each  $k_j$  tends to  $+\infty$  as  $r \rightarrow 0^+$ , we have

$$\lim_{r \rightarrow 0^+} k(r) = +\infty.$$

Moreover, if

$$\int_0^b r k_j(r) dr < \infty, \quad j = 1, \dots, m,$$

then

$$\int_0^b r k(r) dr = \sum_{j=1}^m c_j \int_0^b r k_j(r) dr < \infty.$$

Lower semicontinuity is also preserved under finite positive linear combinations.

It remains to verify the strengthened doubling condition. For each  $j$ , choose constants  $q_j \in (0, 1)$  and  $A_j > 0$  such that

$$k_j(q_j r) \leq A_j k_j(r), \quad q_j A_j < 1,$$

for all sufficiently small  $r > 0$ . Choose a common number  $q \in (0, 1)$  sufficiently small. For each  $j$ , let  $n_j$  be an integer such that

$$q_j^{n_j+1} \leq q < q_j^{n_j}.$$

For sufficiently small  $q$ , we may assume

$$q A_j^{n_j+1} < 1, \quad j = 1, \dots, m,$$

because

$$q A_j^{n_j+1} < q_j^{n_j} A_j^{n_j+1} = A_j (q_j A_j)^{n_j} \rightarrow 0$$

as  $q \rightarrow 0$  and hence  $n_j \rightarrow \infty$ .

Since  $q \geq q_j^{n_j+1}$  and  $k_j$  is non-increasing, we get

$$k_j(qr) \leq k_j(q_j^{n_j+1}r) \leq A_j^{n_j+1}k_j(r)$$

for all sufficiently small  $r > 0$ . Put

$$A = \max_{1 \leq j \leq m} A_j^{n_j+1}.$$

Then  $qA < 1$ , and

$$k(qr) = \sum_{j=1}^m c_j k_j(qr) \leq \sum_{j=1}^m c_j A_j^{n_j+1} k_j(r) \leq A \sum_{j=1}^m c_j k_j(r) = Ak(r).$$

Thus (K4) holds, and  $k$  is admissible.

**Remark 3.** In particular, one may combine logarithmic, power-type, and iterated logarithmic kernels. For example,

$$k(r) = c_1 |\log r|^\alpha + c_2 r^{-\gamma} + c_3 |\log |\log r||^\beta, \quad \alpha, \beta > 0, \quad 0 < \gamma < 1,$$

with  $c_1, c_2, c_3 > 0$ , is an admissible convex kernel for all sufficiently small  $r > 0$ .

## 8. Conclusion

We have verified several explicit classes of admissible convex kernels suitable for generalized potential theory. In particular, logarithmic kernels  $|\log r|^\alpha$ ,  $\alpha > 0$ , power kernels  $r^{-\alpha}$ ,  $0 < \alpha < 1$ , and iterated logarithmic kernels  $|\log |\log r||^\beta$ ,  $\beta > 0$ , satisfy the required monotonicity, convexity, singularity, integrability, lower semicontinuity, and strengthened doubling conditions near the origin. We also showed that the class is stable under finite positive linear combinations. These examples provide a useful supply of kernels for applications to generalized capacities, thin sets, and Wiener-type estimates.

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**INVERSE PROBLEM FOR THE TIME-FRACTIONAL LINEARIZED KURAMOTO-SIVASHINSKY  
EQUATION IN THE CLASS OF DECREASING FUNCTIONS**

**NORKULOVA RUSHANA**

V.I. ROMANOVSKIY INSTITUTE OF MATHEMATICS,  
UZBEKISTAN ACADEMY OF SCIENCES. TASHKENT  
norkulovarushana1988@gmail.com

**ABSTRACT.** We consider an inverse problem associated with the linearized time-fractional Kuramoto-Sivashinsky equation within the Schwartz space of rapidly decreasing functions. The goal is to determine the unknown time-dependent component of the source term, utilizing an integral overdetermination condition. The existence of a unique solution to the corresponding direct problem is established through the application of the Fourier transform. To address the inverse problem, it is reformulated as an operator equation. The unique solvability of this problem is demonstrated by proving the boundedness of the associated resolvent operator, which maps from  $C[0, T]$  to  $C[0, T]$ . This ensures the well-posedness of the inverse problem under the given framework.

**MSC (2020):** 35R11; 35R30; 35K05; 35G05.

**Key words:** Time-fractional Kuramoto-Sivashinsky equation, inverse problem, Caputo fractional derivative, resolvent operator, quasinilpotent operator, existence and uniqueness.

## 1. Introduction

Over the past decade, focuses on detailed interest have been devoted to fractional-order generalizations of classical evolution equations. Replacing the integer-order time derivative with the Caputo fractional derivative introduces the memory effect into the model, that is, the current state of the system depends not only on its rapid structure, but on its entire past history. This hereditary property renders fractional-order models particularly well suited to describe anomalous diffusion, viscoelastic behavior, and subdiffusive processes that arise in complex physical, biological, and engineering systems. Research on direct and inverse problems for fractional differential equations has provided important opportunities for the analysis and control of various natural phenomena. Studying inverse problems allows us to investigate and regulate physical processes described by fractional differential equations.

We should note contemporary creations associated to some inverse problems connected with fourth order parabolic equations. In [7] K.Cao investigated a fourth order parabolic PDE and established the inverse source identification problem associated with it. More accurately, the author studied the inverse problem of identification of the source function  $f(x)$  from observation  $v(\cdot, \tau)$  or the time-integral observation  $\int_0^\tau \omega(t)u(x, t)dt$ .

The paper by Mansur I. Ismailov [15] examines an inverse problem based on the Mullins surface grooving equation in a finite domain, where the time-dependent surface diffusivity  $D$  is unknown. Using Fourier series analysis, the Mullins coefficient is recovered from total mass data under self-adjoint spectral boundary conditions. In Ibrahim Tekin and He Yang's paper [25] the inverse coefficient problem for the time-fractional Euler-Bernoulli beam equation is investigated: leveraging and additional integral measurement condition, the existence and uniqueness of the solution to the problem of improving the unfamiliar time-dependent coefficient is established. The theoretical results are validated through numerical experiments.

The paper [14] considers the inverse problem for a fourth-order parabolic equation with a complex-valued coefficient. Using the method of separation of variables, the existence and uniqueness of the solution of the Dirichlet and Neumann boundary value problems for a non-self-adjoint differential operator in space  $L_2(-1, 1)$  are investigated.

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In [1], an inverse problem of determining a time dependent coefficient is investigated for the elastic beam equation. By applying the Fourier method and the contraction mapping principle, the existence and uniqueness of the solution are established.

The author's paper [16] investigates the inverse problem of determining the time-dependent heat capacity coefficient in addition to the solution of the heat equation under nonlocal boundary conditions. Assuming suitable regularity and compatibility conditions on the given data, the existence, uniqueness, and stability of the solution with respect to the input data are established.

This article [19] proposes a convenient approximate method for solving the complex mathematical equation describing beam vibrations-the Bernoulli-Euler equation. The method of based on special Trefftz functions, which are used to effectively solve both direct problems and inverse problems-that is, cases where boundary conditions or the load acting on the beam are unknown.

This paper [21] establishes the existence and uniqueness of the solution to the fractional diffusion equation. Moreover, it shows that the initial condition and source term can be recovered from the observation data. Ashurov and Shakarova had studied the inverse problems of determining the time dependent or spatially dependent source function for the subdiffusion equation under different observation conditions ([3], [4]). Durdiyev and Jumayev [8] investigated the inverse problem of determining an unknown kernel in integro-differential diffusion equations with a fractional time derivative and obtained the local existence and global uniqueness of the solution. In these papers ([6],[22],[23]) the inverse problem for subdiffusion equation on a metric star graph is transformed into an operator equation, and the existence and continuity of the associated resolvent operator are established.

Most existing studies focus on fourth-order parabolic equations or low-order models. Inverse problems with integral conditions for the linearized Kuramoto-Sivashinsky equation have not been studied yet. This paper studies the inverse problem in the class of decreasing functions, where the right-hand side has the form  $f(t)g(x, t)$ . The corresponding Cauchy problem is also investigated. Therefore, this paper presents a detailed study of the inverse problem for the linearized Kuramoto-Sivashinsky equation.

$$\partial_t^\beta u + a^2 u_{xxxx} + bu_{xxx} + cu_{xx} + du_x + pu = f(t)g(x, t) + h(x, t), \quad x \in \mathbb{R}, \quad 0 < t \leq T, \quad (1.1)$$

with initial condition

$$u(x, 0) = \varphi(x), \quad x \in \mathbb{R}, \quad 0 \leq t \leq T, \quad (1.2)$$

and integral overdetermination condition

$$\int_{-\infty}^{+\infty} \eta(x)u(x, t)dx = \psi(t), \quad 0 \leq t \leq T, \quad (1.3)$$

where  $0 < \beta < 1$ ;  $a > 0$ ,  $b, c, d$ , and  $p$  are real numbers,  $\eta(x)$  and  $\psi(t)$  are given functions,  $\varphi(x) \in S(\mathbb{R})$ .

Inverse source problems of this type arise naturally in a wide range of physical and engineering applications, where the source term driving the system cannot be accessed through direct measurement and must therefore be reconstructed from indirect observations. The additional integral condition represents a weighted spatial average of the system's state, which constitutes a physically meaningful and practically relevant type of observation data. The problem is formulated in the Schwartz space  $S(\mathbb{R})$  in the class of infinitely differentiable, rapidly decreasing functions. This functional framework is particularly well adapted to the analysis of problems posed on the whole real line. These results extend the existing theory and contribute new developments to the theory of inverse problems for higher-order differential equations.

The remainder of the paper is structured as follows. Section 2 is committed to the following necessary definitions and important lemmas for our framework. In section 3 we proved the existence- uniqueness of a solution for the direct problem. In section 4 we proved the existence-uniqueness and stability of a solution for the inverse problem. The conclusions are provided in the final section.

## 2. Preliminaries

In this section, we recall the definitions and properties that will be used in the remainder of the paper.

### 2.1.The Schwartz space.

**Definition 2.1.**([13], [24],p.135) The Schwartz space  $S(\mathbb{R})$  is the topological vector space of functions  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that  $g \in C^\infty(\mathbb{R})$  and

$$\sup_{x \in \mathbb{R}} |x|^k \left| \frac{d^n g(x)}{dx^n} \right| < \infty, \quad (2.1)$$

for every  $k, n \in \mathbb{N} \cup \{0\}$ .

We present the space of continuous functions  $\mathbf{u}(t) = u(\cdot, t)$  as  $SC(T) = C([0, T], S(\mathbb{R}))$ . Following [26] we introduce the topological space  $SC_\beta(T) = \{u \in SC(T) : \partial_{0t}^\beta u \in SC(T)\}$ . The Schwartz space is a natural one to use for the Fourier transform. Differentiations and multiplication exchange roles under the Fourier transform due to the properties of smoothness and rapid decrease. As a result, the Fourier transform is an automorphism of the Schwartz space.

## 2.2.Fourier transform.

**Definition 2.2.**([13], [24]) The Fourier transform of a function  $f \in L^1(\mathbb{R})$  is a function  $\hat{f} : \mathbb{R} \rightarrow \mathbb{C}$  defined by

$$\hat{f}(\lambda) \equiv \mathbf{F}[f](\lambda) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\lambda x} dx. \quad (2.2)$$

The inverse Fourier transform defined by

$$f(x) \equiv \mathbf{F}^{-1}[\hat{f}](x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{f}(\lambda) e^{i\lambda x} d\lambda, \quad (2.3)$$

provided that the right-hand side exists. We use  $x$  to denote the spatial variable and  $\lambda$  the variable in the Fourier transform.

**Theorem 2.3.**([24],p.137) The Fourier transform and its inverse map the space  $S$  on to itself in a one-to-one, linear and continuous manner.

## 2.3.Fractional integral and derivatives.

The Riemann-Liouville fractional integral of order  $\beta > 0$ , of a function  $g$  is determined by (see [17], p. 69)

$$I_{0t}^\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{g(\xi)}{(t-\xi)^{1-\beta}} d\xi, \quad t > 0,$$

provided that the integral on the right-hand side of equality exists.

If  $f \in L^p(0, T)$ ,  $1 \leq p \leq +\infty$ , then (see Lemma 2.3 in [17])

$$(I_{0t}^\alpha I_{0t}^\beta f)(t) = (I_{0t}^{\alpha+\beta} f)(t).$$

For  $0 < \beta < 1$  an operator defined by

$$\partial_{0t}^\beta g(t) = \frac{d}{dt} I_{0t}^{1-\beta} (g(t) - g(0)) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{g(\xi) - g(0)}{(t-\xi)^\beta} d\xi, \quad t > 0, \quad (2.4)$$

is called the Caputo fractional derivative, where  $\Gamma(\beta) = \int_0^{+\infty} x^{\beta-1} e^{-x} dx$  is Euler's Gamma function. Here we suppose that the right-hand side of the equality (2.4) exists.

For absolute continuous functions on  $[0, T]$  one has [17]

$$\partial_{0t}^\beta g(t) = I_{0t}^{1-\beta} \frac{d}{dt} g(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{g'(\xi)}{(t-\xi)^\beta} d\xi, \quad t > 0.$$

Next, we introduce the two-parameter function of the Mittag-Leffler type, which plays a very important role in the fractional calculus.

#### 2.4. The Mittag-Leffler function.

**Definition 2.4.** ([11], [20]) For  $0 < \alpha < 1$ , a two-parameter function of the Mittag-Leffler type is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}. \quad (2.5)$$

If  $\beta = 1$ , then the Mittag-Leffler function is called the one-parameter or classical Mittag-Leffler function and it is denoted by  $E_{\alpha}(z) = E_{\alpha,1}(z)$ .

**Theorem 2.5.** ([12], [20], p.58) If  $0 < \alpha < 1$  and  $\mu$  is a real number such that

$$\frac{\pi\alpha}{2} < \mu < \min\{\pi, \pi\alpha\}. \quad (2.6)$$

Then we have the following asymptotic expansion:

$$E_{\alpha,\beta}(z) = - \sum_{r=1}^n \frac{1}{\Gamma(\beta - \alpha r)} \frac{1}{z^r} + O\left[\frac{1}{z^{n+1}}\right], \quad (2.7)$$

as  $|z| \rightarrow \infty$ ,  $\mu \leq |\arg z| \leq \pi$ , where  $\beta \in \mathbb{C}$ ,  $n \in \mathbb{N}$ .

The following differentiation formula is an immediate consequence of the definition of the two-parametric Mittag-Leffler function.

$$\left(\frac{d}{dz}\right)^m [z^{\beta-1} E_{\alpha,\beta}(z^{\alpha})] = z^{\beta-m-1} E_{\alpha,\beta-m}(z^{\alpha}), \quad m \geq 1. \quad (2.8)$$

**Lemma 2.6.** ([9], p.136) Let  $0 < \alpha < 2$  and  $\mu$  is defined by (2.6). Then for any  $\beta, z \in \mathbb{C}$  with  $\mu \leq |\arg z| \leq \pi$  one has

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1 + |z|}, \quad (2.9)$$

where  $C$  is constant.

#### 2.5. Fundamental inequalities.

Now we present several fundamental inequalities that are frequently employed in our analysis ([10], p.706).

The following two statements, originally founded by Alikhanov, are next presented in a form suitable for our framework.

**Lemma 2.7.** [2] For any function  $v(t)$  absolutely continuous on  $[0, T]$ , one has the inequality

$$v(t) \partial_{0t}^{\beta} v(t) \geq \frac{1}{2} \partial_{0t}^{\beta} v^2(t), \quad 0 < \beta < 1, \quad (2.10)$$

**Lemma 2.8.** [2] Let a nonnegative absolutely continuous function  $y(t)$  satisfy the inequality

$$\partial_{0t}^{\beta} y(t) \leq c_1 y(t) + c_2(t), \quad 0 < \beta \leq 1, \quad (2.11)$$

for almost all  $t$  in  $[0, T]$ , where  $c_1 > 0$  and  $c_2(t)$  is an integrable nonnegative function on  $[0, T]$ . Then

$$y(t) \leq y(0) E_{\beta}(c_1 t^{\beta}) + \Gamma(\beta) E_{\beta,\beta}(c_1 t^{\beta}) I_{0t}^{\beta} c_2(t). \quad (2.12)$$

### 3. THE DIRECT PROBLEM

We represent the solution of problem (1.1)-(1.2) as  $u(x, t) = v(x, t) + w(x, t)$ , where  $w$  satisfies the following equation.

$$\partial_{0t}^{\beta} w + a^2 w_{xxxx} + b w_{xxx} + c w_{xx} + d w_x + p w = h(x, t), \quad x \in \mathbb{R}, \quad 0 < t \leq T, \quad (3.1)$$

with initial condition

$$w(x, 0) = \varphi(x), \quad x \in \mathbb{R}, \quad 0 \leq t \leq T, \quad (3.2)$$

where  $0 < \beta < 1$ ,  $a > 0$ ,  $b, c, d$ , and  $p$  are real numbers.

The problem (3.1)-(3.2) is called the direct problem.

We obtained an exact analytical solution of the direct problem by employing the Fourier transform method and properties of the Caputo fractional derivative ([5], [17]).

**Theorem 3.1** Let  $\varphi(x) \in S(\mathbb{R})$  and  $h(t, x) \in SC(T)$ , there is a unique solution to the Cauchy problem (3.1)-(3.2) in the class of functions  $SC_\beta(T)$ .

**Proof.** We prove that the Cauchy problem has a solution using Fourier transforms. Applying the Fourier transform on  $x$  (2.2), we get the following result

$$\partial_{0t}^\beta \hat{w}(\lambda, t) + P(\lambda) \hat{w}(\lambda, t) = \hat{h}(\lambda, t), \quad (3.3)$$

$$\hat{w}(\lambda, 0) = \hat{\varphi}(\lambda), \quad (3.4)$$

where  $P(\lambda) = a^2 \lambda^4 - ib \lambda^3 - c \lambda^2 + id \lambda + p$ , functions  $\hat{w}(\lambda, t)$ ,  $\hat{h}(\lambda, t)$ ,  $\hat{\varphi}(\lambda)$  are images of functions  $w(x, t)$ ,  $h(t, x)$ ,  $\varphi(x)$ , respectively.

The unique solution to the Cauchy problem (3.3)-(3.4) is as follows ([17], p.141)

$$\hat{w}(\lambda, t) = \hat{\varphi}(\lambda) \cdot E_{\beta,1}(-P(\lambda)t^\beta) + \int_0^t \hat{h}(\lambda, \tau) (t-\tau)^{\beta-1} E_{\beta,\beta}(-P(\lambda)(t-\tau)^\beta) d\tau. \quad (3.5)$$

It is clear that the real part of the function  $P(\lambda)$  is positive for large  $|\lambda|$ . According to (2.8) we have

$$\left| \frac{d^n}{dz^n} (z^{\beta-1} E_{\beta,\beta}(z^\beta)) \right| = |z^{\beta-n-1} E_{\beta,\beta-n}(z^\beta)| \leq \frac{C}{1+|z|^{1+n}}, \quad (3.6)$$

for large  $|z|$  and  $Re(z) < 0$ .

The derivatives of the function  $E_{\beta,1}(z)$  can be estimated in a similar way. So, we can find that the derivatives of the Mittag-Leffler functions  $E_{\beta,\beta}(z)$  and  $E_{\beta,1}(z)$  are bounded if  $Re(z) < 0$ .

So, taking into account  $\hat{\varphi}(\lambda) \in S(\mathbb{R})$  and  $\hat{h}(\lambda, t) \in SC[0, T]$  from the representation (3.5), we resolve  $\hat{w} \in SC[0, T]$ . According to equation (3.3), it follows that  $\partial_{0t}^\beta \hat{w} = -P(\lambda) \hat{w} + \hat{h}(\lambda, t)$ . The right hand side of this equality is in  $SC(T)$ . So,  $\partial_{0t}^\beta \hat{w} \in SC(T)$ , and consequently,  $\hat{w} \in SC_\beta(T)$ .

Now, based on the properties of direct and inverse Fourier transforms (Theorem 2.3), we conclude that the Cauchy problem (3.1)-(3.2) has a unique solution  $w(x, t) \in SC_\beta(T)$ .

#### 4. SOLVABILITY OF THE INVERSE PROBLEM

Now, we consider the following inverse problem of finding the pair of functions  $(v, f)$ :

$$\partial_{0t}^\beta v(x, t) + a^2 v_{xxxx}(x, t) + b v_{xxx}(x, t) + c v_{xx}(x, t) + d v_x(x, t) + p v(x, t) = f(t) g(x, t) \quad (4.1)$$

with initial conditions

$$v(x, 0) = 0, \quad (4.2)$$

and an additional condition given by

$$\int_{-\infty}^{+\infty} \eta(x) v(x, t) dx = E(t) := \psi(t) - \int_{-\infty}^{+\infty} \eta(x) w(x, t) dx, \quad 0 < t \leq T. \quad (4.3)$$

We multiply both sides of equation (4.1) by the function  $\eta(x)$  and integrate over  $\mathbb{R}$ , we obtain

$$\begin{aligned} & \partial_{0t}^\beta \int_{-\infty}^{+\infty} \eta(x) v(x, t) dx + a^2 \int_{-\infty}^{+\infty} \eta(x) v_{xxxx}(x, t) dx + b \int_{-\infty}^{+\infty} \eta(x) v_{xxx}(x, t) dx \\ & + c \int_{-\infty}^{+\infty} \eta(x) v_{xx}(x, t) dx + d \int_{-\infty}^{+\infty} \eta(x) v_x(x, t) dx + p \int_{-\infty}^{+\infty} \eta(x) v(x, t) dx = f(t) \int_{-\infty}^{+\infty} \eta(x) g(x, t) dx. \end{aligned} \quad (4.4)$$

We denote

$$g^*(t) = \int_{-\infty}^{+\infty} \eta(x)g(x,t)dx, \quad \max_t |g^*(t)| = m > 0.$$

After introducing the notation, we have the following second kind  $f(t)$  operator equation

$$f(t) = (Bf)(t) + \frac{\partial_{0t}^\beta E(t)}{g^*(t)}, \quad (4.5)$$

where  $E(t)$  is defined in (4.3) and

$$\begin{aligned} (Bf)(t) = \frac{1}{g^*(t)} & \left[ a^2 \int_{-\infty}^{+\infty} \eta(x)v_{xxxx}dx + b \int_{-\infty}^{+\infty} \eta(x)v_{xxx}dx \right. \\ & \left. + c \int_{-\infty}^{+\infty} \eta(x)v_{xx}dx + d \int_{-\infty}^{+\infty} \eta(x)v_xdx + p \int_{-\infty}^{+\infty} \eta(x)vdx \right]. \end{aligned} \quad (4.6)$$

**Theorem 4.1.** Let  $\eta \in L^2(\mathbb{R})$ ,  $E, \partial_{0t}^\beta E \in C[0, T]$ ,  $g \in SC(T)$ ,  $g^*(t) > 0$ , then the inverse problem (4.1)-(4.3) has a unique solution  $(v, f) \in SC_\beta(T) \times C[0, T]$  and the following stability estimate holds

$$\|f\|_{C[0, T]} \leq M \|\partial_{0t}^\beta E\|_{C[0, T]},$$

where  $M$  is a positive constant.

**Proof.** We prove that equation (4.5) admits a solution. We need to study the operator  $B$ . The operator  $B : C[0, T] \rightarrow C[0, T]$ . Indeed, if  $f \in C[0, T]$ , then, by Theorem 3.1,  $v \in SC_\beta[0, T]$ . Since the function  $v$  and its derivatives in the expression under the integral of (4.6) belong to the class  $SC(T)$ , they are continuous in  $t$ . Hence, the integral in (4.6) is uniformly convergent with respect to  $t$ . Therefore,  $Bf \in C[0, T]$ .

Now we need to obtain some a-priori estimates. For this, first we prove the following lemma.

**Lemma 4.2.** Let  $T > 0$  and hold the conditions of Theorem 4.1. Then

$$\left\| \frac{\partial^n}{\partial x^n} v(\cdot, t) \right\|^2 \leq C\Gamma(\beta)E_{\beta, \beta}(\tilde{D}t^\beta)I_{0t}^\beta |f(t)|^2,$$

for all  $0 \leq t \leq T$ , where  $n$  is non-integer positive number,  $C$  is a positive constant, and  $\tilde{D} = \frac{2+a^2}{a^2}$ .

Let us now carry out an induction on the order of the derivative. We put

$$|||v|||_n^2 = \int_{-\infty}^{+\infty} \left( \frac{d^n v(x)}{dx^n} \right)^2 dx,$$

and  $|||v|||_0 = \|v\|_{L^2(\mathbb{R})}$ .

Now we can take the  $n$ th derivative of each term of the equation with respect to  $x$ , then multiply by  $2\frac{\partial^n v}{\partial x^n}$ , and integrate with respect to  $x$  in  $\mathbb{R}$ .

$$2 \int_{-\infty}^{+\infty} \partial_{0t}^\beta \frac{\partial^n v}{\partial x^n} \frac{\partial^n v}{\partial x^n} dx + 2 \int_{-\infty}^{+\infty} \frac{\partial^n}{\partial x^n} Lv \frac{\partial^n v}{\partial x^n} dx = 2f(t) \int_{-\infty}^{+\infty} \frac{\partial^n g}{\partial x^n} \frac{\partial^n v}{\partial x^n} dx \quad (4.7)$$

where  $Lv = a^2 v_{xxxx} + bv_{xxx} + cv_{xx} + dv_x + pv$ .

Now we estimate the terms on the left-hand side of this equality.

$$2 \int_{-\infty}^{+\infty} \frac{\partial^n v}{\partial x^n} \partial_{0t}^\beta \frac{\partial^n v}{\partial x^n} dx \geq \partial_{0t}^\beta \int_{-\infty}^{+\infty} \left( \frac{\partial^n v}{\partial x^n} \right)^2 dx = \partial_{0t}^\beta |||v|||_n^2, \quad (4.8)$$

$$2a^2 \int_{-\infty}^{+\infty} \frac{\partial^{n+4} v}{\partial x^{n+4}} \frac{\partial^n v}{\partial x^n} dx = 2a^2 \int_{-\infty}^{+\infty} \left( \frac{\partial^{n+2} v}{\partial x^{n+2}} \right)^2 dx = 2a^2 |||v|||_{n+2}^2, \quad (4.9)$$

$$\int_{-\infty}^{+\infty} \frac{\partial^{n+3} v}{\partial x^{n+3}} \frac{\partial^n v}{\partial x^n} dx = 0, \quad \int_{-\infty}^{+\infty} \frac{\partial^{n+1} v}{\partial x^{n+1}} \frac{\partial^n v}{\partial x^n} dx = 0, \quad (4.10)$$

$$\left| 2 \int_{-\infty}^{+\infty} \frac{\partial^{n+2} v}{\partial x^{n+2}} \frac{\partial^n v}{\partial x^n} dx \right| = 2 |||v|||_{n+1}^2 \leq \sigma |||v|||_{n+2}^2 + \frac{1}{\sigma} |||v|||_n^2. \quad (4.11)$$

The last integral on the right-hand side of (4.7) is estimated using the Cauchy inequality:

$$\left| 2 \int_{-\infty}^{+\infty} f(t) \frac{\partial^n g}{\partial x^n} \frac{\partial^n v}{\partial x^n} dx \right| \leq (f(t))^2 \int_{-\infty}^{+\infty} \left( \frac{\partial^n g}{\partial x^n} \right)^2 dx + \int_{-\infty}^{+\infty} \left( \frac{\partial^n v}{\partial x^n} \right)^2 dx. \quad (4.12)$$

Summarizing the above estimate, using the appropriate choice of  $\sigma = \frac{a^2}{2}$ , we have

$$\partial_{0t}^\beta |||v|||_n^2 \leq \tilde{D} |||v|||_n^2 + C|f(t)|^2. \quad (4.13)$$

Using Gronwall's inequality (see Lemma 2.8) we rich the desired estimate

$$|||v|||_n^2 \leq C\Gamma(\beta)E_{\beta,\beta}(\tilde{D}t^\beta)I_{0t}^\beta |f(t)|^2. \quad (4.14)$$

Now we study the properties of the operator  $B$ . For that, we need the following estimates.

$$I_k = \int_{-\infty}^{+\infty} \eta(x) \frac{\partial^k v(x, t)}{\partial x} dx, \quad k = 0, 1, 2, 3, 4.$$

Using Cauchy's inequality, we obtain the following estimate for each  $I_k$

$$|I_k| \leq \|\eta\|_{L^2} \cdot \left\| \frac{\partial^k v}{\partial x} \right\|_{L^2} \leq C(T) \sqrt{I_{0t}^\beta |f(t)|^2}. \quad (4.15)$$

Using the definition of  $Bf$  and expression (4.15), we obtain the following

$$|Bf|^2 \leq \frac{5}{m} \left( \sum_{k=0}^4 I_k^2 \right) \leq \tilde{C} I_{0t}^\beta |f(t)|^2.$$

Second iteration

$$|B^2 f|^2 = |B(B(f))|^2 \leq \tilde{C} I_{0t}^\beta |Bf|^2 \leq \tilde{C} I_{0t}^\beta \circ (\tilde{C} I_{0t}^\beta |f(t)|^2) \leq \tilde{C}^2 I_{0t}^{2\beta} |f(t)|^2.$$

Continuing in the same manner, we repeat this process  $n$  times. From this we find the norm of the operator  $B^n$ .

$$|B^n f|^2 = |B(B^{n-1} f)|^2 \leq \tilde{C}^n I_{0t}^{n\beta} |f(t)|^2. \quad (4.16)$$

Next, we estimate the norm of  $Bf$ .

$$\|Bf\|_{C[0,T]}^2 \leq \tilde{C} I_{0t}^\beta |f|^2 \leq \tilde{C} \max_{0 \leq t \leq T} |f|^2 \cdot I_{0t}^\beta 1 \leq \frac{\tilde{C} T^\beta}{\Gamma(\beta+1)} \|f\|_{C[0,T]}^2. \quad (4.17)$$

Now, iterating the inequality (4.16)

$$\|B^n f\|_{C[0,T]}^2 \leq \frac{\tilde{C}^n T^{n\beta}}{\Gamma(n\beta+1)} \cdot \|f\|_{C[0,T]}^2.$$

By definition

$$\|B^n\| = \sup_{\|f\|=1} \frac{\|B^n f\|_{C[0,T]}}{\|f\|_{C[0,T]}} \leq \sqrt{\frac{\tilde{C}^n T^{n\beta}}{\Gamma(n\beta + 1)}}.$$

According to Stirling's formula,

$$\sqrt{\frac{\tilde{C}^n T^{n\beta}}{\Gamma(n\beta + 1)}} \sim \sqrt{\frac{(\tilde{C} T^\beta)^n}{\sqrt{2\pi n\beta} \cdot \left(\frac{n\beta}{e}\right)^{n\beta}}}.$$

Now, let us calculate the spectral radius

$$r(B) = \lim_{n \rightarrow +\infty} \sqrt[n]{\|B^n\|} \leq \lim_{n \rightarrow +\infty} \frac{\tilde{C} T^{\frac{\beta}{2}}}{(2\pi\beta)^{\frac{1}{4n}} \sqrt[n]{n} \left(\frac{n\beta}{e}\right)^{\frac{\beta}{2}}} = 0.$$

Therefore,  $B$  is a quasinilpotent operator [18]. Consequently, the spectrum of the operator  $B$  consists only of the point  $\{\lambda = 0\}$ , so it follows that the resolvent operator  $(I - B)^{-1}$  exists. Hence,

$$\|(I - B)^{-1} f\|_{C[0,T]} \leq \sum_{k=0}^{+\infty} \sqrt{\frac{\tilde{C}^k T^{k\beta}}{\Gamma(k\beta + 1)}} \|f\|_{C[0,T]}.$$

So, summarizing the above, the resolvent operator

$$(I - B)^{-1} : C[0, T] \rightarrow C[0, T]$$

is bounded and continuous. From this, according to (4.5),  $f$  can be determined

$$f = (I - B)^{-1} \frac{\partial_{0t}^\beta E}{g^*}.$$

Then, according to the estimate for the resolvent operator, we have

$$\|f\|_{C[0,T]} \leq M \|\partial_{0t}^\beta E\|_{C[0,T]},$$

where  $M = \sum_{k=0}^{+\infty} \sqrt{\frac{\tilde{C}^k T^{k\beta}}{\Gamma(k\beta + 1)}} < +\infty$ , since  $r(B) = 0$ .

We proved the existence of unique  $f \in C[0, T]$ . The existence of unique  $v \in SC_\beta(T)$  follows from Theorem 3.1. Hence, the theorem is proved.

## 5. CONCLUSION

In this work we investigated the inverse problem of determining the unknown time-dependent component of the source term in the linearized Kuramoto-Sivashinsky equation, subject to an integral overdetermination condition. The existence of a unique solution to the corresponding direct problem is established through the application of the Fourier transform. To solve the inverse problem, it is reformulated as an operator equation. Using a-priori estimates, it is demonstrated that the resolvent operator, defined with the aid of the overdetermination condition, is well-posed. These findings contribute significantly to the theoretical understanding of the Kuramoto-Sivashinsky equation and lay a robust foundation for future studies, particularly in exploring the nonlinear case.

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## ASYMPTOTICALLY EFFICIENT RECURSIVE ESTIMATION FOR INCOMPLETE SAMPLES UNDER RIGHT CENSORING

NURMUKHAMEDOVA NARGIZA\*

DEPARTMENT OF ECONOMETRICS & ECONOMICAL MODELLING,  
NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M.ULUGBEK, TASHKENT, UZBEKISTAN  
n.nurmuhamedova@nuu.uz

BERDIMURADOV MIRKAMOL

DEPARTMENT OF APPLIED MATHEMATICS & INTELLECTUAL TECHNOLOGIES,  
NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M.ULUGBEK, TASHKENT, UZBEKISTAN  
mirkamol1403@gmail.com

**ABSTRACT.** This paper is devoted to Fisher-type recursive inference for parametric statistical models under incomplete observations arising from Type-I right censoring. We consider a one-parameter family of distributions and construct a recursive estimation procedure based on successive contributions of the censored score function normalized by the corresponding Fisher information. The proposed algorithm belongs to the class of stochastic approximation methods with a decreasing step size and can be interpreted as a recursive analogue of the classical Fisher scoring method adapted to incomplete data. The asymptotic properties of the recursive estimator are discussed under standard regularity conditions, including almost sure convergence and asymptotic normality.

**MSC (2020):** 62F12; 62F10; 62N01; 62L12.

**Key words:** Fisher information, recursive estimation, Fisher scoring, stochastic approximation, Type-I right censoring, strong consistency, asymptotic normality, asymptotic efficiency.

## Introduction

*Background and Related Works.* The problem of estimating unknown parameters of probability distributions is one of the fundamental topics of mathematical statistics. A wide variety of estimation methods has been developed, among which the maximum likelihood method occupies a central position due to its strong theoretical properties. In particular, under standard regularity conditions, maximum likelihood estimators are known to be asymptotically efficient and to attain the Cramer-Rao lower bound [1–4].

However, in many practically relevant situations, the likelihood equations do not admit closed-form solutions. This difficulty becomes especially pronounced in incomplete statistical models, such as models with censored observations, truncated samples, or missing data. In such cases, numerical and recursive procedures play a crucial role in constructing feasible estimators [5–7].

One of the classical approaches to recursive estimation is based on stochastic approximation methods. The foundational theory of stochastic approximation and recurrent estimation was developed in the seminal monograph by Nevelson and Khasminskii [1], where convergence properties and asymptotic behavior of recursive estimators were systematically investigated. A prominent example of such recursive procedures is the Fisher scoring method, which can be viewed as a stabilized variant of the Newton-Raphson algorithm using Fisher information instead of the observed Hessian [2,8,9].

Recursive estimation techniques are particularly attractive in problems where data arrive sequentially or where re-estimation based on the full sample is computationally expensive. Moreover, recursive estimators naturally arise in signal processing, reliability theory, and survival analysis, where censored observations are ubiquitous [6,7,10,11].

In recent years, parameter estimation under censoring has received renewed attention. Various authors have studied estimation problems for different distributions in incomplete models, including exponential, gamma,

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\*Corresponding author

and normal distributions, using both classical and Bayesian approaches. In particular, comparative analyses of estimation methods under right-censoring for gamma and normal distributions were carried out in [12–16], highlighting the impact of censoring on estimator bias, variability, and efficiency.

*Positioning and Contribution.* Motivated by the above developments, the present paper focuses on recursive estimation based on Fisher information in incomplete statistical models with right-censored observations. We consider a parametric family of distributions depending on an unknown parameter  $\theta$ , and study recursive estimators constructed via score-type updating rules of the form

$$\theta_{n+1} = \varphi(n, \theta_n, X_{n+1}), \quad n = 0, 1, 2, \dots, \quad (1)$$

where the update function  $\varphi$  is defined through the score function and Fisher information.

Special attention is paid to the Type-I right-censoring scheme, in which observations exceeding a fixed threshold are censored. In this framework, classical maximum likelihood estimation typically leads to implicit likelihood equations that are difficult or impossible to solve analytically. To overcome this difficulty, we employ Fisher-type recursive formulas of the form

$$\theta_{n+1} - \theta_n = \frac{1}{(n+1)I(\theta_n)} \frac{\partial}{\partial \theta} \log L_n(X^{(n)}, \theta), \quad n = 1, 2, \dots,$$

where  $I(\theta)$  denotes the Fisher information associated with the underlying statistical model.

We adapt this recursive scheme to the incomplete-data setting induced by right-censoring and investigate how the censoring mechanism affects both the score function and the Fisher information. Theoretical analysis is complemented by numerical illustrations, demonstrating that Fisher-type recursive procedures provide a stable and computationally efficient alternative to direct likelihood maximization in censored models.

The main contributions of the paper can be summarized as follows. First, we derive recursive Fisher-type estimation formulas tailored to Type-I right-censored samples and establish their theoretical properties. Second, we study the influence of different censoring levels on the behavior and accuracy of the resulting estimators. Finally, This paper extends Fisher-type recursive estimation to Type-I right-censored samples and establishes strong consistency, asymptotic normality, and asymptotic efficiency under mild regularity conditions.

## 2. Model under Type-I Right Censoring and Fisher Information

*2.1. Type-I right-censoring scheme and observable sample.* Assume that the values  $X_i$  of a random variable  $\xi$  are observed only on the interval  $X_i \leq T$ , and for those observations that exceed  $T$  the value  $T$  is recorded instead. Such an experimental model is called the *Type-I right-censoring model* with threshold  $T$ .

Hence, instead of the random variable  $\xi$ , we observe the vector  $(\eta, \delta)$ , where

$$\eta = \xi\delta + T(1 - \delta), \quad \delta = I(\xi \leq T),$$

and  $I(A)$  denotes the indicator of an event  $A$ . Therefore, in  $n$  independent trials the observed statistical sample is

$$C^{(n)} = \{(\eta_i, \delta_i), i = 1, \dots, n\}. \quad (2)$$

Here

$$\eta_i = X_i\delta_i + T(1 - \delta_i), \quad \delta_i = I(X_i \leq T).$$

In the sample (2), the number of non-censored observations is a random variable

$$\nu_n = \sum_{i=1}^n \delta_i.$$

*2.2. Distribution function.* Let the distribution function of  $\xi$  be

$$F(x, \theta) = \int_{-\infty}^x f(u, \theta) \mu(du) = P_\theta(\xi < x)$$

which can be written as

$$F(x, \theta) = \begin{cases} \int_{-\infty}^x f(u, \theta) du, & \text{if } \mu(\cdot) \text{ is the Lebesgue measure,} \\ \sum_{\{i: x_i < x\}} [F(x_i + 0; \theta) - F(x_i; \theta)], & \text{if } \mu(\cdot) \text{ is a discrete measure.} \end{cases}$$

2.3. *Density of the observable variable under censoring.* In this case, the density function of  $\eta$  is defined by

$$g(x, \theta) = \begin{cases} f(x, \theta), & x \leq T, \\ 1 - F(T; \theta), & x > T, \end{cases} \quad \text{that is,} \quad g(x, \theta) = [f(x, \theta)]^y [1 - F(T; \theta)]^{1-y}. \quad (3)$$

Here  $y$  denotes the fixed value of  $\delta$  (i.e.,  $y \in \{0, 1\}$ ).

2.4. *Derivative of  $g(x, \theta)$  with respect to  $\theta$ .* Under regularity conditions,

$$\frac{\partial g(x; \theta)}{\partial \theta} = \begin{cases} \frac{\partial f(x; \theta)}{\partial \theta}, & x \leq T, \\ \int_{-\infty}^T \frac{\partial f(u; \theta)}{\partial \theta} \mu(du), & x > T, \end{cases} = \left[ \frac{\partial f(u; \theta)}{\partial \theta} \right]^y \left[ \int_{-\infty}^T \frac{\partial f(u; \theta)}{\partial \theta} \cdot \mu(du) \right]^{1-y}. \quad (4)$$

2.5. *Regularity conditions.* When applying (4), we use the following regularity conditions:

- (A)  $N_F = \{x : 0 < F(x, \theta) < 1\}$  does not depend on  $\theta$ ;
- (B) the function  $f(x, \theta)$  is twice differentiable with respect to  $\theta$ ;
- (C) the identity

$$\int_{-\infty}^{+\infty} f(x, \theta) \mu(dx) = 1$$

can be differentiated twice under the integral sign with respect to  $\theta$ , and

$$\int_{-\infty}^{+\infty} \left| \frac{\partial^k f(x, \theta)}{\partial \theta^k} \right| \mu(dx) < \infty, \quad k = 1, 2.$$

2.6. *Fisher information for the censored observation.* Under conditions (A)–(C), the Fisher information corresponding to the density  $g(x, \theta)$  can be computed as

$$E_{\theta} \left( \frac{\partial}{\partial \theta} \log g(\eta, \theta) \right)^2 = E_{\theta} \left( \frac{1}{g(\eta, \theta)} \frac{\partial g(\eta, \theta)}{\partial \theta} \right)^2.$$

Taking into account the structure of  $g(x, \theta)$ , we obtain

$$E_{\theta} \left( \delta \left[ \frac{1}{f(\eta, \theta)} \frac{\partial f(\eta, \theta)}{\partial \theta} \right]^2 \right) + E_{\theta} \left( (1 - \delta) \left[ \frac{1}{1 - F(T; \theta)} \int_T^{+\infty} \frac{\partial f(u, \theta)}{\partial \theta} \mu(du) \right]^2 \right) = I(\theta). \quad (5)$$

Since

$$E_{\theta} \delta = \int_{-\infty}^T f(x, \theta) \mu(dx), \quad E_{\theta} (1 - \delta) = 1 - F(T; \theta),$$

it follows from (4) that

$$I(\theta) = \int_{-\infty}^T \left( \frac{\partial}{\partial \theta} \log f(u, \theta) \right)^2 f(u, \theta) \mu(du) + \frac{\left( \int_T^{+\infty} \frac{\partial f(u, \theta)}{\partial \theta} \mu(du) \right)^2}{1 - F(T; \theta)}. \quad (6)$$

The quantity  $I(\theta)$  defined by (6) represents the Fisher information about the unknown parameter  $\theta$  contained in a single observation  $(\eta, \delta)$  from the sample (2). Due to the additivity property of Fisher information, the total information contained in the sample  $C^{(n)}$  equals

$$I_n(\theta) = n I(\theta).$$

### 3. Recursive Fisher-type Estimation and Asymptotic Properties

3.1. *Auxiliary function and a basic inequality.* Introduce the function

$$R(\theta) = \int_{-\infty}^{+\infty} \left[ \ln \frac{g(x, \theta)}{g(x, \theta_0)} \right] g(x, \theta_0) \mu(dx). \quad (7)$$

Clearly,  $R(\theta_0) = 0$ . Moreover, since for  $z \neq 1$  we have  $\ln z < z - 1$ , we obtain

$$R(\theta_0) = E_{\theta_0} \left\{ \ln \left[ \frac{g(\eta; \theta)}{g(\eta; \theta_0)} \right] \right\} \leq \int_{-\infty}^{+\infty} \left( \frac{g(x, \theta)}{g(x, \theta_0)} - 1 \right) g(x, \theta_0) \mu(dx) = 1 - 1 = 0.$$

3.2. *Additional condition.* Hence, for all  $\theta \in \Theta$  we impose the following condition:

(D) the function  $R(\theta)$  is twice differentiable and the identity (7) can be differentiated twice under the integral sign.

3.3. *Recursive sequence.* Define the following recursive sequence:

$$\theta_{n+1} - \theta_n = \frac{1}{(n+1)I(\theta_n)} \cdot \frac{\partial}{\partial \theta} \ln g(\eta_{n+1}; \theta) \Big|_{\theta=\theta_n}. \quad (8)$$

**Theorem 1.** Assume that, together with regularity conditions (A)–(D), the following conditions hold:

(E) for  $r(\theta) = R'(\theta)$  and  $\theta \neq \theta_0$  one has  $r(\theta)(\theta - \theta_0) < 0$ ;

(F) as  $|\theta| \rightarrow \infty$ , the function

$$Q(\theta) = I^2(\theta) E_{\theta_0} \left[ \frac{\partial}{\partial \theta} \ln g(\eta; \theta) \right]^2 = I^2(\theta) \int_{-\infty}^{+\infty} \left( \frac{\partial g(x, \theta)}{\partial \theta} \right)^2 \frac{g(x, \theta_0)}{g^2(x, \theta)} \mu(dx)$$

does not grow faster than a quadratic function;

(J) as  $L \rightarrow \infty$ , for every  $\varepsilon > 0$ , uniformly in  $|\theta - \theta_0| < \varepsilon$ ,

$$\int_{\{x: |\frac{\partial \ln g(x; \theta)}{\partial \theta}| > L\}} \left[ \frac{\partial \ln g(x; \theta)}{\partial \theta} \right]^2 g(x, \theta_0) \mu(dx) \rightarrow 0;$$

(K) the functions  $I(\theta)$  and  $Q(\theta)$  are continuous and positive for  $\theta \in \Theta$ .

Then the recursive sequence (8) is strongly consistent, asymptotically normal, and asymptotically efficient for  $\theta_0$ :

$$P \left( \lim_{n \rightarrow \infty} \theta_n = \theta_0 \right) = 1, \quad (9)$$

$$\sqrt{n}(\theta_n - \theta_0) \Rightarrow N(0; I^{-1}(\theta_0)). \quad (10)$$

**Proof.** Introduce the function

$$\Phi(\eta_{m+1}; \theta) = \frac{1}{I(\theta)} \frac{\partial}{\partial \theta} \ln g(\eta_{m+1}; \theta).$$

Then

$$E_{\theta}\Phi(\eta_{n+1};\theta) = \frac{r(\theta)}{I(\theta)},$$

and

$$E_{\theta}\left[\Phi(\eta_{n+1};\theta) - \frac{r(\theta)}{I(\theta)}\right]^2 \leq E_{\theta}\Phi^2(\eta_{n+1};\theta) - \left(\frac{r(\theta)}{I(\theta)}\right)^2 \leq E_{\theta}\Phi^2(\eta_{n+1};\theta) = Q(\theta).$$

By Theorem 4.1.1 in [1], relation (7) follows. Moreover, taking into account that  $r(\theta_0) = 0$  and the theorem conditions, we obtain

$$\left(\frac{r(\theta)}{I(\theta)}\right)' \Big|_{\theta=\theta_0} = \frac{r'(\theta_0)}{I(\theta_0)} = I^{-1}(\theta_0) \left[ \int_{-\infty}^T \frac{\partial^2 g(x, \theta_0)}{\partial \theta^2} \mu(dx) - \int_{-\infty}^T \left( \frac{\partial^2 g(x, \theta_0)}{\partial \theta^2} \right) \frac{g(x, \theta_0)}{g(x, \theta_0)} \mu(dx) \right] = -1.$$

Checking similarly the remaining assumptions of Theorem 6.6.2 in [1], as  $n \rightarrow \infty$  we have

$$\sqrt{n}(\theta_n - \theta_0) \Rightarrow N(0; E_{\theta}\Phi^2(\eta; \theta_0)). \quad (11)$$

But

$$E_{\theta}\Phi^2(\eta; \theta_0) = I(\theta_0) I^{-2}(\theta_0) = I^{-1}(\theta_0). \quad (12)$$

Hence, (10) follows from (11) and (12). The theorem is proved.

#### 4. Analytical Consequences of the Main Theorem

*4.1. Strong consistency.* Theorem 1 establishes that the recursive sequence  $\{\theta_n\}$  defined by (8) is strongly consistent, that is,

$$P\left(\lim_{n \rightarrow \infty} \theta_n = \theta_0\right) = 1.$$

This result guarantees that, despite the incompleteness of the sample caused by Type-I right-censoring, the proposed Fisher-type recursive estimator converges almost surely to the true value of the unknown parameter  $\theta_0$ . In particular, censoring affects neither the existence nor the consistency of the estimator, provided that the Fisher information is properly modified according to (6).

*4.2. Asymptotic normality.* The asymptotic distribution of the recursive estimator is given by

$$\sqrt{n}(\theta_n - \theta_0) \Rightarrow N(0, I^{-1}(\theta_0)).$$

Hence, the limiting variance of the estimator is completely determined by the Fisher information corresponding to the censored observation. This highlights the key role of the information functional (6), which incorporates both the contribution of uncensored observations and the correction term induced by censoring.

*4.3. Asymptotic efficiency.* Since the asymptotic variance coincides with the inverse Fisher information, the recursive estimator (8) attains the Cramer–Rao lower bound. Consequently, the estimator is asymptotically efficient. This property is particularly important in incomplete statistical models, where classical maximum likelihood estimators often lack explicit representations or require iterative numerical optimization.

*4.4. Interpretation of the recursive update.* The recursion

$$\theta_{n+1} - \theta_n = \frac{1}{(n+1)I(\theta_n)} \frac{\partial}{\partial \theta} \ln g(\eta_{n+1}; \theta) \Big|_{\theta=\theta_n}$$

can be interpreted as a stochastic approximation scheme with a decreasing step size  $(n+1)^{-1}$ . The presence of the Fisher information  $I(\theta_n)$  stabilizes the update and compensates for the information loss due to censoring. As a result, the recursion naturally balances bias and variability and leads to optimal asymptotic behavior.

*4.5. Role of censoring.* Formula (6) shows explicitly how the censoring mechanism modifies the Fisher information. The first term corresponds to the contribution of observed values below the threshold  $T$ , while the second term reflects the information contained in censored observations. This decomposition provides a transparent analytical explanation of how increasing censoring affects the efficiency of estimation.

## Discussion

The main objective of this work was to formulate and justify a Fisher-type recursive estimation scheme in an incomplete statistical model generated by Type-I right censoring. In this setting, the original observation  $\xi$  is not always available; instead, the observed data take the form of pairs  $(\eta, \delta)$ , where  $\eta = \xi\delta + T(1 - \delta)$  and  $\delta = I(\xi \leq T)$ . Consequently, the statistical model is determined not by the original density  $f(x; \theta)$  but by the induced density  $g(x; \theta)$  corresponding to the censored experiment.

A key methodological step is the explicit construction of the censored-data density  $g(x; \theta)$  and its derivative with respect to  $\theta$ . In particular, for the Type-I censoring mechanism, the derivative given in equation (4) captures, in a compact form, how censoring transforms the score structure: the case  $x \leq T$  corresponds to the usual parametric contribution of fully observed values, while the case  $x > T$  summarizes the information content of censored observations through an integral quantity.

Under the regularity conditions (A)–(C), this construction leads to a well-defined Fisher information functional associated with the censored observation. Its decomposed representation given in equation (6) makes the effect of censoring analytically transparent: the first term accounts for the contribution of observations below the threshold  $T$ , whereas the second term represents the contribution of the censored part through the tail probability  $1 - F(T; \theta)$ . Therefore, the loss or redistribution of information induced by censoring is encoded directly in the Fisher information functional.

On this basis, the recursive sequence is introduced via a Fisher-type update rule of the form (8), which belongs to the classical family of stochastic approximation procedures [8–10] with a decreasing step size  $(n+1)^{-1}$ . The normalization by  $I(\theta_n)$  plays a stabilizing role and yields a natural recursive analogue of the Fisher scoring method.

The asymptotic justification of this construction is provided by Theorem 1 under conditions (A)–(D) together with (E)–(K). In particular, the theorem establishes three fundamental properties of the proposed recursive estimator: strong consistency, asymptotic normality, and asymptotic efficiency, as formalized in Section 4.

From a statistical viewpoint, these results imply that the proposed recursion achieves the best possible asymptotic precision permitted by the incomplete-data model. Moreover, the dependence of the limiting variance on  $I(\theta_0)$  emphasizes that estimation efficiency is governed by the information content of the censored observation rather than that of the complete-data model. Hence, the effect of censoring on estimation accuracy is fully characterized through the modified Fisher information given in equation (6).

It is also worth noting that the proof strategy relies on the auxiliary functional defined in equation (7), which satisfies  $R(\theta_0) = 0$  and  $R(\theta) \leq 0$  for  $\theta \neq \theta_0$ . This functional plays the role of an information-type contrast measuring the discrepancy between the true model and its parametric perturbations. Condition (E), formulated in terms of  $r(\theta) = R'(\theta)$ , ensures a stabilizing drift toward  $\theta_0$  and guarantees almost sure convergence of the recursive sequence. In this sense, the established asymptotic properties follow naturally from the interaction between the censored-data Fisher structure and the imposed regularity framework. Similar censored-data methodologies have recently been employed in engineering reliability analysis and operational lifetime modelling [17].

Overall, the presented results provide a rigorous theoretical foundation for Fisher-type recursive estimation in right-censored incomplete statistical models. The framework is formulated at the level of a general parametric family and is not restricted to a specific distribution, making it suitable for subsequent specialization and numerical investigation in concrete models.

## Conclusion

In this paper, we investigated recursive estimation of an unknown parameter  $\theta$  in an incomplete statistical model arising under Type-I right censoring. Instead of complete observations, the available data consist of censored pairs  $(\eta, \delta)$ , which induce a modified density  $g(x; \theta)$  for the observed experiment. We provided an explicit representation of  $g(x; \theta)$  and its derivative with respect to  $\theta$ , thereby clarifying how censoring transforms the score function and the information structure.

Under regularity conditions (A)–(C), we derived the Fisher information functional associated with the censored observation and obtained its analytical decomposition into contributions from the observed and

censored parts of the sample. Based on this information functional, we introduced a Fisher-type recursive scheme of the form (8), which extends the classical Fisher scoring philosophy to the recursive estimation setting for censored data.

The main theoretical result (Theorem 1) establishes that the proposed recursive estimator is strongly consistent, asymptotically normal, and asymptotically efficient. In particular, the asymptotic variance is given by  $I^{-1}(\theta_0)$ , where  $I(\theta)$  is the Fisher information corresponding to the censored experiment. This confirms that the recursion attains the optimal asymptotic precision permitted by the incomplete-data model, and that the impact of censoring on estimation efficiency is fully captured by the modified Fisher information.

The obtained results form a rigorous basis for further investigation of Fisher-type recursive procedures in concrete parametric families and for numerical studies of their finite-sample performance under different censoring levels. Such numerical illustrations and model-specific applications are natural continuations of the present theoretical framework.

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ON THE EIGENFUNCTION EXPANSION OF THE  $B$ -POLYHARMONIC OPERATOR.

PIRMATOV SHAMSHOD

TASHKENT STATE TECHNICAL UNIVERSITY NAMED AFTER ISLAM KARIMOV, TASHKENT, UZBEKISTAN  
shamshod@rambler.ru

**ABSTRACT.** The paper investigates the problem of eigenfunction expansion for a  $B$ -polyharmonic operator in an arbitrary  $N$ -dimensional domain adjacent to a singularity hypersurface. It is proved that if the Riesz partial sums of an arbitrary function converge at a certain point on this hypersurface, then its mean value over a hemisphere centered at that point possesses generalized continuity.

**MSC (2020):** 34L10; 34L15; 47A10; 42A16; 42A20; 42B08.

**Key words:** eigenfunctions, eigenvalues, spectrum, Fourier series, convergence, summability.

## Introduction

The problem of convergence of spectral expansions associated with elliptic operators has been extensively investigated by numerous researchers; nevertheless, it continues to attract considerable attention due to its ongoing relevance and complexity [1-4]. Although sufficient conditions ensuring the convergence of spectral expansions for elliptic operators with singular coefficients have been studied in substantial detail, the corresponding necessary conditions remain comparatively less explored.

The work [2] examines eigenvalues and eigenfunctions, as well as their asymptotic behavior. It also provides estimates for spectral expansions and investigates local convergence properties of expansions in eigenfunctions. In all the aforementioned studies, sufficient conditions are established for the convergence of spectral expansions corresponding to various classes of functions.

In [5-9], questions of convergence and Riesz summability of spectral expansions associated with elliptic operators were studied for piecewise smooth functions. The research of Alimov [10] and Pirmatov [11] examines spectral expansions associated with the self-adjoint extension of the Laplace operator and the biharmonic operator in an  $N$ -dimensional domain. It was established that if the spectral expansion of an arbitrary function at a certain point is Riesz summable, then its spherical mean centered at that point possesses generalized continuity. Furthermore, necessary conditions were obtained for the Riesz summability of spectral expansions in terms of the eigenfunctions of an elliptic operator containing a Bessel operator with respect to one of the variables in an arbitrary  $N$ -dimensional domain adjacent to the singularity hypersurface [13-16].

## Problem Formulation

Consider the Euclidean half-space  $R_N^+$ , the points of which are denoted by  $(x', x_N)$ , where  $x' = (x_1, x_2, \dots, x_{N-1})$  and  $x_N > 0$ . Let a bounded domain  $\Omega^+$  with a piecewise smooth boundary  $\Gamma$  be located in the half-space  $R_N^+$  and be adjacent to the hyperplane  $x_N = 0$ . The part of the boundary  $\Gamma$  located in  $R_N^+$  is denoted by  $\Gamma^+$ , while the part of the boundary  $\Gamma$  lying on the hyperplane  $x_N = 0$  is denoted by  $\Gamma^0$ .

In the domain  $\Omega^+$ , we consider the differential operator

$$\Delta_B = \frac{1}{x_N^k} \frac{\partial}{\partial x_N} x_N^k \frac{\partial}{\partial x_N} + \sum_{i=1}^{N-1} \frac{\partial^2}{\partial x_i^2}, \quad (1)$$

where  $k$  is a fixed positive constant.

We define the domain of the operator  $\Delta_B$  to be the set of functions  $u \in C_0^\infty(\Omega^+ \cup \Gamma^0)$  satisfying the condition

$$\frac{\partial u}{\partial x_N} = 0, \quad x \in \Gamma^0.$$

Consider an orthonormal system of eigenfunctions  $u_n(x)$  of the polyharmonic operator, corresponding to the eigenvalues  $\{\lambda_n\}$ :

$$\Delta_B^m u_n(x) - (-1)^m \lambda_n u_n(x) = 0, \quad x \in \Omega^+. \quad (2)$$

Assume that the regular boundary conditions are self-adjoint and such that all eigenvalues are positive:  $\lambda_n > 0$ .

We introduce the Hilbert space  $L_{2,k}(\Omega^+)$  with dot product

$$(u, v)_k = \int_{\Omega^+} u(x)v(x)x_N^k dx,$$

and denote the associated norm by  $\|u\|_k = \sqrt{(u, u)_k}$ .

In the domain  $\Omega^+$ , we expand a function  $f \in L_{2,k}(\Omega^+)$  into a Fourier series in terms of the eigenfunctions  $\{u_n(x)\}_{n=1}^\infty$ :

$$f(x) \sim \sum_{n=1}^{\infty} f_n u_n(x),$$

where

$$f_n = \int_{\Omega^+} f(x)u_n(x)x_N^k dx$$

are the Fourier coefficients. Denote  $\mu_n = \lambda_n^{1/2m}$ .

Consider the partial sums of this series

$$E_\mu f(x) = \sum_{\mu_n < \mu} f_n u_n(x), \quad f_n = (f, u_n)_k. \quad (3)$$

We define the mean value of order  $\alpha > 0$  for a function  $f \in L_{2,k}(\Omega^+)$  over a hemisphere of radius  $r$  centered at a point  $x \in \Gamma^0$  as follows:

$$S_r^\alpha f(x) = \frac{1}{\omega_N(k, \alpha) r^{N+k}} \int_{|y| < r, y_N > 0} \left(1 - \frac{|y|^2}{r^2}\right)^{\alpha-1} f(x+y) y_N^k dy, \quad (4)$$

where

$$\omega_N(k, \alpha) = \int_{|y| < 1, y_N > 0} (1 - |y|^2)^{\alpha-1} y_N^k dy.$$

The main result of the present paper is stated as follows.

**Theorem.** If the spectral expansion (3) of a function  $f \in L_{2,k}(\Omega^+)$  converges at a point  $x \in \Gamma^0$ , then for any  $\alpha > 1/2$ , the following equality holds:

$$\lim_{r \rightarrow 0} S_r^\alpha f(x) = f(x). \quad (5)$$

The case  $k = 0$  of the theorem was proved for the Laplace operator in [12]. Similar theorems have been established for the  $B$ -elliptic operator [18], as well as for the operator  $-\Delta + q(x)$  in an arbitrary  $N$ -dimensional domain [19].

The proof of the theorem relies substantially on the results of Kipriyanova [17], in which a mean value formula for the  $B$ -polyharmonic operator was obtained. Specifically, for any solution to equation (2), the following formula holds:

$$\int_{\omega_1^+} u_n(x + r\theta, r\theta_N) \theta_N^k d\omega_1 = \pi^{\frac{N-1}{2}} \Gamma\left(\frac{k+1}{2}\right) u_n(x) J_{\frac{N+k-2}{2}}(r\mu_n) \left(\frac{r\mu_n}{2}\right)^{-\frac{N+k-2}{2}} + \psi_n(x, r) \quad (6)$$

where the function  $\psi_n(x, r)$ , for any compact set  $K \subset \Omega^+ \cap \Gamma^0$ , satisfies the estimate

$$|\psi_n(x, r)| \leq Ce^{-\gamma\mu_n}. \quad (7)$$

with some positive constants  $C = C(K)$  and  $\gamma = \gamma(K)$ .

From (6) it follows that  $\psi_n(x, 0) = 0$ . Indeed, setting  $r = 0$  in (6), the left-hand side becomes  $u_n(x) \int_{\omega_1^+} \theta_N^k d\omega_1$ , while the right-hand side gives

$$\pi^{\frac{N-1}{2}} \Gamma\left(\frac{k+1}{2}\right) u_n(x) \lim_{r \rightarrow 0} J_{\frac{N+k-2}{2}}(r\mu_n) \left(\frac{r\mu_n}{2}\right)^{-\frac{N+k-2}{2}} = \frac{\pi^{\frac{N-1}{2}} \Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{N+k}{2}\right)} u_n(x),$$

which coincides with the left-hand side because  $\int_{\omega_1^+} \theta_N^k d\omega_1 = \pi^{\frac{N-1}{2}} \Gamma\left(\frac{k+1}{2}\right) / \Gamma\left(\frac{N+k}{2}\right)$ . Hence  $\psi_n(x, 0) = 0$ .

Moreover, for each fixed  $n$ , the function  $\psi_n(x, r)$  is continuous in  $r$  (or at least tends to zero as  $r \rightarrow 0$ ) due to the smoothness of eigenfunctions and the explicit form of the Bessel function. Consequently,

$$\lim_{r \rightarrow 0} \psi_n(x, r) = 0.$$

### Preliminary Results

Fix a point  $x \in \Gamma^0$  and consider the following function:

$$g(y) = \begin{cases} \left(1 - \frac{|y-x|^2}{r^2}\right)^{\alpha-1}, & \text{if } |y-x| \leq r, \\ 0, & \text{if } |y-x| > r. \end{cases}$$

where  $|y-x|^2 = \sum_{i=1}^{N-1} (y_i - x_i)^2 + y_N^2$ ,  $x = (x_1, x_2, \dots, x_{N-1})$ .

For  $\alpha > 1/2$ , the function  $g$  belongs to  $L_{2,k}(\Omega^+)$ . We calculate the Fourier coefficients of the function  $g$  by transforming to spherical coordinates:

$$\begin{aligned} g_n &= (g, u_n) = \int_{\Omega^+} \left(1 - \frac{|y-x|^2}{r^2}\right)^{\alpha-1} u_n(y) y_N^k dy = \\ &= \int_{\rho \leq r} \left(1 - \frac{\rho^2}{r^2}\right)^{\alpha-1} u_n(x + \rho\theta, \rho\theta_N) \rho^{N-1} \rho^k \theta_N^k d\rho d\omega_1 = \\ &= \int_0^r \left(1 - \frac{\rho^2}{r^2}\right)^{\alpha-1} \rho^{N+k-1} d\rho \int_{\omega_1^+} u_n(x + \rho\theta, \rho\theta_N) \theta_N^k d\omega_1. \end{aligned}$$

Further, we use the mean value formula (6):

$$\begin{aligned} g_n &= \pi^{\frac{N-1}{2}} 2^{(N+k)/2-1} \Gamma\left(\frac{k+1}{2}\right) \mu_n^{-(N+k-2)/2} u_n(x) \int_0^r \left(1 - \frac{\rho^2}{r^2}\right)^{\alpha-1} \rho^{(N+k)/2} J_{(N+k)/2-1}(\rho\mu_n) d\rho + \\ &\quad + \int_0^r \left(1 - \frac{\rho^2}{r^2}\right)^{\alpha-1} \rho^{N+k-1} \psi_n(x, \rho) d\rho. \end{aligned} \quad (8)$$

The first integral is reduced to Sonine's integral (see [19], formula 12.11.(1)) by simple transformations, and the second integral is evaluated using estimate (7). Then, the Fourier coefficient of the function  $g(y)$  takes the following form:

$$g_n = \pi^{\frac{N-1}{2}} 2^{(N+k)/2-1} \Gamma\left(\frac{k+1}{2}\right) r^{N+k} J_{(N+k)/2+\alpha-1}(r\mu_n) (r\mu_n)^{-(N+k)/2-\alpha+1} + r^{N+k} \phi_n(r), \quad (9)$$

where

$$\phi_n(r) = O(1)e^{-\gamma\mu_n}. \quad (10)$$

From the above, for each fixed  $n$ , we have  $\lim_{r \rightarrow 0} \phi_n(r) = 0$ . Indeed, using the estimate (7) and the fact that  $\psi_n(x, \rho) \rightarrow 0$  as  $\rho \rightarrow 0$ , for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $|\psi_n(x, \rho)| < \varepsilon$  for  $\rho < \delta$ . Then for  $r < \delta$  the second integral in (8) is bounded by  $\varepsilon \int_0^r \rho^{N+k-1} d\rho = C\varepsilon r^{N+k}$ , which implies  $|\phi_n(r)| \leq C\varepsilon$ . Hence  $\phi_n(r) \rightarrow 0$  as  $r \rightarrow 0$ .

Applying Parseval's identity to the functions  $f$  and  $g$ , and taking into account the obtained expression (9) for the function  $g$ , we arrive at the following representation for the mean value (4) of the function  $f$  at the point  $x$ :

$$S_r^\alpha f(x) = B(N, k, \alpha) \sum_{n=1}^{\infty} \frac{J_{(N+k)/2+\alpha-1}(r\mu_n)}{(r\mu_n)^{(N+k)/2+\alpha-1}} f_n u_n(x) + \sum_{n=1}^{\infty} f_n \phi_n(r), \quad (11)$$

where  $B(N, k, \alpha) = 2^{(N+k)/2+\alpha-1} \Gamma((N+k)/2 + \alpha)$ , and the function  $\phi_n(r)$  satisfies the estimate (10).

Let us consider each of the two terms on the right-hand side of (11) separately:

$$T_r^\alpha f(x) = \sum_{n=1}^{\infty} \varphi_\alpha(r\mu_n) f_n u_n(x), \quad (12)$$

where

$$\varphi_\alpha(t) = B(N, k, \alpha) \frac{J_{(N+k)/2+\alpha-1}(t)}{t^{(N+k)/2+\alpha-1}}, \quad (13)$$

and

$$I_r^\alpha f(x) = \sum_{n=1}^{\infty} f_n \phi_n(r). \quad (14)$$

Note that

$$I_0^\alpha f(x) = \sum_{n=1}^{\infty} f_n \phi_n(0) = 0.$$

First, let us estimate the series (14).

**Lemma 1.** For any  $\sigma > N + k$ , the series

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n^\sigma} < \infty.$$

converges.

*Proof.* Since  $A$  is a self-adjoint extension of the  $B$ -polyharmonic operator, for any  $\tau > 0$ , the operator  $A^{-\tau}$  is an integral operator

$$A^{-\tau} u(x) = \int_{\Omega^+} G_\tau(x, y) u(y) y^k dy$$

whose kernel  $G_\tau(x, y)$  has the following Fourier series expansion in terms of the eigenfunctions:

$$G_\tau(x, y) = \sum_{n=1}^{\infty} \frac{u_n(x) u_n(y)}{\lambda_n^\tau}, \quad x, y \in \Omega^+.$$

In particular, for  $\tau = 1$ , this kernel coincides with the Green's function, and the following estimate holds for the kernel  $G_\tau(x, y)$  (see [1], p. 117, formula (4.1.9)):

$$|G_\tau(x, y)| \leq C|x - y|^{2m\tau - (N+k)}. \quad (15)$$

Then, applying Parseval's identity and taking into account estimate (15), we obtain

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{2\tau}} = \int_{\Omega^+} \int_{\Omega^+} |G_\tau(x, y)|^2 x_N^k y_N^k dx dy \leq C \int_{\Omega^+} \int_{\Omega^+} |x - y|^{4m\tau - 2(N+k)} x_N^k y_N^k dx dy$$

Since the domain  $\Omega^+$  is bounded, the last integral converges if  $4m\tau > N + k$ . Setting  $\sigma = 4m\tau$  and noting that  $\lambda_n^{2\tau} = \mu_n^{4m\tau}$ , we obtain the assertion of the lemma.  $\square$

**Corollary 1.** For any  $p > 0$ , the series

$$\sum_{n=1}^{\infty} e^{-\gamma\mu_n} \mu_n^p < \infty. \quad (16)$$

converges.

*Proof.* We use the estimate of the exponential function, which follows from its series expansion. For any  $m \in \mathbb{N}$  and  $t > 0$ , we have the equality

$$e^{\gamma t} \geq \frac{(\gamma t)^m}{m!},$$

from where it follows

$$e^{-\gamma t} \leq \frac{C_m(\gamma)}{t^m}.$$

Therefore

$$t^p e^{-\gamma t} \leq \frac{C_m(\gamma)}{t^{m-p}}.$$

In this case, taking  $m > p + N + k$ , we obtain from Lemma 1

$$\sum_{n=1}^{\infty} e^{-\gamma\mu_n} \mu_n^p < \sum_{n=1}^{\infty} \frac{C_m(\gamma)}{\mu_n^{m-p}} < \infty.$$

$\square$

**Corollary 2.** For the quantity  $I_r^\alpha f$  defined by equality (14), the following estimate holds:

$$|I_r^\alpha f(x)| \leq C \|f\|_{L_{2,k}(\Omega^+)}. \quad (17)$$

*Proof.* The proof of the lemma follows from the Cauchy-Bunyakovsky inequality and from estimates (10) and (16):

$$\begin{aligned} |I_r^\alpha f(x)| &\leq \left( \sum_{n=1}^{\infty} f_n^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} \phi_n^2(r) \right)^{1/2} \leq \\ &\leq \|f\|_{L_{2,k}(\Omega^+)} C \left( \sum_{n=1}^{\infty} e^{-2\gamma\mu_n} \right)^{1/2} \leq C \|f\|_{L_{2,k}(\Omega^+)}. \end{aligned}$$

$\square$

**Lemma 2.** If the spectral expansion of the function  $f \in L_{2,k}(\Omega)$  converges at the point  $x \in \Gamma^0$ , then for any  $\alpha > 1/2$ , the following equality holds:

$$\lim_{r \rightarrow 0} T_r^\alpha f(x) = f(x).$$

*Proof.* We represent the sum (12) as an integral and integrate by parts

$$T_r^\alpha f(x) = \int_0^\infty \varphi_\alpha(r\mu) dE_\mu f(x) = \varphi_\alpha(r\mu) E_\mu f(x) \Big|_{\mu=0}^{\mu=\infty} - r \int_0^\infty E_\mu f(x) \varphi'_\alpha(r\mu) d\mu.$$

Using the equalities  $E_0 f = 0$ ,  $\varphi_\alpha(+\infty) = 0$  and the change of variable  $r\mu = t$ , we obtain the following expression:

$$T_r^\alpha f(x) = -r \int_0^\infty E_\mu f(x) \varphi'_\alpha(r\mu) d\mu = - \int_0^\infty E_{t/r} f(x) \varphi'_\alpha(t) dt = B(N, k, \alpha) \int_0^\infty E_{t/r} f(x) \frac{J_{l+1}(t)}{t^l} dt,$$

where  $l = (N + k)/2 + \alpha - 1$ .

According to the condition of the lemma,  $E_\mu f(x) \rightarrow f(x)$  as  $\mu \rightarrow \infty$ . Then, for any  $\varepsilon > 0$ , there exists  $T > 0$  such that

$$|E_\mu f(x) - f(x)| < \varepsilon, \quad \mu \geq T.$$

From here,

$$T_r^\alpha f(x) = B(N, k, \alpha) \left\{ \int_0^{rT} E_{t/r} f(x) \frac{J_{l+1}(t)}{t^l} dt + f(x) \int_{rT}^\infty \frac{J_{l+1}(t)}{t^l} dt + \int_{rT}^\infty [E_{t/r} f(x) - f(x)] \frac{J_{l+1}(t)}{t^l} dt \right\}$$

As  $rT \rightarrow 0$ , the first term admits the estimate

$$\int_0^{rT} E_{t/r} f(x) \frac{J_{l+1}(t)}{t^l} dt = O \left( \int_0^{rT} t dt \right) = O(r^2 T^2),$$

while the last term satisfies the estimate

$$\int_{rT}^\infty [E_{t/r} f(x) - f(x)] \frac{J_{l+1}(t)}{t^l} dt = O \left( \varepsilon \int_{rT}^\infty \frac{J_{l+1}(t)}{t^l} dt \right) = O(\varepsilon).$$

The integral in the brackets converges if  $l > 1/2$ .

Then, from the preceding estimates, we have

$$T_r^\alpha f(x) = 2^l \Gamma(l+1) f(x) \int_{rT}^\infty \frac{J_{l+1}(t)}{t^l} dt + O(r^2 T^2) + O(\varepsilon)$$

Using the asymptotics

$$\int_{rT}^\infty \frac{J_{l+1}(t)}{t^l} dt = \frac{2^{-l}}{\Gamma(l+1)} + O(r^2 T^2),$$

we obtain

$$T_r^\alpha f(x) = f(x) + O(r^2 T^2) + O(\varepsilon).$$

Taking the limit as  $r \rightarrow 0$ , and given that  $\varepsilon > 0$  is arbitrary, we obtain

$$\lim_{r \rightarrow 0} T_r^\alpha f(x) = f(x).$$

□

**Lemma 3.** Let  $f \in L_{2,k}(\Omega^+)$ . Then

$$\overline{\lim}_{r \rightarrow 0} |S_r^\alpha f(x) - T_r^\alpha f(x)| = 0. \quad (18)$$

*Proof.* According to (14) we have

$$I_r^\alpha f(x) = \sum_{n=1}^\infty f_n \phi_n(r),$$

with  $|\phi_n(r)| \leq C e^{-\gamma \mu_n}$  and, as shown above,  $\lim_{r \rightarrow 0} \phi_n(r) = 0$  for each  $n$ . The series  $\sum_{n=1}^\infty |f_n| C e^{-\gamma \mu_n}$  converges by Corollary 1. By applying the dominated convergence theorem of Lebesgue, we obtain

$$\lim_{r \rightarrow 0} |I_r^\alpha f(x)| = 0.$$

Hence, the desired equality (18) follows from (11).

□

**The proof** of the theorem follows directly from Lemmas 2 and 3.

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## NONLOCAL PROBLEM FOR A FRACTIONAL WAVE-TYPE EQUATION WITH HILFER DERIVATIVE

SADULLAEVA SHAHRIZODA

V. I. ROMANOVSKIY INSTITUTE OF MATHEMATICS UZBEKISTAN ACADEMY OF SCIENCES, UNIVERSITY  
STREET 9, TASHKENT-100174, UZBEKISTAN  
ssh45984@gmail.com

**ABSTRACT.** In this paper, the nonlocal problem is studied for a differential equation with a fractional Hilfer derivative, where the order of the fractional derivative is  $1 < \alpha < 2$ . The nonlocal boundary value problem  $D_t^{\alpha,\beta} u(t) + Au(t) = f(t)$ ,  $0 < t \leq T$ ,  $(I^\delta u)(T) = \gamma I^\delta u(+0) + \varphi$  ( $\gamma$  is a constant), is considered in arbitrary separable Hilbert space  $H$  with positive self-adjoint operator  $A$ . The obtained results demonstrate the conditions for the existence and uniqueness of a solution to this problem, as well as the necessary conditions required for the existence of the solution.

**MSC (2020):** 35R11; 33E12; 35L05.

**Key words:** Nonlocal problem; Cauchy problem; Hilfer derivatives; Mittag-Leffler function.

## 1. Introduction

Fractional differential equations have become an essential tool in modeling various physical and biological phenomena involving memory and hereditary properties, such as anomalous diffusion, viscoelasticity and signal processing (see, [14, 20, 22]). Among the numerous definitions of fractional derivatives, the Riemann-Liouville and Caputo derivatives are the most widely used. However, each has its limitations: a key issue with the Riemann-Liouville definition is its reliance on initial conditions of non-integer order, which are difficult to interpret physically. On the other hand, the Caputo operator addresses this problem, but at the cost of demanding higher regularity from the initial data (see [24]). To overcome these limitations, R. Hilfer introduced a two-parameter fractional derivative that interpolates between the Riemann-Liouville and Caputo derivatives ([14]). For a function  $y(t)$ , the Hilfer derivative of order  $n - 1 < \alpha < n$  and  $\beta \in [0, 1]$  is defined as

$$D^{\alpha,\beta} y(t) = I^{\beta(n-\alpha)} \frac{d^n}{dt^n} I^{(1-\beta)(n-\alpha)} y(t), \quad t > 0.$$

The parameter  $\beta$  provides additional convenience: when  $\beta = 0$ , we recover the Riemann-Liouville derivative; when  $\beta = 1$ , we obtain the Caputo derivative. This interpolation property makes the Hilfer derivative particularly attractive for modeling complex systems where the type of fractional derivative itself may be subject to identification from experimental data (see [13, 15]).

This work is devoted to studying a nonlocal problem for a fractional differential equation involving the Hilfer derivative of order  $1 < \alpha < 2$ , which corresponds to fractional wave-type equations describing intermediate processes between diffusion and wave propagation. The problem is studied in an arbitrary separable Hilbert space  $H$  with a positive self-adjoint operator  $A$ . The aim is to identify conditions for the problem to be well-posed. Furthermore, in the ill-posed case, we seek additional conditions that ensure the existence of a solution (which may not be unique), as well as the explicit form of all possible solutions.

Nonlocal problems for Hilfer fractional differential equations of order  $1 < \alpha < 2$  have been studied in several recent works. In [2], nonlocal boundary value problems for such equations were investigated using fixed point theorems. In [18], a nonlocal boundary value problem for a mixed-type equation involving bi-ordinal Hilfer fractional differential operators was studied. In [19], a non-local problem for a time-fractional differential equation with the Hilfer operator on a metric star graph was considered, where the existence and uniqueness of the solution were established using a priori estimates. In the Caputo sense, a four-parameter non-local problem for a fractional wave equation of the same order was studied in [6].

When  $0 < \alpha < 1$ , nonlocal time conditions for fractional diffusion equations with Caputo derivative were investigated in [8]. The Riemann-Liouville case was treated in [4]. The Hilfer derivative was considered in [5, 28].

Let  $H$  be a separable Hilbert space, and let  $A$  be an arbitrary operator on  $H$  that is self-adjoint, positive, unbounded, and defined on the domain  $D(A) \subset H$ . Suppose that  $A$  has a complete in system of orthonormal eigenfunctions  $\{v_k\}$  and a countable set of positive eigenvalues  $\{\lambda_k\}$ . It is convenient to assume that the eigenvalues do not decrease as their number increases, i.e.  $0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow +\infty$ .

Let  $\varepsilon$  be an arbitrary real number. For  $h \in H$  with Fourier coefficients  $h_k = (h, v_k)$ ,

$$A^\varepsilon h = \sum_{k=1}^{\infty} \lambda_k^{2\varepsilon} h_k v_k,$$

clearly, the domain of this operator is given by

$$D(A^\varepsilon) = \left\{ h \in H : \sum_{k=1}^{\infty} \lambda_k^{2\varepsilon} |h_k|^2 < \infty \right\}.$$

For elements of  $D(A^\varepsilon)$  we introduce the norm

$$\|h\|_\varepsilon^2 = \sum_{k=1}^{\infty} \lambda_k^{2\varepsilon} |h_k|^2 = \|A^\varepsilon h\|^2,$$

and together with this norm  $D(A^\varepsilon)$  turns into a Hilbert space.

For a function  $y(t)$  on  $[0, +\infty)$ , the Riemann-Liouville integral of order  $\alpha$  is defined by the formula below (see, e.g. [20], p. 181):

$$I^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \xi)^{\alpha-1} y(\xi) d\xi.$$

Let  $1 < \alpha < 2$  and  $0 \leq \beta \leq 1$  are a fixed number and let  $C((a, b); H)$  stands for a set of continuous functions  $u(t)$  of  $t \in (a, b)$  with values in  $H$ . Consider the following nonlocal problem:

$$\begin{cases} D_t^{\alpha, \beta} u(t) + Au(t) = f(t), & 0 < t \leq T, \\ (I^\delta u)(T) = \gamma(I^\delta u)(+0) + \varphi, \\ \frac{d}{dt}(I^\delta u)(+0) = \phi, \end{cases} \quad (1.1)$$

where  $f(t) \in C([0, T]; H)$  is a given function,  $\varphi, \phi \in H$  are given elements,  $\gamma$  is a given constant and  $\delta = (1 - \beta)(2 - \alpha)$ .

In the present article, we focus on the case  $\gamma \neq 0$ . The case  $\gamma = 0$  is referred to as the backward problem. This problem has been studied for the Caputo derivative in [9, 19], for the Riemann-Liouville derivative in [1] and for the Hadamard derivative in [25].

**Definition 1.1** A function  $t^\delta u(t) \in C([0, T]; H)$  with the properties  $D_t^{\alpha, \beta} u(t)$ ,  $Au(t) \in C([0, T]; H)$  and satisfying conditions (1.1) is called the solution of the nonlocal problem (1.1).

Suppose  $\lambda_0 > 0$  fulfills  $E_{\alpha, 1}(-\lambda_0 T^\alpha) = \gamma$ . We put  $\lambda_k = \lambda_0$  and let  $m_0$  be the multiplicity of  $\lambda_k$ . The assumptions on the operator  $A$  guarantee that  $m_0$  is finite. The set of indices  $k$  in question is then defined by  $K_0 = \{k_0, k_0 + 1, \dots, k_0 + m_0 - 1\}$ .

**Theorem 1.2** Let  $\varphi, \phi \in H$  and  $f(t) \in C([0, T]; D(A^\varepsilon))$  for some  $\varepsilon \in (0, 1)$ .

- (a) If  $|\gamma| \geq 1$  or  $\gamma \in (-1; 0) \cup (0; 1)$ , but  $\lambda_k \neq \lambda_0$  for all  $k \geq 1$ , then problem (1.1) has a unique solution and this solution has the form

$$\begin{aligned} u(t) = \sum_{k=1}^{\infty} \left[ \left( \frac{\varphi_k - (I^\delta V_k)(T)}{E_{\alpha, 1}(-\lambda_k T^\alpha) - \gamma} - \frac{\phi_k T E_{\alpha, 2}(-\lambda_k T^\alpha)}{E_{\alpha, 1}(-\lambda_k T^\alpha) - \gamma} \right) t^{-\delta} E_{\alpha, 1-\delta}(-\lambda_k t^\alpha) \right. \\ \left. + \phi_k t^{1-\delta} E_{\alpha, 2-\delta}(-\lambda_k t^\alpha) + V_k(t) \right] v_k, \end{aligned}$$

where

$$V_k(t) = \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k \tau^\alpha) f_k(t-\tau) d\tau.$$

- (b) If  $\gamma \in (-1; 0) \cup (0; 1)$  and  $\lambda_k = \lambda_0$  for some  $k \in K_0$ , then for problem (1.1) to have a solution, the fulfillment of the following orthogonality conditions

$$(\varphi, v_k) = ((I^\delta V)(T), v_k) = (\phi, v_k) = 0, \quad K_0 = \{k_0, k_0 + 1, \dots, k_0 + m_0 - 1\} \quad (1.2)$$

are sufficient.

Accordingly, the form of the solution can be written as follows:

$$\begin{aligned} u(t) = & \sum_{k \notin K_0} \left[ \left( \frac{\varphi_k - (I^\delta V_k)(T)}{E_{\alpha,1}(-\lambda_k T^\alpha) - \gamma} - \frac{\phi_k T E_{\alpha,2}(-\lambda_k T^\alpha)}{E_{\alpha,1}(-\lambda_k T^\alpha) - \gamma} \right) t^{-\delta} E_{\alpha,1-\delta}(-\lambda_k t^\alpha) \right. \\ & \left. + \phi_k t^{1-\delta} E_{\alpha,2-\delta}(-\lambda_k t^\alpha) + V_k(t) \right] v_k \\ & + \sum_{k \in K_0} \left( b_k t^{-\delta} E_{\alpha,1-\delta}(-\lambda_k t^\alpha) + \phi_k t^{1-\delta} E_{\alpha,2-\delta}(-\lambda_k t^\alpha) + V_k(t) \right) v_k. \end{aligned}$$

with arbitrary coefficients  $b_k, k \in K_0$ .

Moreover, the following coercive type inequality is valid for some constants  $C_\gamma > 0$  and  $C_\varepsilon > 0$ :

$$\begin{aligned} \| D_t^{\alpha,\beta} u(t) \|^2 + \| u(t) \|^2 \leq & C_\gamma t^{-2(\alpha+\delta)} \| \varphi \|^2 + \left( \frac{C_\gamma}{\lambda_1^2 t^{2(\alpha+\delta)}} + t^{2-2\alpha-2\delta} \right) \| \phi \|^2 \\ & + C_\varepsilon \max_{t \in [0, T]} \| f(t) \|^2, \quad 0 < t \leq T. \end{aligned}$$

Our paper is structured as follows. Section 2 provides basic notations, definitions and lemmas for solving problems. In Section 3, we study the nonlocal problem for homogeneous equation and find solution in which case they uniquely exist. In Section 4, we formulate the Cauchy problem for the non-homogeneous equation and the existence and uniqueness of the solution are proved using the corresponding theorem. In Section 5, we formulate the main problem by using results of the previous two sections. In Section 6, we demonstrate the theoretical results from the previous sections by means of a specific numerical example.

## 2. Preliminaries

In order to find a solution to the nonlocal problem, we introduce some concepts. For  $\alpha$  and an arbitrary complex number  $\beta$ , we denote the Mittag-Leffler function with two parameters by  $E_{\alpha,\beta}(t)$ :

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}.$$

**Lemma 2.1** If  $\alpha > 0$ , then for any  $t \geq 0$  one has:

$$|E_{\alpha,\mu}(-t)| \leq \frac{C}{1+t},$$

where  $C$  - constant does not depend on  $\mu$  and  $t$  (see, e.g. [7], p.136).

**Lemma 2.2** Let  $1 < \alpha < 2$  and  $t \geq 0$  then following inequality holds (see [6]):

$$|E_{\alpha,1}(-t)| < 1.$$

**Lemma 2.3** Let  $0 < \alpha < 2$ , then following relation holds (see [8]):

$$|t^{\alpha-1} E_{\alpha,\mu}(-\lambda t^\alpha)| \leq C \lambda^{\varepsilon-1} t^{\varepsilon\alpha-1}, \quad t > 0,$$

where  $\lambda$  is a positive number,  $C$  is a constant and  $0 < \varepsilon < 1$ .

**Lemma 2.4** Let  $0 < \varepsilon < 1$  be any fixed number and  $f(t) \in C([0, T]; D(A^\varepsilon))$ . Then the following estimate holds (see, [8]):

$$\sum_{k=1}^{\infty} \left| \lambda_k \int_0^t \tau^{\alpha-1} E_{\alpha,\mu}(-\lambda_k \tau^\alpha) f_k(t-\tau) d\tau \right|^2 \leq C_\varepsilon \max_{t \in [0, T]} \|f(t)\|_\varepsilon^2,$$

where  $C_\varepsilon$  is a constant which depends on  $\varepsilon$ .

**Lemma 2.5** If  $\alpha, \delta, \mu > 0$  and  $t > 0$  then following equality holds:

$$I^\delta \left[ t^{\mu-\delta} E_{\alpha,\mu-\delta+1}(-t^\alpha) \right] = t^\mu E_{\alpha,\mu+1}(-t^\alpha).$$

*Proof.* Using the definition of the Riemann-Liouville fractional integral and the series representation of the Mittag-Leffler function, we have

$$\begin{aligned} I^\delta [t^{\mu-\delta} E_{\alpha,\mu-\delta+1}(-t^\alpha)] &= \frac{1}{\Gamma(\delta)} \int_0^t (t-\xi)^{\delta-1} \xi^{\mu-\delta} \sum_{k=0}^{\infty} \frac{(-\xi^\alpha)^k}{\Gamma(\alpha k + \mu - \delta + 1)} d\xi \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\delta) \Gamma(\alpha k + \mu - \delta + 1)} \int_0^t (t-\xi)^{\delta-1} \xi^{\alpha k + \mu - \delta} d\xi \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k t^{\alpha k + \mu}}{\Gamma(\alpha k + \mu + 1)} = t^\mu E_{\alpha,\mu+1}(-t^\alpha). \end{aligned}$$

The proof is complete.  $\square$

### 3. Nonlocal problem for homogeneous equation

Let us consider the following nonlocal problem for homogeneous equation

$$\begin{cases} D^{\alpha,\beta} W(t) + AW(t) = 0, \quad 0 < t \leq T, \\ (I^\delta W)(T) = \gamma(I^\delta W)(+0) + \psi, \\ \frac{d}{dt}(I^\delta W)(+0) = \phi, \end{cases} \quad (3.1)$$

where  $\psi, \phi \in H$  are given functions.

We solve problem (3.1) using the Fourier method and we look for a solution of that, in the form of series  $W(t) = \sum_{k=1}^{\infty} T_k(t) v_k$ ,  $k \geq 1$ . Then we have the following problem:

$$\begin{cases} D^{\alpha,\beta} T_k(t) + \lambda_k T_k(t) = 0, \quad 0 < t \leq T, \\ (I^\delta T_k)(T) = \gamma(I^\delta T_k)(+0) + \psi_k, \\ \frac{d}{dt}(I^\delta T_k)(+0) = \phi_k, \end{cases} \quad (3.2)$$

where  $\psi_k$  and  $\phi_k$  are the Fourier coefficients of functions  $\psi$  and  $\phi$  respectively.

Let us denote  $(I^\delta T_k)(0) = b_k$ . Then, the unique solution to the differential equation (3.2) with this initial condition has the form  $T_k(t) = b_k t^{-\delta} E_{\alpha,1-\delta}(-\lambda_k t^\alpha) + \phi_k t^{1-\delta} E_{\alpha,2-\delta}(-\lambda_k t^\alpha)$  (see, [8]). From the nonlocal conditions of (3.2) we obtain the following equation to find the unknown numbers  $b_k$ :

$$I^\delta \left( b_k t^{-\delta} E_{\alpha,1-\delta}(-\lambda_k t^\alpha) + \phi_k t^{1-\delta} E_{\alpha,2-\delta}(-\lambda_k t^\alpha) \right) \Big|_{t=T} = \gamma b_k + \psi_k,$$

according to Lemma 2.5 one has

$$b_k E_{\alpha,1}(-\lambda_k T^\alpha) + \phi_k T E_{\alpha,2}(-\lambda_k T^\alpha) = \gamma b_k + \psi_k. \quad (3.3)$$

Hence,

$$b_k = \frac{\psi_k - \phi_k T E_{\alpha,2}(-\lambda_k T^\alpha)}{E_{\alpha,1}(-\lambda_k T^\alpha) - \gamma}. \quad (3.4)$$

If  $\lambda_k > 0$  and  $T > 0$ , then according to Lemma 2.2 we have  $E_{\alpha,1}(-\lambda_k T^\alpha) \neq \gamma$  for all  $|\gamma| \geq 1$  and using Lemma 2.1, we obtain following estimate

$$|b_k| \leq C_\gamma \left( |\psi_k| + \frac{|\phi_k|}{\lambda_1} \right), \quad k \geq 1. \quad (3.5)$$

Here and below, by  $C_\gamma$  we will denote a constant depending on  $\gamma$ , not necessarily the same one.

Additionally, let  $\gamma \in (-1; 0) \cup (0, 1)$ . Then, according to Lemma 2.2, there is a unique  $\lambda_0 > 0$  such that  $E_{\alpha,1}(-\lambda_0 T^\alpha) = \gamma$ . If  $\lambda_k \neq \lambda_0$  for all  $k \geq 1$ , then the estimate (3.5) holds with some constant  $C_\gamma$ .

Thus, if  $|\gamma| \geq 1$  or  $\gamma \in (-1; 0) \cup (0, 1)$ , but  $\lambda_k \neq \lambda_0$  for all  $k \geq 1$ , then the formal solution of problem (3.1) has the form

$$W(t) = \sum_{k=1}^{\infty} \left[ \left( \frac{\psi_k}{E_{\alpha,1}(-\lambda_k T^\alpha) - \gamma} - \frac{\phi_k T E_{\alpha,2}(-\lambda_k T^\alpha)}{E_{\alpha,1}(-\lambda_k T^\alpha) - \gamma} \right) t^{-\delta} E_{\alpha,1-\delta}(-\lambda_k t^\alpha) + \phi_k t^{1-\delta} E_{\alpha,2-\delta}(-\lambda_k t^\alpha) \right] v_k. \quad (3.6)$$

Additionally, if  $\gamma \in (-1; 0) \cup (0, 1)$  and  $\lambda_k = \lambda_0$  for  $k = k_0, k_0 + 1, \dots, k_0 + m_0 - 1$ , where  $m_0$  is the multiplicity of the eigenvalue  $\lambda_k$ . Then, the nonlocal problem (3.1) has a solution if and only if the boundary functions  $\psi$  and  $\phi$  satisfy the following orthogonality conditions.

$$\psi_k = (\psi, v_k) = 0, \quad \phi_k = (\phi, v_k) = 0, \quad k \in K_0; \quad K_0 = \{k_0, k_0 + 1, \dots, k_0 + m_0 - 1\}, \quad (3.7)$$

and for these  $k \in K_0$  arbitrary numbers  $b_k$  are solutions of equation (3.3). For all other  $k$  we have following:

$$b_k = \frac{\psi_k - \phi_k T E_{\alpha,2}(-\lambda_k T^\alpha)}{E_{\alpha,1}(-\lambda_k T^\alpha) - \gamma}, \quad |b_k| \leq C_\gamma (|\psi_k| + \frac{|\phi_k|}{\lambda_1}), \quad k \notin K_0. \quad (3.8)$$

Thus, the formal solution of problem (3.1) in this case has the form

$$W(t) = \sum_{k \notin K_0} \left[ \left( \frac{\psi_k}{E_{\alpha,1}(-\lambda_k T^\alpha) - \gamma} - \frac{\phi_k T E_{\alpha,2}(-\lambda_k T^\alpha)}{E_{\alpha,1}(-\lambda_k T^\alpha) - \gamma} \right) t^{-\delta} E_{\alpha,1-\delta}(-\lambda_k t^\alpha) + \phi_k t^{1-\delta} E_{\alpha,2-\delta}(-\lambda_k t^\alpha) \right] v_k + \sum_{k \in K_0} \left( b_k t^{-\delta} E_{\alpha,1-\delta}(-\lambda_k t^\alpha) + \phi_k t^{1-\delta} E_{\alpha,2-\delta}(-\lambda_k t^\alpha) \right) v_k, \quad (3.9)$$

note that the set  $K_0$  has a finite number of  $k$ . Throughout what follows we will assume that whenever  $\gamma \in (-1; 0) \cup (0, 1)$  and  $\lambda_k = \lambda_0$ , then orthogonality condition (3.7) is satisfied.

We now prove that the formal solution obtained above is, in fact, a solution to problem (3.1). To do this, We show that this solution satisfies all the condition of Definition 1.1. For this, we take the partial sum of series (3.6) and check for convergence by applying the operator  $A$ . Applying operator  $A$  to the partial sum  $S_j$ , we get

$$AS_j(t) = \sum_{k=1}^j \lambda_k \left[ \left( \frac{\psi_k}{E_{\alpha,1}(-\lambda_k T^\alpha) - \gamma} - \frac{\phi_k T E_{\alpha,2}(-\lambda_k T^\alpha)}{E_{\alpha,1}(-\lambda_k T^\alpha) - \gamma} \right) t^{-\delta} E_{\alpha,1-\delta}(-\lambda_k t^\alpha) + \phi_k t^{1-\delta} E_{\alpha,2-\delta}(-\lambda_k t^\alpha) \right] v_k.$$

Therefore,

$$\| AS_j(t) \|^2 = \left| \sum_{k=1}^j \lambda_k \left[ \left( \frac{\psi_k}{E_{\alpha,1}(-\lambda_k T^\alpha) - \gamma} - \frac{\phi_k T E_{\alpha,2}(-\lambda_k T^\alpha)}{E_{\alpha,1}(-\lambda_k T^\alpha) - \gamma} \right) t^{-\delta} E_{\alpha,1-\delta}(-\lambda_k t^\alpha) + \phi_k t^{1-\delta} E_{\alpha,2-\delta}(-\lambda_k t^\alpha) \right] v_k \right|^2.$$

According to the Parseval equality

$$\begin{aligned} \| AS_j(t) \|^2 &\leq \sum_{k=1}^j \lambda_k^2 \left| \left( \frac{\psi_k}{E_{\alpha,1}(-\lambda_k T^\alpha) - \gamma} - \frac{\phi_k T E_{\alpha,2}(-\lambda_k T^\alpha)}{E_{\alpha,1}(-\lambda_k T^\alpha) - \gamma} \right) t^{-\delta} E_{\alpha,1-\delta}(-\lambda_k t^\alpha) + \phi_k t^{1-\delta} E_{\alpha,2-\delta}(-\lambda_k t^\alpha) \right|^2 \\ &\leq \sum_{k=1}^j \lambda_k^2 \left| \frac{\psi_k}{E_{\alpha,1}(-\lambda_k T^\alpha) - \gamma} t^{-\delta} E_{\alpha,1-\delta}(-\lambda_k t^\alpha) \right|^2 \\ &+ \sum_{k=1}^j \lambda_k^2 \left| \frac{-\phi_k T E_{\alpha,2}(-\lambda_k T^\alpha)}{E_{\alpha,1}(-\lambda_k T^\alpha) - \gamma} t^{-\delta} E_{\alpha,1-\delta}(-\lambda_k t^\alpha) \right|^2 + \sum_{k=1}^j \lambda_k^2 \left| \phi_k t^{1-\delta} E_{\alpha,2-\delta}(-\lambda_k t^\alpha) \right|^2 = AS_j^1 + AS_j^2 + AS_j^3, \end{aligned}$$

where

$$\begin{aligned} AS_j^1 &= \sum_{k=1}^j \lambda_k^2 \left| \frac{\psi_k}{E_{\alpha,1}(-\lambda_k T^\alpha) - \gamma} t^{-\delta} E_{\alpha,1-\delta}(-\lambda_k t^\alpha) \right|^2, \\ AS_j^2 &= \sum_{k=1}^j \lambda_k^2 \left| \frac{-\phi_k T E_{\alpha,2}(-\lambda_k T^\alpha)}{E_{\alpha,1}(-\lambda_k T^\alpha) - \gamma} t^{-\delta} E_{\alpha,1-\delta}(-\lambda_k t^\alpha) \right|^2, \\ AS_j^3 &= \sum_{k=1}^j \lambda_k^2 \left| \phi_k t^{1-\delta} E_{\alpha,2-\delta}(-\lambda_k t^\alpha) \right|^2. \end{aligned}$$

Using estimates (3.4), (3.7) and Lemma 2.1 we obtain

$$\begin{aligned} AS_j^1 &= \sum_{k=1}^j \lambda_k^2 \left| \frac{\psi_k}{E_{\alpha,1}(-\lambda_k T^\alpha) - \gamma} t^{-\delta} E_{\alpha,1-\delta}(-\lambda_k t^\alpha) \right|^2 \\ &\leq C_\gamma \sum_{k=1}^j \lambda_k^2 t^{-2\delta} \left| \frac{\psi_k}{1 + \lambda_k t^\alpha} \right|^2 \leq \frac{C_\gamma}{t^{2\alpha+2\delta}} \sum_{k=1}^j |\psi_k|^2. \\ AS_j^2 &= \sum_{k=1}^j \lambda_k^2 \left| \frac{-\phi_k T E_{\alpha,2}(-\lambda_k T^\alpha)}{E_{\alpha,1}(-\lambda_k T^\alpha) - \gamma} t^{-\delta} E_{\alpha,1-\delta}(-\lambda_k t^\alpha) \right|^2 \\ &\leq C_\gamma \sum_{k=1}^j \lambda_k^2 t^{-2\delta} \left| \frac{\phi_k}{(1 + \lambda_k t^\alpha) \lambda_1} \right|^2 \leq \frac{C_\gamma}{\lambda_1^2 t^{2\alpha+2\delta}} \sum_{k=1}^j |\phi_k|^2. \\ AS_j^3 &= \sum_{k=1}^j \lambda_k^2 \left| \phi_k t^{1-\delta} E_{\alpha,2-\delta}(-\lambda_k t^\alpha) \right|^2 \leq C \sum_{k=1}^j \lambda_k^2 t^{2-2\delta} \left| \frac{\phi_k}{1 + \lambda_k t^\alpha} \right|^2 \\ &\leq C t^{2-2\delta-2\alpha} \sum_{k=1}^j |\phi_k|^2. \\ \| AS_j(t) \|^2 &\leq \frac{C_\gamma}{t^{2(\alpha+\delta)}} \sum_{k=1}^j |\psi_k|^2 + \left( \frac{C_\gamma}{\lambda_1^2 t^{2(\alpha+\delta)}} + t^{2-2\alpha-2\delta} \right) \sum_{k=1}^j |\phi_k|^2. \end{aligned}$$

Hence, if  $\psi, \phi \in H$  then  $AW(t) \in C((0, T]; H)$ . From equation (3.1) one has  $D_t^{\alpha, \beta} W(t) = -AW(t)$ ,  $t > 0$ . So we have  $D_t^{\alpha, \beta} W(t) \in C((0, T]; H)$ .

**Uniqueness.** Suppose problem (3.1) has two solutions:  $W_1(t), W_2(t)$  and set  $W(t) = W_1(t) - W_2(t)$ . Then we have

$$\begin{cases} D_t^{\alpha, \beta} W(t) + AW(t) = 0, & 0 < t \leq T, \\ (I^\delta W)(T) = \gamma(I^\delta W)(+0), \\ \frac{d}{dt}(I^\delta W)(+0) = 0. \end{cases}$$

We define  $W_k(t) = (W(t), v_k)$ . From the self-adjointness of the operator  $A$ , we have following equality:

$$D_t^{\alpha, \beta} W_k(t) = (D_t^{\alpha, \beta} W(t), v_k) = -(AW(t), v_k) = -(W(t), Av_k) = -\lambda_k W_k(t) \quad (3.10)$$

and the nonlocal condition implies

$$(I^\delta W_k)(T) = \gamma I^\delta W_k(+0). \quad (3.11)$$

Let us denote  $W_k(0) = b_k$ . Then, the unique solution to the differential equation (3.10) with this initial condition has the form  $W_k(t) = b_k E_{\alpha, 1}(-\lambda_k t^\alpha)$  (see, e.g. [7], p.174). From the nonlocal conditions of (3.11) we obtain the following equation to find the unknown numbers  $b_k$

$$I^\delta \left( b_k t^{-\delta} E_{\alpha, 1-\delta}(-\lambda_k t^\alpha) \right) \Big|_{t=T} = \gamma b_k,$$

and by Lemma 2.5 we have

$$b_k E_{\alpha, 1}(-\lambda_k T^\alpha) = \gamma b_k. \quad (3.12)$$

Let  $|\gamma| \geq 1$  or  $\gamma \in (-1; 0) \cup (0; 1)$ , but  $\lambda_k \neq \lambda_0$  for all  $k \geq 1$ . Then,  $E_{\alpha, 1}(-\lambda_k T^\alpha) \neq \gamma$  for all  $k$ . Consequently, in this case all  $b_k = 0$ , therefore,  $W_k(t) = 0$ , and by virtue of completeness of the set of eigenfunctions  $\{v_k\}$ , we conclude that  $W(t) \equiv 0$ . Thus, problem (3.1) in this case has a unique solution.

Now suppose that  $\gamma \in (-1; 0) \cup (0; 1)$  and  $\lambda_k = \lambda_0$ ,  $k \in K_0$ . Then,  $E_{\alpha, 1}(-\lambda_k T^\alpha) = \gamma$ ,  $k \in K_0$  and therefore equation (3.12) has the following solution:  $b_k = 0$ , if  $k \notin K_0$  and  $b_k$  is an arbitrary number for  $k \in K_0$ . Thus, in this case, there is no uniqueness of the solution to problem (3.1).

Thus, we obtain the following statement.

**Theorem 3.1.** Let  $\psi, \phi \in H$ .

If  $|\gamma| \geq 1$  or  $\gamma \in (-1; 0) \cup (0; 1)$ , but  $\lambda_k \neq \lambda_0$  for all  $k \geq 1$ , then problem (3.1) has a unique solution and this solution has the form (3.6).

If  $\gamma \in (-1; 0) \cup (0; 1)$  and  $\lambda_k = \lambda_0$ , for  $k \in K_0$ , then problem (3.1) to have a solution, the fulfillment of the orthogonality conditions (3.7) are sufficient. In this case the solution of the problem (3.1) has the form (3.9) with arbitrary coefficients  $b_k$ ,  $k \in K_0$ .

Moreover, there is constant  $C_\gamma > 0$  such that the following coercive type inequality holds:

$$\begin{aligned} & \| D_t^{\alpha, \beta} W(t) \|^2 + \| W(t) \|^2 \leq \frac{C_\gamma}{t^{2(\alpha+\delta)}} \| \psi \|^2 \\ & + \left( \frac{C_\gamma}{\lambda_1^2 t^{2(\alpha+\delta)}} + t^{2-2\alpha-2\delta} \right) \| \phi \|^2, \quad 0 < t \leq T. \end{aligned}$$

#### 4. Cauchy problem for non-homogeneous equation

We now study the following Cauchy problem

$$\begin{cases} D_t^{\alpha, \beta} V(t) + AV(t) = f(t), & 0 < t \leq T; \\ (I^\delta V)(0) = 0, \\ \frac{d}{dt}(I^\delta V)(0) = 0. \end{cases} \quad (4.1)$$

**Theorem 4.1** Let  $f(t) \in C([0, T]; D(A^\varepsilon))$  for some  $\varepsilon \in (0, 1)$ . Then, problem (4.1) has a unique solution and this solution has the following form:

$$V(t) = \sum_{k=1}^{\infty} \left( \int_0^t \tau^{\alpha-1} E_{\alpha, \alpha}(-\lambda_k \tau^\alpha) f_k(t - \tau) d\tau \right) v_k. \quad (4.2)$$

Moreover, there is a constant  $C_\varepsilon > 0$  such that the following coercive type inequality holds

$$\| D_t^{\alpha, \beta} V(t) \|^2 + \| V(t) \|_1^2 \leq C_\varepsilon \max_{t \in [0, T]} \| f(t) \|_\varepsilon^2, \quad 0 < t \leq T. \quad (4.3)$$

where  $f_k(t)$  are the Fourier coefficients of the function  $f(t)$ .

*Proof. Existence.* Assume that a solution to problem (4.1) exists. Then, due to the completeness of the system  $v_k$ , the solution can be written in the form:

$$V(t) = \sum_{k=1}^{\infty} T_k(t) v_k, \quad (4.4)$$

where  $T_k(t)$  are the Fourier coefficients of the function  $V(t)$ . Then we put the equation (4.4) to the problem (4.1), and obtain following problem:

$$\begin{cases} D_t^{\alpha, \beta} T_k(t) + \lambda_k T_k(t) = f_k(t), \\ I^\delta T_k(0) = 0, \\ \frac{d}{dt} (I^\delta T_k)(0) = 0, \quad k \geq 1. \end{cases} \quad (4.5)$$

The solution of the problem (4.5) has the form (see, for example [7]):

$$T_k(t) = \int_0^t \tau^{\alpha-1} E_{\alpha, \alpha}(-\lambda_k \tau^\alpha) f_k(t - \tau) d\tau. \quad (4.6)$$

Thus, according to the equalities (4.4) and (4.6), we find the formal solution of the problem (4.1) as the form (4.2).

To prove the uniqueness of the solution, we use the standart technique, that is, the solution of the problem (4.1) with the homogeneous condition is identically zero. Taking into account  $f(t) = 0$ , then the Fourier coefficients of that function  $f_k(t)$  is zero (i.e.  $f_k(t) = 0$ ). Then it follows  $T_k(t) \equiv 0$ , for all  $k \geq 1$ . In that case,  $u(t) \equiv 0$  derives from the equality (4.4) and the completeness of the system  $\{v_k\}$ .

We now verify that the formal solution satisfies the conditions of Definition 1.1. We denote the partial sum of series (4.2) by  $S_n(t)$ .

Now we apply the operator  $A$  on the partial sum  $S_n(t)$ , then we have:

$$AS_n(t) = \sum_{k=1}^n \left( \int_0^t \tau^{\alpha-1} E_{\alpha, \alpha}(-\lambda_k \tau^\alpha) f_k(t - \tau) d\tau \right) \lambda_k v_k.$$

Due to the Parseval equality we may write

$$\| AS_n(t) \|^2 = \sum_{k=1}^n \lambda_k^2 \left| \int_0^t \tau^{\alpha-1} E_{\alpha, \alpha}(-\lambda_k \tau^\alpha) f_k(t - \tau) d\tau \right|^2.$$

According to Lemma 2.4, we have:

$$AS_n^2 = \sum_{n=1}^j \lambda_k^2 \left| \int_0^t \tau^{\alpha-1} E_{\alpha, \alpha}(-\lambda_k \tau^\alpha) f_k(t - \tau) d\tau \right|^2 \leq C_\varepsilon \max_{t \in [0, T]} \| f(t) \|_\varepsilon^2.$$

Therefore,

$$\| AS_n(t) \|^2 \leq C_\varepsilon \max_{t \in [0, T]} \| f(t) \|_\varepsilon^2.$$

Hence, if  $f(t) \in C([0, T]; D(A^\varepsilon))$  we obtain  $AV(t) \in C((0, T]; H)$  and in particular  $V(t) \in C([0, T]; H)$ .

Further, from equation (4.1) one has  $D_t^{\alpha, \beta} S_n(t) = -AS_n(t) + \sum_{k=1}^n f_k(t)v_k$ ,  $t > 0$ . Therefore, from the above reasoning, we have  $D^{\alpha, \beta} V(t) \in C((0, T]; H)$  and

$$\|D_t^{\alpha, \beta} S_n(t)\|^2 \leq C_\varepsilon \max_{t \in [0, T]} \|f(t)\|_\varepsilon^2 + \|f(t)\|^2, \quad t > 0.$$

Thus, we have completed the rationale that (4.2) is a solution to problem (4.1). The last two inequalities imply estimate (4.3).

## 5. Solution of the nonlocal problem

Now we turn attention to the solution of the main problem (1.1).

Let  $\varphi \in H$  and  $f(t) \in C((0, T]; D(A^\varepsilon))$  for some  $\varepsilon \in (0, 1)$ . If in problem (3.1) we take the function  $\psi$  in a nonlocal condition in the form  $\psi = (\varphi - (I^\delta V)(T)) \in H$ , then  $W(t) + V(t)$  is a solution of the main problem (1.1), where  $W(t)$  is a solution of (3.1) and  $V(t)$  is a solution of (4.1). Thus, if  $|\gamma| \geq 1$  or  $\gamma \in (-1; 0) \cup (0; 1)$ , but  $\lambda_k \neq \lambda_0$  for all  $k \geq 1$ , then

$$u(t) = \sum_{k=1}^{\infty} \left[ \left( \frac{\varphi_k - (I^\delta V_k)(T)}{E_{\alpha, 1}(-\lambda_k T^\alpha) - \gamma} - \frac{\phi_k T E_{\alpha, 2}(-\lambda_k T^\alpha)}{E_{\alpha, 1}(-\lambda_k T^\alpha) - \gamma} \right) t^{-\delta} E_{\alpha, 1-\delta}(-\lambda_k t^\alpha) + \phi_k t^{1-\delta} E_{\alpha, 2-\delta}(-\lambda_k t^\alpha) + V_k(t) \right] v_k, \quad (5.1)$$

where

$$V_k(t) = \int_0^t \tau^{\alpha-1} E_{\alpha, \alpha}(-\lambda_k \tau^\alpha) f_k(t - \tau) d\tau.$$

If  $\gamma \in (-1; 0) \cup (0; 1)$  and  $\lambda_k = \lambda_0$ ,  $k \in K_0$ , then

$$u(t) = \sum_{k \notin K_0} \left[ \left( \frac{\varphi_k - (I^\delta V_k)(T)}{E_{\alpha, 1}(-\lambda_k T^\alpha) - \gamma} - \frac{\phi_k T E_{\alpha, 2}(-\lambda_k T^\alpha)}{E_{\alpha, 1}(-\lambda_k T^\alpha) - \gamma} \right) t^{-\delta} E_{\alpha, 1-\delta}(-\lambda_k t^\alpha) + \phi_k t^{1-\delta} E_{\alpha, 2-\delta}(-\lambda_k t^\alpha) + V_k(t) \right] v_k + \sum_{k \in K_0} \left( b_k t^{-\delta} E_{\alpha, 1-\delta}(-\lambda_k t^\alpha) + \phi_k t^{1-\delta} E_{\alpha, 2-\delta}(-\lambda_k t^\alpha) + V_k(t) \right) v_k.$$

The corresponding orthogonality conditions have the form

$$(\varphi, v_k) = (I^\delta V(t), v_k) = (\phi, v_k), \quad k \in K_0; \quad K_0 = \{k_0, k_0 + 1, \dots, k_0 + m_0 - 1\}. \quad (5.2)$$

In particular, if

$$(\varphi, v_k) = 0, \quad (\phi, v_k) = 0, \quad ((I^\delta V)(T), v_k) = 0,$$

for all

$$k \in K_0; \quad K_0 = \{k_0, k_0 + 1, \dots, k_0 + m_0 - 1\},$$

then the orthogonality conditions (5.2) are satisfied. Theorem 1.2 is completely proved.

## 6. Numerical illustration

In this section, we present a concrete numerical example to illustrate the theoretical results obtained in the previous sections. In particular, we demonstrate:

1. The loss of uniqueness when some eigenvalue  $\lambda_k$  equals the critical value  $\lambda_0$ ;
2. The influence of the Hilfer parameter  $\beta$  on the regularity of the solution;
3. The behavior of the solution for different choices of the arbitrary constants  $b_k$ .

**6.1 Choice of the operator and space.** It should be emphasized that the main results of this work are proved by the Fourier method, which is applicable to any positive self-adjoint operator with a discrete spectrum. To illustrate these results, we replace the abstract operator  $A$  by a concrete operator with Dirichlet boundary conditions.

Let  $\Omega = (0, \pi) \subset \mathbb{R}$  and consider the operator

$$A = -\frac{d^2}{dx^2},$$

with the domain

$$D(A) = \{u \in H^2(0, \pi) : u(0) = u(\pi) = 0\}.$$

It is well known that  $A$  is positive and self-adjoint operator in  $H = L_2(0, \pi)$ , with eigenvalues

$$\lambda_k = k^2, \quad k = 1, 2, 3, \dots$$

and corresponding orthonormal eigenfunctions

$$v_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx), \quad x \in (0, \pi).$$

With this choice, problem (1.1) takes the following explicit form:

$$\begin{cases} D_t^{\alpha, \beta} u(x, t) - \frac{d^2}{dx^2} u(x, t) = f(x, t), & 0 < x < \pi, \quad 0 < t \leq T, \\ u(0, t) = u(\pi, t) = 0, \\ I^\delta u(x, t)|_{t=T} = \gamma I^\delta u(x, +0) + \varphi(x), \\ \lim_{t \rightarrow +0} \frac{d}{dt} I^\delta u(x, t) = \phi(x). \end{cases} \quad (6.1)$$

Here, the eigenvalues and eigenfunctions of  $A$  will be used in the Fourier series expansion of the solution.

**6.2 Choice of the functions  $f(x, t)$ ,  $\varphi(x)$  and  $\phi(x)$**

For the numerical illustration, we need to specify the source term  $f(x, t)$  and the functions  $\varphi(x)$  and  $\phi(x)$  appearing in the nonlocal conditions. Their Fourier coefficients will be computed using the orthonormal eigenfunctions  $v_k(x) = \sqrt{\frac{2}{\pi}} \sin(kx)$ .

We take the source function

$$f(x, t) = 0.3 t \sin 2x, \quad x \in (0, \pi), \quad t \in [0, T].$$

Its Fourier coefficients are computed as follows:

$$f_k(t) = \int_0^\pi f(x, t) v_k(x) dx = 0.3 t \int_0^\pi \sin(2x) \cdot \sqrt{\frac{2}{\pi}} \sin(kx) dx.$$

Due to orthogonality of the sine functions on  $(0, \pi)$ , the integral is nonzero only when  $k = 2$ :

$$f_2(t) = 0.3 t \sqrt{\frac{2}{\pi}} \int_0^\pi \sin^2(2x) dx = 0.3 t \sqrt{\frac{\pi}{2}},$$

and  $f_k(t) = 0$  for  $k \neq 2$ . Thus, the source term activates only the second mode, with a coefficient proportional to  $t$ .

Additionally, we take the same function for both  $\varphi(x)$  and  $\phi(x)$ :

$$\varphi(x) = \phi(x) = 0.5 \sin(x) + \sin(2x), \quad x \in (0, \pi).$$

Their Fourier coefficients are computed in the same way as for the source function  $f(x, t)$ , using the orthonormal eigenfunctions:

$$\varphi_1 = \phi_1 = \int_0^\pi (0.5 \sin x + \sin 2x) \cdot \sqrt{\frac{2}{\pi}} \sin x dx = 0.5 \sqrt{\frac{\pi}{2}},$$

$$\varphi_2 = \phi_2 = \int_0^\pi (0.5 \sin x + \sin 2x) \cdot \sqrt{\frac{2}{\pi}} \sin(2x) dx = \sqrt{\frac{\pi}{2}}.$$

For  $k \neq 1, 2$ , both terms are orthogonal to  $v_k(x)$ , hence

$$\varphi_k = \phi_k = 0, \quad k = 3, 4, \dots$$

Therefore, now we present graphical and numerical references of our results that illustrate the theoretical findings obtained in the previous sections.

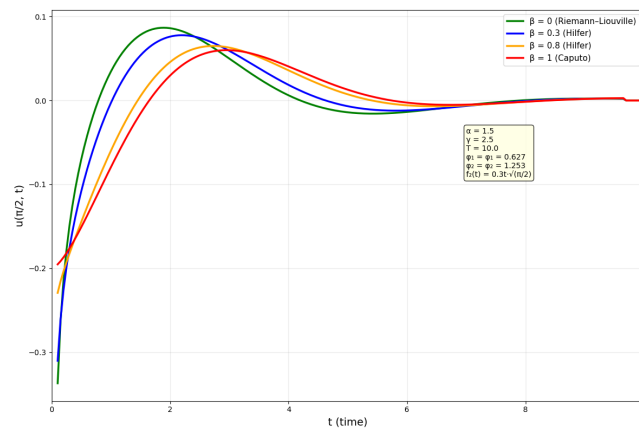


Рис. 4: (A) Solution for the case  $\gamma \geq 1$ , where no critical value exists (unique solution).

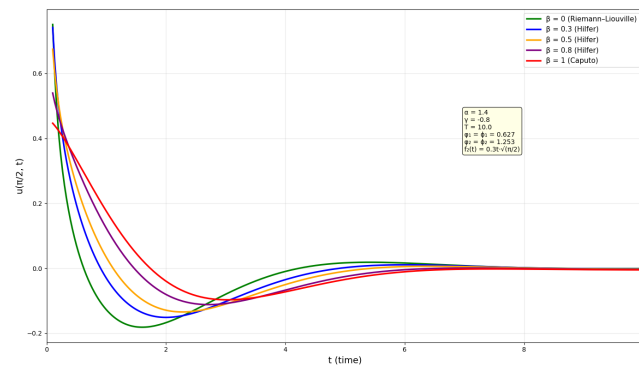


Рис. 5: (B) Solution for the case  $-1 < \gamma < 0 \cup 0 < \gamma < 1$ , but with  $\lambda_k \neq \lambda_0$  for all  $k$ , so the solution remains unique.

Figures 1 and 2 demonstrate the pointwise traces  $u(\frac{\pi}{2}, t)$  in the admissible regions. The solution for the case  $\gamma \geq 1$  with parameter  $\alpha = 1.5$  and for  $-1 < \gamma < 0 \cup 0 < \gamma < 1$ , but with  $\lambda_k \neq \lambda_0$  for all  $k$ . In the first case, there is no critical value, and the solution is unique. In the second case, although the critical value  $\lambda_0$  exists (since  $\gamma < 1$ ), it does not coincide with any eigenvalue  $\lambda_k$ , hence the solution remains unique. The figure also shows the influence of the Hilfer parameter  $\beta$ : as  $\beta$  increases from 0 to 1, the solution transitions smoothly from the Riemann-Liouville case ( $\beta = 0$ , slower decay) to the Caputo case ( $\beta = 1$ , faster decay), confirming that  $\beta$  serves as an interpolation parameter between the two classical fractional derivatives.

Figure 3 illustrates the solution for the case  $-1 < \gamma < 0 \cup 0 < \gamma < 1$ , with  $\lambda_k = \lambda_0$  for some  $k \in K_0$ . The parameters are  $\alpha = 1.5$ ,  $T = 7.0$ ,  $\beta = 0.5$ , chosen so that  $\lambda_0 = \lambda_1 = 1$ . Different choices of the arbitrary constant

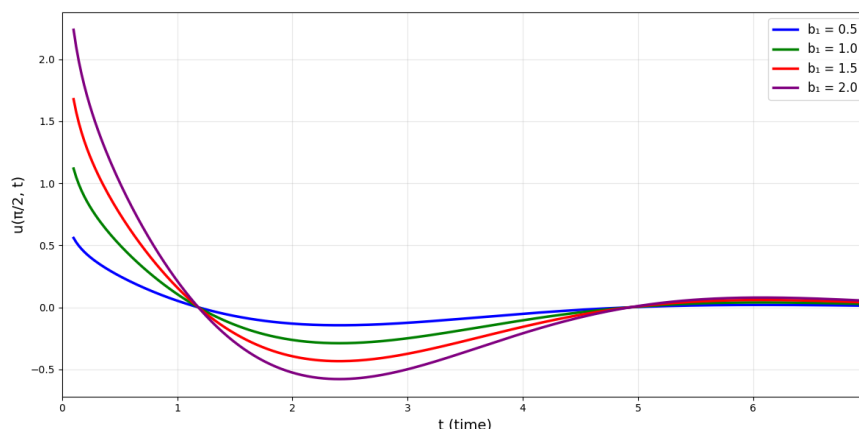


Рис. 6: Solution for the case  $-1 < \gamma < 0 \cup 0 < \gamma < 1$ , with  $\lambda_k = \lambda_0$  for some  $k \in K_0$ .

$b_1 = 0.5, 1.0, 1.5, 2.0$  lead to different solutions, performing the loss of uniqueness when the spectrum contains the critical value. The solutions decay to zero as  $t$  increases. This decline is primarily due to the asymptotic behavior of the Mittag-Leffler function (see Lemma 2.3).

The factor  $t^{-\delta}$  with  $\delta = 0.25$  has a mild influence, but the main decreasing mechanism comes from the Mittag-Leffler function itself. This behavior is physically natural, as the equation models diffusion-like processes where the system tends to an equilibrium state (zero) over time.

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# CONTROLLABILITY VIA KILLING VECTOR FIELDS ON THE SPHERE AND TRANSVERSE SINGULAR FOLIATIONS

SAITOVA SAYYORA\*

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M.ULUGBEK, TASHKENT, UZBEKISTAN  
sayyorass1985@gmail.com

ABDUSALOMOVA OZODA

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M.ULUGBEK, TASHKENT, UZBEKISTAN  
ozodaa226@gmail.com

**ABSTRACT.** In this paper, we investigate the controllability on sphere and transverse singular foliations generated by family of Killing vector fields. We construct an explicit family of vector fields on  $R^n$  for corresponding Sphere, compute the Lie algebra generated by them, and describe the corresponding orbits. It is shown that the regular leaves are spheres, while the origin is a singular leaf. Such construction provides a concrete local model for transverse singular foliations and illustrates their relation with orbit decompositions and sphere fibrations.

**Key words:** singular foliation, transversal singular foliation, Killing vector fields, Lie bracket.

## Introduction

Although singular foliations are less studied than regular foliations, they appear frequently in differential geometry. In the 1970s, the definition of singular foliations was the subject of intense debate [4]. Subsequently, the definition proposed by Hermann [1] was almost universally adopted: a singular foliation is a Lie-bracket-closed, locally finitely generated submodule of the module of vector fields. The theory of foliations plays an important role in differential geometry, as it allows one to study the local and global structure, dynamics, and topological properties of high-dimensional smooth manifolds by decomposing them into leaves. In relativity, spacetime is modeled as a four-dimensional manifold that admits foliations, providing a mathematical framework for describing the interplay between space and time. Particle trajectories can be interpreted within such foliated structures.

In this research, orbits of a family of Killing vector fields on the sphere can be regarded as orbits of an isometric action, i.e., rotations, of a connected subgroup  $G \subset SO(n+1)$ . Each Killing field has the form  $X_A(x) = Ax$ , where  $A \in SO(n+1)$ , and the whole family generates a Lie algebra  $G \subset SO(n+1)$ . Therefore, the orbit of a point  $x \in S^n$  is equal to  $G \cdot x$ , and the dimension of tangent space is given by the formula  $\dim(G \cdot x) = \dim g - \dim g_x$ . For the algebra  $SO(n+1)$ , the action is transitive, and the whole sphere is one orbit. For smaller subalgebras, the sphere decomposes into regular and singular orbits, which is naturally interpreted as a singular Riemannian foliation. In the context of Brockett's control systems, reachable sets on the sphere coincide with such orbits; controllability means that the orbit coincides with the whole sphere. In the case of Killing foliations, modern transverse sphere theorems show that spaces of leaf closures can be described as quotient spaces of actions on spheres. [14-17]

## Preliminaries

We introduce the notion of formal singular foliation along a leaf  $L$  by statements in [4]. Such singular foliation on a manifold  $M$  admitting  $L$  as a leaf induces some transversal singular foliation.

Let  $B$  be a commutative algebra and  $B = [x_1, x_2, \dots, x_d]$  be the algebra of formal power series in  $d$  variables, i.e. the algebra of formal sums

$$F = \sum F_{i_1 \dots i_d} x_1^{i_1} \dots x_d^{i_d}$$

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\*Corresponding autor

with  $F_{i_1, \dots, i_d} \in B$ . This algebra is filtered by the decreasing sequence  $(B_{\geq n})_{n \in \mathbb{N}}$  with  $B_{\geq n}$  being the ideal of formal functions of valuation  $\geq n$  the ideal generated by monomials of degree  $\geq n$ . It is a complete algebra with respect to the topology induced by this filtration. We call a formal neighborhood of a manifold  $L$  a sheaf of filtered commutative algebras

$$\mathcal{U} \mapsto C^{formal}(\mathcal{U})$$

over  $L$  that satisfies the property that each point  $l \in L$  admits a neighborhood  $\mathcal{U} \subset L$  such that

$$C^{formal}(\mathcal{U} \simeq C^\infty(\mathcal{U})[x_1, \dots, x_d].$$

Elements of this sheaf shall be called formal functions. This sheaf is a sheaf of filtered algebras  $(C_{\geq k}^{formal})$  with respect to a decreasing filtration given by the valuation of the formal power series. In other words, the  $k$ -th ideal of this filtration is made of formal functions which in any neighborhood as in read:

$$\sum_{i_1 + \dots + i_d} f_{i_1 \dots i_d} x_1 \dots x_d$$

with  $f_{i_1 \dots i_d}$  being smooth functions on  $\mathcal{U}$ . We denote by  $C_{\geq k}^{formal}$  this ideal and call it the ideal of formal functions vanishing along  $L$  at order at least  $k$ . In particular,  $I_L := C_{\geq 1}^{formal}$  is the kernel of the natural map  $C^{formal}(L) \rightarrow C^\infty(L)$  and will be referred to as the ideal of formal functions vanishing along  $L$ .

We call vector field on a formal neighborhood  $C^{formal}$  of  $L$  a derivation of the sheaf  $C^{formal}$ . The space  $\mathfrak{X}^{formal}$  of all vector fields is a sheaf of Lie algebra and of  $C^{formal}$ -modules - both structures being related by the usual Leibniz identity:

$$[X, FY] = F[X, Y] + X[F]Y$$

for all  $F \in C^{formal}$  and  $X, Y \in \mathfrak{X}$ . We say that a formal vector field  $X \in \mathfrak{X}^{formal}$  is tangent to  $L$  if it preserves the ideal  $C_{\geq 1}^{formal}$ . Such a vector field can be evaluated at a point  $l \rightarrow L$  give an element in the tangent space  $T_l L$  that we denote by  $X_l$ .

**Definition 1.** [4.Def.1.9.p.7] Let  $L$  be a manifold. We call a formal singular foliation on a formal neighborhood  $C^{formal}$  of the manifold  $L$  a subsheaf  $\mathcal{F}$  of  $\mathfrak{X}^{formal}$  that satisfies the three following conditions:

- 1)  $\mathcal{F}$  is stable under Lie bracket,
- 2)  $\mathcal{F}$  is stable under multiplication by  $C^{formal}$ ,
- 3)  $\mathcal{F}$  is a locally finitely generated  $C^{formal}$ -module.

When, in addition,

- a) every  $X \in \mathcal{F}$  is tangent to  $L$
- b) for every  $u \in T_l L$ , there exists  $X \in \mathcal{F}$  such that  $X_l = u$  then speak of a formal singular foliation along the leaf  $L$ .

Let us first define the transverse singular foliation.

**Definition 2.** [4] Let  $\mathcal{F}$  be a formal singular foliation along the leaf  $L$ .

(1) formal vector fields in  $X \in \mathcal{F} \cap \text{Vert}_\pi$  shall be called vertical vector fields in  $\mathcal{F}$  and denoted by  $\mathcal{T}$ . By construction,  $\mathcal{T}$  is a  $C^{formal}$ -module stable under Lie bracket. It is locally finitely generated, so is a singular foliation along  $L$  that we call the transverse singular foliation.

(2) For every  $l \in L$ , any formal vector field in  $\mathcal{T}$  restricts to yield a formal vector field on the normal bundle  $T_l$ . The image of this restriction is a formal singular foliation on the finite dimensional vector space  $T_l$  that we denote by  $(T_l, \mathcal{T}_l)$ . By construction, it is made of vector fields that vanish at  $0 \in T_l$ . We call  $(T_l, \mathcal{T}_l)$  the transverse singular foliation at  $l \in L$ .

Investigating the definitions above we have the following definition as corollary for our investigations:

**Definition 3 (Transverse singular foliation).** Let  $L \subset M$  be a submanifold which is a leaf of the singular foliation  $F$ .

Let  $\pi : T \rightarrow L$  denote the normal bundle of  $L$ . Define

$$\mathfrak{T} = \text{Vert}_\pi \cap F,$$

where  $\text{Vert}_\pi$  is the family of vertical vector fields tangent to the fibers of  $\pi$ . Then  $\mathfrak{T}$  consists of the vertical vector fields that are tangent to the leaves of  $F$ .

For a point  $l \in L$ , let

$$T_l = \pi^{-1}(l)$$

be the fiber over  $l$ . If

$$\pi^{-1}(l) \perp T_l L,$$

then  $\mathfrak{T}$  induces a singular foliation  $\mathfrak{T}_l$  on the fiber  $T_l$ . The pair

$$(T_l, \mathfrak{T}_l)$$

is called the *transversal singular foliation at the point  $l$* .

**Remark.** For a transversal singular foliation to be well defined, the tangent space  $T_l L$  of the leaf and the normal fiber  $T_l = \pi^{-1}(l)$  must be transversal, that is, orthogonal at every point. Thus, a transversal singular foliation describes the local structure of the foliation in directions normal to the leaf.

### Main part

Let

$$S^n = \left\{ x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1 \right\}.$$

For  $1 \leq i < j \leq n+1$ , denote by

$$R_{ij} = -x_j \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial x_j}$$

the infinitesimal rotation in the coordinate plane  $(x_i, x_j)$ . Each  $R_{ij}$  is tangent to  $S^n$  and is a Killing vector field, since it is generated by a one-parameter subgroup of orthogonal transformations of  $\mathbb{R}^{n+1}$ .

The vector fields for our statement can be written in the form

$$X = R_{12} + R_{34} + R_{56} + \dots, \quad Y = R_{23} + R_{45} + R_{67} + \dots.$$

Hence  $X$  and  $Y$  are Killing vector fields on  $S^n$ .

The research on the properties of orbits and Lie algebras given in [15,16,19] by definitions in [18] provide the following

**Theorem 1.** Let  $\mathcal{A}(D)$  be the minimal Lie algebra of vector fields containing  $D = \{X, Y\}$ . Since the Lie bracket of Killing vector fields is again a Killing vector field, all elements of  $\mathcal{A}(D)$  are Killing vector fields.

We show that the orbit of  $D$  has dimension  $n$ . Consider the point

$$p = (1, 0, \dots, 0) \in S^n.$$

At this point,

$$X(p) = \frac{\partial}{\partial x_2}.$$

Using the standard commutation relations of infinitesimal rotations,

$$[R_{ij}, R_{jk}] = R_{ik}, \quad i < j < k,$$

we obtain successively vector fields from  $\mathcal{A}(D)$  whose values at  $p$  are

$$\frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_3}, \quad \dots, \quad \frac{\partial}{\partial x_{n+1}}.$$

Indeed, repeated Lie brackets of  $X$  and  $Y$  generate, at the point  $p$ , the directions corresponding to the coordinate rotations  $R_{12}, R_{13}, \dots, R_{1,n+1}$ . Therefore

$$\mathcal{A}_p(D) = \{Z(p) : Z \in \mathcal{A}(D)\} = T_p S^n.$$

Consequently,

$$\dim L(p) = \dim T_p S^n = n.$$

By the orbit theorem for families of smooth vector fields, the orbit  $L(p)$  is an immersed submanifold whose tangent space at  $p$  is  $\mathcal{A}_p(D)$ . Since this tangent space has maximal possible dimension, the orbit  $L(p)$  is open in  $S^n$  (see [10,13]).

On the other hand, the vector fields under consideration are Killing vector fields on the sphere. By the theorem on orbits of Killing vector fields, every orbit of such a family is a closed subset of the Riemannian manifold. Hence  $L(p)$  is both open and closed in  $S^n$  ([13,15]).

Since  $S^n$  is connected, any nonempty subset which is simultaneously open and closed must coincide with the whole sphere. Therefore,

$$L(p) = S^n.$$

So we proved the following

**Theorem 2.** For given Killing vector fields  $X = \{-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7\}$  and  $Y = \{0, -x_3, x_2, -x_5, x_4, -x_7, x_6, 0\}$  the orbit of the family  $D = \{X, Y\}$  is the whole sphere.

Now using Theorem 2, we construct the singular foliation with leaves as 4-Spheres.

Let be  $X = \{-x_2, x_1, -x_4, x_3, 0\}$  and  $Y = \{0, -x_3, x_2, -x_5, x_4, \}$  vector fields defined on the  $R^5$  Euclidean space. Consider the minimal algebra generated by these vector fields. Since the distribution constructed from this algebra is integrable, it defines a corresponding foliation. One determines what type of manifolds its leaves are, and identifies the singular foliation that serves as a transversal model to this foliation. Solution: In order to construct the algebra, we compute the Lie brackets of the given vector fields as follows.

$$\begin{aligned} X &= \{-x_2, x_1, -x_4, x_3, 0\} \text{ or } R_{12} + R_{34} \\ Y &= \{0, -x_3, x_2, -x_5, x_4\} \text{ or } R_{23} + R_{45} \\ Z &= [X, Y] = \sum (X(Y^i) - Y(X^i))\vec{e}_i = \sum_i \sum_j (X^j \frac{\partial Y^i}{\partial X^j} - Y^j \frac{\partial X^i}{\partial Y^j})\vec{e}_i \end{aligned}$$

Substituting the corresponding functions into the above formulas, we obtain a vector field. Performing the computations using the resulting vector field, we arrive at the following results:

$$\begin{aligned} Z &= [X, Y] = \{-x_3, x_4, x_1 - x_5, -x_2, x_3\} \\ A &= [X, Z] = \{2x_4, 2x_3, 2x_2, -2x_1 + x_5, -x_4\}, \\ B &= [Y, Z] = \{-x_2, x_1, -2x_4, -2x_3, 0\} \\ D &= [A, B] = \{3x_3, 3x_4, 3x_1 - x_5, -3x_2, x_3\} \\ F &= [B, D] = \{9x_4, 3x_3, -3x_2, -9x_1 + 2x_5, 2x_4\} \\ B_1 &= B - X = \{0, 0, -x_4, x_3, 0\} \\ D_1 &= D - 3Z = \{0, 0, 2x_5, 0, -2x_3\} \\ X_1 &= X - B_1 = \{-x_2, x_1, 0, 0, 0\} \\ C_1 &= [B_1, D_1] = \{0, 0, 0, x_5, -x_4\} \\ Y_1 &= Y + C_1 = \{0, -x_3, x_2, 0, 0\} \\ F_1 &= F - 2C_1 = \{9x_4, x_3, -3x_2, -9x_1, 0\} \\ F_2 &= F_1 + 3Y_1 = \{x_4, 0, 0, -x_1, 0, \} \end{aligned}$$

These vector fields, namely  $B_1, D_1, X_1, C_1, Y_1, F_2$  form a basis of vector fields hence, the minimal algebra can be completely described using them. Using the basis vector fields, we construct a matrix and compute its rank. Since the rank of the matrix is 4, the minimal algebra generated by the given vector fields also has dimension 4, and the corresponding orbit is a 4-dimensional sphere.

Embedding  $R^5$  together with the 4-dimensional sphere constructed into  $R^8$  we obtain the following vector fields:

$$\begin{aligned} V_1 &= \{0, 0, 0, 0, -x_6, x_5, 0, 0\} \\ V_2 &= \{0, 0, 0, 0, -x_7, 0, x_5, 0\} \\ V_3 &= \{0, 0, 0, 0, -x_8, 0, 0, x_5\} \end{aligned}$$

We compute the Lie bracket of these vector fields:

$$W_1 = \{0, 0, 0, 0, 0, -x_7, x_6, 0\}$$

$$W_2 = \{0, 0, 0, 0, 0, -x_8, x_7\}$$

$$W_3 = \{0, 0, 0, 0, 0, -x_8, 0, x_6\}$$

Later these vector fields are vertical vector fields for a transversal singular foliation. Let's check the dimension

$$(V_1, V_2, V_3, W_1, W_2, W_3) = \begin{pmatrix} 0 & 0 & 0 & 0 & -x_6 & x_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_7 & 0 & x_5 & 0 \\ 0 & 0 & 0 & 0 & -x_8 & 0 & 0 & x_5 \\ 0 & 0 & 0 & 0 & 0 & -x_7 & x_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_8 & x_7 \\ 0 & 0 & 0 & 0 & 0 & -x_8 & 0 & x_6 \end{pmatrix}$$

$$M = \begin{pmatrix} 0 & 0 & 0 & x_5 & -x_4 & 0 & 0 & 0 \\ 0 & -x_4 & x_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_5 & 0 & x_3 & 0 & 0 & 0 \\ -x_2 & -x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_4 & 0 & 0 & -x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -x_6 & x_5 & 0 \\ 0 & 0 & 0 & 0 & -x_8 & 0 & 0 & x_5 \\ 0 & 0 & 0 & 0 & -x_8 & 0 & 0 & x_5 \end{pmatrix}$$

If  $x_i = 0$ , then  $\text{rank}(M) = 0$ .

For  $x_1 \neq 0, x_6 \neq 0$ :  $M_1 = x_1 \neq 0$ ,

$$M_2 = \begin{vmatrix} x_1 & 0 \\ 0 & -x_1 \end{vmatrix} = -x_1^2 \neq 0,$$

$$M_3 = \begin{vmatrix} x_1 & 0 & 0 \\ 0 & -x_1 & 0 \\ 0 & 0 & x_6 \end{vmatrix} = x_1^2 x_6 \neq 0$$

And continue we introduce a modification to the base matrix in order to compute higher-order ranks. More precisely, instead of the last two vector fields, we take the result of their pairwise Lie bracket, that is  $W_1, W_2$ . Then correspondingly, using similar transformations  $M_4 \neq 0, M_5 \neq 0, M_6 \neq 0, M_7 \neq 0$  remain, and only  $M_8 = 0$ .

The rank of the matrix we constructed is  $\text{rank} = 7$ . From this, we can draw the following conclusion: in the Euclidean space  $R^8$  the spheres  $S^3$  and  $S^4$  are transversal to each other. Conclusion. The singular foliation consisting of the 4- and 3-dimensional spheres that we constructed, which are mutually transversal, provides a transversal singular model.

Now we investigate the properties of transversal singular model using the Theorem 3 from [9, Th 2, p3] and riemannian submersions generated by the Killing vector fields.

**Theorem 3.** [9] If  $F_j$  are sets of vector fields on manifolds  $M_j$  for  $(j = 0, 1)$   $\varphi : M_0 \rightarrow M_1$  satisfies  $\varphi_* F_0 = F_1$  then  $\varphi$  takes  $F_0$ -orbits into  $F_1$ -orbits. On each orbit  $\varphi$  constant rank. If the vector fields in both families are complete, then  $\varphi$  is a fiber bundle mapping on each orbit.

Using the method developed for  $S^4$ , we now consider a geometric realization of a transversal singular foliation on the sphere  $S^7$ .

Let

$$D_1 = \{X_1, X_2\}$$

be a family of Killing vector fields (rotations) on  $S^7$ , where

$$X_1 = \{-x_2, x_1, -x_4, x_3, 0, 0, 0, 0\} = R_{12} + R_{34},$$

and

$$X_2 = \{0, -x_3, x_2, 0, 0, 0, 0\} = R_{23}.$$

Denote by

$$A_1 = \text{minAlg}(D_1)$$

the minimal Lie algebra generated by  $D_1$ .

Let  $P_1$  be the distribution generated by  $A_1$ , that is,

$$P_1 : x \mapsto A_1(x).$$

By direct computation, the distribution  $P_1(x) = A_1(x)$  is involutive in the sense of Hermann [6]. Therefore, it is integrable and admits corresponding integral manifolds.

These integral manifolds coincide with the orbits  $L(x)$  of the family  $D_1$ , that is, with the leaves of the foliation.

The corresponding orbits are 3-dimensional spheres  $S^3$ . Consequently, they form a singular foliation consisting of concentric spheres centered at the origin. This foliation has maximal dimension 3 on the set

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 > 0,$$

and dimension 0 elsewhere.

Consider the map

$$\varphi : x \mapsto L(x),$$

or equivalently,

$$\varphi : M \longrightarrow M/F \tag{1}$$

where  $M = S^7$  and  $F$  is the foliation generated by the distribution  $P_1$ , whose leaves are the orbits  $L(x)$ .

Now we need the result of Detlef Gromoll and Gerard Walschap stated in Section (iv) on page 8 of [?]:

**Theorem 4 [20, Th 1.2.1. p6].** Let  $\pi : M \rightarrow B$  be a submersion with connected fibers, where  $M$  is a Riemannian manifold. Then there exists a metric on  $B$  for which  $\pi$  becomes Riemannian iff the Lie derivative  $\mathcal{L}_U g^h$  of  $g^h$  vanishes any vertical direction  $U$ .

In our suggestions, vertical directions  $U$  are given by Killing vector fields, so,  $\mathcal{L}_U g^h = 0$ . Then the map

$$\varphi : x \mapsto L(x)$$

is a Riemannian submersion and

$$\text{Vert } \varphi_* = A_1.$$

Where The differential of  $\varphi$  is  $\varphi_* = d\varphi$ ,

and satisfies

$$\ker(\varphi_*) = A_1,$$

or equivalently,

$$\ker(\varphi_*)(x) = A_1(x).$$

Now consider the family  $D = \{Y_1, Y_2\}$ ,

where

$$Y_1 = \{-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7\},$$

and

$$Y_2 = \{0, -x_3, x_2, -x_5, x_4, -x_7, x_6, 0\}.$$

The orbits of this family cover the entire sphere  $S^7$ .

For the map  $\varphi$ , we have

$$\text{Im } \varphi_*(x) = A/A_1 = A_2,$$

where  $A$  denotes the minimal Lie algebra generated by the family  $D$ .

To determine  $A_2$ , consider the irreducible Killing vector fields  $R_{ij}$  generating the algebras  $A$  and  $A_1$ . Every element of these algebras can be expressed as a linear combination of such vector fields.

The irreducible elements of  $A_1$  are

$$R_{ij}, \quad 1 \leq i < j \leq 4,$$

while the irreducible elements of  $A$  are

$$R_{ij}, \quad 1 \leq i < j \leq 8.$$

Therefore, the irreducible elements of the quotient algebra  $A/A_1$  are

$$B_{ij} = R_{ab}, \quad 4 \leq i < j \leq 8.$$

Setting

$$a = i - 3, \quad b = j - 3,$$

we obtain

$$1 \leq a < b \leq 5.$$

Hence,

$$A_2 = \{R_{ab} \mid 1 \leq a < b \leq 5\}.$$

The orbits corresponding to the distribution generated by  $A_2$  are 4-dimensional spheres  $S^4$  through every point of  $S^7$ .

Thus, we have that

$$\varphi : S^7 \longrightarrow S^4$$

is a Riemannian submersion.

Moreover,

$$\varphi_*(R_{ij}) = R_{ab}, \quad 4 \leq i < j \leq 8,$$

where

$$a = i - 3, \quad b = j - 3,$$

and

$$\varphi_*(R_{ij}) = 0$$

in all other cases.

Therefore, the differential of  $\varphi$  maps the family

$$D = \{R_{ij} \mid 1 \leq i < j \leq 8\}$$

on  $S^7$  to the family

$$D_2 = \{R_{ab} \mid 1 \leq a < b \leq 5\}$$

on  $S^4$ .

So on every orbit we have the fibration with fibers  $L(x) = S^3$ .

Consequently,  $\varphi$  induces the structure

$$S^3 \longrightarrow S^7 \longrightarrow S^4,$$

which coincides with the classical Hopf fibration.

Thus, we have proved the following theorem

**Theorem 5.** The map

$$\varphi : x \mapsto L(x),$$

defined in (1), determines a transversal singular foliation on  $S^7$  that is equivalent to the Hopf fibration.

### Conclusion

In this work, starting from the given vector fields  $X$  and  $Y$  on  $\mathbb{R}^5$ , we constructed the minimal Lie algebra generated by them through successive Lie brackets. It was shown that the resulting system of vector fields defines an involutive and integrable distribution, and therefore induces a foliation.

By analyzing the obtained vector fields and their linear combinations, a basis of the minimal algebra was determined. The rank computation of the corresponding matrix showed that the dimension of the generated algebra is 4, and hence its orbits are 4-dimensional manifolds, identified as spheres  $S^4$ .

Further, embedding this structure into  $\mathbb{R}^8$ , we constructed additional vector fields  $V_i$  and their Lie brackets  $W_i$ , which generate a transversal singular foliation. The rank analysis of the extended matrix demonstrated that its rank is 7, implying that the corresponding orbits are 7-dimensional spheres  $S^7$ .

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OPTIMAL QUADRATURE FORMULA FOR THE APPROXIMATE CALCULATION OF THE SINGULAR  
INTEGRAL WITH THE HILBERT KERNEL IN THE SPACE  $L_2^{(3)}(0,1)$

SHADIMETOV KHALMATVAY

V.I.ROMANOVSKIY INSTITUTE OF MATHEMATICS, UZBEKISTAN ACADEMY OF SCIENCES, TASHKENT,  
UZBEKISTAN,

TASHKENT STATE TRANSPORT UNIVERSITY, TASHKENT, UZBEKISTAN

kholmatshadimetov@mail.ru

JABBOROV KHURSHID\*

V.I.ROMANOVSKIY INSTITUTE OF MATHEMATICS, UZBEKISTAN ACADEMY OF SCIENCES, TASHKENT,  
UZBEKISTAN,

UNIVERSITY OF WORLD ECONOMY AND DIPLOMACY, TASHKENT, UZBEKISTAN

jabborovx@bk.ru

**ABSTRACT.** In this article, an optimal quadrature formula is constructed for the approximate calculation of the solution of the Fredholm type singular integral equation of the second kind with a Hilbert kernel in the space  $L_2^{(3)}(0,1)$ . In the quadrature formula under consideration, the Sard problem is generated, in which the interval  $[0,1]$  is equidistantly distributed through any  $N$  points, and this problem is solved using the Sobolev method. For this, an analytical representation of the optimal coefficients of the quadrature formula is obtained by minimizing the norm of the error functional in the conjugate space  $L_2^{(3)*}(0,1)$ .

**MSC (2020):** 65D32; 65R20.

**Key words:** Hilbert kernel, singular integral, error functional, extreme function, optimal quadrature formula, optimal coefficients.

### Problem statement

We will consider the following quadrature formula

$$\int_0^1 \varphi(x) \cot \pi(x-t) dx \cong \sum_{\beta=0}^N C_{\beta} \varphi(x_{\beta}), \quad (1)$$

where  $C_{\beta}$  are coefficients,  $x_{\beta}$  are nodes,  $x_{\beta} \in [0,1]$ . The functions  $\varphi(x)$  belong to the Hilbert space  $L_2^{(3)}(0,1)$ , defined as

$$L_2^{(3)}(0,1) = \{\varphi(x) : [0,1] \rightarrow \mathbb{R}, \varphi''(x) - \text{abs. cont}, \varphi'''(x) \in L_2(0,1)\}.$$

The scalar product of the functions  $\varphi$  and  $\psi$  in the space  $L_2^{(3)}(0,1)$  is defined as follows

$$\langle \varphi, \psi \rangle = \int_0^1 \varphi'''(x) \psi'''(x) dx \quad (2)$$

and the norm of the function  $\varphi$  in this space is calculated using the scalar product (2) as follows

$$\|\varphi\| = \sqrt{\int_0^1 \left( \varphi'''(x) \right)^2 dx}. \quad (3)$$

The difference between the integral and the quadrature sum in the quadrature formula (1)

$$\begin{aligned} \int_0^1 \varphi(x) \cot \pi(x-t) dx - \sum_{\beta=0}^N C_{\beta} \varphi(x_{\beta}) &= \int_{-\infty}^{\infty} (\varepsilon_{[0,1]}(x) \cot \pi(x-t) - \sum_{\beta=0}^N C_{\beta} \delta(x-x_{\beta})) \varphi(x) dx = \\ &= \int_{-\infty}^{\infty} \ell(x) \varphi(x) dx = (\ell, \varphi) \end{aligned} \quad (4)$$

the difference is called the *error* of the quadrature formula.

The generated (4) corresponds to the following *error functional* from the conjugate space  $L_2^{(3)*}(0,1)$

$$\ell(x) = \varepsilon_{[0,1]}(x) \cot \pi(x-t) - \sum_{\beta=0}^N C_{\beta} \delta(x-x_{\beta}), \quad (5)$$

where  $\varepsilon_{[0,1]}(x)$  is the characteristic function of the interval  $[0,1]$ , and  $\delta(x)$  is the Dirac delta-function.

According to the definition of the norm of a linear continuous functional, the following is known (see [5], p. 82)

$$\|\ell\|_{L_2^{(3)*}} = \sup_{\|\varphi\| \neq 0} \frac{|(\ell, \varphi)|}{\|\varphi\|}.$$

From this equality follows the Cauchy-Schwarz inequality

$$|(\ell, \varphi)| \leq \|\ell\|_{L_2^{(3)*}} \|\varphi\|_{L_2^{(3)}}. \quad (6)$$

According to the Cauchy-Schwarz inequality, according to the norm of the function  $\varphi(x)$ , the following conditions hold

$$(\ell, x^{\alpha}) = 0, \quad \alpha = 0, 1, 2. \quad (7)$$

According to the conditions (7), the quadrature formula (1) is defined for the function  $d_2 x^2 + d_1 x + d_0$ . These conditions ensure that the error functional is bounded.

Thus, the upper bound of the error (4) for the quadrature formula (1) is estimated using the norm of the error functional  $\ell(x)$ . This gives rise to the following problem.

**Problem 1.** To find the form of the norm for the error functional  $\ell(x)$  belonging to the conjugate space  $L_2^{(3)*}(0,1)$ .

It can be seen that the error functional depends on the coefficients  $C_{\beta}$  and the nodes  $x_{\beta}$ . The minimization of the norm of the error functional for the coefficients  $C_{\beta}$  and the nodes  $x_{\beta}$  is called the *Nikolsky problem*, and this problem was first studied in the work of Nikolsky [6, 7]. The problem of minimizing the norm of the error functional only with respect to the coefficients  $C_{\beta}$  by prefixing the nodes  $x_{\beta}$  is called the *Sard problem*, and this problem was first studied in the works of Sard [8, 9].

In this work, we will solve the Sard problem. For this, we construct a quadrature formula for points  $x_{\beta}$  where the nodes  $x_{\beta} = h\beta$  ( $h = \frac{1}{N}$ ,  $N \in \mathbb{N}$ ) are equidistantly distributed.

which minimizes the norm of the error functional  $\ell(x)$ , i.e.

$$\inf_{C_{\beta}} \|\ell\|_{L_2^{(3)*}(0,1)} \quad (8)$$

The coefficients  $C_{\beta}$  that achieve the expression are called *optimal coefficients* and are denoted as  $\overset{\circ}{C}_{\beta}$ . This quadrature formula built on optimal coefficients is called *optimal quadrature formula*. Therefore, to construct an optimal quadrature formula in the sense of Sard in the space  $L_2^{(3)}(0,1)$ , it is necessary to solve the following problem.

**Problem 2.** Find the optimal coefficients  $\hat{C}_\beta$  to get (8).

The main purpose of this article is to construct an optimal quadrature formula that calculates the integral (1) with high accuracy in the space  $L_2^{(3)}(0, 1)$ . Using the functional method in Hilbert spaces, optimal quadrature formulas were constructed in the case of singular integrals with Cauchy kernels [11, 18, 19, 20] and in the case of singular integrals with Hilbert kernels [12, 13]. In addition, optimal quadrature formulas for regular integrals in various other spaces were constructed using the Sobolev method.

The rest of this article is structured as follows. In the third section, the extremal function corresponding to the error functional  $\ell(x)$  is determined, and with its help, the expression of the norm of the error functional (5) is found, i.e., problem 1 is solved. In the fourth section, to find the minimum of the quantity  $\|\ell\|^2$  on the coefficients  $C_\beta$ , a system of linear equations for the coefficients of the optimal quadrature formula in the Sard sense in the space  $L_2^{(3)}(0, 2\pi)$  is obtained and solved using the Sobolev method. In the fifth section, the analytical form of the optimal coefficients of the quadrature formula (1) is found, i.e., problem 2 is solved.

### Finding the extreme function and expressing the error functional norm

To solve problem 1, i.e., to find the form of the norm of the error functional (5), we use the concept of an extremal function.

The function  $\psi_\ell(x)$  is called *extreme function* corresponding to the error functional  $\ell(x)$  if the following equality holds (see, [3, 4]).

$$(\ell, \psi_\ell) = \|\ell\|_{L_2^{(3)*}} \cdot \|\psi_\ell\|_{L_2^{(3)}} \quad (9)$$

The concept of an extremal function was first introduced by S.L. Sobolev and was defined as follows.

$$\psi_\ell = -\ell(x) * G_3(x) + P_2(x),$$

here  $G_3(x) = \frac{|x|^5}{2 \cdot 5!}$ ,  $P_2(x)$  is a second-degree polynomial.

Note that  $*$  is a convolution operation and a convolution of two functions

$$\varphi(x) * \psi(x) = \int_{-\infty}^{\infty} \varphi(x-y)\psi(y)dy = \int_{-\infty}^{\infty} \varphi(y)\psi(x-y)dy$$

determined by the formula.

Since the space  $L_2^{(3)}(0, 1)$  is a Hilbert space, we present the Riesz theorem on the general form of a linear continuous functional.

**Theorem 1.** (Rice's theorem) Let  $H$  be some Hilbert space. Then for each linear continuous functional  $\ell \in H^*$  in the space  $H$  there exists a unique function  $\psi_\ell$  called the Riesz element, which satisfies the condition  $(\ell, \varphi) = \langle \psi_\ell, \varphi \rangle$ , and for this function the following equality holds

$$\|\ell\|_{H^*} = \|\psi_\ell\|_H$$

and vice versa.

For any  $\psi_\ell \in H$ , there exists a unique function  $\ell \in H^*$  such that  $(\ell, \varphi) = \langle \psi_\ell, \varphi \rangle$  and  $\|\ell\|_{H^*} = \|\psi_\ell\|_H$ . According to the theorem, the extremal function  $\psi_\ell(x)$  is a Riesz element and

$$(\ell, \psi_\ell) = \|\ell\|^2 = \|\psi_\ell\|^2.$$

According to the theorem, we calculate the square of the norm of the error functional.

$$\begin{aligned} \|\ell\|^2 &= - \int_0^1 \int_0^1 \frac{|x-y|^5}{2 \cdot 5!} \cot \pi(x-t) \cot \pi(x-t) dx dy + \\ &+ 2 \sum_{\beta=0}^N C_\beta \int_0^1 \frac{|x-h\beta|^5}{2 \cdot 5!} \cot \pi(x-t) dx - \sum_{\beta=0}^N \sum_{\gamma=0}^N C_\beta C_\gamma \frac{|h\beta-h\gamma|^5}{2 \cdot 5!} \end{aligned}$$

Thus, the square of the norm of the error functional  $\ell(x)$  is found, and problem 1 is solved.

### Find conditional minimum of error functional

Now we will deal with solving the second problem. The square of the norm of the error functional generated in (10) is a multi-variable function with respect to coefficients  $C_\beta$ . According to the rule for finding the conditional extremum of a function of several variables, we find its local minimum together with the conditions (7). For this, we construct the Lagrange function

$$\Phi(C_\beta, d_2, d_1, d_0) = \|ell\|^2 + 2 \sum_{\alpha=0}^2 d_\alpha(\ell, x^\alpha), \quad \beta = 0, 1, \dots, N.$$

Here, according to (7), the following applies

$$(\ell, x^\alpha) = \int_0^1 x^\alpha \cot \pi(x-t) dx - \sum_{\beta=0}^N C_\beta (h\beta)^\alpha = 0, \quad \alpha = 0, 1, 2.$$

Equating the partial derivatives of the resulting function  $\Phi$  with respect to the coefficients  $C_\beta$  and the unknowns  $d_2, d_1, d_0$  to zero, we obtain the following system of linear equations.

$$\sum_{\gamma=0}^N C_\gamma \frac{|h\beta - h\gamma|^5}{2 \cdot 5!} + d_2(h\beta)^2 + d_1(h\beta) + d_0 = f_3(h\beta), \quad \beta = 0, 1, \dots, N, \quad (11)$$

$$\sum_{\gamma=0}^N C_\gamma (h\gamma)^2 = g_2, \quad (12)$$

$$\sum_{\gamma=0}^N C_\gamma (h\gamma) = g_1, \quad (13)$$

$$\sum_{\gamma=0}^N C_\gamma = g_0. \quad (14)$$

here

$$\begin{aligned} f_3(h\beta) &= \int_0^1 \frac{|x - h\beta|^5}{2 \cdot 5!} \cot \pi(x-t) dx = \\ &= \frac{1}{2 \cdot 5!} \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k} 2^{2k} \pi^{2k-1}}{(2k)!} \left[ \frac{1}{2k+5} \left( (1-t)^{2k+5} - 2(h\beta-t)^{2k+5} - t^{2k+5} \right) + \right. \\ &\quad + \frac{5(t-h\beta)}{(2k+4)} \left( (1-t)^{2k+4} - 2(h\beta-t)^{2k+4} + t^{2k+4} \right) + \\ &\quad + \frac{10(t-h\beta)^2}{(2k+3)} \left( (1-t)^{2k+3} - 2(h\beta-t)^{2k+3} - t^{2k+3} \right) + \\ &\quad + \frac{10(t-h\beta)^3}{(2k+2)} \left( (1-t)^{2k+2} - 2(h\beta-t)^{2k+2} + t^{2k+2} \right) + \\ &\quad \left. + \frac{5(t-h\beta)^4}{(2k+1)} \left( (2\pi-t)^{2k+1} - 2(h\beta-t)^{2k+1} - t^{2k+1} \right) \right] + \\ &\quad + \frac{(t-h\beta)^5}{120 \cdot \pi} \ln \left| \frac{\sin \pi \cdot t}{\sin \pi(h\beta-t)} \right|, \end{aligned}$$

$$g_2 = \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k} 2^{2k} \pi^{2k-1}}{(2k)!} \left( \frac{1}{2k+2} \left( (1-t)^{2k+2} - t^{2k+2} \right) + \frac{2t}{2k+1} \left( (1-t)^{2k+1} + t^{2k+1} \right) \right),$$

$$g_1 = \sum_{k=0}^{\infty} (-1)^k \frac{B_{2k} 2^{2k} \pi^{2k-1}}{(2k+1)(2k)!} \left( (1-t)^{2k+1} + t^{2k+1} \right),$$

$$g_0 = \int_0^{2\pi} \cot \frac{x-t}{2} dx = 0,$$

where  $B_{2k}$  are Bernoulli numbers and the following formula was used (see [16], p. 224)

$$\int x^p \cot x dx = \sum_{k=0}^{\infty} (-1)^k \frac{2^{2k} B_{2k}}{(2k+p)(2k)!} x^{2k+p}, \quad [p \geq 1, |x| < \pi].$$

The system (11)-(14) consists of  $(N+4)$  unknowns and  $(N+4)$  linear equations with respect to  $C_\gamma, \gamma = 0, 1, \dots, N, d_0, d_1, d_2$ . To solve this system, we will use the functional method proposed by Sobolev. For this method, the use of a discrete analogue of the differential operator  $d^6/dx^6$  is very important. In [10], a discrete analog of the differential operator  $d^6/dx^6$  is constructed.

**Theorem 2.** The discrete analog of the differential operator  $d^6/dx^6$  satisfying the equality  $D_3(h\beta) * G_3(h\beta) = \delta(h\beta)$  has the form

$$D_3(h\beta) = \frac{5!}{h^6} \begin{cases} \sum_{t=1}^2 K_t \cdot q_t^{|\beta|-1}, & |\beta| \geq 2, \\ 1 + \sum_{t=1}^2 K_t, & |\beta| = 1, \\ -2^5 + \sum_{t=1}^2 \frac{K_t}{q_t}, & \beta = 0, \end{cases} \quad (15)$$

where  $\delta(h\beta)$  is a discrete delta function,  $K_1 = \frac{(1-q_1)^7}{E_5(q_1)}$ ,  $K_2 = \frac{(1-q_2)^7}{E_5(q_2)}$ ,  $E_5(x)$  is an Euler-Frobenius polynomial of the 5th degree, and  $q_1$  and  $q_2$  are roots of the Euler-Frobenius polynomial with  $E_4(q)$  satisfying the inequality  $|q_t| < 1$ ,  $(t = 1, 2)$ .

**Theorem 3.** The discrete operator  $D_3(h\beta)$  has the following properties:

$$\begin{aligned} 1) \quad & D_3(h\beta) * 1 = 0 \\ 2) \quad & D_3(h\beta) * (h\beta) = 0 \\ 3) \quad & D_3(h\beta) * (h\beta)^2 = 0 \\ 4) \quad & hD_3(h\beta) * G_3(h\beta) = \delta(h\beta) \end{aligned} \quad (16)$$

The proofs of the theorems are given in [10].

Since we will now use discrete-argument functions and operations on them, we will present some concepts about them.

**Definition 1.** A function  $\varphi(h\beta)$  is called a function with a discrete argument if it is defined on the set of some integer values of the variable  $\beta$ .

**Definition 2.** The scalar product of two discrete-argument functions  $\varphi(h\beta)$  and  $\psi(h\beta)$  is defined as follows

$$[\varphi(h\beta), \psi(h\beta)] = \sum_{\beta=-\infty}^{\infty} \varphi(h\beta) \cdot \psi(h\beta)$$

This equality holds in the case when the series on the right is absolutely convergent.

**Definition 3.** The convolution of two discrete-argument functions  $\varphi(h\beta)$  and  $\psi(h\beta)$  is the following scalar product

$$\varphi(h\beta) * \psi(h\beta) = [\varphi(h\gamma), \psi(h\beta - h\gamma)] = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\beta - h\gamma).$$

We define the left side of (11)

$$u(h\beta) = \sum_{\gamma=0}^N C_{\gamma} \frac{|h\beta - h\gamma|^5}{2 \cdot 5!} + d_2(h\beta)^2 + d_1(h\beta) + d_0. \quad (17)$$

Let's convert this (17) expression to convolutional form using the discrete argument function concept mentioned above

$$u(h\beta) = C_{\beta} * G_3(h\beta) + d_2(h\beta)^2 + d_1(h\beta) + d_0. \quad (18)$$

Applying the discrete operator  $D_3(h\beta)$  to this expression and using its properties, we get

$$C_{\beta} = hD_3(h\beta) * u(h\beta) \quad (19)$$

Thus, to find the coefficients of the optimal quadrature formula, it is necessary to find the general form of the function  $u(h\beta)$  at all integer points. From the equality (11) we see that  $u(h\beta) = f_3(h\beta)$  for  $\beta = 0, 1, \dots, N$ . Now we need to find the expression (18) for the function  $u(h\beta)$  when  $\beta < 0$  and  $\beta > N$  under the assumption  $C_{\beta} = 0$ .

when  $\beta < 0$

$$\begin{aligned} u(h\beta) &= - \sum_{\gamma=0}^N C_{\gamma} \frac{(h\beta - h\gamma)^5}{2 \cdot 5!} + d_2(h\beta)^2 + d_1(h\beta) + d_0 = \\ &= - \frac{g_0}{240}(h\beta)^5 + \frac{g_1}{48}(h\beta)^4 - \frac{g_2}{24}(h\beta)^3 + p_2(h\beta)^2 - p_1(h\beta) + p_0 + d_2(h\beta)^2 + d_1(h\beta) + d_0, \end{aligned}$$

where  $p_2, p_1, p_0$  are coefficients equal to

$$p_2 = \frac{1}{24} \sum_{\gamma=0}^N C_{\gamma}(h\gamma)^3, \quad p_1 = \frac{1}{48} \sum_{\gamma=0}^N C_{\gamma}(h\gamma)^4, \quad p_0 = \frac{1}{240} \sum_{\gamma=0}^N C_{\gamma}(h\gamma)^5.$$

When  $\beta > N$

$$\begin{aligned} u(h\beta) &= \sum_{\gamma=0}^N C_{\gamma} \frac{(h\beta - h\gamma)^5}{2 \cdot 5!} + d_2(h\beta)^2 + d_1(h\beta) + d_0 = \\ &= \frac{g_0}{240}(h\beta)^5 - \frac{g_1}{48}(h\beta)^4 + \frac{g_2}{24}(h\beta)^3 - p_2(h\beta)^2 + p_1(h\beta) - p_0 + d_2(h\beta)^2 + d_1(h\beta) + d_0, \end{aligned}$$

Thus,  $u(h\beta)$  has the following form.

$$u(h\beta) = \begin{cases} -\frac{g_0}{240}(h\beta)^5 + \frac{g_1}{48}(h\beta)^4 - \frac{g_2}{24}(h\beta)^3 + (d_2 + p_2)(h\beta)^2 + (d_1 - p_1)(h\beta) + (d_0 + p_0), & \beta < 0, \\ f_3(h\beta), & 0 \leq \beta \leq N, \\ \frac{g_0}{240}(h\beta)^5 - \frac{g_1}{48}(h\beta)^4 + \frac{g_2}{24}(h\beta)^3 + (d_2 - p_2)(h\beta)^2 + (d_1 + p_1)(h\beta) + (d_0 - p_0), & \beta > N. \end{cases}$$

Here, to simplify the appearance of  $u(h\beta)$ , we introduce the following notations

$$\begin{aligned} d_2 + p_2 &= a_2^-, & d_1 - p_1 &= a_1^-, & d_0 + p_0 &= a_0^-, \\ d_2 - p_2 &= a_2^+, & d_1 + p_1 &= a_1^+, & d_0 - p_0 &= a_0^+, \end{aligned}$$

from which it is possible to establish the following

$$\begin{aligned} d_2 &= \frac{a_2^- + a_2^+}{2}, & d_1 &= \frac{a_1^- + a_1^+}{2}, & d_0 &= \frac{a_0^- + a_0^+}{2}, \\ p_2 &= \frac{a_2^- - a_2^+}{2}, & p_1 &= \frac{a_1^- - a_1^+}{2}, & p_0 &= \frac{a_0^- - a_0^+}{2}. \end{aligned}$$

Then, considering that  $g_0 = 0$ , we write the form of  $u(h\beta)$  as follows

$$u(h\beta) = \begin{cases} \frac{g_1}{48}(h\beta)^4 - \frac{g_2}{24}(h\beta)^3 + a_2^-(h\beta)^2 + a_1^-(h\beta) + a_0^-, & \beta < 0, \\ f_3(h\beta), & 0 \leq \beta \leq N, \\ -\frac{g_1}{48}(h\beta)^4 + \frac{g_2}{24}(h\beta)^3 + a_2^+(h\beta)^2 + a_1^+(h\beta) + a_0^+, & \beta > N. \end{cases} \quad (20)$$

Here,  $a_2^-, a_1^-, a_0^-, a_2^+, a_1^+, a_0^+$  are unknown quantities. If these unknowns are found, the general form of  $u(h\beta)$  is determined and optimal coefficients  $C_\beta$ ,  $\beta = 0, 1, \dots, N$  are found from the formula (19). Now let's find these unknowns.

First, we define  $a_0^-$  and  $a_0^+$  from (20).

At  $\beta = 0$

$$a_0^- = f_3(0),$$

in  $\beta = N$

$$a_0^+ = \frac{g_1}{48} - \frac{g_2}{24} - a_2^+ - a_1^+ + f_3(1)$$

We determine that it is equal to. Then  $u(h\beta)$  will look like this

$$u(h\beta) = \begin{cases} \frac{g_1}{48}(h\beta)^4 - \frac{g_2}{24}(h\beta)^3 + a_2^-(h\beta)^2 + a_1^-(h\beta) + f_3(0), & \beta < 0, \\ f_3(h\beta), & 0 \leq \beta \leq N, \\ -\frac{g_1}{48}(1 - (h\beta)^4) + \frac{g_2}{24}((h\beta)^3 - 1) + a_2^+((h\beta)^2 - 1) + a_1^+(h\beta - 1) + f_3(1), & \beta > N. \end{cases} \quad (21)$$

Now let's determine the remaining unknowns  $a_2^-, a_1^-, a_2^+, a_1^+$ . According to (19) since  $\beta < 0$  and  $\beta > N$  has  $C_\beta = 0$  based on our assumption

$$C_\beta = hD_3(h\beta) * u(h\beta) = 0, \quad h\beta \notin [0, 2\pi]$$

$$D_3(h\beta) * u(h\beta) = 0,$$

$$\sum_{\gamma=-\infty}^{\infty} D_3(h\beta - h\gamma)u(h\gamma) = 0.$$

Taking (21) into account, we write it down as follows

$$\begin{aligned} \sum_{\gamma=-\infty}^{-1} D_3(h\beta - h\gamma)u(h\gamma) + \sum_{\gamma=0}^N D_3(h\beta - h\gamma)u(h\gamma) + \sum_{\gamma=N+1}^{\infty} D_3(h\beta - h\gamma)u(h\gamma) &= 0, \\ \sum_{\gamma=1}^{\infty} D_3(h\beta + h\gamma)u(-h\gamma) + \sum_{\gamma=0}^N D_3(h\beta - h\gamma)u(h\gamma) + \sum_{\gamma=1}^{\infty} D_3(h\beta - h(\gamma + N))u(h(\gamma + N)) &= 0. \end{aligned}$$

Substituting the function  $u(h\beta)$  into the last expression, we obtain an equation with respect to the unknowns  $a_2^-, a_1^-, a_2^+, a_1^+$

$$\begin{aligned} &a_2^- \sum_{\gamma=1}^{\infty} D_3(h\beta + h\gamma)(h\gamma)^2 - a_1^- \sum_{\gamma=1}^{\infty} D_3(h\beta + h\gamma)(h\gamma) + \\ &+ a_2^+ \sum_{\gamma=1}^{\infty} D_3(h\beta - h(\gamma + N))h^2(\gamma^2 + 2N\gamma) + a_1^+ \sum_{\gamma=1}^{\infty} D_3(h\beta - h(\gamma + N))(h\gamma) = \\ &= - \sum_{\gamma=0}^N D_3(h\beta - h\gamma)f_3(h\gamma) - \sum_{\gamma=1}^{\infty} D_3(h\beta + h\gamma) \left( \frac{g_1}{48}(h\gamma)^4 + \frac{g_2}{24}(h\gamma)^3 + f_3(0) \right) - \\ &- \sum_{\gamma=1}^{\infty} D_3(h\beta - h(\gamma + N)) \left( -\frac{g_1}{48}(h\gamma)^4 + \left( \frac{g_2 h^3}{24} - \frac{g_1 h^3}{12} \right) \gamma^3 + \left( \frac{g_2 h^2}{8} - \frac{g_1 h^2}{8} \right) \gamma^2 + \left( \frac{g_2 h}{8} - \frac{g_1 h}{12} \right) \gamma + f_3(1) \right). \end{aligned} \quad (22)$$

From this, to find the unknowns  $a_2^-, a_1^-, a_2^+, a_1^+$ , we calculate the equation (22) when  $\beta = -1, \beta = -2, \beta = N + 1$  and  $\beta = N + 2$ , and obtain the following system of linear equations

$$u(h\beta) = \begin{cases} A_{11}a_2^- + A_{12}a_1^- + A_{13}a_2^+ + A_{14}a_1^+ = B_1 \\ A_{21}a_2^- + A_{22}a_1^- + A_{23}a_2^+ + A_{24}a_1^+ = B_2 \\ A_{31}a_2^- + A_{32}a_1^- + A_{33}a_2^+ + A_{34}a_1^+ = B_3 \\ A_{41}a_2^- + A_{42}a_1^- + A_{43}a_2^+ + A_{44}a_1^+ = B_4 \end{cases} \quad (23)$$

here

$$\begin{aligned}
 A_{11} &= \frac{5!}{h^4} \left( -28 + \sum_{i=1}^2 \frac{K_i D_{i2}}{q_i^2} \right); \\
 A_{12} &= -\frac{5!}{h^5} \left( -30 + \sum_{i=1}^2 \frac{K_i D_{i1}}{q_i^2} \right); \\
 A_{13} &= \frac{5!}{h^4} \left( \sum_{i=1}^2 K_i q_i^N (D_{i2} + 2ND_{i1}) \right); \\
 A_{14} &= \frac{5!}{h^5} \left( \sum_{i=1}^2 K_i D_{i1} q_i^N \right); \\
 A_{21} &= \frac{5!}{h^4} \left( -118 + \sum_{i=1}^2 K_i \left( 1 - \frac{1}{q_i^2} + \frac{D_{i2}}{q_i^3} \right) \right); \\
 A_{22} &= -\frac{5!}{h^5} \left( -60 + \sum_{i=1}^2 K_i \left( 1 - \frac{1}{q_i^2} + \frac{D_{i1}}{q_i^3} \right) \right); \\
 A_{23} &= -\frac{5!}{h^5} \left( \sum_{i=1}^2 K_i q_i^{N+1} (D_{i2} + 2ND_{i1}) \right); \\
 A_{24} &= \frac{5!}{h^5} \left( \sum_{i=1}^2 K_i D_{i1} q_i^{N+1} \right); \\
 A_{31} &= \frac{5!}{h^4} \left( \sum_{i=1}^2 K_i D_{i2} q_i^N \right); \\
 A_{32} &= -\frac{5!}{h^5} \left( \sum_{i=1}^2 K_i D_{i1} q_i^N \right); \\
 A_{33} &= \frac{5!}{h^4} \left( -28 - 60N + \sum_{i=1}^2 \frac{K_i}{q_i^2} (D_{i2} + 2ND_{i1}) \right); \\
 A_{34} &= \frac{5!}{h^5} \left( -30 + \sum_{i=1}^2 \frac{K_i D_{i1}}{q_i^2} \right); \\
 A_{41} &= \frac{5!}{h^4} \left( \sum_{i=1}^2 K_i D_{i2} q_i^{N+1} \right); \\
 A_{42} &= -\frac{5!}{h^5} \left( \sum_{i=1}^2 K_i D_{i1} q_i^{N+1} \right); \\
 A_{43} &= \frac{5!}{h^4} \left( -118 - 120N + \sum_{i=1}^2 K_i \left( \left( 1 - \frac{1}{q_i^2} \right) (1 + 2N) + \frac{D_{i2} + 2ND_{i1}}{q_i^3} \right) \right); \\
 A_{44} &= \frac{5!}{h^5} \left( -60 + \sum_{i=1}^2 K_i \left( 1 - \frac{1}{q_i^2} + \frac{D_{i1}}{q_i^3} \right) \right); \\
 B_1 &= -\frac{5!}{h^6} \left( -\frac{g_1 h^4}{3} - g_2 h^3 - 30f_3(0) + \sum_{i=1}^2 K_i \left( F_i + \frac{1}{q_i^2} M_i + q_i^N N_i \right) \right);
 \end{aligned}$$

$$\begin{aligned}
 B_2 &= -\frac{5!}{h^6} \left( -\frac{215g_1h^4}{24} - \frac{19g_2h^3}{2} - 30f_3(0) + \sum_{i=1}^2 K_i \left( F_i q_i + \left( 1 - \frac{1}{q_i^2} \right) \left( \frac{g_1h^4}{48} + \frac{g_2h^3}{24} + f_3(0) \right) + \frac{1}{q_i^3} M_i + q_i^{N+1} N_i \right) \right); \\
 B_3 &= -\frac{5!}{h^6} \left( \frac{g_1h^4}{3} - 24h^3 \left( \frac{g_2}{24} - \frac{g_1}{12} \right) - 28h^2 \left( \frac{g_2}{8} - \frac{g_1}{8} \right) - 30h \left( \frac{g_2}{8} - \frac{g_1}{12} \right) - 30f_3(1) + \right. \\
 &\quad \left. + \sum_{i=1}^2 K_i \left( \sum_{\gamma=0}^N q_i^{N-\gamma} f_3(h\gamma) + q_i^N M_i + \frac{1}{q_i^2} N_i \right) \right); \\
 B_4 &= -\frac{5!}{h^6} \left( \frac{215g_1h^4}{24} - 228h^3 \left( \frac{g_2}{24} - \frac{g_1}{12} \right) - 118h^2 \left( \frac{g_2}{8} - \frac{g_1}{8} \right) - 60h \left( \frac{g_2}{8} - \frac{g_1}{12} \right) - 30f_3(1) + \right. \\
 &\quad \left. + \sum_{i=1}^2 K_i \left( \sum_{\gamma=0}^N q_i^{N+1-\gamma} f_3(h\gamma) + q_i^{N+1} M_i + \frac{1}{q_i^3} N_i + \left( 1 - \frac{1}{q_i^2} \right) \left( -\frac{g_1h^4}{48} + \right. \right. \right. \\
 &\quad \left. \left. \left. + h^3 \left( \frac{g_2}{24} - \frac{g_1}{12} \right) + h^2 \left( \frac{g_2}{8} - \frac{g_1}{8} \right) + h \left( \frac{g_2}{8} - \frac{g_1}{12} \right) + f_3(1) \right) \right) \right);
 \end{aligned}$$

where

$$\begin{aligned}
 D_{10} &= \sum_{\gamma=1}^{\infty} q_1^{\gamma}, & D_{11} &= \sum_{\gamma=1}^{\infty} q_1^{\gamma} \gamma, & D_{12} &= \sum_{\gamma=1}^{\infty} q_1^{\gamma} \gamma^2, & D_{13} &= \sum_{\gamma=1}^{\infty} q_1^{\gamma} \gamma^3, & D_{14} &= \sum_{\gamma=1}^{\infty} q_1^{\gamma} \gamma^4, \\
 D_{20} &= \sum_{\gamma=1}^{\infty} q_2^{\gamma}, & D_{21} &= \sum_{\gamma=1}^{\infty} q_2^{\gamma} \gamma, & D_{22} &= \sum_{\gamma=1}^{\infty} q_2^{\gamma} \gamma^2, & D_{23} &= \sum_{\gamma=1}^{\infty} q_2^{\gamma} \gamma^3, & D_{24} &= \sum_{\gamma=1}^{\infty} q_2^{\gamma} \gamma^4, \\
 F_1 &= \sum_{\gamma=0}^N q_1^{\gamma} f_3(\gamma), & F_2 &= \sum_{\gamma=0}^N q_2^{\gamma} f_3(\gamma),
 \end{aligned}$$

$$M_i = \frac{D_{i4}g_1h^4}{48} + \frac{D_{i3}g_2h^3}{24} + D_{i0}f_3(0),$$

$$N_i = -\frac{D_{i4}g_1h^4}{48} + D_{i3}h^3 \left( \frac{g_2}{24} - \frac{g_1}{12} \right) + D_{i2}h^2 \left( \frac{g_2}{8} - \frac{g_1}{8} \right) + D_{i1}h \left( \frac{g_2}{8} - \frac{g_1}{12} \right) + D_{i0}f_3(1).$$

Solving the system of linear equations using the Kramer method, we determine the unknowns  $a_2^-, a_1^-, a_2^+, a_1^+$ .

Thus, the general form of the function  $u(h\beta)$  is defined.

### Optimal Coefficients of a Quadrature Formula

In this section, we present the main result for the quadrature formula (1).

**Theorem 3.** The optimal coefficients of the quadrature formula (1), the nodes of which are equidistant in the space  $L_2^{(3)}(0, 2\pi)$ , have the form

$$\begin{aligned}
 C_0 &= \frac{5!}{h^5} \left[ \frac{g_1h^4}{48} + \frac{g_2h^3}{24} + a_2^- h^2 - a_1^- h - 31 \cdot f_3(0) + f_3(1) + \sum_{n=1}^2 \frac{K_n}{q_n} \left( F_n + U_n + q_n^N V_n \right) \right]; \\
 C_{\beta} &= \frac{5!}{h^5} \left[ \sum_{n=1}^2 K_n \left( q_n^{\beta-1} U_n + q_n^{N-\beta-1} V_n + \sum_{\gamma=0}^N q_n^{|\beta-\gamma|-1} f_3(\gamma) \right) + f_3(\beta-1) - 32f_3(\beta) + f_3(\beta+1) \right]; \\
 &\quad \beta = \overline{1, \dots, N-1} \\
 C_N &= \frac{5!}{h^5} \left[ -\frac{g_1h^4}{48} + \frac{g_2h^3}{24} - \frac{g_1h^3}{12} + \frac{g_2h^2}{8} - \frac{g_1h^2}{8} + \frac{g_2h}{8} - \frac{g_1h}{12} + 2a_2^+ h + a_2^+ h^2 + a_1 h - \right. \\
 &\quad \left. - 31 \cdot f_3(N) + f_3(N-1) + \sum_{n=1}^2 \frac{K_n}{q_n} \left( q_n^N U_n + V_n + \sum_{\gamma=0}^N q_n^{N-\gamma} f_3(\gamma) \right) \right];
 \end{aligned}$$

here

$$U_n = \frac{D_{n4}g_1h^4}{48} + \frac{D_{n3}g_2h^3}{24} + D_{n2}a_2^-h^2 - D_{n1}a_1^-h + D_{n0}f_3(0);$$

$$V_n = -\frac{D_{n4}g_1h^4}{48} + D_{n3}h^3\left(\frac{g_2}{24} - \frac{g_1}{12}\right) + D_{n2}h^2\left(\frac{g_2}{8} - \frac{g_1}{8} + a_2^+\right) + D_{n1}h\left(\frac{g_2}{8} - \frac{g_1}{12} + 2a_2^+ + a_1^+\right) + D_{n0}f_3(N);$$

**Proof.** To prove the theorem, we use the expression (19). For this, the discrete operator  $D_3(h\beta)$  is defined by the formula (15), and the function  $u(h\beta)$  is defined by the expression (21). So,

$$\begin{aligned} C_\beta &= hD_3(h\beta) * u(h\beta) = h \sum_{\gamma=-\infty}^{\infty} D_3(h\beta - h\gamma)u(h\gamma) = \\ &= h \left( \sum_{\gamma=-\infty}^{-1} D_3(h\beta - h\gamma)u(h\gamma) + \sum_{\gamma=0}^N D_3(h\beta - h\gamma)u(h\gamma) + \sum_{\gamma=N+1}^{\infty} D_3(h\beta - h\gamma)u(h\gamma) \right) = \\ &= h \left( \sum_{\gamma=1}^{\infty} D_3(h\beta + h\gamma)u(-h\gamma) + \sum_{\gamma=0}^N D_3(h\beta - h\gamma)u(h\gamma) + \sum_{\gamma=1}^{\infty} D_3(h\beta - h(\gamma + N))u(h(\gamma + N)) \right) = \\ &= h \left( \sum_{\gamma=1}^{\infty} D_3(h\beta + h\gamma) \left( \frac{g_1h^4}{48}\gamma^4 + \frac{g_2h^3}{24}\gamma^3 + a_2^-h^2\gamma^2 - a_1^-h\gamma + f_3(0) \right) + \right. \\ &\quad \left. + \sum_{\gamma=0}^N D_3(h\beta - h\gamma)f_3(\gamma) + \sum_{\gamma=1}^{\infty} D_3(h\beta - h(\gamma + N)) \left( -\frac{g_1h^4}{48}\gamma^4 + \left( \frac{g_2h^3}{24} - \frac{g_1h^3}{12} \right) \gamma^3 + \right. \right. \\ &\quad \left. \left. + \left( \frac{g_2h^2}{8} - \frac{g_1h^2}{8} + a_2^+h^2 \right) \gamma^2 + \left( \frac{g_2h}{8} - \frac{g_1h}{12} + 2a_2^+h + a_1^+h \right) \gamma + f_3(N) \right) \right). \end{aligned}$$

From the last expression follows the coefficient  $C_0$  when  $\beta = 0$ , the coefficients  $C_\beta$  when  $\beta = 1, 2, \dots, N-1$ , and the coefficient  $C_N$  when  $\beta = N$ .

The theorem is proven.

Problem 2 is solved.

## Conclusion

In this article, using the Sobolev method, an optimal quadrature formula was developed for the approximate calculation of a singular integral with a Hilbert kernel with high accuracy in the space  $L_2^{(3)}(0, 1)$  and an analytical expression of its optimal coefficients was found.

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# MORERA BOUNDARY THEOREM FOR THE MATRIX BALL OF THE FIRST TYPE IN THE CLASS OF INTEGRABLE FUNCTIONS

SHOIMKULOV BAKHODIR

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M. ULUGBEK, TASHKENT, UZBEKISTAN  
shoimkba@rambler.ru

OTEMURATOV BAYRAM

KARAKALPAK STATE UNIVERSITY NAMED AFTER BERDAKH, NUKUS, UZBEKISTAN  
bayram\_utmurato@mail.ru

KUTLIMURATOV BAYMURAT\*

KARAKALPAK STATE UNIVERSITY NAMED AFTER BERDAKH, NUKUS, UZBEKISTAN  
bayko-2020@mail.ru

**ABSTRACT.** We establish an analogue of the multidimensional Morera boundary theorem for the matrix ball of the first type defined in the space of vectors of square matrices. In contrast to the known result of S. Kosbergenov for continuous functions, in the present paper we consider integrable functions of class  $L^p(X_{mn})$  ( $p \geq 1$ ). The obtained result generalizes the well-known Nagel–Rudin theorem to the case of integrable functions on the skeleton of the matrix ball.

**MSC (2020):** 32A10, 32Q02, 32A35, 32A26

**Key words:** *matrix ball of the first type, integrable functions, Shilov boundary*

## Introduction

Morera’s theorem is one of the fundamental theorems of complex analysis. In the one-dimensional case, it states that if a continuous function on a domain of the complex plane has zero integrals over the boundaries of all triangles lying in that domain, then the function is holomorphic.

Multidimensional generalizations of Morera’s theorem are an actively developing direction of modern complex analysis. Of particular interest are boundary versions of Morera’s theorem for integrable functions, where conditions are imposed on special families of complex curves related to automorphisms of the domain ([1]–[8]). In the present work, all functions on the Shilov boundary are assumed to belong to the class  $L^p$  ( $p \geq 1$ ).

The starting point was the classical result of Nagel and Rudin [3] established for the unit ball in  $\mathbb{C}^n$ : if a continuous function  $f$  on the sphere satisfies

$$\int_0^{2\pi} f(\psi(e^{i\phi}, 0, \dots, 0)) e^{i\phi} d\phi = 0$$

for all holomorphic automorphisms  $\psi$  of the ball, then  $f$  extends holomorphically into the ball.

An alternative proof of the Nagel–Rudin theorem was proposed by S. Kosbergenov in [4]. This approach proved to be very fruitful: it allowed generalizing the Morera boundary theorem to classical domains for continuous functions ([5]).

For the matrix ball, a Morera boundary theorem in the class of continuous functions  $C(X_{mn})$  was proved by S. Kosbergenov in [6]. The main difference of the present study is the consideration of integrable functions  $L^p(X_{mn})$  ( $p \geq 1$ ). For this class, the proof requires a more subtle approach: unlike the continuous case, here functions are defined only almost everywhere.

In this paper we consider the *matrix ball* of the first type — an important particular case of the classical domain of the first type. The matrix ball  $B_{mn}$  is defined as the set of vectors  $Z = (Z_1, \dots, Z_n)$ , where each component  $Z_j$  is a square matrix of order  $m$ , satisfying  $I^{(m)} - \sum_{j=1}^n Z_j Z_j^* > 0$ . Its Shilov boundary (skeleton)

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\*Corresponding autor

$X_{mn}$  consists of matrix tuples for which  $\sum_{j=1}^n Z_j Z_j^* = I^{(m)}$ . A detailed exposition of the theory of the matrix ball, including the description of its automorphisms and invariant kernels, is contained in the monograph [7] (see also [9]).

Automorphisms of the matrix ball are of fractional-linear form and act transitively on the domain. An essential role is played by the invariant Poisson kernel  $P(Z, W)$ , which allows recovering a holomorphic function from its boundary values. For the matrix ball this kernel has an explicit expression in terms of determinants of matrix blocks [7, 9].

The aim of this paper is to prove a Morera boundary theorem for the matrix ball for integrable functions of class  $L^p(X_{mn})$  ( $p \geq 1$ ), generalizing the result of Nagel–Rudin [3] and the results of Kosbergenov [4, 5, 6] to this case.

### Preliminaries

Let  $Z = (Z_1, \dots, Z_n)$  be a vector consisting of square matrices  $Z_j$  of order  $m$  over the field of complex numbers. We introduce the matrix inner product ([7]):

$$\langle Z, W \rangle = \sum_{j=1}^n Z_j W_j^*,$$

where  $W_j^*$  is the conjugate transpose matrix.

**Definition 1 ([7]).** The matrix ball of the first type is the domain

$$B_{mn} = \left\{ Z \in \mathbb{C}^{m^2 n} : I^{(m)} - \langle Z, Z \rangle > 0 \right\},$$

where  $I^{(m)}$  is the identity matrix of order  $m$ , and the inequality means positive definiteness of the Hermitian matrix.

The skeleton (Shilov boundary) of the domain  $X_{mn}$  is the manifold

$$X_{mn} = \left\{ Z \in \mathbb{C}^{m^2 n} : \langle Z, Z \rangle = I^{(m)} \right\}.$$

The dimension of  $X_{mn}$  is  $m^2(2n-1)$ ; for  $m > 1$  it does not coincide with the dimension of the topological boundary. In particular, when  $m = n$  the manifold  $X_{mn}$  coincides with the set of unitary matrices  $U(n)$ .

Denote by  $H^p(B_{mn})$  the class of all functions  $F$  holomorphic in  $B_{mn}$  for which

$$\sup_{0 < r < 1} \int_{X_{mn}} |F(r\zeta)|^p d\mu(\zeta) < +\infty,$$

where  $d\mu$  is the normalized Haar measure on  $X_{mn}$  invariant under unitary transformations [9]. For  $p \geq 1$ ,  $L^p(X_{mn})$  denotes the class of  $p$ -integrable functions on  $X_{mn}$  with respect to  $d\mu$ .

For any function  $f \in L^p(X_{mn})$  and for any  $\zeta \in X_{mn}$  consider the slice function  $f_\zeta(t) = f(t\zeta)$  for  $t \in \mathbb{C}$ ,  $|t| < 1$ .

The following result belongs to Kosbergenov ([6, 7]).

**Theorem 1.** For any point  $B \in B_{mn}$  there exists an automorphism  $\varphi_B$  of the matrix ball  $B_{mn}$  of the form

$$\varphi_B(Z) = R^{-1} (I^{(m)} - \langle Z, B \rangle)^{-1} \sum_{s=1}^n (Z_s - B_s) Q_{sk}, \quad k = 1, \dots, n,$$

mapping  $B$  to the origin:  $\varphi_B(B) = 0$ . The matrices  $R$  and  $Q_{sk}$  are determined by the conditions

$$R^* (I^{(m)} - \langle B, B \rangle) R = I^{(m)}, \quad Q^* (I^{(mn)} - B^* B) Q = I^{(mn)}.$$

**Corollary.** For any  $Z, W \in B_{mn}$  the following identities hold:

$$\begin{aligned} I^{(m)} - \langle \varphi_B(Z), \varphi_B(W) \rangle &= R^{-1} (I - \langle Z, B \rangle)^{-1} (I - \langle Z, W \rangle) \times \\ &\times (I - \langle B, W \rangle)^{-1} R^{*-1}, \end{aligned}$$

$$\det(I - \langle \varphi_B(Z), \varphi_B(W) \rangle) = \frac{\det(I - \langle B, B \rangle) \det(I - \langle Z, W \rangle)}{\det(I - \langle Z, B \rangle) \det(I - \langle B, W \rangle)}.$$

### Main part

**Lemma 1.** Let  $d\mu$  be the normalized Haar measure on the skeleton  $X_{mn}$  of the matrix ball  $B_{mn}$ . The volume element  $d\mu$  can be written as

$$d\mu = \frac{d\varphi}{2\pi} \wedge d\mu' = \frac{1}{2\pi i} \frac{dt}{t} \wedge d\mu',$$

where  $d\mu'$  is a differential form defining a positive measure on the manifold  $X'_{mn}$ , which is obtained from  $X_{mn}$  by fixing one of the angular variables. Moreover, for any function  $f \in L^p(X_{mn})$  ( $p \geq 1$ ) we have

$$\int_{X_{mn}} |f(\zeta)|^p d\mu(\zeta) = \int_{X'_{mn}} d\mu'(\lambda) \int_0^{2\pi} |f(e^{i\phi}\lambda)|^p \frac{d\phi}{2\pi}. \quad (1)$$

**Proof.** The skeleton  $X_{mn}$  of the matrix ball consists of all vectors  $Z = (Z_1, \dots, Z_n)$  satisfying  $\langle Z, Z \rangle = I^{(m)}$ . Consider the action of the circle  $T = \{t \in \mathbb{C} : |t| = 1\}$  on  $X_{mn}$  by  $(t, Z) \mapsto tZ = (tZ_1, \dots, tZ_n)$ . This action is free, because if  $tZ = Z$  for some  $Z \neq 0$ , then  $t = 1$ . For a fixed  $Z$ , the set  $\{tZ : t \in T\}$  is a circle. On each such circle we choose a unique point by fixing, for example, the argument of the first matrix component. More precisely, we require that in the matrix  $Z_1$  the element in the upper left corner be a positive real number. The set of all points of  $X_{mn}$  satisfying this condition is denoted by  $X'_{mn}$ . Clearly,  $X'_{mn}$  is a smooth manifold of dimension one less than  $X_{mn}$ , and the map

$$\Phi : X'_{mn} \times T \rightarrow X_{mn}, \quad \Phi(\lambda, t) = t\lambda$$

is a diffeomorphism.

The Haar measure  $d\mu$  on  $X_{mn}$  is invariant under the action of the group  $T$ . Consequently, in the coordinates  $(\lambda, t)$  this invariance allows us to write the measure as a direct product of the Haar measure on  $T$  and some measure on  $X'_{mn}$ . The normalized Haar measure on  $T$  is  $\frac{d\phi}{2\pi}$ , where  $\phi = \arg t$ . Thus,

$$d\mu(\zeta) = \frac{d\phi}{2\pi} \wedge d\mu'(\lambda),$$

where  $d\mu'$  is a positive differential form on  $X'_{mn}$ .

Passing to the complex variable  $t = e^{i\phi}$ , we have  $dt = ie^{i\phi}d\phi$ , hence

$$d\phi = \frac{dt}{it}, \quad \frac{d\phi}{2\pi} = \frac{1}{2\pi i} \frac{dt}{t}.$$

Therefore,

$$d\mu = \frac{1}{2\pi i} \frac{dt}{t} \wedge d\mu'.$$

Applying Fubini's theorem, for any integrable function  $g(\zeta)$  we get

$$\int_{X_{mn}} g(\zeta) d\mu(\zeta) = \int_{X'_{mn}} d\mu'(\lambda) \int_T g(t\lambda) \frac{1}{2\pi i} \frac{dt}{t}.$$

Set  $g(\zeta) = |f(\zeta)|^p$ . Since for  $t = e^{i\phi}$  we have  $\frac{1}{2\pi i} \frac{dt}{t} = \frac{d\phi}{2\pi}$ , we obtain (1). Lemma 1 is proved.

**Lemma 2 (invariance of  $L^p$  class).** Let  $f \in L^p(X_{mn})$ ,  $p \geq 1$ . Then for any automorphism  $\psi$  of the matrix ball  $B_{mn}$  the function  $f \circ \psi$  also belongs to  $L^p(X_{mn})$ .

**Proof.** For an arbitrary automorphism  $\psi$  of the domain  $B_{mn}$ , consider the measure  $d\mu(\psi^{-1}(\zeta))$ . It is known ([7]) that

$$d\mu(\psi^{-1}(\zeta)) = |h(\zeta)| d\mu(\zeta),$$

where  $h(\zeta)$  is a smooth function on  $X_{mn}$  and  $|h(\zeta)| \geq \delta > 0$  for some  $\delta$  (because the Jacobian of the automorphism is bounded and bounded away from zero due to compactness of  $X_{mn}$ ). Hence,

$$\int_{X_{mn}} |f(\psi(\zeta))|^p d\mu(\zeta) = \int_{X_{mn}} |f(W)|^p |h(W)|^{-1} d\mu(W).$$

Indeed, the change of variables  $W = \psi(\zeta)$  gives  $\zeta = \psi^{-1}(W)$  and  $d\mu(\zeta) = |h(W)|^{-1} d\mu(W)$ . Since  $|h(W)|^{-1} \leq 1/\delta$  and  $f \in L^p(X_{mn})$ , the integral is finite. Lemma 2 is proved.

**Theorem 2.** Let  $f \in L^p(X_{mn})$ ,  $p \geq 1$ . If  $f$  satisfies the Morera condition

$$\int_T f(\psi(t\lambda)) dt = 0 \quad (2)$$

for all automorphisms  $\psi$  of the matrix ball  $B_{mn}$  and almost all  $\lambda \in X_{mn}$  (here  $T = \{t \in \mathbb{C} : |t| = 1\}$ ,  $dt$  is the differential of the complex variable), then  $f$  is the radial boundary value on  $X_{mn}$  of some function  $F \in H^p(B_{mn})$ .

**Proof.** Transform the Morera condition into an integral over the skeleton. Multiplying equality (2) by  $d\mu'(\lambda)$  and integrating over  $X'_{mn}$ , using Lemma 1 for  $f \in L^p(X_{mn})$  we obtain

$$\int_{X_{mn}} f(\psi(\zeta)) d\mu(\zeta) = 0. \quad (3)$$

Consider the  $k$ -th component of the vector  $\zeta = (\zeta_1, \dots, \zeta_{m^2n})$ . Integrating (2) with weight  $\lambda_k$  over  $X'_{mn}$ , similarly we get

$$\int_{X_{mn}} f(\psi(\zeta)) \zeta_k d\mu(\zeta) = 0, \quad k = 1, \dots, m^2n. \quad (4)$$

Let  $\varphi_B$  be the automorphism sending  $B \in B_{mn}$  to 0. Substitute  $\psi = \varphi_B^{-1}$  into (4). Make the change of variables  $W = \varphi_B^{-1}(\zeta)$ , which is valid because  $\varphi_B$  is a diffeomorphism. Then  $\zeta = \varphi_B(W)$ , and (4) becomes

$$\int_{X_{mn}} f(W) [\varphi_B^{-1}(W)]_k d\mu(\varphi_B(W)) = 0. \quad (5)$$

Express the measure  $d\mu(\varphi_B(W))$  via the Poisson kernel. By Lemma 3.4 from [10] we have

$$d\mu(\varphi_B(W)) = P(W, B) d\mu(W),$$

where

$$P(W, B) = \left( \frac{\det(I^{(m)} - \langle B, B \rangle)}{|\det(I^{(m)} - \langle W, B \rangle)|^2} \right)^{mn}$$

is the Poisson kernel for the matrix ball [7]. Note also that, up to a nondegenerate matrix  $M(B)$  depending only on  $B$ , the components of the automorphism satisfy

$$[\varphi_B^{-1}(W)]_k = \varphi_B^k(W) \cdot M(B).$$

Since  $M(B)$  is nondegenerate, from (5) we obtain

$$\int_{X_{mn}} f(W) \varphi_B^k(W) P(W, B) d\mu(W) = 0, \quad \forall B \in B_{mn}, \forall k. \quad (6)$$

Consider the Poisson integral of  $f \in L^p(X_{mn})$ :

$$F(B) = \int_{X_{mn}} f(W) P(W, B) d\mu(W), \quad B \in B_{mn}. \quad (7)$$

The function  $F(B)$  is real analytic in  $B_{mn}$ .

We first prove the following lemma.

**Lemma 3.** For the automorphism  $\varphi_B$  and the Poisson kernel  $P(W, B)$  the following identities hold:

$$\sum_{s=1}^m \sum_{p=1}^m \bar{b}_{sp} \varphi_B^{sp}(W) = \text{Sp} \left[ (I - \langle B, B \rangle)^{-1} - (I - \langle W, B \rangle)^{-1} \right], \quad (8)$$

$$\sum_{s=1}^m \sum_{p=1}^m \bar{b}_{sp} \frac{\partial P(W, B)}{\partial \bar{b}_{sp}} = mn P(W, B) \left[ \text{Sp}(I - \langle W, B \rangle)^{-1} - \text{Sp}(I - \langle B, B \rangle)^{-1} \right]. \quad (9)$$

**Proof.** From the formula for the automorphism of the matrix ball ([7]) we have

$$\varphi_B(W) = R^{-1}(I - \langle W, B \rangle)^{-1}(W - B)Q,$$

where

$$R^*(I - \langle B, B \rangle)R = I, \quad QB^* = (I - \langle B, B \rangle)^{-1}R^{*-1}.$$

Then

$$\begin{aligned} \text{Sp}(\varphi_B(W)B^*) &= \text{Sp}[R^{-1}(I - \langle W, B \rangle)^{-1}(W - B)(I - \langle B, B \rangle)^{-1}R^{*-1}] = \\ &= \text{Sp}[(I - \langle W, B \rangle)^{-1}(W - B)(I - \langle B, B \rangle)^{-1}]. \end{aligned}$$

Since

$$(W - B)(I - \langle B, B \rangle)^{-1} = (I - \langle W, B \rangle)^{-1} - (I - \langle B, B \rangle)^{-1},$$

we obtain

$$\text{Sp}(\varphi_B(W)B^*) = \text{Sp}[(I - \langle W, B \rangle)^{-1} - (I - \langle B, B \rangle)^{-1}].$$

For the left-hand side we have  $\text{Sp}(\varphi_B(W)B^*) = \sum_{s,p} \bar{b}_{sp} \varphi_B^{sp}(W)$ . This yields (8).

Now we prove (9). The Poisson kernel has the form

$$P(W, B) = \frac{\det(I - \langle B, B \rangle)^{mn}}{\det(I - \langle W, B \rangle)^{mn} \cdot \det(I - \langle W, B \rangle)^{mn}}.$$

Differentiate  $P$  with respect to  $\bar{b}_{sp}$ . We get

$$\begin{aligned} \frac{\partial P}{\partial \bar{b}_{sp}} &= mn \det(I - \langle B, B \rangle)^{mn-1} \frac{\partial \det(I - \langle B, B \rangle)}{\partial \bar{b}_{sp}} \cdot \det(I - \langle W, B \rangle)^{-mn} \overline{\det(I - \langle W, B \rangle)}^{-mn} - \\ &- mn \det(I - \langle B, B \rangle)^{mn} \det(I - \langle W, B \rangle)^{-mn-1} \frac{\partial \det(I - \langle W, B \rangle)}{\partial \bar{b}_{sp}} \cdot \overline{\det(I - \langle W, B \rangle)}^{-mn} - \\ &- mn \det(I - \langle B, B \rangle)^{mn} \det(I - \langle W, B \rangle)^{-mn} \overline{\det(I - \langle W, B \rangle)}^{-mn-1} \frac{\partial \overline{\det(I - \langle W, B \rangle)}}{\partial \bar{b}_{sp}}. \end{aligned}$$

Factoring out  $P(W, B)$ , we have

$$\begin{aligned} \frac{\partial P}{\partial \bar{b}_{sp}} &= P \left( mn \frac{1}{\det(I - \langle B, B \rangle)} \frac{\partial \det(I - \langle B, B \rangle)}{\partial \bar{b}_{sp}} - \right. \\ &- mn \frac{1}{\det(I - \langle W, B \rangle)} \frac{\partial \det(I - \langle W, B \rangle)}{\partial \bar{b}_{sp}} - mn \frac{1}{\det(I - \langle W, B \rangle)} \frac{\partial \overline{\det(I - \langle W, B \rangle)}}{\partial \bar{b}_{sp}} \left. \right). \end{aligned}$$

Multiply both sides by  $\bar{b}_{sp}$  and sum over  $s, p$ :

$$\sum_{s,p} \bar{b}_{sp} \frac{\partial P}{\partial \bar{b}_{sp}} = mn P \left( \frac{1}{\det(I - \langle B, B \rangle)} \sum_{s,p} \bar{b}_{sp} \frac{\partial \det(I - \langle B, B \rangle)}{\partial \bar{b}_{sp}} - \right.$$

$$-\frac{1}{\det(I - \langle W, B \rangle)} \sum_{s,p} \bar{b}_{sp} \frac{\partial \det(I - \langle W, B \rangle)}{\partial \bar{b}_{sp}} - \frac{1}{\det(I - \langle W, B \rangle)} \sum_{s,p} \bar{b}_{sp} \frac{\partial \overline{\det(I - \langle W, B \rangle)}}{\partial \bar{b}_{sp}} \Bigg).$$

Using the rule for differentiating determinants, we find

$$\begin{aligned} \frac{1}{\det(I - \langle B, B \rangle)} \sum_{s,p} \bar{b}_{sp} \frac{\partial \det(I - \langle B, B \rangle)}{\partial \bar{b}_{sp}} &= \text{Sp}((I - \langle B, B \rangle)^{-1} \langle B, B \rangle), \\ \frac{1}{\det(I - \langle W, B \rangle)} \sum_{s,p} \bar{b}_{sp} \frac{\partial \det(I - \langle W, B \rangle)}{\partial \bar{b}_{sp}} &= \text{Sp}((I - \langle W, B \rangle)^{-1} \langle W, B \rangle), \\ \frac{1}{\det(I - \langle W, B \rangle)} \sum_{s,p} \bar{b}_{sp} \frac{\partial \overline{\det(I - \langle W, B \rangle)}}{\partial \bar{b}_{sp}} &= \text{Sp}((I - \langle B, W \rangle)^{-1} \langle B, W \rangle). \end{aligned}$$

Substituting these expressions, we obtain

$$\begin{aligned} \sum_{s,p} \bar{b}_{sp} \frac{\partial P}{\partial \bar{b}_{sp}} &= mn P \left[ \text{Sp}((I - \langle B, B \rangle)^{-1} \langle B, B \rangle) - \right. \\ &\quad \left. - \text{Sp}((I - \langle W, B \rangle)^{-1} \langle W, B \rangle) - \text{Sp}((I - \langle B, W \rangle)^{-1} \langle B, W \rangle) \right]. \end{aligned}$$

For each term we use the identity  $\text{Sp}((I - X)^{-1} X) = \text{Sp}((I - X)^{-1} - I)$ . Then

$$\begin{aligned} \text{Sp}((I - \langle B, B \rangle)^{-1} \langle B, B \rangle) &= \text{Sp}(I - \langle B, B \rangle)^{-1} - \text{Sp}(I), \\ \text{Sp}((I - \langle W, B \rangle)^{-1} \langle W, B \rangle) &= \text{Sp}(I - \langle W, B \rangle)^{-1} - \text{Sp}(I), \\ \text{Sp}((I - \langle B, W \rangle)^{-1} \langle B, W \rangle) &= \text{Sp}(I - \langle B, W \rangle)^{-1} - \text{Sp}(I). \end{aligned}$$

Substituting, we get

$$\begin{aligned} \sum_{s,p} \bar{b}_{sp} \frac{\partial P}{\partial \bar{b}_{sp}} &= mn P \left[ \text{Sp}(I - \langle B, B \rangle)^{-1} - \text{Sp}(I) - \right. \\ &\quad \left. - (\text{Sp}(I - \langle W, B \rangle)^{-1} - \text{Sp}(I)) - (\text{Sp}(I - \langle B, W \rangle)^{-1} - \text{Sp}(I)) \right]. \end{aligned}$$

After simplification we obtain

$$\sum_{s,p} \bar{b}_{sp} \frac{\partial P}{\partial \bar{b}_{sp}} = mn P [\text{Sp}(I - \langle W, B \rangle)^{-1} - \text{Sp}(I - \langle B, B \rangle)^{-1}].$$

Lemma 3 is proved.

Continuing the proof of the theorem. Comparing (8) and (9), we notice that

$$\sum_{s=1}^m \sum_{p=1}^m \bar{b}_{sp} \varphi_B^{sp}(W) = \frac{1}{mn} \sum_{s=1}^m \sum_{p=1}^m \bar{b}_{sp} \frac{\partial P(W, B)}{\partial \bar{b}_{sp}} \cdot P(W, B)^{-1}.$$

Substituting this expression into (6), we obtain

$$\sum_{s=1}^m \sum_{p=1}^m \bar{b}_{sp} \int_{X_{mn}} f(W) \frac{\partial P(W, B)}{\partial \bar{b}_{sp}} d\mu(W) = 0.$$

Since  $P(W, B)$  is real analytic in  $B$  and  $f \in L^p(X_{mn})$ , differentiation under the integral sign is legitimate. Hence,

$$\sum_{s=1}^m \sum_{p=1}^m \bar{b}_{sp} \frac{\partial F(B)}{\partial \bar{b}_{sp}} = 0, \quad \forall B \in B_{mn}. \quad (10)$$

Expand  $F(B)$  in a Taylor series near zero:

$$F(B) = \sum_{|\alpha|, |\beta| \geq 0} c_{\alpha, \beta} B^\alpha \bar{B}^\beta,$$

where  $|\alpha| = \sum_{q=1}^m \sum_{j=1}^n \alpha_{qj}$ ,  $B^\alpha = \prod_{q=1}^m \prod_{j=1}^n b_{qj}^{\alpha_{qj}}$ .

Substituting this expansion into (10) gives

$$\sum_{|\alpha| \geq 0, |\beta| > 0} |\beta| c_{\alpha, \beta} B^\alpha \bar{B}^\beta = 0,$$

hence  $c_{\alpha, \beta} = 0$  for all  $|\beta| > 0$ . Consequently,  $F(B)$  does not depend on  $\bar{B}$ , i.e.,  $F$  is holomorphic in  $B_{mn}$ .

From definition (7) and Lemma 2 it follows that  $F \in H^p(B_{mn})$ . By the Fatou theorem on radial limits (generalized to the matrix ball [4, 5]), for almost all  $\zeta \in X_{mn}$  the function  $f$  coincides with the radial boundary value of  $F$ :

$$\lim_{r \rightarrow 1-0} F(r\zeta) = f(\zeta).$$

Thus,  $f$  is the radial boundary value on  $X_{mn}$  of  $F \in H^p(B_{mn})$ .

Theorem 2 is completely proved.

**Corollary.** If a function  $f \in L^p(X_{mn})$  ( $p \geq 1$ ) extends holomorphically (in  $t$ ) to the analytic disks

$$\Delta_\psi = \{\zeta : \zeta = \psi(t\lambda), |t| < 1\}$$

for all automorphisms  $\psi$  of the matrix ball  $B_{mn}$  (or for all automorphisms sending the origin to points of some fixed open set  $V \subset B_{mn}$ ) to functions of class  $H^p(\Delta_\psi)$ , then  $f$  extends holomorphically to  $B_{mn}$ .

**Proof.** The holomorphy condition in  $t$  implies the integral Morera condition (2) (see [5]). Then by the proved theorem,  $f$  is the boundary value of a holomorphic function in  $B_{mn}$ .

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## DIRECT AND INVERSE PROBLEMS WITH INTEGRAL CONDITIONS FOR THE STRING VIBRATION EQUATION

USMONOV DONIYOR

FERGANA STATE UNIVERSITY, FERGANA, UZBEKISTAN  
dusmonov909@gmail.com

JURAYEVA DILNAVOZ\*

FERGANA STATE UNIVERSITY, FERGANA, UZBEKISTAN  
jorayevadilnavoz683@gmail.com

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**ABSTRACT.** In this paper, direct and inverse nonlocal problems with integral conditions for the string vibration equation are investigated. The solution is obtained by reducing the problem to the Cauchy problem for the string vibration equation, and the general form of the solution is constructed using auxiliary differential equations. The main theorems establish the existence and uniqueness of the solution. As a result, an explicit solution formula is derived for a new type of problem with integral conditions for the string vibration equation.

**MSC (2020):** 35L05; 35L20; 35R30.

**Key words:** String vibration equation, nonlocal problem, integral condition, Cauchy problem, existence of solution, uniqueness of solution, delay differential equation, differential equation.

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## Introduction

At present, initial, boundary, and spectral problems for the string vibration equation have been widely investigated by many researchers (see, for example, [1]–[3]). However, problems with integral conditions are distinguished by their strong physical significance. The study of vibrations propagating in rigid bodies or elastic media is one of the most important directions in mathematical physics. In modeling such processes, classical initial and boundary value problems for vibration equations are usually considered. However, in real physical processes, nonlocal (integral or functional) conditions frequently arise. This restricts the applicability of classical solution methods and requires the development of new approaches. In this paper, direct and inverse nonlocal problems with integral conditions for the string vibration equation are studied. To determine the solution, the Cauchy problem for the string vibration equation is first considered, and then its solution is subjected to an integral condition. As a result, a nonhomogeneous differential equation with a delayed argument is obtained. By employing auxiliary differential equations, the structure of the fundamental solution is constructed. Under appropriate assumptions, the existence and uniqueness of the solution are proved. Let us discuss the physical interpretation of the problem. The initial conditions determine the initial shape and velocity of the string, whereas the nonlocal conditions indicate that the state of the system depends on other points of the domain. This reflects global interactions and reflection effects. The delayed argument indicates the presence of time delay in the system, i.e., the current state depends on past states. The integral condition represents the cumulative effect of all influences. Thus, the proposed model adequately describes complex vibration processes occurring in real physical systems. Typical examples include signal transmission with delay, echo phenomena, and internal delays in elastic systems.

The development of this field has been significantly influenced by the works [4]–[12]. Below, we briefly review studies on differential equations with delayed arguments that are closely related to the topic of this paper. Joseph Wiener investigated boundary value problems for differential equations involving argument reflection and obtained solutions in parametric form [4]. Will Watkins studied boundary value problems for modified Wiener-type equations and their solutions. In deriving the solution of a homogeneous differential equation with delayed argument, we make use of the methods developed in [5]–[6]. The works of Marina Shamilevna Burluskaya

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\*Corresponding autor

are also of considerable importance in this area. She obtained the classical solution of a general two-point mixed boundary value problem for a first-order differential equation with involution. In the case of a symmetric potential, the solution was obtained explicitly; for an arbitrary potential, the corresponding spectral problem was studied in terms of eigenvalues and eigenfunctions. Refined asymptotic formulas enabled the application of the Fourier method. In her other works, new properties of involutive functional differential operators and equations were established, and their close relationship with the Dirac system and the Sturm–Liouville equation was described. This represents the simplest particular case of involutive equations [7]–[9]. Spectral problems for such equations were also considered in the works of Abdizhahan Sarsenbi [10]–[11]. Mixed nonlocal spectral problems were studied in the works of Leonid Kritskov [12]. In general, the study of such boundary value problems for differential equations is important not only from a theoretical point of view but also from a practical standpoint.

### Problem Statement

Let  $D$  denote the domain bounded by the lines  $t-x=0$ ,  $t+x=1$  and  $t=0$ . In the domain  $D$ , consider the following string vibration equation:

$$u_{tt}(x, t) - u_{xx}(x, t) = f(x, t). \quad (1)$$

For this equation, we consider the following nonlocal boundary value problem.

**$I_1$  problem.** Find a function belonging to the class  $u, u_t \in C(\overline{D})$ ,  $u_{xx}, u_{tt} \in C(D)$  that satisfies equation (1) in the domain  $D$  and the following conditions:

$$u(x/2, x/2) + \alpha \int_0^x u(1-\xi, 0) d\xi = \varphi(x), \quad 0 \leq x \leq 1, \quad (2)$$

$$u_t(x, 0) = v(x), \quad 0 \leq x \leq 1, \quad (3)$$

satisfying the conditions  $u(x, t)$  find a function, where  $\varphi(x)$ ,  $v(x)$ -given functions,  $\alpha$ - a given nonzero real number.

#### $I_1$ investigation of the problem

To solve this problem

$$u(x, 0) = \tau(x), \quad 0 \leq x \leq 1, \quad (4)$$

introduce the notation, where  $\tau(x)$  - unknown function.

Now, for equation (1), the Cauchy problem satisfying conditions (3) and (4) is obtained.

**Theorem 1.** If  $\tau(x) \in C^1[0, 1] \cap C^2(0, 1)$ ,  $f(x, t) \in L(D)$  and  $v(x) \in C[0, 1]$  holds, then the Cauchy problem admits a unique solution

$$u(x, t) = \frac{1}{2} [\tau(x-t) + \tau(x+t)] + \frac{1}{2} \int_{x-t}^{x+t} v(z) dz + \frac{1}{2} \int_0^t d\eta \int_{x-t+\eta}^{x+t-\eta} f(s, \eta) ds \quad (5)$$

is of the form. [1]

Now, the unknown  $\tau(x)$  to find the function, we impose condition (2) on solution (5), and

$$\frac{1}{2} [\tau(0) + \tau(x)] + \frac{1}{2} \int_0^x v(z) dz + \frac{1}{2} \int_0^{x/2} d\eta \int_{\eta}^{x-\eta} f(s, \eta) ds + \alpha \int_0^x \tau(1-\xi) d\xi = \varphi(x), \quad 0 \leq x \leq 1, \quad (6)$$

we obtain an integral equation.

in (6), denoted by  $x=0$ ,  $\tau(x) \in C[0, 1]$  and  $v(x) \in C[0, 1]$  noting that,

$$\tau(0) = \varphi(0) \quad (7)$$

we obtain the condition, and rewrite (6) as follows:

$$\tau(x) + 2\alpha \int_0^x \tau(1-\xi) d\xi = F(x) \quad 0 \leq x \leq 1, \quad (8)$$

Here, the value of  $F(x)$  is equal to the following.

$$F(x) = \varphi(x) - \varphi(0) - \int_0^x v(z) dz - \int_0^{x/2} d\eta \int_\eta^{x-\eta} f(s, \eta) ds, \quad 0 \leq x \leq 1,$$

First, we show that integral equation (8) has a unique solution. For this purpose, we use Banach's fixed-point theorem.

**Definition 1.**  $X$  a metric space and a self-mapping on it  $T$  let a mapping be given. If  $\delta \in (0, 1)$  there exists a number such that for all  $x_1, x_2 \in X$  for all  $\rho(Tx_1, Tx_2) \leq \delta \rho(x_1, x_2)$  if the inequality holds,  $T$  is called a contraction mapping, where  $\rho(\cdot, \cdot)$  -  $X$  the metric of the space. [13]

**Banach fixed-point theorem.** Any contraction mapping defined on a complete metric space has a unique fixed point.

the space of continuous functions  $C[a, b]$  denote by. The metric of this space  $\rho(x_1, x_2) = \max_{a \leq x \leq b} |x_1 - x_2|$  is of the form. [13]

**Theorem 2.**  $C[a, b]$  the space is a complete metric space. [13]

$T : C[0, 1] \rightarrow C[0, 1]$  we define the operator as follows:

$$T\tau = F(x) - 2\alpha \int_{1-x}^1 \tau(s) ds.$$

To apply Banach's fixed-point theorem, we need to prove the following:

- a)  $\tau(x) \in C[0, 1]$  since that  $T\tau(x) \in C[0, 1]$  holds;
- b) arbitrary  $\tau_1(x) \in C[0, 1]$ ,  $\tau_2(x) \in C[0, 1]$  for functions

$$|T\tau_1(x) - T\tau_2(x)| \leq \delta |\tau_1(x) - \tau_2(x)|, \quad 0 < \delta < 1$$

We need to show that the inequality holds.

First, we show that condition a) is satisfied.

Let us assume that  $\varphi(x) \in C[0, 1]$ ,  $f(x, t) \in L(D)$  and  $v(x) \in C[0, 1]$  If this is the case, then  $\tau(x) \in C[0, 1]$  since  $T\tau(x) \in C[0, 1]$  It is not difficult to show that it follows that.

Now we show that condition b) is satisfied.

$T$  the operator  $C[a, b]$  We show that the operator is a contraction on the space. An arbitrary  $\tau_1(x) \in C[0, 1]$ ,  $\tau_2(x) \in C[0, 1]$  Let us consider the following equality for functions:

$$|T\tau_1 - T\tau_2| = 2|\alpha| \left| \int_{1-x}^1 [\tau_1(s) - \tau_2(s)] ds \right|.$$

Since  $\tau_1(x) \in C[0, 1]$ ,  $\tau_2(x) \in C[0, 1]$ , using the last equality, we obtain the following inequality:

$$\begin{aligned} |T\tau_1 - T\tau_2| &\leq \max_{0 \leq x \leq 1} 2|\alpha| \left| \int_{1-x}^1 [\tau_1(s) - \tau_2(s)] ds \right| \leq \max_{0 \leq x \leq 1} 2|\alpha| \int_{1-x}^1 |\tau_1(s) - \tau_2(s)| ds \leq \\ &\leq \max_{0 \leq x \leq 1} 2|\alpha| |\tau_1(x) - \tau_2(x)| x \leq \max_{0 \leq x \leq 1} 2|\alpha| |\tau_1(x) - \tau_2(x)|. \end{aligned}$$

If  $|\alpha| < \frac{1}{2}$  holds, then the  $T$  operator is a contraction.

By Banach's fixed-point theorem,  $T$  the operator has a unique fixed point, that is, integral equation (8)  $\tau(x) \in C[0, 1]$  has a unique solution.

Now, differentiating equation (8) once with respect to  $x$ ,

$$\tau'(x) + 2\alpha\tau(1-x) = F'(x), \quad (9)$$

we obtain a differential equation with a delayed argument.

If we find a unique function satisfying conditions (7) and (9), then we can uniquely determine the solution of problem  $I_1$  Therefore, we consider the following auxiliary problems.

### Auxiliary problems

$K_1$  **masala.**  $y(x) \in C[0, 1] \cap C^1(0, 1)$  belongs to the class

$$y'(x) + \beta y(1-x) = 0, 0 < x < 1 \quad (10)$$

the differential equation,

$$y(0) = k_1 \quad (11)$$

satisfying the initial condition  $y(x)$  find a function such that  $k_1, \beta$ - given non-zero real numbers.

To solve this problem, in equation (10) we replace  $x$  with  $1-x$ ,

$$y'(1-x) + \beta y(x) = 0, 0 < x < 1 \quad (12)$$

we obtain a differential equation. Now, differentiating equation (10) once,

$$y''(x) - \beta y'(1-x) = 0, 0 < x < 1 \quad (13)$$

we obtain a differential equation.

It is not difficult to show that from (12) and (13)

$$y''(x) + \beta^2 y(x) = 0 \quad (14)$$

we obtain a differential equation. It is known that the general solution of differential equation (14) is

$$y(x) = C_1 \cos(\beta x) + C_2 \sin(\beta x) \quad (15)$$

is of the form, where  $C_1, C_2$ - arbitrary constants.

However, solution (15) for arbitrary constants  $C_1, C_2$  (10) may not satisfy the differential equation. Therefore, substituting (15) into (10), we obtain the following equality:

$$-C_1 \beta \sin(\beta x) + C_2 \beta \cos(\beta x) + C_1 \beta \cos(\beta - \beta x) + C_2 \beta \sin(\beta - \beta x) = 0.$$

Simplifying this equality, we obtain

$$(C_1 \cos \beta + C_2 \sin \beta + C_2) \cos(\beta x) + (C_1 \sin \beta - C_1 - C_2 \cos \beta) \sin(\beta x) = 0$$

we obtain an equality. This equality must hold for arbitrary  $x \in (0, 1)$  therefore

$$\begin{cases} C_1 \cos \beta + (1 + \sin \beta) C_2 = 0 \\ (\sin \beta - 1) C_1 - C_2 \cos \beta = 0 \end{cases}$$

a system of equations must be satisfied. It is not difficult to show that this system has infinitely many solutions for arbitrary  $\beta$ . Therefore  $C_2$  can be expressed in terms of  $C_1$ :

$$C_2 = \frac{\sin \beta - 1}{\cos \beta} C_1, \beta \neq \pi/2 + \pi n, n \in \mathbb{Z}. \quad (16)$$

we substitute (16) into (15):

$$y(x) = C_1 \cos(\beta x) + C_1 \frac{\sin \beta - 1}{\cos \beta} \sin(\beta x).$$

Simplifying the last equality, we rewrite it as follows:

$$y(x) = \frac{C_1}{\cos \beta} (\cos(\beta - \beta x) - \sin(\beta x)) \quad (17)$$

we obtain an equality. It can be directly verified that function (17) satisfies differential equation (10).

Now, substituting (17) into condition (11), we obtain,  $C_1 = k_1$  we obtain an equality. Substituting it into (17),  $y(x)$  the function

$$y(x) = \frac{k_1}{\cos \beta} (\cos(\beta - \beta x) - \sin(\beta x)) \quad (18)$$

**Theorem 3.** If  $\beta \neq \pi/2 + \pi n$ ,  $n \in \mathbb{Z}$  holds, then  $K_1$  the problem has a unique solution, which is given by equality (18).

Now we consider another auxiliary problem.

**$K_2$  problem.**  $y(x) \in C[0, 1] \cap C^1(0, 1)$  belongs to the class

$$y'(x) + \beta y(1 - x) = f_1(x), 0 < x < 1 \quad (19)$$

the differential equation

$$y(0) = 0 \quad (20)$$

satisfying the initial condition  $y(x)$  find a function such that  $\beta$ - a given non-zero real number,  $f_1(x)$  - the given function.

the solution of this problem

$$y(x) = \int_0^1 G_1(x, s) f_1(s) ds + \int_0^1 G_2(x, s) f_1(s) ds \quad (21)$$

we look for a solution in the form, where  $G_1(x, s)$  and  $G_2(x, s)$  are unknown functions for now, which are respectively

$$G_1(x, s) = \begin{cases} R_1(x, s), & 0 \leq s \leq x, \\ R_2(x, s), & x \leq s \leq 1; \end{cases}$$

$$G_2(x, s) = \begin{cases} R_3(x, s), & 0 \leq s \leq 1 - x, \\ R_4(x, s), & 1 - x \leq s \leq 1; \end{cases}$$

we seek a solution in the form,  $R_j(x, s)$ ,  $j = 1, 2, 3, 4$  - are also sufficiently smooth functions that are unknown for now

Initially, from (21) we

$$y(x) = \int_0^x R_1(x, s) f_1(s) ds + \int_x^1 R_2(x, s) f_1(s) ds + \int_0^{1-x} R_3(x, s) f_1(s) ds + \int_{1-x}^1 R_4(x, s) f_1(s) ds \quad (22)$$

we rewrite it in the following form.

$$R_2(0, s) + R_3(0, s) = 0 \quad (23)$$

if (23) holds, then  $y(0) = 0$  holds.

First, differentiating (22) once and then replacing  $x$  with  $1 - x$ , we obtain the following equality:

$$\begin{aligned} & y'(x) + \beta y(1 - x) = \\ & = [R_1(x, x - 0) - R_2(x, x + 0)] f_1(x) + [R_4(x, 1 - x + 0) - R_3(x, 1 - x - 0)] f_1(1 - x) + \\ & + \int_0^x \frac{\partial}{\partial x} R_1(x, s) f_1(s) ds + \int_x^1 \frac{\partial}{\partial x} R_2(x, s) f_1(s) ds + \int_0^{1-x} \frac{\partial}{\partial x} R_3(x, s) f_1(s) ds + \int_{1-x}^1 \frac{\partial}{\partial x} R_4(x, s) f_1(s) ds + \\ & + \beta \left( \int_0^{1-x} R_1(1 - x, s) f_1(s) ds + \int_{1-x}^1 R_2(1 - x, s) f_1(s) ds + \int_0^x R_3(1 - x, s) f_1(s) ds + \int_x^1 R_4(1 - x, s) f_1(s) ds \right). \end{aligned}$$

Here, this

$$\frac{\partial}{\partial x} R_1(x, s) + \beta R_3(1 - x, s) = 0,$$

$$\frac{\partial}{\partial x} R_3(x, s) + \beta R_1(1 - x, s) = 0,$$

$$\begin{aligned}\frac{\partial}{\partial x} R_2(x, s) + \beta R_4(1 - x, s) &= 0, \\ \frac{\partial}{\partial x} R_4(x, s) + \beta R_2(1 - x, s) &= 0 \\ [R_1(x, x - 0) - R_2(x, x + 0)] &= 1, \\ [R_4(x, 1 - x + 0) - R_3(x, 1 - x - 0)] &= 0,\end{aligned}$$

Let the relations hold.

First, we consider the first two of these relations:

$$\begin{cases} \frac{\partial}{\partial x} R_1(x, s) + \beta R_3(1 - x, s) = 0, \\ \frac{\partial}{\partial x} R_3(x, s) + \beta R_1(1 - x, s) = 0. \end{cases}$$

Adding and subtracting this system once,

$$\begin{cases} \frac{\partial}{\partial x} R_1(x, s) + \frac{\partial}{\partial x} R_3(x, s) + \beta R_1(1 - x, s) + \beta R_3(1 - x, s) = 0, \\ \frac{\partial}{\partial x} R_1(x, s) - \frac{\partial}{\partial x} R_3(x, s) - \beta R_1(1 - x, s) + \beta R_3(1 - x, s) = 0, \end{cases} \quad (24)$$

we obtain a system. This

$$\begin{bmatrix} F_1(x, s) = R_1(x, s) + R_3(x, s) \\ F_2(x, s) = R_1(x, s) - R_3(x, s) \end{bmatrix} \quad (25)$$

Let us introduce the following notations. Then, from (24)

$$\begin{aligned}\frac{\partial}{\partial x} F_1(x, s) + \beta F_1(1 - x, s) &= 0, \\ \frac{\partial}{\partial x} F_2(x, s) - \beta F_2(1 - x, s) &= 0,\end{aligned}$$

we obtain differential equations. The general solutions of these differential equations, based on (12) and (17), are respectively

$$\begin{aligned}F_1(x, s) &= A(s) (\cos(\beta - \beta x) - \sin(\beta x)) \\ F_2(x, s) &= B(s) (\cos(\beta - \beta x) + \sin(\beta x))\end{aligned}$$

can be written in the form, where  $A(s), B(s)$  are unknown functions for now.

Found  $F_1(x, s), F_2(x, s)$  substituting the obtained functions into notation (25), the functions  $R_1(x, s), R_3(x, s)$

$$\begin{aligned}R_1(x, s) &= \frac{A(s) + B(s)}{2} \cos(\beta - \beta x) - \frac{A(s) - B(s)}{2} \sin(\beta x), \\ R_3(x, s) &= \frac{A(s) - B(s)}{2} \cos(\beta - \beta x) - \frac{A(s) + B(s)}{2} \sin(\beta x),\end{aligned}$$

we obtain in the form.

This

$$a(s) = \frac{A(s) + B(s)}{2}, \quad b(s) = \frac{A(s) - B(s)}{2}$$

Introducing these notations, the functions  $R_1(x, s), R_3(x, s)$

$$R_1(x, s) = a(s) \cos(\beta - \beta x) - b(s) \sin(\beta x), \quad (26)$$

$$R_3(x, s) = b(s) \cos(\beta - \beta x) - a(s) \sin(\beta x), \quad (27)$$

we obtain in the form.

Using the above method, the functions  $R_2(x, s), R_4(x, s)$

$$R_2(x, s) = c(s) \cos(\beta - \beta x) - d(s) \sin(\beta x), \quad (28)$$

$$R_4(x, s) = d(s) \cos(\beta - \beta x) - c(s) \sin(\beta x), \quad (29)$$

we obtain in the form.

This

$$[R_1(x, x - 0) - R_2(x, x + 0)] = 1,$$

using the relation

$$[a(x) - c(x)] \cos(\beta - \beta x) + [d(x) - b(x)] \sin(\beta x) = 1, \quad (30)$$

we obtain the equality.

Now, this

$$[R_4(x, 1 - x + 0) - R_3(x, 1 - x - 0)] = 0$$

using the relation

$$[d(1 - x) - b(1 - x)] \cos(\beta - \beta x) + [a(1 - x) - c(1 - x)] \sin(\beta x) = 0$$

we obtain the equality. Here, replacing  $x$  with  $1 - x$ , we rewrite the last equality as.

$$[d(x) - b(x)] \cos(\beta x) + [a(x) - c(x)] \sin(\beta - \beta x) = 0 \quad (31)$$

we obtain the equality.

It is not difficult to show that using (30) and (31),

$$a(x) - c(x) = \frac{\cos(\beta x)}{\cos \beta}$$

$$b(x) - d(x) = \frac{\sin(\beta - \beta x)}{\cos \beta}$$

we obtain equalities.

Now, this

$$R_2(0, s) + R_3(0, s) = 0$$

using the condition

$$[b(s) + c(s)] \cos \beta = 0$$

we obtain equalities.

Using the above equalities,

$$c(x) = a(x) - \frac{\cos(\beta x)}{\cos \beta}$$

$$b(x) = -a(x) + \frac{\cos(\beta x)}{\cos \beta}$$

$$d(x) = -a(x) + \frac{\cos(\beta x) - \sin(\beta - \beta x)}{\cos \beta}$$

we obtain equalities. Substituting these equalities into (26), (27), (28), and (29), the functions  $R_1(x, s)$ ,  $R_2(x, s)$ ,  $R_3(x, s)$ ,  $R_4(x, s)$

$$R_1(x, s) = a(s) [\cos(\beta - \beta x) + \sin(\beta x)] - \frac{\sin(\beta x) \cos(\beta s)}{\cos \beta},$$

$$R_2(x, s) = a(s) [\cos(\beta - \beta x) + \sin(\beta x)] - \frac{\cos(\beta - \beta x) \cos(\beta s) - \sin(\beta - \beta s) \sin(\beta x) + \cos(\beta s) \sin(\beta x)}{\cos \beta},$$

$$R_3(x, s) = -a(s) [\cos(\beta - \beta x) + \sin(\beta x)] + \frac{\cos(\beta - \beta x) \cos(\beta s)}{\cos \beta},$$

$$R_4(x, s) = -a(s) [\cos(\beta - \beta x) + \sin(\beta x)] +$$

$$+ \frac{\sin(\beta x) \cos(\beta s) + \cos(\beta - \beta x) \cos(\beta s) - \cos(\beta - \beta x) \sin(\beta - \beta s)}{\cos \beta},$$

we obtain in the form

Now, substituting the obtained  $R_1(x, s)$ ,  $R_2(x, s)$ ,  $R_3(x, s)$ ,  $R_4(x, s)$  functions into (22), the function  $y(x)$

$$\begin{aligned} y(x) = & \int_0^x a(s) [\cos(\beta - \beta x) + \sin(\beta x)] f_1(s) ds - \int_0^x \frac{\sin(\beta x) \cos(\beta s)}{\cos \beta} f_1(s) ds \\ & + \int_x^1 a(s) [\cos(\beta - \beta x) + \sin(\beta x)] f_1(s) ds - \\ & - \int_x^1 \frac{\cos(\beta - \beta x) \cos(\beta s) - \sin(\beta - \beta s) \sin(\beta x) + \cos(\beta s) \sin(\beta x)}{\cos \beta} f_1(s) ds \\ & - \int_0^{1-x} a(s) [\cos(\beta - \beta x) + \sin(\beta x)] f_1(s) ds + \int_0^{1-x} \frac{\cos(\beta - \beta x) \cos(\beta s)}{\cos \beta} f_1(s) ds \\ & - \int_{1-x}^1 a(s) [\cos(\beta - \beta x) + \sin(\beta x)] f_1(s) ds + \\ & + \int_{1-x}^1 \frac{\sin(\beta x) \cos(\beta s) + \cos(\beta - \beta x) \cos(\beta s) - \cos(\beta - \beta x) \sin(\beta - \beta s)}{\cos \beta} f_1(s) ds \end{aligned}$$

we obtain in the form. After some simplifications, we rewrite the last equality as follows

$$\begin{aligned} y(x) = & \frac{\cos(\beta - \beta x) - \sin(\beta x)}{\cos \beta} \int_0^1 \cos(\beta s) f_1(s) ds - \\ & - \int_x^1 \frac{\cos(\beta - \beta x) \cos(\beta s) - \sin(\beta - \beta s) \sin(\beta x)}{\cos \beta} f_1(s) ds + \\ & + \int_{1-x}^1 \frac{\sin(\beta x) \cos(\beta s) - \cos(\beta - \beta x) \sin(\beta - \beta s)}{\cos \beta} f_1(s) ds. \end{aligned} \quad (32)$$

**Theorem 4.** If  $\beta \neq \pi/2 + \pi n$ ,  $n \in \mathbb{Z}$  and  $f_1(x) \in C(0, 1) \cap L_1(0, 1)$ , then the problem has a unique  $K_2$  solution, which is given by equality (32).

Now, using the above problems  $K_1$  and  $K_2$ , we obtain the solution of problem (7), (9) in the following form:

$$\begin{aligned} \tau(x) = & \frac{\varphi(0)}{\cos 2\alpha} (\cos(2\alpha - 2\alpha x) - \sin(2\alpha x)) + \\ & + \frac{\cos(2\alpha - 2\alpha x) - \sin(2\alpha x)}{\cos 2\alpha} \int_0^1 \cos(2\alpha s) \left[ 2\varphi'(s) - v(s) - \int_0^{s/2} f(s - \eta, \eta) d\eta \right] ds - \\ & - \int_x^1 \frac{\cos(2\alpha - 2\alpha x) \cos(2\alpha s) - \sin(2\alpha - 2\alpha s) \sin(2\alpha x)}{\cos 2\alpha} \left[ 2\varphi'(s) - v(s) - \int_0^{s/2} f(s - \eta, \eta) d\eta \right] ds + \\ & + \int_{1-x}^1 \frac{\sin(2\alpha x) \cos(2\alpha s) - \cos(2\alpha - 2\alpha x) \sin(2\alpha - 2\alpha s)}{\cos 2\alpha} \left[ 2\varphi'(s) - v(s) - \int_0^{s/2} f(s - \eta, \eta) d\eta \right] ds. \end{aligned} \quad (33)$$

integrating (33) by parts, we rewrite it in the following form:

$$\begin{aligned} \tau(x) = & 2\varphi(x) + \frac{\cos(2\alpha - 2\alpha x) - \sin(2\alpha x)}{\cos 2\alpha} \int_0^1 [4\alpha \cdot \varphi(s) \sin(2\alpha s)] ds - \\ & - \frac{\cos(2\alpha - 2\alpha x) - \sin(2\alpha x)}{\cos 2\alpha} \int_0^1 \cos(2\alpha s) \left[ v(s) + \int_0^{s/2} f(s - \eta, \eta) d\eta \right] ds - \end{aligned}$$

$$\begin{aligned}
 & - \int_x^1 \frac{\cos(2\alpha - 2\alpha x) \sin(2\alpha s) - \cos(2\alpha - 2\alpha s) \sin(2\alpha x)}{\cos 2\alpha} 4\alpha \cdot \varphi(s) ds + \\
 & + \int_x^1 \frac{\cos(2\alpha - 2\alpha x) \cos(2\alpha s) - \sin(2\alpha - 2\alpha s) \sin(2\alpha x)}{\cos 2\alpha} \left[ v(s) + \int_0^{s/2} f(s - \eta, \eta) d\eta \right] ds + \\
 & - \int_{1-x}^1 \frac{\sin(2\alpha x) \cos(2\alpha s) - \cos(2\alpha - 2\alpha x) \sin(2\alpha - 2\alpha s)}{\cos 2\alpha} \left[ v(s) + \int_0^{s/2} f(s - \eta, \eta) d\eta \right] ds
 \end{aligned} \quad (34)$$

Substituting (34) into (5), we obtain the  $I_1$  solution of the problem.

**Theorem 5.** If  $|\alpha| < \frac{1}{2}$ ,  $\varphi(x) \in C^1[0, 1] \cap C^2(0, 1)$ ,  $f(x, t) \in L(D) \cap C(D)$ ,  $\tau(x) \in C^1[0, 1] \cap C^2(0, 1)$  and  $v(x) \in C[0, 1] \cap C^1(0, 1)$ , then the  $I_1$  problem has a unique solution, which is determined by formulas (5) and (34).

Now, we consider the following inverse problem for equation (1).

### Inverse problem

**T problem.**  $D$  in the domain

$$u_{tt}(x, t) - u_{xx}(x, t) = f(t) \quad (35)$$

satisfying the equation  $u \in C(\overline{D})$ ,  $u_{tt}$ ,  $u_{xx} \in C(D)$ ,  $f(t) \in C[0, 1/2]$  belonging to the class and

$$u(x, 0) + \alpha_1 \int_0^x u(1 - \xi, 0) d\xi = \varphi(x), \quad 0 \leq x \leq 1, \quad (36)$$

(36), (3) and

$$u(t, t) = \psi(t), \quad 0 \leq t \leq 1/2 \quad (37)$$

satisfying the conditions  $\{u(x, t), f(t)\}$  find a pair of functions such that, where  $\varphi(x)$ ,  $v(x)$ ,  $\psi(t)$  - given functions, and  $\alpha_1$  is a given non-zero real number.

To solve this problem, let us introduce notation (4), where  $\tau(x)$  - unknown function.

Now, for equation (35), we obtain a Cauchy problem satisfying conditions (3) and (4).

**Theorem 6.** If  $\tau(x) \in C^1[0, 1] \cap C^2(0, 1)$  and  $\varphi(t) \in C[0, \frac{1}{2}]$  then the Cauchy problem has a unique solution

$$u(x, t) = \frac{1}{2} [\tau(x - t) + \tau(x + t)] + \frac{1}{2} \int_{x-t}^{x+t} v(z) dz + \frac{1}{2} \int_0^t d\eta \int_{x-t+\eta}^{x+t-\eta} f(\eta) ds \quad (38)$$

is obtained in the form. [1]

Now, to find the unknown function  $\tau(x)$ , we substitute solution (38) into condition (36) and obtain,

$$\tau(x) + \alpha_1 \int_0^x \tau(1 - \xi) d\xi = \varphi(x), \quad 0 \leq x \leq 1, \quad (39)$$

we obtain an integral equation.

In (39), taking into account that  $\tau(x) \in C[0, 1]$  and  $v(x) \in C[0, 1]$  we obtain condition (7). Now, differentiating equation (39) once with respect to  $x=0$  we obtain  $x$ ,

$$\tau'(x) + \alpha_1 \tau(1 - x) = \varphi'(x), \quad (40)$$

we obtain a differential equation with a delayed argument.

Now, for equation (40), we find the  $\tau(x)$  function satisfying condition (7) using the same method as in the above direct problem. The value of  $\tau(x)$

$$\begin{aligned}
 \tau(x) = & \varphi(x) + \frac{\cos(\alpha_1 - \alpha_1 x) - \sin(\alpha_1 x)}{\cos \alpha_1} \int_0^1 \alpha_1 \cdot \varphi(s) \sin(\alpha_1 s) ds - \\
 & - \int_x^1 \frac{\cos(\alpha_1 - \alpha_1 x) \sin(\alpha_1 s) - \cos(\alpha_1 - \alpha_1 s) \sin(\alpha_1 x)}{\cos \alpha_1} \alpha_1 \cdot \varphi(s) ds +
 \end{aligned} \quad (41)$$

$$+ \int_{1-x}^1 \frac{\sin(\alpha_1 x) \sin(\alpha_1 s) - \cos(\alpha_1 - \alpha_1 x) \cos(\alpha_1 - \alpha_1 s)}{\cos \alpha_1} \alpha_1 \cdot \varphi(s) ds$$

Now, from (38)

$$\Phi(x, t, \eta) = \int_{x-t+\eta}^{x+t-\eta} f(\eta) ds$$

we compute the integral.

$$\Phi(x, t, \eta) = 2f(\eta)(t - \eta)$$

we obtain an equality.

Substituting this equality into (38), we obtain

$$u(x, t) \frac{1}{2} = \frac{1}{2} [\tau(x-t) + \tau(x+t)] + \frac{1}{2} \int_{x-t}^{x+t} v(z) dz + \int_0^t f(\eta)(t - \eta) d\eta \quad (42)$$

we obtain an equality.

substituting (42) into condition (37), we obtain,

$$\frac{1}{2} [\tau(0) + \tau(2t)] + \frac{1}{2} \int_0^{2t} \nu(z) dz + \int_0^t (t - \eta) f(\eta) d\eta = \psi(t) \quad (43)$$

we obtain an equation, where

$$\tau(0) = \psi(0).$$

Now, from (43), we obtain the following equation

$$\int_0^t (t - \eta) f(\eta) d\eta = \psi(t) - \frac{1}{2} \psi(0) - \frac{1}{2} \tau(2t) - \frac{1}{2} \int_0^{2t} \nu(z) dz \quad (44)$$

Here, differentiating once with respect to  $t$

$$\int_0^t f(\eta) d\eta = \psi'(t) - \tau'(2t) - \nu(2t) \quad (45)$$

we obtain an equality. Differentiating equality (45) once more with respect to  $t$ , we obtain the function  $f(t)$

$$f(t) = \psi''(t) - 2\tau''(2t) - 2\nu'(2t) \quad (46)$$

we obtain in the form  $t = 0$  If we set  $\psi'(0) = \tau'(0) + \nu(0)$  in equality (45), we obtain the equality and here, to find  $\tau'(0)$ , we use equality (41). We obtain  $\tau'(x)$

$$\begin{aligned} \tau'(x) = & \varphi'(x) + \frac{\sin(\alpha_1 - \alpha_1 x) - \cos(\alpha_1 x)}{\cos \alpha_1} \int_0^1 \alpha_1^2 \cdot \varphi(s) \sin(\alpha_1 s) ds + \\ & + \frac{\cos(\alpha_1 - \alpha_1 x) \sin(\alpha_1 x) - \cos(\alpha_1 - \alpha_1 x) \sin(\alpha_1 x)}{\cos \alpha_1} \alpha_1 \cdot \varphi(s) - \\ & - \int_x^1 \frac{\sin(\alpha_1 - \alpha_1 x) \sin(\alpha_1 s) - \cos(\alpha_1 - \alpha_1 s) \cos(\alpha_1 x)}{\cos \alpha_1} \alpha_1^2 \cdot \varphi(s) ds + \\ & + \frac{\sin(\alpha_1 x) \sin(\alpha_1 - \alpha_1 x) - \cos(\alpha_1 - \alpha_1 x) \cos(\alpha_1 x)}{\cos \alpha_1} \alpha_1 \cdot \varphi(s) + \\ & + \int_{1-x}^1 \frac{\cos(\alpha_1 x) \sin(\alpha_1 s) - \sin(\alpha_1 - \alpha_1 x) \cos(\alpha_1 - \alpha_1 s)}{\cos \alpha_1} \alpha_1^2 \cdot \varphi(s) ds \end{aligned}$$

we obtain in the form, where  $x = 0$  if we set, the value of  $\tau'(0)$  is obtained as follows.

$$\tau'(0) = \alpha_1 \varphi(0) + \varphi'(0) + \frac{\alpha_1^2}{\cos \alpha_1} \int_0^1 [\cos(\alpha_1 - \alpha_1 s) - \sin(\alpha_1 s)] \varphi(s) ds$$

Now  $\psi(0) = \tau(0)$ ,  $\psi'(0) = \tau'(0) + \nu(0)$ ,  $\tau(x) \in C^2[0, 1]$ ,  $\nu(x) \in C^2[0, 1]$  in order for  $\psi(0) = \varphi(0)$ ,  $\varphi(x) \in C^2[0, 1]$  and

$$\psi'(0) = \nu(0) + \alpha_1 \varphi(0) + \varphi'(0) + \frac{\alpha_1^2}{\cos \alpha_1} \int_0^1 [\cos(\alpha_1 - \alpha_1 s) - \sin(\alpha_1 s)] \varphi(s) ds$$

must be.

**Theorem 7.** If  $|\alpha_1| < \frac{1}{2}$ ,  $\psi(0) = \tau(0)$ ,  $\tau(x) \in C^2[0, 1]$ ,  $\nu(x) \in C^2[0, 1]$ ,  $\varphi(x) \in C^2[0, 1]$ ,  $\psi(0) = \varphi(0)$ ,  $\psi(t) \in C^2[0, 1/2]$  and

$$\psi'(0) = \nu(0) + \alpha_1 \varphi(0) + \varphi'(0) + \frac{\alpha_1^2}{\cos \alpha_1} \int_0^1 [\cos(\alpha_1 - \alpha_1 s) - \sin(\alpha_1 s)] \varphi(s) ds$$

then the  $T$  problem has a unique solution, which is determined by formulas (41), (42), and (46).

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## ON GEODESIC MAPPINGS OF SURFACES OF REVOLUTION

USMONXUJAYEV ZOKIRXUJA

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M. ULUGBEK, TASHKENT, UZBEKISTAN  
zokusm@gmail.com

**ABSTRACT.** It is known that in one of the directions of global geometry, problems such as reconstructing surfaces from a given extrinsic curvature, Gaussian curvature, or intrinsic curvature are studied. In particular, when solving reconstruction problems based on Gaussian curvature, the sign of the Gaussian curvature of a surface plays an important role. In this paper studies geodesic mappings of surfaces of revolution and investigates how the sign of Gaussian curvature changes under such mappings. Explicit formulas for the curvature of the image are obtained, conditions for preserving or reversing its sign are established, and examples of surfaces with positive and negative curvature are provided.

**MSC (2020):** 53C20; 53C22; 53C21.

**Key words:** geodesic mappings, surfaces of revolution, Gaussian curvature, curvature sign, nontrivial geodesic mappings, first fundamental form, deformation parameter.

## Introduction

Geodesic mappings play a fundamental role in differential geometry, serving as a tool for comparing geometric structures, classifying manifolds, and studying their intrinsic invariants [1].

This paper is devoted to the study of geodesic mappings of surfaces of revolution. The behavior of the sign of Gaussian curvature under nontrivial geodesic mappings is investigated. Based on the analysis of the coefficients of the first fundamental forms of the original surface and its image, explicit expressions for the Gaussian curvature of the image under a geodesic mapping are obtained. Conditions on the intervals of the deformation parameter for which the curvature sign is preserved or changes to the opposite are determined. Meaningful examples for surfaces of revolution with positive and negative curvature are presented.

Geodesic mappings establish connections between many applied disciplines such as geometric analysis, physics, and optimal control. Their overall value lies in the ability to transfer dynamic and geometric structures while preserving key trajectories, making them a universal tool for both theoretical research and engineering applications.

## Preliminaries

In this section, the general definition of a geodesic mapping is introduced, and a theorem on the necessary and sufficient condition for the existence of a geodesic mapping is formulated. An example of a geodesic mapping is also provided.

Let  $M$  and  $N$  be smooth manifolds of dimension  $n$  with affine connections.

**Definition 1** [2]. A geodesic mapping  $f$  of a manifold  $M$  onto a manifold  $N$  is a diffeomorphism that sends geodesic curves of  $M$  into geodesic curves of  $N$ .

**Example 1.** Consider a semi-cylinder

$$M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 = r^2, 0 < z \leq r, r > 0\}$$

and the plane

$$E^2 = \{(x, y, z) \in \mathbb{R}^3 : z = \text{const}\}.$$

Then the mapping

$$f(x, y, z) = \left( \frac{rx}{z}, \frac{ry}{z}, r \right)$$

is a geodesic mapping, where  $r \in \mathbb{R}$ .

Consider these manifolds in a coordinate system  $x^1, x^2, \dots, x^n$  common to the mapping  $f$ . Denote the components of the connection objects of the manifolds  $M$  and  $N$  at corresponding points  $M(x)$  and  $N(x)$  by  $\Gamma_{ij}^h(x)$  and  $\bar{\Gamma}_{ij}^h(x)$ , respectively, assuming they are symmetric, and set

$$\bar{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + P_{ij}^h(x) \quad (h, i, j = 1, 2, \dots, n), \quad (1)$$

where  $P_{ij}^h$  is the deformation tensor.

**Theorem 1** [3, p. 72]. For a mapping  $f$  of an affine connection manifold  $M$  onto an affine connection manifold  $N$  to be geodesic, it is necessary and sufficient that the deformation tensor  $P_{ij}^h$  of the connection of the mapping  $f$  be representable in the form

$$P_{ij}^h(x) = \psi_i(x)\delta_j^h + \psi_j(x)\delta_i^h, \quad (2)$$

where  $\delta_i^h$  are the Kronecker symbols, and  $\psi_i$  is some covariant vector.

Conditions (2) are tensorial in nature and therefore invariant with respect to the choice of the coordinate system  $x^1, x^2, \dots, x^n$  common to the mapping  $f$ . Based on these conditions, equations (1) take the form:

$$\bar{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \psi_i(x)\delta_j^h + \psi_j(x)\delta_i^h. \quad (3)$$

**Remark 1** [3, p. 74]. When the vector  $\psi_i(x)$  is identically zero, the geodesic mapping  $f$  is called trivial.

**Example 2.** Let the surface  $\bar{F}$  be the image of the surface  $F$  under a homothety in three-dimensional space and let  $k$  be the homothety coefficient. Then the metric tensors of the surfaces are related by  $\bar{g}_{ij} = k^2 g_{ij}$ .

### Main part

In this section, we study special types of Riemannian spaces of revolution using the tools of the theory of geodesic mappings of Riemannian manifolds.

Let  $F$  be a surface of revolution with a first fundamental form of the type:

$$ds^2 = a(\omega)d\omega^2 + b(\omega)dt^2, \quad (4)$$

where  $a(\omega)$  and  $b(\omega)$  are positive functions, and belongs to the smoothness class  $C^k (k \geq 1)$ . Coordinates are  $\omega \in [\omega_1, \omega_2]$  and  $t \in [0, 2\pi)$ , and parameters  $\omega_1, \omega_2$  can reach values  $\pm\infty$ .

Let  $f : F \rightarrow \bar{F}$  be a nontrivial geodesic mapping, where  $\bar{F}$  is the surface of revolution that is the image of the surface of revolution  $F$  under the geodesic mapping  $f$ . In works [4, 5], it was proved that the images of surfaces of revolution under a nontrivial geodesic mapping are surfaces of revolution. In [6], it was derived that the first fundamental form of the surface  $\bar{F}$  has the following form:

$$d\bar{s}^2 = \frac{pa(\omega)}{(1 + qb(\omega))^2} d\omega^2 + \frac{pb(\omega)}{1 + qb(\omega)} dt^2, \quad (5)$$

where the parameter  $p$  corresponds to homothetic transformations and is always positive [6],  $q$  is limited by the condition  $1 + qb(\omega) \neq 0$ , and  $\omega$  is the same parameter as on the surface  $F$  and  $t \in [0, 2\pi)$ . When  $1 + qb(\omega) < 0$ , the space with the first fundamental form (5) may have a different signature, but we will focus on the Riemannian case where  $1 + qb(\omega) > 0$ .

We will examine the changes of surfaces of revolution under geodesic mapping and begin by investigating surfaces of revolution  $F$  with positive Gaussian curvature  $K$ .

**Theorem 2.** The images of surfaces of revolution with positive Gaussian curvature have positive Gaussian curvature when

$$q \in \left( -\frac{4abK}{4ab^2K + (b')^2}; +\infty \right),$$

and have negative Gaussian curvature when

$$q \in \left( -b^{-1}; -\frac{4abK}{4ab^2K + (b')^2} \right).$$

**Proof.** To prove the theorem, we must find expressions for the Gaussian curvature of the surfaces of revolution  $F$  and  $\bar{F}$  having the first fundamental forms (4) and (5). Let's start by finding the expression for the Gaussian curvature  $K$  of the surface  $F$ . According to [8, p. 499], the Gaussian curvature of a two-dimensional manifold is defined as follows:

$$K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2} = \frac{\left( \frac{\partial \Gamma_{11}^i}{\partial t} - \frac{\partial \Gamma_{21}^i}{\partial \omega} + \sum_j \Gamma_{2j}^i \cdot \Gamma_{11}^j - \sum_j \Gamma_{1j}^i \cdot \Gamma_{21}^j \right) \cdot g_{i2}}{g_{11}g_{22} - g_{12}^2}, \quad (6)$$

where  $\Gamma_{ij}^k$  are the Christoffel symbols, and  $R_{1212}$  is a component of the Riemann tensor.

The Christoffel symbols of the surface of revolution  $F$  have the form:

$$\begin{aligned} \Gamma_{11}^1 &= \frac{1}{2a(\omega)} \frac{\partial a(\omega)}{\partial \omega}; & \Gamma_{12}^1 &= \Gamma_{21}^1 = 0; & \Gamma_{22}^1 &= -\frac{1}{2a(\omega)} \frac{\partial b(\omega)}{\partial \omega}; \\ \Gamma_{11}^2 &= 0; & \Gamma_{12}^2 &= \Gamma_{21}^2 = \frac{1}{2b(\omega)} \frac{\partial b(\omega)}{\partial \omega}; & \Gamma_{22}^2 &= 0. \end{aligned}$$

The Gaussian curvature of the surface  $F$  has the following form:

$$K = \frac{1}{4a^2(\omega)b(\omega)} \left( b(\omega) \frac{\partial a(\omega)}{\partial \omega} \frac{\partial b(\omega)}{\partial \omega} + a(\omega) \left( \frac{\partial b(\omega)}{\partial \omega} \right)^2 - 2a(\omega)b(\omega) \frac{\partial^2 b(\omega)}{\partial \omega^2} \right).$$

Let us denote the Gaussian curvature of the surface  $\bar{F}$  by  $\bar{K}$ . Now we find the Christoffel symbols of the surface  $\bar{F}$  whose first fundamental form is given by (5):

$$\begin{aligned} \bar{\Gamma}_{11}^1 &= \frac{1}{2a(\omega)(1+qb(\omega))} \left( (1+qb(\omega)) \frac{\partial a(\omega)}{\partial \omega} - 2qa(\omega) \frac{\partial b(\omega)}{\partial \omega} \right); \\ \bar{\Gamma}_{12}^1 &= \bar{\Gamma}_{21}^1 = 0; & \bar{\Gamma}_{22}^1 &= -\frac{1}{2a(\omega)} \frac{\partial b(\omega)}{\partial \omega}; \\ \bar{\Gamma}_{11}^2 &= 0; & \bar{\Gamma}_{12}^2 &= \bar{\Gamma}_{21}^2 = \frac{1}{2b(\omega)(1+qb(\omega))} \frac{\partial b(\omega)}{\partial \omega}; & \bar{\Gamma}_{22}^2 &= 0. \end{aligned}$$

Using formula (6), the Gaussian curvature  $\bar{K}$  of the surface  $\bar{F}$  has the form:

$$\begin{aligned} \bar{K} &= \frac{(1+qb(\omega))}{4pa^2(\omega)b^2(\omega)} \left[ b(\omega) \frac{\partial a(\omega)}{\partial \omega} \frac{\partial b(\omega)}{\partial \omega} + a(\omega) \left( \frac{\partial b(\omega)}{\partial \omega} \right)^2 - 2a(\omega)b(\omega) \frac{\partial^2 b(\omega)}{\partial \omega^2} \right] + \\ &+ \frac{1}{4pa^2(\omega)b^2(\omega)} \left[ qa(\omega)b(\omega) \left( \frac{\partial b(\omega)}{\partial \omega} \right)^2 \right]. \end{aligned}$$

Consequently, we obtain:

$$\bar{K} = \frac{1}{p} \left( K \cdot (1+qb(\omega)) + \frac{q}{4a(\omega)b(\omega)} \cdot \left( \frac{\partial b(\omega)}{\partial \omega} \right)^2 \right). \quad (7)$$

Expanding the inner bracket in (7), we can show that:

$$\bar{K} > 0, \text{ when } q \in \left( -\frac{4abK}{4ab^2K + (b')^2}; +\infty \right), \quad \bar{K} < 0, \text{ when } q \in \left( -b^{-1}; -\frac{4abK}{4ab^2K + (b')^2} \right).$$

The theorem is proved.

**Example 3.** Consider a paraboloid of revolution  $F$  in Euclidean space  $\mathbb{R}^3$ , given in Cartesian coordinates  $(x; y; z)$  by the equations:

$$x = \omega \cos t, \quad y = \omega \sin t, \quad z = \omega^2,$$

where  $\omega \in (-\infty; +\infty)$ ,  $t \in [0, 2\pi)$ . The first fundamental form of the paraboloid of revolution has the following form:

$$ds^2 = (1 + 4\omega^2)d\omega^2 + \omega^2 dt^2.$$

The Gaussian curvature of  $F$  is:

$$K = \frac{4}{(1 + 4\omega^2)^2}.$$

The paraboloid of revolution  $F$  is mapped under a non-trivial geodesic mapping  $f$  to surfaces of revolution  $\bar{F}$ , whose first fundamental form is:

$$d\bar{s}^2 = \frac{p(1 + 4\omega^2)}{(1 + q\omega^2)^2} d\omega^2 + \frac{p\omega^2}{1 + q\omega^2} dt^2.$$

The Christoffel symbols for these surfaces are:

$$\begin{aligned} \bar{\Gamma}_{11}^1 &= \frac{2\omega(2 - q - 2q\omega^2)}{(1 + 4\omega^2)(1 + q\omega^2)}, \quad \bar{\Gamma}_{12}^1 = \bar{\Gamma}_{21}^1 = 0, \quad \bar{\Gamma}_{22}^1 = -\frac{\omega}{1 + 4\omega^2}, \\ \bar{\Gamma}_{11}^2 &= 0, \quad \bar{\Gamma}_{12}^2 = \bar{\Gamma}_{21}^2 = \frac{1}{\omega(1 + q\omega^2)}, \quad \bar{\Gamma}_{22}^2 = 0. \end{aligned}$$

Denoting by  $\bar{K}$  the Gaussian curvature of the image of the paraboloid of revolution under the geodesic mapping, according to (7) we obtain:

$$\bar{K} = \frac{1}{p} \cdot \left( K \cdot (1 + q\omega^2) + \frac{q}{1 + 4\omega^2} \right) = \frac{1}{p} \cdot \frac{q + 4 + 8q\omega^2}{(1 + 4\omega^2)^2}, \quad (8)$$

where  $K$  is the Gaussian curvature of the paraboloid of revolution. It can be shown that:

$$\begin{aligned} \bar{K} &> 0 \text{ when } q \in \left( -\frac{4}{1 + 8\omega^2}; +\infty \right); \\ \bar{K} &= 0 \text{ when } q = -\frac{4}{1 + 8\omega^2}; \\ \bar{K} &< 0 \text{ when } q \in \left( -\frac{1}{\omega^2}; -\frac{4}{1 + 8\omega^2} \right). \end{aligned}$$

Now consider surfaces of revolution with negative Gaussian curvature having the first fundamental form (5).

**Remark 2.** The images of surfaces of revolution with negative Gaussian curvature under a non-trivial geodesic mapping have always negative Gaussian curvature if  $q \in \left( -\frac{1}{b(\omega)}; 0 \right)$ .

Indeed, if  $F$  is a surface of revolution with negative Gaussian curvature and  $\bar{F}$  is the image of  $F$  under a geodesic mapping, then  $K$  and  $\bar{K}$ , the Gaussian curvatures of  $F$  and  $\bar{F}$ , are related by formula (7), and we obtain:

$$\bar{K} = \frac{1}{p} \left( K \cdot (1 + qb(\omega)) + \frac{q}{4a(\omega)b(\omega)} \cdot \left( \frac{\partial b(\omega)}{\partial \omega} \right)^2 \right).$$

It is clear that  $\bar{K}$  is negative, since  $a(\omega)$ ,  $b(\omega)$  are positive functions and  $q \in \left( -\frac{1}{b(\omega)}; 0 \right)$ .

**Example 4.** Consider a one-sheeted hyperboloid of revolution in Euclidean space  $\mathbb{R}^3$ , given in Cartesian coordinates  $(x; y; z)$  by the equations:

$$x = \cosh \omega \cos t, \quad y = \cosh \omega \sin t, \quad z = \sinh \omega,$$

where  $\omega$  and  $t$  ( $t \in [0, 2\pi)$ ) are parameters. The first fundamental form of the one-sheeted hyperboloid of revolution has the following form:

$$ds^2 = \cosh 2\omega d\omega^2 + \cosh^2 \omega dt^2.$$

The Christoffel symbols of this surface are:

$$\Gamma_{11}^1 = \tanh 2\omega; \quad \Gamma_{12}^1 = \Gamma_{21}^1 = 0; \quad \Gamma_{22}^1 = -\frac{1}{2} \tanh 2\omega;$$

$$\Gamma_{11}^2 = 0; \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \tanh \omega; \quad \Gamma_{22}^2 = 0.$$

Denoting the Gaussian curvature of the one-sheeted hyperboloid of revolution by  $K$ , using formula (6), we have:

$$K = -\frac{1}{\cosh^2 2\omega}.$$

The one-sheeted hyperboloid of revolution  $F$  is mapped under a non-trivial geodesic mapping  $f$  to a surface of revolution  $\bar{F}$  with the first fundamental form:

$$d\bar{s}^2 = \frac{p \cosh 2\omega}{(1 + q \cosh^2 \omega)^2} d\omega^2 + \frac{p \cosh^2 \omega}{1 + q \cosh^2 \omega} dt^2.$$

Denote the Gaussian curvature of the image of the one-sheeted hyperboloid of revolution under the non-trivial geodesic mapping by  $\bar{K}$ . According to (6), we have:

$$\bar{K} = \frac{1}{p} \left( -\frac{1}{\cosh^2 2\omega} (1 + q \cosh^2 \omega) + \frac{q \sinh^2 \omega}{\cosh 2\omega} \right). \quad (9)$$

Taking into account the identity  $\cosh 2\omega = \cosh^2 \omega + \sinh^2 \omega$  and  $q \in (-\frac{1}{\cosh^2 \omega}, 0)$ , the images of the one-sheeted hyperboloid of revolution under the non-trivial geodesic mapping have negative Gaussian curvature.

**Remark 3.** The images of surfaces of revolution with zero Gaussian curvature under a non-trivial geodesic mapping have non-zero Gaussian curvature.

Indeed, if  $F$  is a surface of revolution with zero Gaussian curvature and  $\bar{F}$  is the image of  $F$  under a geodesic mapping, then  $K$  and  $\bar{K}$ , the Gaussian curvatures of  $F$  and  $\bar{F}$ , are related by formula (7) and we obtain:

$$\bar{K} = \frac{1}{p} \left( \frac{q}{4a(\omega)b(\omega)} \cdot \left( \frac{\partial b(\omega)}{\partial \omega} \right)^2 \right),$$

where  $q \neq 0$ , since the geodesic mapping is non-trivial.

**Example 5.** Let  $F$  be a circular cone in Euclidean space  $\mathbb{R}^3$ , given in Cartesian coordinates  $(x; y; z)$  by the equations:

$$x = \omega \cos t, \quad y = \omega \sin t, \quad z = \omega,$$

where  $\omega \in (0, \omega_1)$ ,  $t \in [0, 2\pi)$ . The first fundamental form of the circular cone  $F$  has the following form:

$$ds^2 = 2d\omega^2 + \omega^2 dt^2.$$

The circular cone admits a non-trivial geodesic mapping, and the Gaussian curvature of the resulting surfaces is constant and non-zero. Denote the image of the circular cone under the geodesic mapping by  $\bar{F}$  and its Gaussian curvature by  $\bar{K}$ . The metric has the following form:

$$d\bar{s}^2 = \frac{2p}{(1 + q\omega^2)^2} d\omega^2 + \frac{p\omega^2}{1 + q\omega^2} dt^2,$$

where  $p$  is a positive parameter and  $q \in (-\frac{1}{\omega^2}; +\infty)$ ,  $1 + q\omega^2 \neq 0$ ,  $q\omega^2 \neq 0$ , and  $\omega$  is the same parameter as on the surface  $F$  and  $t \in [0, 2\pi]$ . We find the Christoffel symbols for this metric:

$$\bar{\Gamma}_{11}^1 = -\frac{2q\omega}{1 + q\omega^2}, \quad \bar{\Gamma}_{12}^1 = \bar{\Gamma}_{21}^1 = 0, \quad \bar{\Gamma}_{22}^1 = -\frac{\omega}{2};$$

$$\bar{\Gamma}_{11}^2 = 0, \quad \bar{\Gamma}_{12}^2 = \bar{\Gamma}_{21}^2 = \frac{1}{\omega(1 + q\omega^2)}, \quad \bar{\Gamma}_{22}^2 = 0.$$

Using formula (6) we have  $\bar{K} = \frac{q}{2p}$ .

**Remark 4.** The images of surfaces of revolution with zero Gaussian curvature under a trivial geodesic mapping also have zero Gaussian curvature.

Since according to Remark 1, the connection components of manifolds admitting a trivial geodesic mapping are equal, i.e.,

$$\bar{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x).$$

It follows that  $\bar{R}_{ijk}^h = 0$ .

**Example 6.** Consider a circular cylinder with the metric

$$ds^2 = d\omega^2 + dt^2,$$

where  $\omega \in [\omega_1, \omega_2]$ ,  $t \in [0, 2\pi]$ . The parameters  $\omega_1$  and  $\omega_2$  can reach  $\pm\infty$ . The circular cylinder admits a trivial geodesic mapping [5], and the resulting surfaces have zero Gaussian curvature. Denote the Gaussian curvature of the image of the circular cylinder under the geodesic mapping by  $\bar{K}$ . The metric has the following form:

$$d\bar{s}^2 = \frac{p}{(1+q)^2} d\omega^2 + \frac{p}{1+q} dt^2.$$

Here,  $p$  is a positive parameter and  $q \in (-1; +\infty)$ . The Christoffel symbols of this surface consist of zeros, i.e.,  $\bar{\Gamma}_{ij}^k = 0$ , so the Gaussian curvature  $\bar{K} = 0$ .

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# ON THE WEAK GENERALIZED SOLUTION OF THE FOURTH-ORDER TRICOMI EQUATION WITH A SEMI-NONLOCAL BOUNDARY CONDITION

YUSUPOV SHERZOD\*

V.I. ROMANOVSKY INSTITUTE OF MATHEMATICS OF THE ACADEMY OF SCIENCES OF THE REPUBLIC OF UZBEKISTAN

sherzod.yusupov.2020@inbox.ru

TOJINOROVA MAHLIYO

TASHKENT STATE TRANSPORT UNIVERSITY (TSTU)

tojinorovamaxliyo6@gmail.com

**ABSTRACT.** In this paper, we prove the unique solvability of a weak generalized solution to a semi-nonlocal boundary value problem for the fourth-order Tricomi equation in the Sobolev space  $W_2^2(Q)$ . The uniqueness of the solution to the semi-nonlocal boundary value problem for the fourth-order Tricomi equation is proved using the energy integral method, and its existence is proved using the modified Galerkin, a priori estimates, and “ $\varepsilon$ -regularization” methods.

**MSC (2020):** 35M10.

**Key words:** fourth-order Tricomi equation; semi-nonlocal boundary value problem; methods of modified Galerkin, a priori estimates and “ $\varepsilon$ -regularization”; unique solvability of a weak generalized solution.

## Introduction

As is well known, A.V. Bitsadze showed in his work that the Dirichlet problem for a mixed-type equation is ill-posed [1]. The question naturally arises: is it possible to replace the conditions of the Dirichlet problem with other conditions covering the entire boundary, which ensure the well-posedness of the problem? Such boundary value problems (nonlocal boundary value problems) for a second-order mixed-type equation were first proposed and studied in the work of F.I. Frankl [2]. A problem similar in formulation to the one under study for a second-order mixed-type equation of the first kind has been studied in bounded domains in [3-6].

As we know, local boundary value problems for fourth-order mixed-type equations without degeneration were proposed and studied by classical methods in the works of T. Juraev, A. Sopuev [7], Yu. Apakov, S. Mamazhanov [8]. Non-local boundary value problems for high-order partial differential equations without degeneracy have been studied by many scientists; a complete bibliography is available in the book by A. Dezin [9].

Linear inverse problems for model equations of mixed type of the second order in the plane were studied by classical methods in the work [10], and such problems for equations of mixed type of high order with local boundary conditions in different Sobolev spaces were studied in the works [11,12], and with non-local boundary conditions such problems have been very little studied. We will try to partially fill this gap within the framework of this work.

In this paper, using the results of works [5,6] and applying a modified Galerkin method, a priori estimates, and “ $\varepsilon$ -regularization”, the unique solvability of a weak generalized solution of one semi-non-local boundary value problem for a mixed-type equation of the second kind of the fourth order in Sobolev space is studied.

## Problem statement

In the domain

$$Q = (-1, 1) \times (0, T) = \{(x, t); -1 < x < 1; 0 < t < T < +\infty\}$$

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\*Corresponding autor

let's consider the fourth-order Tricomi equation

$$Lu = Pu + Mu = f(x, t), \quad (1)$$

here

$$Pu = \sum_{i=0}^4 K_i(x, t) D_t^i u; \quad Mu = a(x) u_{xxxx} - b(x) u_{xxtt} - c(x) u_{xx},$$

where  $K_4(x, t) = K_4(x) = x$ ,  $K_j(x, 0) = K_j(x, T)$ ,  $(j = \overline{0, 3})$  for every  $x \in [-1, 1]$ ,  $D_t^i u = \frac{\partial^i u}{\partial t^i}$  ( $i = \overline{0, 4}$ ),  $D_t^0 u = u$ , and

$$a(x) \geq a_0 > 0, \quad b(x) \geq b_0 > 0, \\ c(x) \geq c_0 > 0.$$

Let all coefficients of equation (1) be sufficiently smooth functions in  $\overline{Q}$ .

**Semi-nonlocal boundary value problem:** Find a generalized solution  $u(x, t)$  of equation (1) from the Sobolev space  $W_2^2(Q)$  satisfying the following semi-nonlocal boundary conditions

$$\gamma D_t^p u|_{t=0} = D_t^p u|_{t=T}, \quad p = 0, 1, 2, 3, \quad (2)$$

$$u|_{x=-1} = u|_{x=1} = 0; \quad u_{xx}|_{x=-1} = u_{xx}|_{x=1} = 0 \quad (3)$$

where  $\gamma$  is a nonzero quantity, which will be specified below.

Further, we need the following definition and auxiliary propositions [4].

Through  $W_2^l(Q)$  ( $l = 0, 1, 2, \dots$ , at  $l = 0$ ,  $W_2^0(Q) = L_2(Q)$ ) we define Sobolev space with scalar product

$$(u, v)_l = \sum_{|\alpha| \leq l} \int_Q D^\alpha u D^\alpha v dx dt$$

and with norm

$$\|v\|_l^2 = \|v\|_{W_2^l(Q)}^2 = \sum_{|\alpha| \leq l} \int_Q |D^\alpha v|^2 dx dt.$$

Here  $\alpha$  is a multi-index and  $D^\alpha$  is a generalized derivative with respect to the variables  $x$  and  $t$  [13,14]. When obtaining various a priori estimates, we often use the Cauchy inequality with  $\sigma$  [13], i.e.

$$\forall u, v \geq 0; \quad \forall \sigma > 0; \quad 2u v \leq \sigma u^2 + \frac{v^2}{\sigma}.$$

Let's define the following class of functions:

$$W_1(Q) = \{v \in W_2^4(Q); \quad \gamma D_t^p v|_{t=0} = D_t^p v|_{t=T}; \quad p = 0, 1, 2, 3; \\ v|_{x=-1} = v|_{x=1} = 0, \quad v_x|_{x=-1} = v_x|_{x=1} = 0 \}.$$

Let's define the following bilinear forms using the function  $v \in W_1(Q)$ :

$$A(u, e^{-\lambda t} v) \equiv (Pu, e^{-\lambda t} v) = (xu_{tt}, e^{-\lambda t} v_{tt})_0 - ((K_3 + 2\lambda x) u_{tt}, e^{-\lambda t} v_t)_0 + \left( \sum_{i=0}^2 b_i D_t^i u, e^{-\lambda t} v \right)_0,$$

$$m(u, v) \equiv (Mu, v) \equiv ((au_{xx} - bu_{tt}), v_{xx})_0 + 2((a'(x) u_{xx} - b'(x) u_{tt}), v_x) + \\ + ((a''(x) u_{xx} - b''(x) u_{tt}), v) + (cu_{xx}, v)_0,$$

where  $b_i \in C^2(Q) \cap C(\overline{Q})$ ;  $b_2(x, t) = \lambda K_3 - K_{3t} + K_2 + \lambda^2 x$ ;  $b_1 = K_1$ ;  $b_0 = K_0$ .

**Definition 1.** If for any function  $v(x, t) \in W_1(Q)$ , is valid the following integral identity

$$A(u, e^{-\lambda t} v) + m(u, e^{-\lambda t} v) = (f, e^{-\lambda t} v)_0, \quad \lambda = \frac{2}{T} \ln |\gamma| > 0, \quad |\gamma| > 1, \quad (4)$$

then the function  $u(x, t) \in W_2^2(Q)$  satisfying the boundary conditions (2),(3) is called the weak generalized solution of the problem (1) - (3).

# Main part

**Theorem 1.** Let the above conditions be satisfied for the coefficients of the equation (1), in addition, let the following conditions be satisfied for the coefficients of the equation (1):  $-(2K_3 + 3\lambda x) \geq \delta_3 > 0$ ,  $2K_1 - K_{2t} + \lambda K_2 \geq \delta_2 > 0$ ,  $\lambda K_0 - K_{0t} \geq \delta_1 > 0$ , for any  $(x, t) \in \bar{Q}$  where  $\lambda = \frac{2}{T} \ln |\gamma| > 0$ ,  $|\gamma| > 1$ . Then for any  $f(x, t) \in L_2(Q)$  if there exists a generalized solution of the problem (1) - (3) from the Sobolev space  $W_2^2(Q)$ , then it is unique and the following estimate is valid

$$\|u\|_{W_2^2(Q)}^2 \leq c \|f\|_0^2.$$

**Proof.** First, we prove the uniqueness of the solution to problem (1)-(3). We prove the uniqueness of the solution to the problem using the energy integral method. Let there exist a generalized solution  $u(x, t)$  to problem (1)-(3) from the Sobolev space  $W_2^2(Q)$ . Consider the following identity

$$2 \int_Q Lu \cdot e^{-\lambda t} \cdot u_t \, dx dt = 2 \int_Q f \cdot e^{-\lambda t} \cdot u_t \, dx dt. \quad (5)$$

By virtue of the conditions of Theorem 1 and the Cauchy inequality with  $\sigma$  [8], from the boundary conditions (2), (3) by integrating identity (5) we can easily obtain the following inequality

$$\begin{aligned} 2 \int_Q Lu \cdot e^{-\lambda t} \cdot u_t dQ &\geq e^{-\lambda T} \int_Q \{ -(2K_3 + 3\lambda x) \cdot u_{tt}^2 + \lambda (a(x) u_{xx}^2 + b(x) u_{xt}^2 + c(x) u_x^2) + \\ &+ (2K_1 + \lambda K_2 - K_{2t}) \cdot u_t^2 + (\lambda K_0 - K_{0t}) \cdot u^2 \} dQ - \\ &- \left\{ +3\sigma \|u_{tt}\|_0^2 + \sigma^{-1} K \cdot \|u_t\|_0^2 + \sigma \|u_{xx}\|_0^2 + 2\|a'\|_{C[-1;1]} \|u_{xx}\|_0^2 + \sigma^{-1} \|b'\|_{C[-1;1]}^2 \|u_{xt}\|_0^2 + \right. \\ &+ 2\|a'\|_{C[-1;1]} \|u_{xt}\|_0^2 + \sigma \|u_x\|_0^2 \left. \right\} \cdot e^{-\lambda T} \\ &+ \int_{\partial Q} (K_0 u^2 + (K_2 + \lambda^2 x) \cdot u_t^2 + \\ &+ (2K_3 + 2\lambda x) \cdot u_t u_{tt} - x u_{tt}^2 + 2x u_t u_{ttt} + a(x) u_{xx}^2 + b(x) u_{xt}^2 + c(x) u_x^2) \cdot e^{-\lambda t} \cdot \nu_t dS + \\ &+ \int_{\partial Q} (2a(x) u_{xxx} u_t - 2a(x) u_{xx} u_{tx} - 2a'(x) u_{xx} u_t - 2b(x) u_t u_{xtt} - 2b'(x) u_{tt} \cdot u_t - 2c(x) u_t \cdot u_x) \cdot e^{-\lambda t} \nu_x dS. \end{aligned} \quad (6)$$

where

$$K = \max \left\{ \|K_3\|_{C^1(\bar{Q})}^2, \|a''\|_{C[-1;1]}^2, \|b''\|_{C[-1;1]}^2 \right\},$$

$\vec{\nu} = (\nu_t, \nu_x)$  is a unit inward normal vector to the boundary  $\partial Q$ .

The conditions of Theorem 1 ensure the non-negativity of the integral over the domain and the conversion of boundary integrals to zero. From the inequality (6) we obtain

$$\begin{aligned} \|f\|_0^2 &\geq e^{-\lambda T} \int_Q \{ \delta_3 \cdot u_{tt}^2 + \delta_2 u_t^2 + \delta_1 \cdot u^2 \} \, dx \, dt - \\ &- 3\sigma \cdot \|u_{tt}\|_0^2 - \sigma^{-1} K \cdot \|u_t\|_0^2 - \sigma \int_Q u_{xx}^2 dQ + \int_Q \lambda a(x) u_{xx}^2 dQ + \int_Q \lambda b(x) u_{xt}^2 dQ + \int_Q \lambda c(x) u_x^2 dQ - \\ &- 2\|a'\| \int_Q u_{xx}^2 e^{-\lambda t} dQ - \sigma^{-1} \|b'\|^2 \int_Q u_{xt}^2 e^{-\lambda t} dQ - 2\|a'\|^2 \int_Q u_{xt}^2 e^{-\lambda t} dQ - \sigma \int_Q u_x^2 e^{-\lambda t} dQ = \end{aligned} \quad (7)$$

$$= e^{-\lambda T} \left( \int_Q (\delta_3 - 3\sigma) u_{tt}^2 dQ + \int_Q (\lambda a(x) - 2\|a'\|_C - \sigma) u_{xx}^2 dQ + \int_Q (\lambda b(x) - \sigma^{-1} \|b'\|_C^2 - 2\|a'\|_C) u_{xt}^2 dQ + \right. \\ \left. + \int_Q (\lambda c(x) - \sigma) u_x^2 dQ + \int_Q (\delta_2 - \sigma^{-1} K) u_t^2 dQ + \int_Q \delta_1 u^2 dQ \right).$$

In inequality (7), we choose the number  $\sigma$  so that the following inequalities are true

$$\delta_3 - 3\sigma = \frac{\delta_3}{2} \Rightarrow \sigma = \frac{\delta_3}{6} \Rightarrow \sigma^{-1} = \frac{6}{\delta_3},$$

$$\delta_2 - \sigma^{-1} K = \delta_2 - \frac{6}{\delta_3} K > \delta_{02} > 0,$$

$$a(x) \geq a_0 = (\sigma + 2\|a'\|_C) = \left( \frac{\delta_3}{6} + 2\|a'\|_C \right) > 0,$$

$$b(x) \geq b_0 = \left( \sigma^{-1} \|b'\|_C^2 + 2\|a'\|_C \right) = \left( \frac{6}{\delta_3} \|b'\|_C^2 + 2\|a'\|_C \right) > 0,$$

$$c(x) \geq c_0 = \frac{\delta_3}{6} > 0.$$

Further, denoting by  $\delta = e^{-\lambda T} \min\{\frac{\delta_3}{2}, a_0, b_0, c_0, \delta_{02}, \delta_1\}$  and dividing (7) by  $\delta > 0$ , we obtain the first a priori estimate for solving the problem (1) - (3)

$$\|u\|_{W_2^2(Q)}^2 \leq c \|f\|_{L_2(Q)}^2.$$

From this estimate, the uniqueness of the solution of the problem (1) - (3) in  $W_2^2(Q)$ .

Let us prove the theorem from the opposite, let the problem (1) - (3) have two solutions  $u_1(x, t)$ ,  $u_2(x, t)$ . Then the new function  $\vartheta(x, t) = u_1(x, t) - u_2(x, t)$  satisfies the homogeneous equation (1) with the conditions (2) - (3) and the first estimate  $\|\vartheta\|_2^2 \leq 0$  is valid for it. Hence, the uniqueness of the solution of the problem (1) - (3)  $u_1(x, t) = u_2(x, t)$  from the space  $W_2^2(Q)$ .

Now we will prove the solvability of the problem (1) - (3).

### Fifth-order equation with a small parameter (auxiliary task).

We will prove the solvability of the problem (1) - (3) by the “ $\varepsilon$ -regularization” method in combination with the modified Galerkin method and the method of a priori estimates, namely: in the domain  $Q = (0, 1) \times (0, T)$  we consider a family of fifth-order equations with a small parameter

$$L_\varepsilon u_\varepsilon = -\varepsilon \frac{\partial \Delta^2 u_\varepsilon}{\partial t} + L u_\varepsilon = f(x, t) \quad (8)$$

with semi-non-local boundary conditions

$$\gamma D_t^q u_\varepsilon|_{t=0} = D_t^q u_\varepsilon|_{t=T}, \quad q = 0, 1, 2, 3, 4, \quad (9)$$

$$u_\varepsilon|_{x=-1} = u_\varepsilon|_{x=1} = 0, \quad u_{\varepsilon xx}|_{x=-1} = u_{\varepsilon xx}|_{x=1} = 0, \quad (10)$$

where  $\varepsilon$  is a small positive number  $D_z^q w = \frac{\partial^q w}{\partial z^q}$ ,  $q = 1, 2, 3, 4, 5$ ;  $D_z^0 w = w$ ,  $\Delta^2 u = \left( \frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} \right)^2 u = \left( \frac{\partial^4 u}{\partial t^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial t^2} + \frac{\partial^4 u}{\partial x^4} \right)$  is a biharmonic operator.

Below, we use the fifth-order equation with a small parameter (8) as the “ $\varepsilon$ -regulating” equation for the first-order mixed-type fourth-order equation (1) [5,11,12].

Let's by  $V(Q)$  we denote the class of functions such that  $u_\varepsilon(x, t) \in W_2^4(Q)$ ,  $\frac{\partial \Delta^2 u_\varepsilon}{\partial t} \in L_2(Q)$ , and satisfying the corresponding conditions (9), (10).

**Definition 2.** The function  $u_\varepsilon(x, t) \in V(Q)$  is called a weak generalized solution of problem (8)-(10) if the following integral identity holds

$$\varepsilon (u_{\varepsilon ttt}, (e^{-\lambda t} \vartheta)_{tt}) + \varepsilon (u_{\varepsilon xxt}, (e^{-\lambda t} \vartheta)_{tt}) + \varepsilon (u_{\varepsilon xt}, (e^{-\lambda t} \vartheta)_{tt}) + \\ + A (u_\varepsilon, e^{-\lambda t} \vartheta) + m (u_\varepsilon, e^{-\lambda t} \vartheta) = (f, e^{-\lambda t} \vartheta)_0.$$

**Theorem 2.** Let all conditions of theorem 1 be satisfied. Then, for any function  $f(x, t) \in L_2(Q)$ , there exists a unique weakly generalized solution  $u_\varepsilon(x, t)$  of the problem (8) - (10) from space  $V(Q)$ , and the following estimate is valid for it:

$$I) \quad \varepsilon (\|u_{\varepsilon ttt}\|_0^2 + \|u_{\varepsilon xxt}\|_0^2 + \|u_{\varepsilon xt}\|_0^2) + \|u_\varepsilon\|_2^2 \leq c \|f\|_0^2.$$

**Proof.** The proof of the inequality I) is carried out in the same way as the first estimate of theorem 1, from which it follows that the weak generalized solution of the problem (8) - (10) is unique [6,11,13].

Let's present the proof of the a priori estimate. Let the function  $\phi_j(x, t) \in W_2^4(Q)$  be a eigenfunction of a self-adjoint, positively defined elliptic operator of the fourth order, with boundary conditions

$$D_t^p \phi_j|_{t=0} = D_t^p \phi_j|_{t=T}, \quad p = 0, 1, 2, 3, \quad (11)$$

$$\phi_j|_{x=-1} = \phi_j|_{x=1} = 0, \quad \phi_{jxx}|_{x=-1} = \phi_{jxx}|_{x=1} = 0. \quad (12)$$

From the general theory of linear self-adjoint elliptic operators [15] it is known that all eigenfunctions of a self-adjoint, positively defined, fourth-order elliptic operator with the boundary conditions (11), (12) form a complete orthonormal system in  $L_2(Q)$ . Now, using these sequences of functions, we will construct a solution to the auxiliary problem

$$P\omega_j \equiv \exp(-\frac{\lambda t}{2}) \frac{\partial \omega_j}{\partial t} = \phi_j, \quad (13)$$

$$\gamma \omega_j(x, 0) = \omega_j(x, T), \quad (14)$$

where  $\gamma = \text{const} \neq 0$  and  $|\gamma| > 1$ . It is clear that the problem (13), (14) is uniquely solvable and its solution has the form

$$P^{-1}\phi_j = \omega_j = \int_0^t \exp(\frac{\lambda \tau}{2}) \phi_j d\tau + \frac{1}{\gamma - 1} \int_0^T \exp(\frac{\lambda t}{2}) \phi_j dt.$$

It is clear that the functions  $\omega_j(x, t) \in W_2^5(Q)$  are linearly independent. Indeed, if  $\sum_{j=1}^N c_j \omega_j = 0$  for some set of

function sequences  $\omega_1, \omega_2, \dots, \omega_N$ , then, acting on this sum by the operator  $P$ , we obtain  $\sum_{j=1}^N c_j P\omega_j = \sum_{j=1}^N c_j \phi_j =$

0, and from this it follows that  $c_j = 0$  for all the coefficients.  $j = \overline{1, N}$ . Note that the following conditions arise from the construction of the function  $\phi_j(x, t)$  on the function  $\omega_j(x, t) \in W_2^5(Q)$

$$\gamma \cdot D_t^q \omega_j|_{t=0} = D_t^q \omega_j|_{t=T}, \quad q = 0, 1, 2, 3, 4, \quad (15)$$

$$\omega_j|_{x=-1} = \omega_j|_{x=1} = 0; \quad \omega_{jxx}|_{x=-1} = \omega_{jxx}|_{x=1} = 0. \quad (16)$$

Now we seek the approximate solution of the problem (8) - (10) in the form  $u_\varepsilon^N(x, t) = \sum_{j=1}^N c_j \omega_j(x, t)$  where the coefficients  $c_j$  for any  $j$  from 1 to  $N$  are defined as the solution of the following linear algebraic system

$$2 \int_Q L_\varepsilon u_\varepsilon^N \cdot \exp(-\frac{\lambda t}{2}) \phi_j dx dt = 2 \int_Q f \cdot \exp(-\frac{\lambda t}{2}) \phi_j dx dt. \quad (17)$$

We will prove the uniqueness of the algebraic system (17). Multiplying each equation from (17) by and summing from to, considering the boundary conditions (15) - (16) and the algebraic system (17), we obtain the identity

$$2 \int_Q L_\varepsilon u_\varepsilon^N \cdot \exp(-\lambda t) u_{\varepsilon t}^N dx dt = 2 \int_Q f \cdot \exp(-\lambda t) u_{\varepsilon t}^N dx dt, \quad (18)$$

from which, according to the condition of theorem 2, by integrating the identity (18) in parts, we obtain the first estimate for the approximate solution of the problem (8) - (10), i.e.

$$\varepsilon(\|u_{\varepsilon ttt}\|_0^2 + \|u_{\varepsilon ttx}\|_0^2 + \|u_{\varepsilon txx}\|_0^2) + \|u_{\varepsilon}\|_2^2 \leq c_1 \|f\|_0^2. \quad (19)$$

From this follows the solvability of the algebraic system (17)[13]. Estimate (19) allows, by virtue of Theorem 2 and the weak compactness theorem [6,13,16], to perform the limit passage with respect to  $N \rightarrow \infty$  and conclude that some subsequence  $\{u_{\varepsilon}^{N_k}(x, t)\}$  converges weakly, by virtue of the uniqueness of the solution (Theorem 1) in the space  $V(Q)$  to the desired solution  $u_{\varepsilon}(x, t)$  of problem (8)-(10), which has the properties specified in Theorem 2 [6,13]. For  $u_{\varepsilon}(x, t)$ , by virtue of (19), the following inequality is valid

$$\varepsilon(\|u_{\varepsilon ttt}\|_0^2 + \|u_{\varepsilon ttx}\|_0^2 + \|u_{\varepsilon txx}\|_0^2) + \|u_{\varepsilon}\|_2^2 \leq c_1 \|f\|_0^2. \quad (20)$$

Hence, we obtain the uniqueness of the weakly generalized solution of the problem (8) - (10). Thus, Theorem 2 proved. Let us proceed to the proof of the solvability of the problem (1) - (3).

### Existence of a generalized solution of the problem (1) - (3).

**Theorem 3.** Let all conditions of theorem 1 and 2 be satisfied. Then the solution of the problem (1)-(3) of  $W_2^2(Q)$  exists and is unique.

**Proof.** The uniqueness of the solution of the problem (1) - (3) in space  $W_2^2(Q)$  is proven in Theorem 1. Now we will prove the existence of a generalized solution of the problem (1) - (3) in  $W_2^2(Q)$ . For this, we consider the equation (8) in the domain  $Q$  with the boundary conditions (9), (10) for  $\varepsilon > 0$ . Since all conditions of theorem 2 are satisfied, then there exists a unique generalized solution of the problem (8) - (10) for  $\varepsilon > 0$  and for it the first estimate is valid. From this, it follows that from the set of functions  $\{u_{\varepsilon}(x, t)\}$ ,  $\varepsilon > 0$ , it is possible to extract a weakly convergent subsequence in  $V(Q)$ ,  $u_{\varepsilon_i}(x, t) \rightarrow u(x, t)$  of functions such that  $\varepsilon_i \rightarrow 0$  [6-16]. We will show that the limit function  $u(x, t)$  satisfies the equation  $Lu = f$  almost everywhere in  $Q$ . Indeed, since the subsequence  $\{u_{\varepsilon_i}(x, t)\}$  converges weakly in  $W_2^2(Q)$ , and the subsequences

$$\{\sqrt{\varepsilon_i} u_{\varepsilon_i ttt}(x, t)\}, \quad \{\sqrt{\varepsilon_i} u_{\varepsilon_i ttx}(x, t)\}, \quad \{\sqrt{\varepsilon_i} u_{\varepsilon_i txx}(x, t)\}$$

are uniformly bounded in  $L_2(Q)$ , and the operator  $L$  is linear, then we have

$$\begin{aligned} \varepsilon_i(u_{\varepsilon_i ttt}, (e^{-\lambda t} \vartheta)_{tt})_0 + \varepsilon_i(u_{\varepsilon_i ttx}, (e^{-\lambda t} \vartheta)_{xt})_0 + \varepsilon_i(u_{\varepsilon_i txx}, (e^{-\lambda t} \vartheta)_{xx})_0 + A(u_{\varepsilon_i}, e^{-\lambda t} \vartheta) + \\ + m(u_{\varepsilon_i}, e^{-\lambda t} \vartheta) = (f, e^{-\lambda t} \vartheta)_0. \end{aligned} \quad (21)$$

From the equation (21), passing to the limit at  $\varepsilon_i \rightarrow 0$ , we obtain the only generalized solution of the problem (1) - (3) [6,16]. Thus, Theorem 3 is proven.

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