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LOCAL AND 2-LOCAL ANTI-DERIVATIONS ON NULL-FILIFORM AND FILIFORM LEIBNIZ ALGEBRAS AND THEIR SOLVABLE EXTENSIONS

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ABSTRACT. In this paper, we study anti-derivations, local anti-derivations and 2-local anti-derivations on several finite-dimensional complex Leibniz algebras. More precisely, we consider the null-filiform Leibniz algebra NF_n , the naturally graded non-Lie filiform Leibniz algebras F_n^1 and F_n^2 , and their associated solvable extensions. Here, an anti-derivation is understood as a (-1) -derivation satisfying

$$D([x, y]) = -[D(x), y] - [x, D(y)].$$

We first determine the spaces of anti-derivations for all algebras under consideration. We then obtain complete descriptions of their local anti-derivations. In particular, every local anti-derivation is an anti-derivation precisely for the algebras R , $R(F_n^1)$ and $\mathfrak{L}_2$, whereas NF_n , F_n^1 , F_n^2 and $\mathfrak{L}_1$ admit local anti-derivations which are not anti-derivations. Furthermore, we prove that every 2-local anti-derivation on NF_n and on each of the four solvable extensions is an anti-derivation. For F_n^1 and F_n^2 , we construct families of nonlinear 2-local anti-derivations with values in the center. The members of these families are anti-derivations exactly when the corresponding defining maps are linear.

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Key words: Leibniz algebras, anti-derivations, local anti-derivations, 2-local anti-derivations, filiform algebras.

Introduction

Derivations are useful tools for studying algebraic structures and their symmetries. Their local and 2-local analogues lead to a natural question: when does a mapping that agrees with a derivation at each point, or at each pair of points, necessarily satisfy the derivation identity on the whole algebra? The answer depends on the structure of the algebra and can change when ordinary derivations are replaced by other classes of linear mappings.

The notion of a local derivation was introduced independently by Kadison [18] and by Larson and Sourour [21]. A local derivation is a linear mapping whose value at each element coincides with the value of some derivation, which may depend on that element. Subsequently, Šemrl [22] introduced 2-local derivations: for every pair of elements, one derivation realizes both prescribed values. In the 2-local setting, linearity of the mapping is not assumed. These definitions have motivated the study of when pointwise or pairwise agreement forces a mapping to belong to the original class.

For finite-dimensional Lie algebras over an algebraically closed field of characteristic zero, Ayupov and Kudaybergenov [9] proved that every local derivation on a semisimple Lie algebra is a derivation and constructed nilpotent examples admitting local derivations that are not derivations. Ayupov, Kudaybergenov and Rakhimov [10] established the corresponding result for 2-local derivations on semisimple Lie algebras and showed that every finite-dimensional nilpotent Lie algebra of dimension greater than two admits a 2-local derivation that is not a derivation. The solvable case also exhibits different behaviour: Ayupov and Khudoyberdiyev [7] obtained examples of both possibilities and proved that local derivations are derivations for solvable Lie algebras with model nilradicals and maximal-dimensional complementary spaces.

Related questions have been investigated in infinite-dimensional and non-Lie settings. Local derivations on Witt algebras were studied in [15], while 2-local derivations on generalized Witt algebras were considered in

[11]. For solvable Leibniz algebras, Ayupov, Khudoyberdiyev and Yusupov [8] showed that the nilradical and the dimension of its complement influence whether a local derivation is a derivation. In particular, their results include solvable Leibniz algebras for which every 2-local derivation is a derivation, although local derivations need not be derivations.

Another direction is the study of δ -derivations, developed for Lie algebras by Filippov [16,17]. For a scalar δ , a linear map $D : L \rightarrow L$ is a δ -derivation if

$$D([x, y]) = \delta([D(x), y] + [x, D(y)]) \quad (x, y \in L).$$

The values $\delta = 1$, $\delta = -1$ and $\delta = \frac{1}{2}$ are of particular interest. The first gives ordinary derivations, while a (-1) -derivation is called an anti-derivation. This is the meaning of anti-derivation used throughout the present paper. Local and 2-local δ -derivations were introduced by Khudoyberdiyev and Yusupov [19], who studied the case $\delta = \frac{1}{2}$ on several finite-dimensional Lie algebras. Further results for solvable Lie algebras with a filiform nilradical were obtained by Atajonov, Vaisova and Yusupov [4].

The study of local and 2-local anti-derivations on solvable Lie algebras was initiated by Ayupov, Atajonov and Yusupov [6]. They investigated algebras with filiform, Heisenberg and abelian nilradicals and established both results asserting that local mappings are anti-derivations and examples where this fails. This direction was subsequently developed for solvable Lie algebras with a filiform nilradical [1], the perfect Lie algebras $\mathfrak{L}^m$ [5], and certain Witt-type infinite-dimensional Lie algebras [2]. Analogous questions for null-filiform, filiform and naturally graded quasi-filiform associative algebras were considered in [3].

The present paper contributes to the emerging study of local and 2-local anti-derivations on non-Lie Leibniz algebras. We consider the null-filiform algebra NF_n , the naturally graded non-Lie filiform algebras F_n^1 and F_n^2 , and their maximal solvable extensions. The nilpotent algebras belong to the classes studied by Ayupov and Omirov [12]. The relevant solvable extensions are described in the classification results of Casas, Ladra, Omirov and Karimjanov [13,14], together with the corrigendum by Ladra, Masutova and Omirov [20]. Biderivations of these classes were investigated by Yusupov, Jumaniyozov and Azizov [23]. In that setting, the term anti-derivation is used for the identity $A([x, y]) = [A(x), y] - [A(y), x]$; the present work concerns the $\delta = -1$ identity displayed above.

We first give complete descriptions of anti-derivations and local anti-derivations on the seven algebras under consideration. Among them, every local anti-derivation is an anti-derivation precisely for the solvable algebras R , $R(F_n^1)$ and $\mathfrak{L}_2$. The algebras NF_n , F_n^1 , F_n^2 and $\mathfrak{L}_1$ admit local anti-derivations which are not anti-derivations. For 2-local anti-derivations, the result is different: every 2-local anti-derivation on NF_n and on each of the four solvable extensions is an anti-derivation. On F_n^1 and F_n^2 , we construct families of nonlinear 2-local anti-derivations with values in the center. These families are defined by homogeneous mappings on the two-dimensional abelianization, and their members are anti-derivations exactly when the defining mappings are linear.

The proofs use the multiplication tables, coefficient comparisons and pointwise interpolation by anti-derivations. After recalling the necessary definitions and algebra presentations, we determine the anti-derivations in two propositions. The two subsequent theorems describe all local anti-derivations and establish the stated results for 2-local anti-derivations.

Preliminaries

All algebras and linear maps are over $\mathbb{C}$. The arguments also work over any field of characteristic zero. We assume $n \geq 2$ for NF_n and R , and $n \geq 4$ in the filiform cases. All products not explicitly listed are zero; skew-symmetry is not assumed. A sum with an empty index set is understood to be zero. A right Leibniz algebra is a vector space $\mathfrak{L}$ equipped with a bilinear multiplication satisfying

$$[x, [y, z]] = [[x, y], z] - [[x, z], y] \quad (x, y, z \in \mathfrak{L}).$$

Throughout the paper, by a Leibniz algebra we mean a right Leibniz algebra.

For a Leibniz algebra $\mathfrak{L}$, consider its lower central and derived series:

$$\begin{aligned} \mathfrak{L}^1 &= \mathfrak{L}, & \mathfrak{L}^{k+1} &= [\mathfrak{L}^k, \mathfrak{L}^1], & k &\geq 1, \\ \mathfrak{L}^{[1]} &= \mathfrak{L}, & \mathfrak{L}^{[s+1]} &= [\mathfrak{L}^{[s]}, \mathfrak{L}^{[s]}], & s &\geq 1. \end{aligned}$$

Definition 1. A Leibniz algebra $\mathfrak{L}$ is called nilpotent (respectively, solvable) if $\mathfrak{L}^p = 0$ (respectively, $\mathfrak{L}^{[q]} = 0$) for some $p, q \in \mathbb{N}$. The least such p (respectively, q) is called the nilpotency index (respectively, the solvability index) of $\mathfrak{L}$.

Note that any Leibniz algebra $\mathfrak{L}$ contains a unique maximal solvable (resp. nilpotent) ideal, called the radical (resp. nilradical) of the algebra. It is known that the maximal nilindex for the n -dimensional Leibniz algebras is equal to $n + 1$, and such Leibniz algebras are called null-filiform. In [12] it is proved that up to isomorphism, there exists a unique null-filiform Leibniz algebra, and the multiplication of this algebra is the following

$$NF_n : [e_i, e_1] = e_{i+1} \quad 1 \leq i \leq n-1.$$

Definition 2. An n -dimensional Leibniz algebra $\mathfrak{L}$ is said to be filiform if $\dim \mathfrak{L}^i = n - i$ for $2 \leq i \leq n$.

In [12], it is shown that up to isomorphism, there exist two non-Lie naturally graded filiform Leibniz algebras

- $F_n^1 : [e_1, e_1] = e_3, \quad [e_i, e_1] = e_{i+1}, \quad 2 \leq i \leq n-1$
- $F_n^2 : [e_1, e_1] = e_3, \quad [e_i, e_1] = e_{i+1}, \quad 3 \leq i \leq n-1.$

In the following theorems, the classification of solvable Leibniz algebras with null-filiform and naturally graded filiform nilradicals is presented.

Theorem 1.[14] Let R be a solvable Leibniz algebra whose nilradical is NF_n . Then there exists a basis $\{e_1, e_2, \dots, e_n, h\}$ of the algebra R such that the multiplication table of R with respect to this basis has the following form:

$$R : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-1, \\ [h, e_1] = -e_1, \\ [e_i, h] = ie_i, & 1 \leq i \leq n \end{cases}$$

Theorem 2.[20] Let $R(F_n^1)$ be $(n+2)$ -dimensional solvable Leibniz algebra with nilradical F_n^1 . Then it is isomorphic to the Leibniz algebra with a basis $\{e_1, e_2, \dots, e_n, h_1, h_2\}$ and the multiplication table:

$$\begin{aligned} [e_i, e_1] &= e_{i+1}, & 2 \leq i \leq n-1, & \quad [e_1, h_2] = e_1, \\ [e_i, h_1] &= e_i, & 2 \leq i \leq n, & \quad [e_i, h_2] = (i-1)e_i, \quad 2 \leq i \leq n, \\ & & & \quad [h_2, e_1] = -e_1. \end{aligned}$$

Although the product $[e_1, e_1]$ does not occur in this presentation, the nilradical is isomorphic to the standard algebra F_n^1 . Indeed, putting

$$f_1 = e_1 + e_2, \quad f_2 = e_2, \quad f_i = e_i \quad (i \geq 3),$$

we obtain

$$[f_1, f_1] = f_3, \quad [f_i, f_1] = f_{i+1} \quad (2 \leq i < n).$$

Thus the difference is only a choice of basis.

Theorem 3. [13] For $s \in \{1, 2\}$, put $\varepsilon_1 = 1, \varepsilon_2 = 0$. After interchanging h_1 and h_2 in the standard presentation, the algebra $\mathfrak{L}_s$ has basis $\{e_1, \dots, e_n, h_1, h_2\}$ and nonzero products

$$\mathfrak{L}_s : \begin{cases} [e_1, e_1] = e_3, \\ [e_i, e_1] = e_{i+1}, & 3 \leq i \leq n-1, \\ [e_1, h_2] = e_1, & [h_2, e_1] = -e_1, \\ [e_i, h_2] = (i-1)e_i, & 3 \leq i \leq n, \\ [e_2, h_1] = e_2, & [h_1, e_2] = -\varepsilon_s e_2. \end{cases}$$

Definition 3. Let $(\mathfrak{L}, [-, -])$ be an algebra over a field $\mathbb{F}$. A linear map $D : \mathfrak{L} \rightarrow \mathfrak{L}$ is called a δ -derivation if it satisfies

$$D([x, y]) = \delta([D(x), y] + [x, D(y)]),$$

for all $x, y \in \mathfrak{L}$, where $\delta \in \mathbb{F}$ is fixed.

The values $\delta = 1$, $\delta = -1$, and $\delta = \frac{1}{2}$ are especially significant. The case $\delta = 1$ gives the usual derivations, whereas $\delta = -1$ leads to anti-derivations.

For a fixed δ , we denote by $\text{Der}_\delta(\mathfrak{L})$ the set of all δ -derivations of $\mathfrak{L}$.

Remark 1. In the literature on Leibniz biderivations, anti-derivation also denotes a map satisfying $A([x, y]) = [A(x), y] - [A(y), x]$ for a right Leibniz algebra; see [23, Definition 2.2]. That identity is different from

$$D([x, y]) = -([D(x), y] + [x, D(y)]). \quad (1)$$

All the results below concern precisely (1).

Definition 4. A linear map Δ is called a local δ -derivation, if for any $x \in \mathfrak{L}$, there exists a δ -derivation $D_x : \mathfrak{L} \rightarrow \mathfrak{L}$ (depending on x) such that $\Delta(x) = D_x(x)$. The set of all local δ -derivations on $\mathfrak{L}$ we denote by $\text{LocDer}_\delta(\mathfrak{L})$.

Definition 5. A map $\nabla : \mathfrak{L} \rightarrow \mathfrak{L}$ (not necessarily linear) is called a 2-local δ -derivation, if for any $x, y \in \mathfrak{L}$, there exists a δ -derivation $D_{x,y} \in \text{Der}_\delta(\mathfrak{L})$ such that

$$\nabla(x) = D_{x,y}(x), \quad \nabla(y) = D_{x,y}(y).$$

It should be noted that a 2-local δ -derivation is not necessarily linear. However, for any $x \in \mathfrak{L}$ and any scalar λ , we have

$$\nabla(\lambda x) = D_{x,\lambda x}(\lambda x) = \lambda D_{x,\lambda x}(x) = \lambda \nabla(x).$$

Main part

In this section, we describe anti-derivations on null-filiform and filiform Leibniz algebras and their solvable extensions. We then apply these descriptions to local and 2-local anti-derivations.

Proposition 1. Let L be one of the complex Leibniz algebras NF_n , F_n^1 , and F_n^2 , with basis $\{e_1, \dots, e_n\}$. A linear map $D : L \rightarrow L$ is an anti-derivation if and only if it has the form specified below for the corresponding algebra. In each case, the parameters are arbitrary and independent.

(i)

$$\text{NF}_n : \begin{cases} D(e_1) = \sum_{j=1}^n a_j e_j, \\ D(e_i) = \kappa_i a_1 e_i + (-1)^{i-1} \sum_{j=i+1}^n a_{j-i+1} e_j, \quad 2 \leq i \leq n. \end{cases}$$

where

$$\kappa_i = \begin{cases} 1, & i \text{ is odd}, \\ -2, & i \text{ is even}. \end{cases} \quad (2)$$

In particular, $\dim \text{Der}_{-1}(\text{NF}_n) = n$.

(ii)

$$F_n^1 : \begin{cases} D(e_1) = \sum_{j=1}^n a_j e_j, \\ D(e_2) = (a_1 + a_2)e_2 + \sum_{j=3}^{n-1} a_j e_j + b e_n, \\ D(e_i) = \theta_i e_i + (-1)^i \sum_{j=i+1}^n a_{j-i+2} e_j, \quad 3 \leq i \leq n, \end{cases}$$

where

$$\theta_i = \begin{cases} a_1 + a_2, & i \text{ is even}, \\ -2a_1 - a_2, & i \text{ is odd}. \end{cases}$$

In particular, $\dim \text{Der}_{-1}(F_n^1) = n + 1$.

(iii)

$$F_n^2 : \begin{cases} D(e_1) = \sum_{j=1}^n a_j e_j, \\ D(e_2) = b e_2 + c e_n, \\ D(e_i) = \kappa_{i-1} a_1 e_i + (-1)^i \sum_{j=i+1}^n a_{j-i+2} e_j, \quad 3 \leq i \leq n, \end{cases}$$

where κ_i is defined by (2). In particular, $\dim \text{Der}_{-1}(F_n^2) = n + 2$.

Proof. We first obtain a common reduction for the three algebras. Define

$$\phi \left(\sum_{j=1}^n x_j e_j \right) = x_1, \quad Tx = [x, e_1].$$

Their multiplication tables give $[x, y] = \phi(y)Tx$, with $T \neq 0$. Consequently, (1) is equivalent to

$$\phi(y)(DT + TD)x + \phi(Dy)Tx = 0 \quad (x, y \in L).$$

For $y \in \ker \phi$, choosing x with $Tx \neq 0$ yields $\phi(Dy) = 0$. If $D(e_1) = \sum_{j=1}^n a_j e_j$, it follows that $\phi D = a_1 \phi$. Taking $y = e_1$ then gives the second identity in

$$\phi D = a_1 \phi, \quad DT + TD = -a_1 T. \quad (3)$$

Conversely, these identities imply (1) for every ordered pair x, y , including all pairs with zero product. Thus they are necessary and sufficient. In particular, $D(e_j)$ has no e_1 component for $j \geq 2$. Whenever $Te_i = e_{i+1}$, they give

$$D(e_{i+1}) = -TD(e_i) - a_1 e_{i+1}. \quad (4)$$

Case (i): $L = \text{NF}_n$ Here $Te_i = e_{i+1}$ for $i < n$ and $Te_n = 0$. Applying (4) to e_1 gives

$$D(e_2) = -2a_1 e_2 - \sum_{j=3}^n a_{j-1} e_j.$$

If d_i denotes the coefficient of e_i in $D(e_i)$, then $d_1 = a_1$ and $d_{i+1} = -d_i - a_1$. Hence $d_i = \kappa_i a_1$. Under (4), each remaining term changes sign and moves one step along the chain. Induction therefore gives the formula in (i).

Case (ii): $L = F_n^1$ Here $Te_1 = Te_2 = e_3$, $Te_i = e_{i+1}$ for $3 \leq i < n$, and $Te_n = 0$. Write

$$D(e_1) = \sum_{j=1}^n a_j e_j, \quad D(e_2) = \sum_{j=2}^n b_j e_j.$$

Applying (3) to e_1 and e_2 gives, respectively,

$$\begin{aligned} D(e_3) &= -(2a_1 + a_2)e_3 - \sum_{j=3}^{n-1} a_j e_{j+1}, \\ D(e_3) &= -(b_2 + a_1)e_3 - \sum_{j=3}^{n-1} b_j e_{j+1}. \end{aligned}$$

Comparing coefficients, we obtain

$$b_2 = a_1 + a_2, \quad b_j = a_j \quad (3 \leq j \leq n-1).$$

Since $Te_n = 0$, the coefficient b_n remains free; denote it by b . For $3 \leq i < n$, relation (4) determines $D(e_{i+1})$. Its diagonal coefficient alternates between $-2a_1 - a_2$ and $a_1 + a_2$, starting with $D(e_3)$. The other coefficients change sign and shift one place at each step. This proves all formulas in (ii).

Case (iii): $L = F_n^2$ In this case,

$$Te_1 = e_3, \quad Te_2 = 0, \quad Te_i = e_{i+1} \quad (3 \leq i < n), \quad Te_n = 0.$$

Thus $\ker T = \text{span}\{e_2, e_n\}$. Applying (3) to e_2 gives $TD(e_2) = 0$, and hence $D(e_2) = be_2 + ce_n$. Applying it to e_1 gives

$$D(e_3) = -2a_1e_3 - \sum_{j=3}^{n-1} a_j e_{j+1}.$$

Relation (4) now determines the remaining values. The diagonal coefficients of $D(e_3), D(e_4), \dots$ alternate between $-2a_1$ and a_1 , while the other terms change sign and shift along the chain. This gives (iii).

We verify sufficiency for the three cases together. Every displayed map satisfies $\phi D = a_1 \phi$. In (i), relation (4) verifies the second identity of (3) on $e_1, \dots, e_{n-1}$. In (ii), the two expressions for $D(e_3)$ verify it on e_1, e_2 , and the recurrence verifies it on $e_3, \dots, e_{n-1}$. In (iii), the expression for $D(e_3)$ and the recurrence verify it along the chain, while $D(e_2) \in \ker T$ verifies it on e_2 . Finally, in all three cases $D(e_n)$ is a scalar multiple of e_n , so the identity holds on e_n as well. By the reduction above, every displayed map is an anti-derivation.

The parameters are independent: $D(e_1)$ determines $a_1, \dots, a_n$; in (ii), the e_n coefficient of $D(e_2)$ determines b ; in (iii), the two coefficients of $D(e_2)$ determine b, c . The dimension assertions follow.

Proposition 2. Let L be one of the solvable Leibniz algebras $R, R(F_n^1), \mathfrak{L}_1$, and $\mathfrak{L}_2$ described above. A linear map $D : L \rightarrow L$ is an anti-derivation if and only if it has the corresponding form below.

- (i) $\text{Der}_{-1}(R) = \{0\}, \quad \dim \text{Der}_{-1}(R) = 0.$
- (ii) $\text{Der}_{-1}(R(F_n^1)) = \{0\}, \quad \dim \text{Der}_{-1}(R(F_n^1)) = 0.$
- (iii)

$$\mathfrak{L}_s : \begin{cases} D(e_2) = \alpha e_2, \\ D(h_1) = -2\alpha h_1 + \beta e_2, \\ D(h_2) = 0, \\ D(e_i) = 0, \quad 1 \leq i \leq n, \quad i \neq 2, \end{cases}$$

where $\alpha \in \mathbb{C}$ is arbitrary, while $\beta \in \mathbb{C}$ is arbitrary for $s = 1$ and $\beta = 0$ for $s = 2$. Consequently,

$$\dim \text{Der}_{-1}(\mathfrak{L}_1) = 2, \quad \dim \text{Der}_{-1}(\mathfrak{L}_2) = 1.$$

Proof. Equation (1) immediately gives $D([L, L]) \subseteq [L, L]$. For every algebra in the statement, $[L, L] = N = \text{span}\{e_1, \dots, e_n\}$; hence $D(N) \subseteq N$.

Case (i): $L = R$ Write

$$D(e_1) = \sum_{j=1}^n a_j e_j, \quad D(h) = \beta h + \sum_{j=1}^n u_j e_j.$$

The zero product $[h, h] = 0$ gives

$$0 = [D(h), h] + [h, D(h)] = \sum_{j=2}^n j u_j e_j.$$

Therefore $D(h) = \beta h + \gamma e_1$. Applying (1) to $[e_1, h] = e_1$ and $[h, e_1] = -e_1$ gives

$$\begin{aligned} D(e_1) &= -\sum_{j=1}^n j a_j e_j - \beta e_1 - \gamma e_2, \\ -D(e_1) &= (\beta + a_1)e_1 - \gamma e_2. \end{aligned}$$

Comparing coefficients, we obtain

$$2a_1 = -\beta, \quad 3a_2 = -\gamma, \quad a_2 = \gamma, \quad a_j = 0 \quad (3 \leq j \leq n).$$

Thus $a_2 = \gamma = 0$, and

$$D(e_1) = a_1 e_1, \quad D(h) = -2a_1 h.$$

Since $[e_1, e_1] = e_2$, we have $D(e_2) = -2a_1 e_2$. Applying (1) to $[e_2, h] = 2e_2$ now gives

$$-4a_1 e_2 = -[-2a_1 e_2, h] - [e_2, -2a_1 h] = 8a_1 e_2.$$

Hence $12a_1 = 0$, so $a_1 = 0$. We have $D(h) = D(e_1) = 0$; induction along $e_{i+1} = [e_i, e_1]$ shows that D vanishes everywhere. The zero map satisfies (1).

Case (ii): $L = R(F_n^1)$ Initially write

$$\begin{aligned} D(h_1) &= ah_1 + bh_2 + \sum_{i=1}^n u_i e_i, \\ D(h_2) &= ch_1 + dh_2 + \sum_{i=1}^n v_i e_i. \end{aligned}$$

The zero products $[h_1, h_1] = [h_2, h_2] = 0$ yield

$$\sum_{i=2}^n u_i e_i = 0, \quad \sum_{i=2}^n (i-1)v_i e_i = 0.$$

Next, $[h_1, h_2] = 0$ gives $u_1 = 0$. Consequently,

$$D(h_1) = ah_1 + bh_2, \quad D(h_2) = ch_1 + dh_2 + ve_1.$$

Write $D(e_1) = \sum_{i=1}^n p_i e_i$. Applying (1) to $[e_1, h_1] = 0$ gives

$$0 = \sum_{i=2}^n p_i e_i + be_1.$$

Thus $b = 0$ and $D(e_1) = pe_1$. For each $2 \leq i \leq n$, the product $[e_i, h_1] = e_i$ yields

$$D(e_i) = -[D(e_i), h_1] - ae_i.$$

The e_1 coefficient on the right is zero. Comparing all other coefficients then gives

$$D(e_i) = qe_i \quad (2 \leq i \leq n), \quad q = -\frac{a}{2}.$$

The product $[e_1, h_2] = e_1$ gives $d = -2p$. Next, $[e_2, h_2] = e_2$ gives

$$qe_2 = -(q + c + d)e_2 - ve_3,$$

so $v = 0$ and $c + d = -2q$. Using $[e_3, h_2] = 2e_3$ then gives

$$2qe_3 = -(2q + c + 2d)e_3,$$

and hence $c + 2d = -4q$. It follows that $d = -2q$, $c = 0$, and $p = q$. Finally, $[e_2, e_1] = e_3$ gives

$$qe_3 = D(e_3) = -[qe_2, e_1] - [e_2, pe_1] = -(p + q)e_3.$$

Thus $p + 2q = 0$, and therefore $p = q = 0$. All coefficients of $D(h_1), D(h_2)$ vanish as well, so $D = 0$. Again, sufficiency is immediate.

Case (iii): $L = \mathfrak{L}_s$, $s = 1, 2$ Set

$$A = \text{span}\{h_2, e_1, e_3, \dots, e_n\}, \quad B_s = \text{span}\{h_1, e_2\}.$$

The multiplication table gives a direct sum of ideals $\mathfrak{L}_s = A \oplus B_s$, with $[A, B_s] = [B_s, A] = 0$. Under the relabelling

$$z_1 = e_1, \quad z_j = e_{j+1} \quad (2 \leq j \leq n-1), \quad h = h_2,$$

the algebra A has the table in (i), with null-filiform nilradical of dimension $n-1$. Thus $\text{Der}_{-1}(A) = 0$ by Case (i).

We justify that D preserves both summands. Their two-sided centers are zero. Indeed, if $z = th_2 + \sum_{j=1}^{n-1} t_j z_j \in Z(A)$, then

$$[z, h_2] = \sum_{j=1}^{n-1} j t_j z_j = 0$$

forces all $t_j = 0$, and $[th_2, z_1] = -tz_1 = 0$ forces $t = 0$. Likewise, for $w = uh_1 + ve_2 \in Z(B_s)$, the equalities $[e_2, w] = ue_2 = 0$ and $[w, h_1] = ve_2 = 0$ give $u = v = 0$.

For $x \in A$ and $y \in B_s$, decompose

$$D(x) = D_A(x) + f(x), \quad D(y) = g(y) + D_B(y),$$

where $D_A(x), g(y) \in A$ and $f(x), D_B(y) \in B_s$. Applying (1) to $[x, y] = 0$ gives

$$0 = [f(x), y] + [x, g(y)].$$

The two terms lie in distinct direct summands, so each is zero. Applying the identity to $[y, x] = 0$ also gives $[y, f(x)] = [g(y), x] = 0$. Since these identities hold for every $x \in A$ and $y \in B_s$, we obtain $f(x) \in Z(B_s) = 0$ and $g(y) \in Z(A) = 0$. Thus D preserves A, B_s , and its restrictions are anti-derivations. Since $\text{Der}_{-1}(A) = 0$, only the restriction to B_s remains to be determined.

Write

$$D(e_2) = \alpha e_2 + \rho h_1, \quad D(h_1) = \beta e_2 + \sigma h_1.$$

The product $[e_2, h_1] = e_2$ yields

$$\alpha e_2 + \rho h_1 = -(\alpha + \sigma)e_2.$$

Hence $\rho = 0$ and $\sigma = -2\alpha$. The zero product $[h_1, h_1] = 0$ gives

$$0 = [D(h_1), h_1] + [h_1, D(h_1)] = (1 - \varepsilon_s)\beta e_2.$$

Therefore β is arbitrary for $s = 1$ and equals zero for $s = 2$.

For sufficiency, the two identities just used hold whenever these conditions are satisfied. For the remaining ordered basis pairs in B_s , both sides of (1) vanish for $[e_2, e_2] = 0$, and

$$D([h_1, e_2]) = -\varepsilon_s \alpha e_2 = -[D(h_1), e_2] - [h_1, D(e_2)].$$

Thus all four ordered basis pairs satisfy (1). Extending these maps by zero on A preserves the identity because all mixed products vanish. This proves the stated classifications. The free parameters α, β for $\mathfrak{L}_1$ and α for $\mathfrak{L}_2$ are independent, giving the dimensions 2 and 1.

Theorem 4. Let L be one of the complex Leibniz algebras considered in Propositions 1 and 2, with the bases and dimension assumptions used there. A linear map $\Delta : L \rightarrow L$ is a local anti-derivation if and only if it has the corresponding form below.

(i)

$$\text{NF}_n : \quad \Delta(e_i) = \sum_{j=i}^n b_{ji} e_j \quad (1 \leq i \leq n),$$

where all $b_{ji} \in \mathbb{C}$ are arbitrary. In particular, $\dim \text{LocDer}_{-1}(\text{NF}_n) = n(n+1)/2$.

(ii)

$$F_n^1 : \begin{cases} \Delta(e_1) = \sum_{j=1}^n a_j e_j, \\ \Delta(e_2) = (a_1 + a_2)e_2 + \sum_{j=3}^{n-1} a_j e_j + b e_n, \\ \Delta(e_i) = \sum_{j=i}^n c_{ji} e_j \quad (3 \leq i \leq n), \end{cases}$$

where $a_1, \dots, a_n, b$ and all c_{ji} are arbitrary and independent. In particular, $\dim \text{LocDer}_{-1}(F_n^1) = n(n-1)/2 + 2$.

(iii)

$$F_n^2 : \begin{cases} \Delta(e_1) = \sum_{j=1}^n a_j e_j, \\ \Delta(e_2) = b e_2 + c e_n, \\ \Delta(e_i) = \sum_{j=i}^n d_{ji} e_j \quad (3 \leq i \leq n), \end{cases}$$

where $a_1, \dots, a_n, b, c$ and all d_{ji} are arbitrary and independent. In particular, $\dim \text{LocDer}_{-1}(F_n^2) = n(n-1)/2 + 3$.

(iv) For $L = R$ or $L = R(F_n^1)$, we have $\Delta = 0$. Thus the corresponding spaces of local anti-derivations have dimension zero.

(v)

$$\mathfrak{L}_s : \begin{cases} \Delta(e_2) = \alpha e_2, \\ \Delta(h_1) = \beta e_2 + \gamma h_1, \\ \Delta(h_2) = 0, \\ \Delta(e_i) = 0 \quad (1 \leq i \leq n, i \neq 2), \end{cases}$$

where $s \in \{1, 2\}$ and $\alpha, \beta, \gamma \in \mathbb{C}$ are arbitrary for $s = 1$, whereas $\beta = 0$ and $\gamma = -2\alpha$ for $s = 2$. Consequently,

$$\dim \text{LocDer}_{-1}(\mathfrak{L}_1) = 3, \quad \dim \text{LocDer}_{-1}(\mathfrak{L}_2) = 1.$$

Among these seven algebras, every local anti-derivation is an anti-derivation precisely for R , $R(F_n^1)$, and $\mathfrak{L}_2$.

Proof. For an algebra L and $x \in L$, write

$$\text{Der}_{-1}(L)x = \{D(x) : D \in \text{Der}_{-1}(L)\}.$$

By definition, a linear map Δ is a local anti-derivation if and only if $\Delta(x) \in \text{Der}_{-1}(L)x$ for every x .

Case (i): $L = \text{NF}_n$. Put $E_k = \text{span}\{e_k, \dots, e_n\}$. We claim that, for every nonzero $x = \sum_{i=k}^n x_i e_i$ with $x_k \neq 0$,

$$\text{Der}_{-1}(\text{NF}_n)x = E_k.$$

The inclusion into E_k follows from Proposition 1(i). Conversely, prescribe $y = \sum_{j=k}^n y_j e_j \in E_k$ and use the parameters $a_1, \dots, a_n$ from that proposition. The coefficient of e_k in $D(x)$ is $\kappa_k a_1 x_k$, so we can choose a_1 to obtain y_k . For each $j > k$, the coefficient of e_j is

$$\kappa_j a_1 x_j + \sum_{i=k}^{j-1} (-1)^{i-1} x_i a_{j-i+1}.$$

The new parameter a_{j-k+1} occurs with nonzero coefficient $(-1)^{k-1} x_k$. Choosing these parameters successively gives $D(x) = y$. This proves the claim, including $k = n$.

Locality at each e_i now forces $\Delta(e_i) \in E_i$. Conversely, a linear map with these values sends every $x \in E_k$ into E_k , and the claim provides an anti-derivation agreeing with it at x . This proves (i).

Case (ii): $L = F_n^1$. Use the auxiliary basis

$$u = e_1 - e_2, \quad v_j = e_{j+1} \quad (1 \leq j \leq m), \quad m = n - 1.$$

The nonzero products in this basis are $[v_j, u] = v_{j+1}$ for $1 \leq j < m$. Proposition 1(ii), expressed in this basis, gives

$$D(u) = au + rv_m, \quad D(v_j) = \lambda_j v_j + (-1)^{j-1} \sum_{\ell=j+1}^m c_{\ell-j+1} v_\ell,$$

where $a, r, b, c_2, \dots, c_m$ are arbitrary and

$$\lambda_j = \begin{cases} b, & j \text{ odd,} \\ -a - b, & j \text{ even.} \end{cases}$$

Indeed, one may choose $D(v_1) = bv_1 + \sum_{\ell=2}^m c_\ell v_\ell$ arbitrarily; the other values follow from $D(v_{j+1}) = -[D(v_j), u] - av_{j+1}$.

Let $W_k = \text{span}\{v_k, \dots, v_m\}$. Locality at u and at the v_j implies

$$\Delta(u) \in \text{span}\{u, v_m\}, \quad \Delta(v_j) \in W_j \quad (1 \leq j \leq m).$$

These conditions are also sufficient. To see this, write $\Delta(u) = au + rv_m$ and take $x = tu + w$. If $w = 0$, any anti-derivation with this value on u agrees with Δ at x . If $w = \sum_{j=k}^m w_j v_j$ with $w_k \neq 0$, then $\Delta(w) \in W_k$. Keep a, r fixed and choose b so that the v_k coefficient of $D(w)$ equals that of $\Delta(w)$. This is possible because the coefficient of b there is $(-1)^{k-1} w_k \neq 0$. For each $\ell > k$, the new parameter $c_{\ell-k+1}$ occurs in the v_ℓ coefficient with nonzero coefficient $(-1)^{k-1} w_k$. Successive choices therefore give $D(w) = \Delta(w)$ and hence $D(x) = \Delta(x)$.

Returning to the original basis, write $\Delta(e_1) = \sum a_j e_j$ and $\Delta(e_2) = \sum_{j=2}^n b_j e_j$. The condition on $\Delta(u) = \Delta(e_1 - e_2)$ gives

$$b_2 = a_1 + a_2, \quad b_j = a_j \quad (3 \leq j < n),$$

while b_n is free. The other conditions are $\Delta(e_i) \in \text{span}\{e_i, \dots, e_n\}$ for $i \geq 3$. These are exactly the formulas in (ii).

Case (iii): $L = F_n^2$. Write

$$F_n^2 = A \oplus \mathbb{C}z, \quad A = \text{span}\{e_1, e_3, \dots, e_n\}, \quad z = e_2.$$

Here z is central and A is a null-filiform algebra of dimension $n - 1$, with ordered basis $e_1, e_3, \dots, e_n$. Proposition 1(iii) shows that every anti-derivation has the form

$$D(x + tz) = D_A(x) + cte_n + (ax_1 + bt)z,$$

where $D_A \in \text{Der}_{-1}(A)$, the parameters a, b, c are arbitrary, and x_1 is the coefficient of e_1 in $x \in A$. In particular, locality at the basis vectors implies all the conditions in (iii).

Conversely, a map with the stated values can be written as

$$\Delta(x + tz) = P(x) + cte_n + (ax_1 + bt)z,$$

where $P : A \rightarrow A$ preserves every tail of the ordered basis of A . By Case (i), P is a local anti-derivation of A . For any fixed $x + tz$, choose D_A with $D_A(x) = P(x)$ and use the same a, b, c in the preceding formula for D . Then $D(x + tz) = \Delta(x + tz)$, proving sufficiency.

Case (iv): $L = R$ or $L = R(F_n^1)$. By Proposition 2, these algebras have only the zero anti-derivation. Locality therefore gives $\Delta(x) = 0$ for every $x \in L$.

Case (v): $L = \mathfrak{L}_s$. Proposition 2 and locality at the basis vectors first give

$$\Delta(e_2) = \alpha e_2, \quad \Delta(h_1) = \beta e_2 + \gamma h_1,$$

and zero values on h_2 and all e_i with $i \neq 2$. For $s = 1$, there are no further conditions. Indeed, write $x = x_0 + te_2 + uh_1$, where $x_0 \in \text{span}\{h_2, e_1, e_3, \dots, e_n\}$. An anti-derivation with parameters p, q from Proposition 2 has

$$D(x) = (pt + qu)e_2 - 2puh_1.$$

If $u \neq 0$, take

$$p = -\frac{\gamma}{2}, \quad q = \beta + \left(\alpha + \frac{\gamma}{2}\right) \frac{t}{u}.$$

If $u = 0$, take $p = \alpha$ and $q = 0$. In either case $D(x) = \Delta(x)$.

For $s = 2$, locality at h_1 forces $\beta = 0$. Locality at $e_2 + h_1$ then gives a scalar p such that

$$\alpha e_2 + \gamma h_1 = pe_2 - 2ph_1,$$

so $\gamma = -2\alpha$. Conversely, each map with these values is already an anti-derivation by Proposition 2.

Finally, the dimensions follow by counting the free parameters. In (ii) and (iii), the free triangular part contributes $(n-2)(n-1)/2$ parameters. Thus the dimensions are, respectively, $n+1+(n-2)(n-1)/2 = n(n-1)/2+2$ and $n+2+(n-2)(n-1)/2 = n(n-1)/2+3$. Comparison with the dimensions in Propositions 1 and 2 proves the last assertion.

Theorem 5. For the algebras in Propositions 1 and 2, the following statements hold.

- (i) Every 2-local anti-derivation on NF_n , R , $R(F_n^1)$, $\mathfrak{L}_1$, or $\mathfrak{L}_2$ is an anti-derivation.
- (ii) Let $L = F_n^1$ or $L = F_n^2$, and let $\Psi : \mathbb{C}^2 \rightarrow Z(L)$ satisfy

$$\Psi(\lambda s, \lambda t) = \lambda \Psi(s, t) \quad (\lambda, s, t \in \mathbb{C}),$$

where

$$Z(F_n^1) = \mathbb{C}e_n, \quad Z(F_n^2) = \text{span}\{e_2, e_n\}.$$

Then the map

$$\nabla_\Psi \left(\sum_{i=1}^n x_i e_i \right) = \Psi(x_1, x_2)$$

is a 2-local anti-derivation. It is an anti-derivation if and only if Ψ is linear. In particular, both F_n^1 and F_n^2 admit nonlinear 2-local anti-derivations.

Proof. We first make an elementary observation. Suppose that an algebra L has an element v for which

$$D \in \text{Der}_{-1}(L), \quad D(v) = 0 \quad \implies \quad D = 0.$$

Let ∇ be a 2-local anti-derivation. Choose an anti-derivation D_0 with $\nabla(v) = D_0(v)$. For any $x \in L$, 2-locality provides an anti-derivation $D_{v,x}$ satisfying

$$D_{v,x}(v) = \nabla(v) = D_0(v), \quad D_{v,x}(x) = \nabla(x).$$

The anti-derivation $D_{v,x} - D_0$ vanishes at v , so it is zero. Hence $\nabla(x) = D_0(x)$ for every x , and $\nabla = D_0$. No linearity assumption on ∇ is needed.

For NF_n , Proposition 1(i) shows that $v = e_1$ has this property, because $D(e_1)$ determines every parameter of D . For $\mathfrak{L}_1$ and $\mathfrak{L}_2$, take $v = h_1$: the values $D(h_1) = -2\alpha h_1 + \beta e_2$ and $D(h_1) = -2\alpha h_1$, respectively, determine all parameters. For R and $R(F_n^1)$, Proposition 2 gives $\text{Der}_{-1}(L) = \{0\}$, so every 2-local anti-derivation is zero. This proves (i).

To prove (ii), observe that for both filiform algebras

$$[L, L] = \text{span}\{e_3, \dots, e_n\}.$$

Their multiplication tables also give the stated two-sided centers. For arbitrary $p, q \in Z(L)$, define a linear map by

$$D_{p,q}(e_1) = p, \quad D_{p,q}(e_2) = q, \quad D_{p,q}(e_i) = 0 \quad (3 \leq i \leq n).$$

This map annihilates $[L, L]$ and takes its values in $Z(L)$. Consequently, both sides of

$$D_{p,q}([x, y]) = -[D_{p,q}(x), y] - [x, D_{p,q}(y)]$$

vanish for all x, y , so $D_{p,q}$ is an anti-derivation.

Now fix $x = \sum x_i e_i$ and $y = \sum y_i e_i$. We seek $p, q \in Z(L)$ satisfying

$$x_1 p + x_2 q = \Psi(x_1, x_2), \quad y_1 p + y_2 q = \Psi(y_1, y_2).$$

If $x_1 y_2 - x_2 y_1 \neq 0$, this system has a unique solution. If the two coordinate pairs are dependent and one is nonzero, homogeneity of Ψ makes the two equations consistent, and one can solve the equation corresponding to the nonzero pair. If both pairs are zero, choose $p = q = 0$, using $\Psi(0, 0) = 0$. Thus in every case an anti-derivation $D_{p,q}$ satisfies

$$D_{p,q}(x) = \nabla_{\Psi}(x), \quad D_{p,q}(y) = \nabla_{\Psi}(y).$$

This proves 2-locality, including pairs with zero or dependent images in $L/[L, L]$.

If Ψ is linear, then $\Psi(s, t) = s\Psi(1, 0) + t\Psi(0, 1)$, and $\nabla_{\Psi} = D_{\Psi(1,0), \Psi(0,1)}$ is an anti-derivation. Conversely, if ∇_{Ψ} is an anti-derivation, its linearity on $\text{span}\{e_1, e_2\}$ implies that Ψ is linear.

For an explicit nonlinear example on either algebra, take

$$\Psi(s, t) = \begin{cases} \frac{t^2}{s} e_n, & s \neq 0, \\ 0, & s = 0. \end{cases}$$

This map satisfies the required homogeneity for all complex scalars. The resulting 2-local anti-derivation satisfies

$$\nabla_{\Psi}(e_1) = \nabla_{\Psi}(e_2) = 0, \quad \nabla_{\Psi}(e_1 + e_2) = e_n,$$

so it is not additive and hence is not an anti-derivation.

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**CYCLIC $\mathcal{R}_2(m)$ LOTKA–VOLTERRA OPERATORS ON SIMPLICES:
CLASSIFICATION ON S^4 AND GLOBAL DYNAMICS FOR ODD m**

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ABSTRACT. We study discrete Lotka–Volterra operators $V : S^{m-1} \rightarrow S^{m-1}$ whose interaction graph is the 2-regular cycle C_m , referred to as $\mathcal{R}_2(m)$ configurations. For $m = 5$ (the simplex S^4) we classify all orientation patterns under the dihedral group D_5 via Burnside’s lemma, obtaining exactly four isomorphism classes (I)–(IV). For each class we determine the $P \rightarrow Q$ repeller–attractor structure, describe invariant polytopes via coordinate signatures, and prove global convergence of trajectories. For general odd $m \geq 3$ we prove that the directed-cycle operator possesses a unique interior fixed point M , construct an explicit Lyapunov function φ_M , and show that every non-fixed interior trajectory converges to the boundary in forward time while its α -limit set is $\{M\}$.

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Key words: discrete Lotka–Volterra operator; simplex; cyclic graph; Burnside’s lemma; coordinate signature; Lyapunov function; trajectory route; α - and ω -limit sets.

Introduction

Lotka–Volterra (LV) mappings on the standard simplex

$$S^{m-1} = \left\{ x \in \mathbb{R}^m : \sum_{i=1}^m x_i = 1, x_i \geq 0 \right\}$$

constitute a fundamental class of nonlinear discrete dynamical systems, widely studied in population dynamics, mathematical biology, and related areas [5,6]. Given a skew-symmetric matrix $A = (a_{ij})$ with $|a_{ij}| \leq 1$, the mapping $V : S^{m-1} \rightarrow S^{m-1}$ is defined by

$$x'_k = x_k \left(1 + \sum_{i=1}^m a_{ki} x_i \right), \quad k = 1, \dots, m. \quad (1)$$

Because A is skew-symmetric, V is a homeomorphism of S^{m-1} [1,2]. The interaction graph $\Gamma(A)$ has vertex set $\{1, \dots, m\}$ with a directed edge $i \rightarrow j$ whenever $a_{ji} > 0$.

Among the most natural sparse structures are the *2-regular cyclic* graphs C_m : each vertex has in-degree and out-degree equal to 1. We denote by $\mathcal{R}_2(m)$ the family of LV operators whose underlying undirected graph is C_m . The study of $\mathcal{R}_2(m)$ operators is motivated by cyclic competition models in ecology and by their analogy with continuous Volterra systems [8].

The *trajectory-route method*—partitioning S^{m-1} into convex polytopes via sign-change hyperplanes of the coordinate increments—was developed in [3,4] for bigraph operators. The present work applies and extends this method to cyclic graphs.

Main results. For the five-dimensional population vector corresponding to the simplex S^4 , we classify all cyclic $\mathcal{R}_2(5)$ Lotka–Volterra operators, up to the natural dihedral symmetries of the cycle, into exactly four isomorphism classes. For each class, we describe the global direction of the trajectories and identify the invariant sets that play the roles of repelling and attracting sets.

For cyclic operators associated with an arbitrary odd number $m \geq 3$ of coordinates, we prove the existence and uniqueness of an interior fixed point. We construct an explicit Lyapunov function showing that this fixed

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point is repelling in forward time. Consequently, every non-fixed trajectory starting in the interior approaches the boundary of the simplex in forward time, whereas in backward time it approaches the unique interior fixed point.

Preliminaries

We collect the main definitions and tools used throughout the paper.

Throughout the paper, $\text{ri } S^{m-1}$ denotes the relative interior of the simplex S^{m-1} , while ∂S^{m-1} denotes its boundary.

Definition 1 (Fixed point). A point $x^* \in S^{m-1}$ is called a *fixed point* of the operator $V : S^{m-1} \rightarrow S^{m-1}$ if $V(x^*) = x^*$. The set of all fixed points is denoted $\text{Fix}(V)$. A fixed point x^* is called an *interior fixed point* if $x^* \in \text{ri } S^{m-1}$, i.e. all coordinates $x_k^* > 0$.

Definition 2 (Transverse repeller and attractor) Let $\Lambda \subset S^{m-1}$ be a compact invariant set for V . Throughout the paper, $\|\cdot\|$ denotes the standard Euclidean norm on $\mathbb{R}^m$.

The set Λ is called a transverse repeller if there exists a neighbourhood U of Λ in S^{m-1} such that, for every $x \in \Lambda$ and every nonzero vector ν transverse to Λ at x ,

$$\|DV(x)\nu\| > \|\nu\|.$$

Equivalently, all eigenvalues of the Jacobian corresponding to directions transverse to Λ have modulus greater than 1.

Similarly, Λ is called a transverse attractor if all eigenvalues of the Jacobian corresponding to transverse directions have modulus less than 1.

Definition 3 (α - and ω -limit sets). For $x^{(0)} \in S^{m-1}$ define:

$$\omega(x^{(0)}) = \bigcap_{n \geq 0} \overline{\{x^{(k)} : k \geq n\}}, \quad \alpha(x^{(0)}) = \bigcap_{n \geq 0} \overline{\{x^{(-k)} : k \geq n\}},$$

the ω -limit set (forward limit) and the α -limit set (backward limit) of the orbit through $x^{(0)}$.

Definition 4 (Coordinate signature). Let V be an LV operator (1) with skew-symmetric matrix A . The *coordinate signature* of a point $x \in S^{m-1}$ is

$$\sigma(x) = (\text{sgn}(x'_1 - x_1), \dots, \text{sgn}(x'_m - x_m)) = (\text{sgn}x_1(Ax)_1, \dots, \text{sgn}x_m(Ax)_m), \quad \text{sgn}(t) \in \{+, -, 0\}.$$

The closure of the set of all points carrying a fixed admissible signature σ is a convex polytope $F_\sigma \subset S^{m-1}$. The ordered sequence of polytopes traversed by the forward orbit of $x^{(0)}$ is called the *trajectory route* of $x^{(0)}$.

Definition 5 (Invariant regions P and Q). For operator (1) define

$$P = \{x \in S^{m-1} : Ax \geq 0\}, \quad Q = \{x \in S^{m-1} : Ax \leq 0\}.$$

These sets are the *positive* and *negative basins* of V , respectively. Note that $P = F_\sigma$ for $\sigma = (+, \dots, +)$ and $Q = F_\sigma$ for $\sigma = (-, \dots, -)$.

Definition 6 (Graph isomorphism of LV operators). Two $\mathcal{R}_2(m)$ operators V and $\tilde{V}$ with interaction matrices A and $\tilde{A}$, respectively, are called *isomorphic* if there exists a permutation $\pi \in S_m$ such that

$$\tilde{A}_{\pi(i)\pi(j)} = A_{ij} \quad \text{for all } i, j.$$

Equivalently, the directed graphs $\Gamma(A)$ and $\Gamma(\tilde{A})$ are isomorphic as directed graphs. The group of symmetries acting on C_m is the dihedral group D_m of order $2m$.

Burnside's lemma [7]. Let a finite group G act on a finite set X . The number of orbits (isomorphism classes) is

$$N = \frac{1}{|G|} \sum_{g \in G} |X^g|, \tag{2}$$

where $X^g = \{x \in X : g \cdot x = x\}$ is the set of elements fixed by g .

Classification on S^4 : $\mathcal{R}_2(5)$ configurations

Let $V : S^4 \rightarrow S^4$ be an $\mathcal{R}_2(5)$ operator. Each of the five edges of C_5 may be oriented in two ways, giving $2^5 = 32$ orientation patterns encoded by binary vectors $\varsigma = (\varepsilon_1, \dots, \varepsilon_5)$, $\varepsilon_i \in \{0, 1\}$ (clockwise = 1, counter-clockwise = 0).

Theorem 1. *Under the action of D_5 the 32 orientations of C_5 split into exactly four isomorphism classes.*

Proof. $|D_5| = 10$. The identity fixes all 32 patterns. Each of the four nontrivial rotations ($k = 1, 2, 3, 4$, $\gcd(k, 5) = 1$) acts as a single 5-cycle on the edges and fixes only the two constant vectors $(0, 0, 0, 0, 0)$ and $(1, 1, 1, 1, 1)$, giving $|X^g| = 2$ in each case. Since C_5 is an odd cycle, every reflection fixes exactly one edge and reverses the orientation of the remaining four; no pattern with a fixed single edge can be fixed globally under such a reflection, so $|X^g| = 0$ for all five reflections. By Burnside's lemma (2):

$$N = \frac{1}{10}(32 + 4 \cdot 2 + 5 \cdot 0) = \frac{40}{10} = 4. \quad \square$$

The four classes are distinguished by their source-sink structure:

- (I) $(1, 1, 1, 1, 1)$ — directed cycle, 0 sources, 0 sinks;
- (II) $(1, 1, 1, 1, 0)$ — 1 source, 1 sink, arc lengths $\{4, 1\}$;
- (III) $(1, 1, 1, 0, 0)$ — 1 source, 1 sink, arc lengths $\{3, 2\}$;
- (IV) $(1, 1, 0, 1, 0)$ — 2 sources, 2 sinks, arc lengths $\{2, 1, 1, 1\}$.

Class (I): directed cycle $(1, 1, 1, 1, 1)$. The representative orientation $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 1$ gives the operator

$$\begin{aligned} x'_1 &= x_1(1 - a_1x_2 + a_5x_5), \\ x'_2 &= x_2(1 + a_1x_1 - a_2x_3), \\ x'_3 &= x_3(1 + a_2x_2 - a_3x_4), \\ x'_4 &= x_4(1 + a_3x_3 - a_4x_5), \\ x'_5 &= x_5(1 + a_4x_4 - a_5x_1), \end{aligned} \tag{3}$$

with skew-symmetric matrix

$$A = \begin{pmatrix} 0 & -a_1 & 0 & 0 & a_5 \\ a_1 & 0 & -a_2 & 0 & 0 \\ 0 & a_2 & 0 & -a_3 & 0 \\ 0 & 0 & a_3 & 0 & -a_4 \\ -a_5 & 0 & 0 & a_4 & 0 \end{pmatrix}.$$

The fixed-point set is

$$\text{Fix}(V) = \Gamma_{13} \cup \Gamma_{24} \cup \Gamma_{35} \cup \Gamma_{14} \cup \Gamma_{25} \cup \{M\},$$

where

$$\Gamma_{ij} = \text{co}\{e_i, e_j\}$$

denotes the edge of S^4 joining the vertices e_i and e_j , and the unique interior fixed point is

$$M = \left(\frac{a_2a_4}{\Delta}, \frac{a_3a_5}{\Delta}, \frac{a_1a_4}{\Delta}, \frac{a_2a_5}{\Delta}, \frac{a_1a_3}{\Delta} \right), \quad \Delta = a_2a_4 + a_3a_5 + a_1a_4 + a_2a_5 + a_1a_3.$$

Theorem 2. *For operator (3) the interior fixed point M is a transverse repeller for V and a transverse attractor for V^{-1} . In particular, $\omega(x^{(0)}) \subset \partial S^4$ for every $x^{(0)} \in \text{ri}S^4 \setminus \{M\}$.*

Proof. For $m = 5$, we apply the Lyapunov-function argument developed in Theorem 6. The corresponding Lyapunov function takes the explicit form

$$\begin{aligned}
 \varphi_M(Vx) &= \prod_{i=1}^5 x_i^{M_i} = (x_1(1 - a_1x_2 + a_5x_5))^{\frac{a_2a_4}{\Delta}} (x_2(1 + a_1x_1 - a_2x_3))^{\frac{a_3a_5}{\Delta}} \cdot \\
 &\cdot (x_3(1 + a_2x_2 - a_3x_4))^{\frac{a_1a_4}{\Delta}} (x_4(1 + a_3x_3 - a_4x_5))^{\frac{a_2a_5}{\Delta}} (x_5(1 + a_4x_4 - a_5x_1))^{\frac{a_1a_3}{\Delta}} = \\
 &= x_1^{\frac{a_2a_4}{\Delta}} x_2^{\frac{a_3a_5}{\Delta}} x_3^{\frac{a_1a_4}{\Delta}} x_4^{\frac{a_2a_5}{\Delta}} x_5^{\frac{a_1a_3}{\Delta}} \left((1 - a_1x_2 + a_5x_5)^{\frac{a_2a_4}{\Delta}} (1 + a_1x_1 - a_2x_3)^{\frac{a_3a_5}{\Delta}} \cdot \right. \\
 &\cdot (1 + a_2x_2 - a_3x_4)^{\frac{a_1a_4}{\Delta}} (1 + a_3x_3 - a_4x_5)^{\frac{a_2a_5}{\Delta}} (1 + a_4x_4 - a_5x_1)^{\frac{a_1a_3}{\Delta}} \Big) \leq \\
 &\leq \varphi_M(x) \left((1 - a_1x_2 + a_5x_5) \cdot \frac{a_2a_4}{\Delta} + (1 + a_1x_1 - a_2x_3) \cdot \frac{a_3a_5}{\Delta} + \right. \\
 &\left. + (1 + a_2x_2 - a_3x_4) \cdot \frac{a_1a_4}{\Delta} + (1 + a_3x_3 - a_4x_5) \cdot \frac{a_2a_5}{\Delta} + (1 + a_4x_4 - a_5x_1) \cdot \frac{a_1a_3}{\Delta} \right) \leq \varphi_M(x)
 \end{aligned}$$

satisfies $\varphi_M(Vx) \leq \varphi_M(x)$ with equality only at $x = M$ (Lemma 1), so forward orbits move away from M and accumulate on ∂S^4 . $\square$

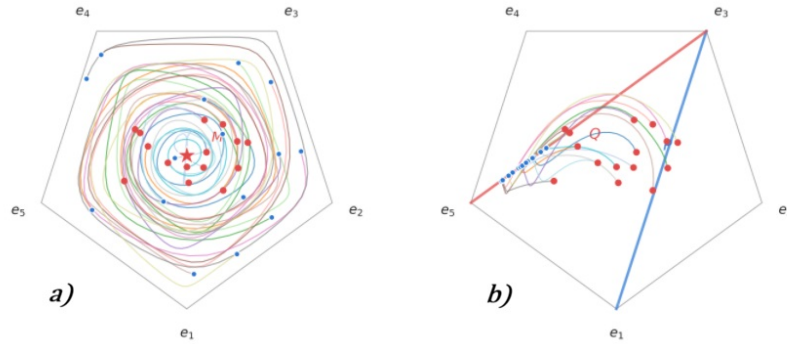


Figure 1: Trajectory dynamics of $\mathcal{R}_2(5)$ operators on S^4 (pentagon projection): *a)* Class (I) — all interior orbits spiral away from the interior repeller M (star) toward ∂S^4 ; *b)* Class (II) — blue segment $P = [e_1, M_1]$ (repeller), red segment $Q = [e_5, M_3]$ (attractor); all orbits follow $P \rightarrow Q$.

Class (II): one source, arc lengths $\{4, 1\}$, representative $(1, 1, 1, 1, 0)$.

This class contains 5 graphs; a representative is the orientation $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \leftarrow 1$ (vertex 1 is the source, vertex 5 is the sink):

$$\begin{aligned}
 x'_1 &= x_1(1 - a_1x_2 - a_5x_5), \\
 x'_2 &= x_2(1 + a_1x_1 - a_2x_3), \\
 x'_3 &= x_3(1 + a_2x_2 - a_3x_4), \\
 x'_4 &= x_4(1 + a_3x_3 - a_4x_5), \\
 x'_5 &= x_5(1 + a_4x_4 + a_5x_1),
 \end{aligned} \tag{4}$$

with skew-symmetric matrix

$$A = \begin{pmatrix} 0 & -a_1 & 0 & 0 & -a_5 \\ a_1 & 0 & -a_2 & 0 & 0 \\ 0 & a_2 & 0 & -a_3 & 0 \\ 0 & 0 & a_3 & 0 & -a_4 \\ a_5 & 0 & 0 & a_4 & 0 \end{pmatrix}.$$

Solving $V(x) = x$ on the boundary gives fixed points

$$M_1 = \left(\frac{a_2}{a_1 + a_2}, 0, \frac{a_1}{a_1 + a_2}, 0, 0 \right), \quad M_2 = \left(0, \frac{a_3}{a_2 + a_3}, 0, \frac{a_2}{a_2 + a_3}, 0 \right), \quad M_3 = \left(0, 0, \frac{a_4}{a_3 + a_4}, 0, \frac{a_3}{a_3 + a_4} \right).$$

The invariant segments are

$$P = [e_1, M_1] \subset \Gamma_{13}, \quad Q = [e_5, M_3] \subset \Gamma_{35}.$$

Theorem 3. For operator (4): every point of riP is a transverse repeller, every point of riQ a transverse attractor, and for every $x^{(0)} \in riS^4$,

$$\alpha(x^{(0)}) \subset P = [e_1, M_1], \quad \omega(x^{(0)}) \subset Q = [e_5, M_3].$$

The global dynamics follows the repeller–attractor scheme $P \rightarrow Q$.

Proof. *Transversality.* For $x \in riP$ the characteristic polynomial of $DV(x)$ factors as

$$(1 - \lambda)^2(1 + a_1k - \lambda) \left(1 + \frac{a_2a_3(1 - k)}{a_1 + a_2} - \lambda\right) \left(1 + a_5k + \frac{a_2a_5(1 - k)}{a_1 + a_2} - \lambda\right), \quad 0 < k < 1,$$

so $\lambda_1 = \lambda_2 = 1$ along P and $\lambda_3, \lambda_4, \lambda_5 > 1$ (repeller). For $x \in riQ$ the analogous computation gives all transverse eigenvalues less than 1 (attractor).

Global dynamics. For $p = (p_1, 0, p_2, 0, p_3) \in P$ (so $Ap \geq 0$), the function $\varphi_p(x) = x_1^{p_1} x_3^{p_2} x_5^{p_3}$ satisfies, by the weighted AM–GM inequality,

$$\varphi_p(Vx) \leq \varphi_p(x)(1 - \langle Ap, x \rangle) \leq \varphi_p(x),$$

since $\langle Ap, x \rangle \geq 0$ on S^4 . Forward orbits are thus driven into Q . Here,

$$\langle Ap, x \rangle = \sum_{i=1}^m (Ap)_i x_i$$

denotes the standard Euclidean inner product in $\mathbb{R}^m$. The symmetric argument for V^{-1} gives $\alpha(x^{(0)}) \subset P$. $\square$

The 8 admissible coordinate signatures for operator (4) are

$$\sigma_1 = (-++++), \sigma_2 = (-++-+), \sigma_3 = (-+---), \sigma_4 = (-+-++), \sigma_5 = (--+++), \sigma_6 = (--+-+),$$

$$\sigma_7 = (----+), \sigma_8 = (---++),$$

with convex polytopes (where $C = \frac{1}{D}(a_2a_4, 0, a_1a_4, 0, a_1a_3)$, $D = a_1a_3 + a_1a_4 + a_2a_4$):

$$\begin{aligned} F_1 &= co\{e_1, e_2, M_1, M_2, C\}, & F_2 &= co\{e_1, e_2, e_5, M_2, C\}, & F_3 &= co\{e_1, e_4, e_5, M_2, C\}, \\ F_4 &= co\{e_1, e_4, M_1, M_2, C\}, & F_5 &= co\{e_2, e_3, M_1, M_2, M_3, C\}, & F_6 &= co\{e_2, e_5, M_2, M_3, C\}, \\ F_7 &= co\{e_4, e_5, M_2, M_3, C\}, & F_8 &= co\{e_3, e_4, M_1, M_2, M_3, C\}. \end{aligned}$$

The transitions across common facets (verified by sign of the corresponding increment) yield an acyclic route diagram terminating at Q . The direction of the transition across each common facet is set by the sign of the corresponding increment; for example, at $T = F_2 \cap F_6 = \{e_2, e_5, M_2, C\}$, $x_2'' = x_2' \left(1 - \frac{a_2a_4k_1k_4}{D}(a_1^2 + a_2^2) - \frac{a_1^2a_3a_4K_3k_4}{D(a_2+a_3)} - \frac{a_1a_2a_3a_4^2a_5k_2k_4}{D^2}\right) < x_2'$, so $T \subset F_6$ and the transition is $F_2 \rightarrow F_6$. Working through all adjacent pairs produces the route diagram, which terminates at Q .

Class (III): one source, arc lengths $\{3, 2\}$, representative $(1, 1, 1, 0, 0)$.

The orientation $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \leftarrow 5 \leftarrow 1$ (single source 1, single sink 4) gives

$$\begin{aligned} x_1' &= x_1(1 - a_1x_2 - a_5x_5), & x_2' &= x_2(1 + a_1x_1 - a_2x_3), \\ x_3' &= x_3(1 + a_2x_2 - a_3x_4), & x_4' &= x_4(1 + a_3x_3 + a_4x_5), \\ x_5' &= x_5(1 - a_4x_4 + a_5x_1), \end{aligned} \tag{5}$$

with skew-symmetric matrix

$$A = \begin{pmatrix} 0 & -a_1 & 0 & 0 & -a_5 \\ a_1 & 0 & -a_2 & 0 & 0 \\ 0 & a_2 & 0 & -a_3 & 0 \\ 0 & 0 & a_3 & 0 & a_4 \\ a_5 & 0 & 0 & -a_4 & 0 \end{pmatrix}.$$

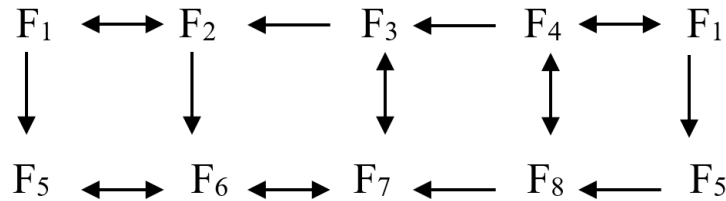


Figure 2: trajectory route for class (II)

Theorem 4. For operator (5): (i) $P = [e_1, M_1] \subset \Gamma_{13}$ and $Q = [e_4, M_2] \subset \Gamma_{24}$, where

$$M_1 = \left(\frac{a_2}{a_1 + a_2}, 0, \frac{a_1}{a_1 + a_2}, 0, 0 \right), \quad M_2 = \left(0, \frac{a_3}{a_2 + a_3}, 0, \frac{a_2}{a_2 + a_3}, 0 \right);$$

(ii) every point of riP is a transverse repeller and every point of riQ a transverse attractor; (iii) for every $x^{(0)} \in riS^4$, $\alpha(x^{(0)}) \subset P$ and $\omega(x^{(0)}) \subset Q$.

Proof.

Step 1 (two constant signs). The increments $(Ax)_1 = -a_1x_2 - a_5x_5 < 0$ and $(Ax)_4 = a_3x_3 + a_4x_5 > 0$ hold for all $x \in riS^4$, so x_1 strictly decreases and x_4 strictly increases along every interior orbit; both converge.

Step 2 (invariant segments). From $Ax \geq 0$: $(Ax)_1 \geq 0$ forces $x_2 = x_5 = 0$; then $(Ax)_3 = -a_3x_4 \geq 0$ forces $x_4 = 0$; and $(Ax)_2 \geq 0$ gives $x_1 \geq a_2/(a_1 + a_2)$. Hence $P = [e_1, M_1] \subset \Gamma_{13}$. From $Ax \leq 0$: $(Ax)_4 \leq 0$ forces $x_3 = x_5 = 0$; then $(Ax)_2 = a_1x_1 \leq 0$ forces $x_1 = 0$; and $(Ax)_3 \leq 0$ gives $x_2 \leq a_3/(a_2 + a_3)$. Hence $Q = [e_4, M_2] \subset \Gamma_{24}$. Thus P and Q lie in distinct 2-faces of S^4 .

Step 3 (absorbing region). On the boundary $\ell_5 : \{a_5x_1 = a_4x_4\}$ one has $x'_5 = x_5$ and $(Ax')_5 < 0$, so the region $\{a_4x_4 \geq a_5x_1\}$ is forward-invariant; since x_4 increases and x_1 decreases, every interior trajectory eventually enters it.

Step 4 (limit sets). On the ω -limit set Ω , x_4 is constant (positive supremum), so $(Ay)_4 = a_3y_3 + a_4y_5 = 0$ forces $y_3 = y_5 = 0$, i.e. $\Omega \subset \Gamma_{124}$. On Γ_{124} the residual dynamics drives $x_1 \rightarrow 0$, so $\Omega \subset \Gamma_{24}$. On Γ_{24} the transverse eigenvalues satisfy $\lambda_1 = 1 - a_1x_2 < 1$, $\lambda_5 = 1 - a_4x_4 < 1$, and $\lambda_3 = 1 + a_2x_2 - a_3x_4 < 1$ iff $x_2 < a_3/(a_2 + a_3)$; hence the attracting part is exactly $[e_4, M_2] = Q$. The symmetric argument for V^{-1} gives $\alpha(x^{(0)}) \subset P$.

Step 5 (acyclic route $P \rightarrow Q$). The simplex riS^4 is partitioned into at most four signature cells by the boundaries $\ell_2 : \{a_1x_1 = a_2x_3\}$ and $\ell_3 : \{a_2x_2 = a_3x_4\}$. Since $\sigma_1 \equiv -$ and $\sigma_4 \equiv +$ (Step 1), the only varying signs are $\sigma_2 = \text{sgn}(a_1x_1 - a_2x_3)$ and $\sigma_3 = \text{sgn}(a_2x_2 - a_3x_4)$. On ℓ_2 one verifies $(Ax')_2 = a_1x'_1(1 + \dots) - a_2x'_3(\dots) < 0$, so the cell $\{a_1x_1 \leq a_2x_3\}$ is forward-invariant once entered. On ℓ_3 , since x_4 strictly increases and $x_2 \rightarrow 0$ (forced by $x_1 \rightarrow 0$ and the residual dynamics), the sign of $a_2x_2 - a_3x_4$ eventually becomes negative and stays so. Hence the route graph has no directed cycles: the four cells can only be traversed in the order dictated by the monotone decrease of x_1 and increase of x_4 , terminating at Q . $\square$

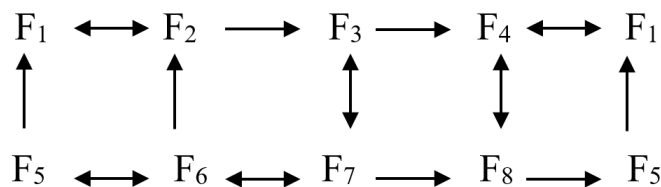


Figure 3: trajectory route for class (III)

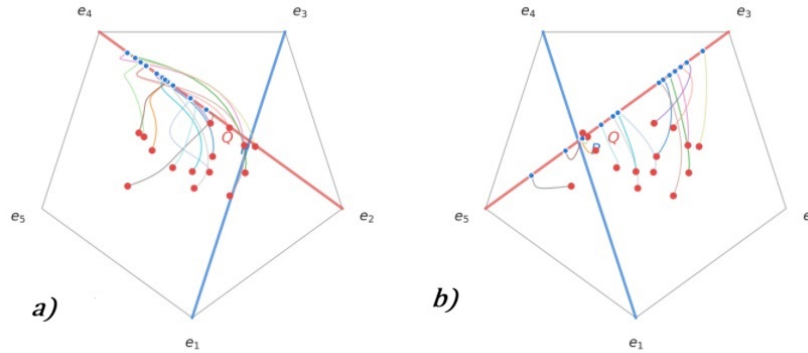


Figure 4: Trajectory dynamics of $\mathcal{R}_2(5)$ operators on S^4 (pentagon projection): a) Class (III) — $P = [e_1, M_1] \subset \Gamma_{13}$ (repeller, blue), $Q = [e_4, M_2] \subset \Gamma_{24}$ (attractor, red); b) Class (IV) — $P = [e_1, e_4]$ (repeller, blue), $Q = [e_3, e_5]$ (attractor, red); single transition $F_+ \rightarrow F_-$ yields terminal signature $(-, -, +, -, +)$.

Class (IV): two sources/two sinks, representative $(1, 1, 0, 1, 0)$.

The orientation $1 \rightarrow 2 \rightarrow 3 \leftarrow 4 \rightarrow 5 \leftarrow 1$ (sources $\{1, 4\}$, sinks $\{3, 5\}$) gives

$$\begin{aligned} x'_1 &= x_1(1 - a_1x_2 - a_5x_5), & x'_2 &= x_2(1 + a_1x_1 - a_2x_3), \\ x'_3 &= x_3(1 + a_2x_2 + a_3x_4), & x'_4 &= x_4(1 - a_3x_3 - a_4x_5), \\ x'_5 &= x_5(1 + a_4x_4 + a_5x_1), \end{aligned} \quad (6)$$

with skew-symmetric matrix

$$A = \begin{pmatrix} 0 & -a_1 & 0 & 0 & -a_5 \\ a_1 & 0 & -a_2 & 0 & 0 \\ 0 & a_2 & 0 & a_3 & 0 \\ 0 & 0 & -a_3 & 0 & -a_4 \\ a_5 & 0 & 0 & a_4 & 0 \end{pmatrix}.$$

Theorem 5. For operator (6): (i) $P = [e_1, e_4]$ and $Q = [e_3, e_5]$; (ii) every point of riP is a transverse repeller and every point of riQ a transverse attractor; (iii) for every $x^{(0)} \in riS^4$, $\alpha(x^{(0)}) \subset P$ and $\omega(x^{(0)}) \subset Q$.

Proof.

Step 1 (four constant signs). For $x \in riS^4$ the two sources and two sinks have definite signs: $(Ax)_1 = -a_1x_2 - a_5x_5 < 0$, $(Ax)_4 = -a_3x_3 - a_4x_5 < 0$ (sources x_1, x_4 decrease), and $(Ax)_3 = a_2x_2 + a_3x_4 > 0$, $(Ax)_5 = a_4x_4 + a_5x_1 > 0$ (sinks x_3, x_5 increase). Only $\sigma_2 = \text{sgn}(a_1x_1 - a_2x_3)$ varies, producing two cells $F_{\pm} = \{a_1x_1 \gtrless a_2x_3\}$ meeting along $\ell_2 : \{a_1x_1 = a_2x_3\}$.

Step 2 (invariant edges). $Ax \geq 0$ forces $x_2 = x_3 = x_5 = 0$ (all sink and mixed coordinates zero), so $P = [e_1, e_4]$. $Ax \leq 0$ forces $x_1 = x_2 = x_4 = 0$, so $Q = [e_3, e_5]$.

Step 3 (transversality). On riP ($x_2 = x_3 = x_5 = 0$): $\lambda_2 = 1 + a_1x_1 > 1$, $\lambda_3 = 1 + a_3x_4 > 1$, $\lambda_5 = 1 + a_4x_4 + a_5x_1 > 1$ (repeller). On riQ ($x_1 = x_2 = x_4 = 0$): $\lambda_1 = 1 - a_5x_5 < 1$, $\lambda_2 = 1 - a_2x_3 < 1$, $\lambda_4 = 1 - a_3x_3 - a_4x_5 < 1$ (attractor).

Step 4 (route $P \rightarrow Q$). On ℓ_2 one verifies $(Ax')_2 = -a_1^2x_1x_2 - a_1a_5x_1x_5 - a_2^2x_2x_3 - a_2a_3x_3x_4 < 0$, so $F_- = \{a_1x_1 \leq a_2x_3\}$ is forward-invariant; since $x_1 \rightarrow 0$ and x_3 increases, every trajectory enters F_- and the single transition $F_+ \rightarrow F_-$ occurs, leaving terminal signature $(-, -, +, -, +)$.

Step 5 (limit sets). On $\omega(x^{(0)})$, x_3 and x_5 are constant, so $(Ay)_5 = a_4y_4 + a_5y_1 = 0$ forces $y_1 = y_4 = 0$ and $(Ay)_3 = a_2y_2 = 0$ forces $y_2 = 0$; hence $\omega(x^{(0)}) \subset [e_3, e_5] = Q$. Symmetrically $\alpha(x^{(0)}) \subset [e_1, e_4] = P$. $\square$

General odd- m cyclic operators

Let $m \geq 3$ be odd and let $V : S^{m-1} \rightarrow S^{m-1}$ be the cyclic operator

$$x'_k = x_k(1 + a_{k-1}x_{k-1} - a_kx_{k+1}), \quad k = 1, \dots, m, \quad 0 < a_k \leq 1, \quad (7)$$

with indices modulo m and skew-symmetric matrix A with entries $A_{k,k-1} = a_{k-1}$, $A_{k,k+1} = -a_k$.

Proposition 1. For odd m , $\det A = 0$ for all $a_k > 0$.

Proof. Since $A^\top = -A$, $\det A = \det(A^\top) = \det(-A) = (-1)^m \det A = -\det A$, so $\det A = 0$. $\square$

Remark. For even m , $\det A = (\text{Pf } A)^2 \geq 0$ $\text{Pf } A = 0 \iff$ interior fixed point exists.

The unique interior fixed point. The equation $(AM)_k = 0$ reads

$$a_{k-1}M_{k-1} - a_kM_{k+1} = 0, \quad \text{i.e.} \quad M_{k+1} = \frac{a_{k-1}}{a_k}M_{k-1}. \quad (8)$$

This is a two-step recurrence. For odd m , $\gcd(2, m) = 1$, so stepping by 2 modulo m visits every index in a single orbit. Thus every M_k is determined by $M_1 > 0$, and normalising $\sum_k M_k = 1$ gives a unique $M \in \text{ri}S^{m-1}$.

Proposition 2. For m odd and all $a_k > 0$, $\dim \text{Ker } A = 1$ and $\text{Ker } A \cap S^{m-1} = \{M\}$.

Proof. Any solution of $Av = 0$ is determined by v_1 alone (single orbit under step-2 shifts for odd m), so $\dim \text{Ker } A = 1$. The unique positive solution, after normalisation, gives $M \in \text{ri}S^{m-1}$. $\square$

Example 1 ($m = 5$). From recurrence (8): $M_3 = (a_1/a_2)M_1$, $M_5 = (a_1a_3)/(a_2a_4) \cdot M_1$, and completing the cycle yields

$$M_1 = \frac{a_2a_4}{\Delta}, \quad M_2 = \frac{a_3a_5}{\Delta}, \quad M_3 = \frac{a_1a_4}{\Delta}, \quad M_4 = \frac{a_2a_5}{\Delta}, \quad M_5 = \frac{a_1a_3}{\Delta},$$

where $\Delta = a_2a_4 + a_3a_5 + a_1a_4 + a_2a_5 + a_1a_3$.

Example 2 ($m = 7$). The same procedure gives

$$M_1 = \frac{a_2a_4a_6}{\Delta}, \quad M_2 = \frac{a_3a_5a_7}{\Delta}, \quad M_3 = \frac{a_1a_4a_6}{\Delta}, \quad M_4 = \frac{a_2a_5a_7}{\Delta},$$

$$M_5 = \frac{a_1a_3a_6}{\Delta}, \quad M_6 = \frac{a_2a_4a_7}{\Delta}, \quad M_7 = \frac{a_1a_3a_5}{\Delta},$$

where Δ is the sum of all seven numerators.

Lemma 1. The function $\varphi_M(x) = \prod_{k=1}^m x_k^{M_k}$ satisfies $\varphi_M(Vx) \leq \varphi_M(x)$ on $\text{ri}S^{m-1}$, with equality if and only if $x = M$.

Proof. Write $u_k = (Ax)_k$, so $x'_k = x_k(1 + u_k)$. Then

$$\frac{\varphi_M(Vx)}{\varphi_M(x)} = \prod_{k=1}^m (1 + u_k)^{M_k}.$$

By skew-symmetry and $AM = 0$: $\sum_k M_k u_k = \langle M, Ax \rangle = -\langle AM, x \rangle = 0$. The weighted AM–GM inequality with weights $M_k > 0$, $\sum M_k = 1$ gives

$$\prod_{k=1}^m (1 + u_k)^{M_k} \leq \sum_{k=1}^m M_k(1 + u_k) = 1 + \sum_{k=1}^m M_k u_k = 1.$$

Equality requires all $1 + u_k$ equal; combined with $\sum M_k u_k = 0$ this forces each $u_k = 0$, i.e. $Ax = 0$, hence $x = M$ by Proposition 2. $\square$

Theorem 6. Let $m \geq 3$ be odd and let M be the unique interior fixed point. For every $x^{(0)} \in \text{ri}S^{m-1} \setminus \{M\}$: (i) $\alpha(x^{(0)}) = \{M\}$; (ii) $\omega(x^{(0)}) \subset \partial S^{m-1}$.

Proof.

Forward direction. For $x^{(0)} \neq M$ Lemma 1 gives strict decrease: $\varphi_M(x^{(n+1)}) < \varphi_M(x^{(n)})$. Being bounded below by 0, the sequence converges to some $L \geq 0$ with $L < \varphi_M(M)$. Suppose $y \in \omega(x^{(0)}) \cap \text{ri}S^{m-1}$. Then φ_M is constant on the forward orbit of y (since $y \in \omega$), yet Lemma 1 gives strict decrease unless $y = M$. Since $L < \varphi_M(M)$ we have $y \neq M$, a contradiction. Hence $\omega(x^{(0)}) \cap \text{ri}S^{m-1} = \emptyset$, proving (ii).

Backward direction. Since V is a homeomorphism of S^{m-1} [1,2], the backward orbit $x^{(-n)}$ is well defined. By Lemma 1, the sequence $\varphi_M(x^{(-n)})$ is strictly increasing and bounded above, and therefore converges to some value L .

Let $z \in \alpha(x^{(0)})$. Then there exists a sequence $n_k \rightarrow \infty$ such that $x^{(-n_k)} \rightarrow z$, and hence, by continuity,

$$\varphi_M(z) = L.$$

Since the α -limit set is invariant under V , we also have $Vz \in \alpha(x^{(0)})$, and therefore

$$\varphi_M(Vz) = L = \varphi_M(z).$$

By the equality condition in Lemma 1, this is possible only if $z = M$. Hence every point of $\alpha(x^{(0)})$ is equal to M , and therefore

$$\alpha(x^{(0)}) = \{M\}.$$

Relation to previously known results. The results of [1] provide a general framework for the qualitative analysis of Volterra quadratic stochastic operators, including Lyapunov functions and estimates for limit sets. The present results develop this framework further for cyclic $R_2(m)$ operators by combining the graph structure with an explicit analysis of the global dynamics.

For $R_2(5)$ operators, Theorems 2–5 give a complete classification of all orientations into four D_5 -isomorphism classes and determine, for each class, the corresponding repeller–attractor structure, invariant sets P and Q , transverse stability properties, and global trajectory routes. Thus, these theorems provide a detailed dynamical classification that goes beyond the general qualitative statements available in [1].

For arbitrary odd m , Theorem 6 gives a further global result for the directed cyclic configuration. Namely, the unique interior fixed point is determined explicitly, and the associated Lyapunov function yields the precise asymptotic relations

$$\alpha(x^{(0)}) = \{M\}, \quad \omega(x^{(0)}) \subset \partial S^{m-1}$$

for every non-fixed interior trajectory. Hence, the results of Theorems 2–6 complement and refine the general theory in [1] by providing an explicit classification and a more detailed description of the global dynamics for cyclic Lotka–Volterra operators.

Proposition 3. *For the cyclic $\mathcal{R}_2(m)$ operator with m odd, $\text{Ker}A \cap \text{ri}S^{m-1} \neq \emptyset$ if and only if the orientation is a directed cycle; otherwise $\text{Ker}A \cap S^{m-1} \subset \partial S^{m-1}$.*

Proof. (If.) If the orientation is a directed cycle, every vertex has in-degree and out-degree 1, so the recurrence $(AM)_k = 0$ takes the form $a_{k-1}M_{k-1} = a_kM_{k+1}$ with all $a_k > 0$. For odd m , $\gcd(2, m) = 1$, so stepping by 2 visits every index; a unique positive solution exists and after normalisation gives $M \in \text{ri}S^{m-1}$.

(Only if.) Suppose $M \in \text{Ker}A \cap \text{ri}S^{m-1}$, so all $M_k > 0$. Then $(AM)_k = 0$ reads $a_{k-1}M_{k-1} = a_kM_{k+1}$ for each k . If vertex k had two outgoing edges (say to $k+1$ and $k-1$), then $(AM)_k = -a_{k,k+1}M_{k+1} - a_{k,k-1}M_{k-1} < 0$ since all terms are positive, contradicting $(AM)_k = 0$. Hence every vertex has exactly one outgoing and one incoming edge, i.e. the orientation is a directed cycle. $\square$

Corollary 1. *For $\mathcal{R}_2(5)$ on S^4 , an interior fixed point exists if and only if the orientation is class (I). In classes (II)–(IV), some vertex has degree imbalance, forcing the unique $\text{Ker}A$ -vector to lie on ∂S^4 .*

Proof. Apply Proposition 3 with $m = 5$. In class (II), vertex 1 has two outgoing edges, so $(AM)_1 = -a_1M_2 - a_5M_5 = 0$ with $a_1, a_5 > 0$ forces $M_2 = M_5 = 0$, hence $M \notin \text{ri}S^4$. The argument is analogous for classes (III) and (IV). $\square$

Conclusion

We have given a complete classification of $\mathcal{R}_2(5)$ Lotka–Volterra operators on S^4 into four D_5 -isomorphism classes (I)–(IV), and for each class proved the $P \rightarrow Q$ repeller–attractor scheme via coordinate signatures and Lyapunov functions. The general odd- m theorem shows that the directed-cycle operator always has a unique interior repeller M and that every other interior trajectory escapes to the boundary, with M as the sole α -limit point. These results connect combinatorial graph data (orientation type) directly to global qualitative dynamics on the simplex. For even m the situation is richer: the Pfaffian $\text{Pf}A$ may be nonzero, and the interior of S^{m-1} may carry no fixed point even for the directed-cycle orientation; a full classification for $m = 6$ into nine D_6 -isomorphism classes, together with the condition $a_1a_3a_5 = a_2a_4a_6$ distinguishing the degenerate regime, is the subject of ongoing work [3,4]. The coordinate-signature framework developed here extends naturally to $\mathcal{R}_r(m)$ operators with $r > 2$ interactions per vertex and to bigraph structures on higher-dimensional simplices.

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ROTA-BAXTER OPERATORS ON A SIX-DIMENSIONAL SIMPLE LEIBNIZ ALGEBRA

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ABSTRACT. We study Rota-Baxter operators of weight zero on a six-dimensional simple Leibniz algebra whose associated Lie algebra is $\mathfrak{sl}_2(\mathbb{C})$ and whose Leibniz kernel is the three-dimensional irreducible $\mathfrak{sl}_2$ -module $V(2)$. Using a fixed module isomorphism between the Lie part and the Leibniz kernel, we represent an arbitrary linear operator in block form and reduce the Rota-Baxter identity to a system of operator equations on $\mathfrak{sl}_2$. We completely describe the case in which the block mapping the Leibniz kernel to the Lie part vanishes. When the Lie component is zero, we obtain an intrinsic description of all admissible operators in terms of the Killing form. For a nonzero Lie component, we use the known classification of weight-zero Rota-Baxter operators on $\mathfrak{sl}_2$ to determine the compatible remaining blocks and the dimensions of the corresponding solution spaces. Several exceptional parameter values at which these dimensions increase are also identified. The cases in which the block from the Leibniz kernel to the Lie part is nonzero are left for further investigation.

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Key words: Leibniz algebra, simple Leibniz algebra, Rota-Baxter operator, $\mathfrak{sl}_2$ -module, Killing form.

Introduction

Leibniz algebras form a non-antisymmetric generalization of Lie algebras. They were first considered by Blokh and later systematically developed by Loday [3], [4]. Their structure theory is closely related to Lie theory, since the quotient by the ideal generated by squares is a Lie algebra. This makes it possible to combine methods from the theory of Lie algebras and their representations with phenomena that arise specifically in the non-antisymmetric setting.

The structure, representations, derivations, and automorphisms of finite-dimensional Leibniz algebras have been investigated extensively. In particular, simple and semisimple Leibniz algebras admit descriptions in terms of simple Lie algebras and their modules; see, for example, [5], [6], [1], [2]. The case in which the associated Lie algebra is $\mathfrak{sl}_2(\mathbb{C})$ is especially suitable for explicit classification problems, since the corresponding Leibniz algebras can be described using finite-dimensional irreducible $\mathfrak{sl}_2$ -modules [6].

Rota-Baxter operators constitute an important class of algebraic operators and are closely connected with classical Yang-Baxter type equations. In the Lie algebra setting, weight-zero Rota-Baxter operators on $\mathfrak{sl}_2(\mathbb{C})$ were completely classified by Pei, Bai and Guo [7]. More recently, Rota-Baxter and relative Rota-Baxter operators have also been studied in the framework of Leibniz algebras. Tang and Sheng established their relation with Leibniz bialgebras, Maurer-Cartan elements, and the classical Leibniz Yang-Baxter equation [8].

In the present paper, we study weight-zero Rota-Baxter operators on the six-dimensional simple Leibniz algebra whose associated Lie algebra is $\mathfrak{sl}_2(\mathbb{C})$ and whose Leibniz kernel is the three-dimensional irreducible module $V(2)$. Since the Lie part and the Leibniz kernel are isomorphic as $\mathfrak{sl}_2$ -modules, a fixed module isomorphism allows us to transfer all blocks of a linear operator to endomorphisms of $\mathfrak{sl}_2$. In this way, the Rota-Baxter identity is reduced to a finite system of operator equations on $\mathfrak{sl}_2$.

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We first consider the case in which the block mapping the Leibniz kernel to the Lie part vanishes. When the Lie component also vanishes, we obtain an intrinsic description of the remaining blocks in terms of the Killing form. For a nonzero Lie component, we use the Pei-Bai-Guo classification of weight-zero Rota-Baxter operators on $\mathfrak{sl}_2$ and determine the corresponding compatible blocks. This yields a complete description of the weight-zero Rota-Baxter operators in the case where the block from the Leibniz kernel to the Lie part is zero.

The remaining cases, in which this block is nonzero, are not treated in the present paper and are left for further investigation.

The paper is organized as follows. We first recall the basic notions concerning Leibniz algebras and Rota-Baxter operators. We then describe the six-dimensional simple Leibniz algebra under consideration and derive the corresponding block equations. The subsequent sections are devoted to the classification in the case where the block from the Leibniz kernel to the Lie part vanishes. Finally, we summarize the obtained results and indicate the remaining open cases.

Preliminaries

Definition 1. A *Lie algebra* is a vector space $\mathfrak{g}$ endowed with a bilinear multiplication $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ such that

$$\begin{aligned} [x, x] &= 0, \\ [x, [y, z]] &= [[x, y], z] + [y, [x, z]] \end{aligned}$$

for all $x, y, z \in \mathfrak{g}$.

Definition 2. A *right Leibniz algebra* is a vector space $\mathcal{L}$ endowed with a bilinear multiplication $[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ satisfying the right Leibniz identity

$$[[x, y], z] = [[x, z], y] + [x, [y, z]]$$

for all $x, y, z \in \mathcal{L}$.

In contrast to Lie algebras, the multiplication of a Leibniz algebra need not be skew-symmetric. The ideal

$$\mathcal{I} = \text{span}\{[x, x] \mid x \in \mathcal{L}\}$$

is called the *Leibniz kernel* of $\mathcal{L}$. For a right Leibniz algebra one has $[\mathcal{L}, \mathcal{I}] = 0$, and the quotient algebra $\mathcal{L}/\mathcal{I}$ is a Lie algebra.

Definition 3. A Leibniz algebra $\mathcal{L}$ is called *simple* if $[\mathcal{L}, \mathcal{L}] \neq \mathcal{I}$ and its only ideals are $\{0\}$, $\mathcal{I}$, $\mathcal{L}$.

In particular, if $\mathcal{L}$ is a simple non-Lie Leibniz algebra, then $\mathcal{L}/\mathcal{I}$ is a simple Lie algebra and $\mathcal{I}$ is an irreducible module over this associated Lie algebra.

Definition 4. Let $\mathcal{L}$ be a Leibniz algebra and let $\lambda \in \mathbb{C}$. A linear operator $R : \mathcal{L} \rightarrow \mathcal{L}$ is called a *Rota-Baxter operator of weight λ* if

$$[R(x), R(y)] = R([R(x), y] + [x, R(y)] + \lambda[x, y])$$

for all $x, y \in \mathcal{L}$.

In the present paper, we consider only Rota-Baxter operators of weight zero. Thus R satisfies

$$[R(x), R(y)] = R([R(x), y] + [x, R(y)]) \tag{1}$$

for all $x, y \in \mathcal{L}$.

Remark 1. Every Lie algebra is a right Leibniz algebra. Hence the notion of a Rota-Baxter operator on a Leibniz algebra naturally extends the usual notion of a Rota-Baxter operator on a Lie algebra.

The six-dimensional simple Leibniz algebra

Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) = \text{span}\{e, f, h\}$ with multiplication

$$[e, h] = -[h, e] = 2e, \quad [f, h] = -[h, f] = -2f, \quad [e, f] = -[f, e] = h.$$

Let $\mathcal{I} = \text{span}\{x_0, x_1, x_2\}$ be the three-dimensional irreducible right $\mathfrak{sl}_2$ -module $V(2)$ with nonzero actions

$$\begin{aligned} [x_0, h] &= 2x_0, & [x_2, h] &= -2x_2, \\ [x_0, f] &= x_1, & [x_1, f] &= x_2, \\ [x_1, e] &= -2x_0, & [x_2, e] &= -2x_1. \end{aligned}$$

Consider the vector space $\mathcal{L} = \mathfrak{sl}_2 \oplus \mathcal{I}$ with multiplication $[x+u, y+v] = [x, y]_{\mathfrak{sl}_2} + [u, v]$, where $x, y \in \mathfrak{sl}_2$, $u, v \in \mathcal{I}$.

Equivalently, $[\mathfrak{sl}_2, \mathcal{I}] = 0$, $[\mathcal{I}, \mathcal{I}] = 0$, $[\mathcal{I}, \mathfrak{sl}_2] = \mathcal{I}$.

Thus, with respect to the ordered basis $\{e, f, h, x_0, x_1, x_2\}$, the nonzero products of $\mathcal{L}$ are

$$\begin{aligned} [e, h] &= 2e, & [h, e] &= -2e, & [e, f] &= h, \\ [f, e] &= -h, & [h, f] &= 2f, & [f, h] &= -2f, \\ [x_0, h] &= 2x_0, & [x_2, h] &= -2x_2, & [x_0, f] &= x_1, \\ [x_1, f] &= x_2, & [x_1, e] &= -2x_0, & [x_2, e] &= -2x_1. \end{aligned}$$

The ideal $\mathcal{I}$ coincides with the Leibniz kernel of $\mathcal{L}$, and $\mathcal{L}/\mathcal{I} \simeq \mathfrak{sl}_2$.

Therefore $\mathcal{L}$ is the six-dimensional simple non-Lie Leibniz algebra associated with the irreducible module $V(2)$ [6].

A module isomorphism.

Lemma 1. The linear map $\theta : \mathfrak{sl}_2 \rightarrow \mathcal{I}$ defined by

$$\theta(e) = 2x_0, \quad \theta(f) = x_2, \quad \theta(h) = 2x_1$$

is an isomorphism of right $\mathfrak{sl}_2$ -modules. In particular,

$$[\theta(x), y] = \theta([x, y])$$

for all $x, y \in \mathfrak{sl}_2$.

Proof. It is enough to verify the identity on the basis $\{e, f, h\}$. For example,

$$\begin{aligned} [\theta(e), h] &= [2x_0, h] = 4x_0 = \theta(2e) = \theta([e, h]), \\ [\theta(f), h] &= [x_2, h] = -2x_2 = \theta(-2f) = \theta([f, h]), \\ [\theta(h), e] &= [2x_1, e] = -4x_0 = \theta(-2e) = \theta([h, e]). \end{aligned}$$

The remaining identities are verified similarly. Since both $\mathfrak{sl}_2$ and $\mathcal{I}$ are three-dimensional and θ maps the chosen basis of $\mathfrak{sl}_2$ onto a basis of $\mathcal{I}$, the map θ is an isomorphism. $\square$

Remark 2. Since $\mathcal{I} = \text{Leib}(\mathcal{L})$, the ideal $\mathcal{I}$ is characteristic. Hence every automorphism of $\mathcal{L}$ induces an automorphism of the quotient $\mathcal{L}/\mathcal{I} \simeq \mathfrak{sl}_2$.

The automorphism group of this simple Leibniz algebra was described in [1]. Since the explicit form of the automorphisms is not needed in the subsequent classification of Rota-Baxter operators, we do not reproduce it here.

Matrix convention. Throughout the paper, matrices of linear operators on $\mathfrak{sl}_2$ are written with respect to the ordered basis $\{e, f, h\}$ and act on coordinate column vectors.

Our convention for $\mathfrak{sl}_2$ is

$$[e, h] = 2e, \quad [f, h] = -2f, \quad [e, f] = h,$$

whereas the Pei-Bai-Guo classification [?] is written with respect to the convention

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

The two bases are related by

$$e_{\text{PBG}} = e, \quad f_{\text{PBG}} = -f, \quad h_{\text{PBG}} = -h.$$

Hence, if

$$J = \text{diag}(1, -1, -1),$$

the corresponding change of basis is represented by J .

Moreover, the matrices P_i in the Pei-Bai-Guo classification are written in the row convention. Therefore, in our column convention, the corresponding operators are represented by

$$A_i = JP_i^T J, \quad 1 \leq i \leq 22.$$

In what follows, A_i always denotes the Pei-Bai-Guo normal form transported to the convention used in the present paper. All matrices appearing below are written in this convention.

Block equations for Rota-Baxter operators. Let $\mathcal{L} = \mathfrak{sl}_2 \oplus \mathcal{I}$ be the six-dimensional simple Leibniz algebra introduced above. Let $R : \mathcal{L} \rightarrow \mathcal{L}$ be a linear operator. With respect to the vector space decomposition $\mathcal{L} = \mathfrak{sl}_2 \oplus \mathcal{I}$, write

$$R = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A : \mathfrak{sl}_2 \rightarrow \mathfrak{sl}_2$, $B : \mathcal{I} \rightarrow \mathfrak{sl}_2$, $C : \mathfrak{sl}_2 \rightarrow \mathcal{I}$, $D : \mathcal{I} \rightarrow \mathcal{I}$.

Using the fixed $\mathfrak{sl}_2$ -module isomorphism $\theta : \mathfrak{sl}_2 \rightarrow \mathcal{I}$, we transfer all four blocks to $\mathfrak{sl}_2$ and write

$$B = G\theta^{-1}, \quad C = \theta F, \quad D = \theta S\theta^{-1},$$

where $F, G, S \in \text{End}(\mathfrak{sl}_2)$. Thus $R = \begin{pmatrix} A & G\theta^{-1} \\ \theta F & \theta S\theta^{-1} \end{pmatrix}$. For $x, z \in \mathfrak{sl}_2$ we have $R(x) = A(x) + \theta(F(x))$ and $R(\theta(z)) = G(z) + \theta(S(z))$. Recall that $[\mathfrak{sl}_2, \mathcal{I}] = 0$, $[\mathcal{I}, \mathcal{I}] = 0$, and $[\theta(x), y] = \theta([x, y])$, $x, y \in \mathfrak{sl}_2$.

Theorem 1. Let $R = \begin{pmatrix} A & G\theta^{-1} \\ \theta F & \theta S\theta^{-1} \end{pmatrix}$ be a linear operator on $L = \mathfrak{sl}_2 \ltimes \mathcal{I}$. Then R is a Rota-Baxter operator of weight zero if and only if the following identities hold for all $x, y, z, w \in \mathfrak{sl}_2$:

$$[A(x), A(y)] = A([A(x), y] + [x, A(y)]) + G([F(x), y]), \quad (2)$$

$$[F(x), A(y)] = F([A(x), y] + [x, A(y)]) + S([F(x), y]), \quad (3)$$

$$[A(x), G(z)] = A([x, G(z)]), \quad (4)$$

$$[F(x), G(z)] = F([x, G(z)]), \quad (5)$$

$$[G(z), A(x)] = A([G(z), x]) + G([S(z), x] + [z, A(x)]), \quad (6)$$

$$[S(z), A(x)] = F([G(z), x]) + S([S(z), x] + [z, A(x)]), \quad (7)$$

$$[G(z), G(w)] = G([z, G(w)]), \quad (8)$$

$$[S(z), G(w)] = S([z, G(w)]). \quad (9)$$

Proof. Since $\mathcal{L} = \mathfrak{sl}_2 \oplus \mathcal{I}$, as a vector space, it is sufficient, by bilinearity, to verify the Rota-Baxter identity on the following four types of pairs:

$$(\mathfrak{sl}_2, \mathfrak{sl}_2), \quad (\mathfrak{sl}_2, \mathcal{I}), \quad (\mathcal{I}, \mathfrak{sl}_2), \quad (\mathcal{I}, \mathcal{I}).$$

Let $x, y, z, w \in \mathfrak{sl}_2$.

Case 1: $X = x$, $Y = y$. We have $R(x) = A(x) + \theta(F(x))$, $R(y) = A(y) + \theta(F(y))$. Since $[\mathfrak{sl}_2, \mathcal{I}] = [\mathcal{I}, \mathcal{I}] = 0$, the left-hand side of the Rota-Baxter identity is

$$[R(x), R(y)] = [A(x), A(y)] + [\theta(F(x)), A(y)].$$

Using $[\theta(u), v] = \theta([u, v])$, we obtain $[R(x), R(y)] = [A(x), A(y)] + \theta([F(x), A(y)])$. Furthermore, $[R(x), y] + [x, R(y)] = [A(x), y] + [x, A(y)] + \theta([F(x), y])$.

Applying R , we obtain

$$R([R(x), y] + [x, R(y)]) = A([A(x), y] + [x, A(y)]) + G([F(x), y]) + \theta(F([A(x), y] + [x, A(y)]) + S([F(x), y])).$$

Comparing the $\mathfrak{sl}_2$ and $\mathcal{I}$ -components gives (2) and (3).

Case 2: $X = x, Y = \theta(z)$. We have $R(\theta(z)) = G(z) + \theta(S(z))$. Hence

$$[R(x), R(\theta(z))] = [A(x), G(z)] + \theta([F(x), G(z)]).$$

On the other hand, $[R(x), \theta(z)] = 0$ because $[\mathfrak{sl}_2, \mathcal{I}] = [\mathcal{I}, \mathcal{I}] = 0$, whereas $[x, R(\theta(z))] = [x, G(z)]$. Therefore

$$R([x, G(z)]) = A([x, G(z)]) + \theta(F([x, G(z)])).$$

Comparison of the two components yields (4) and (5).

Case 3: $X = \theta(z), Y = x$. The left-hand side is

$$\begin{aligned} [R(\theta(z)), R(x)] &= [G(z) + \theta(S(z)), A(x) + \theta(F(x))] \\ &= [G(z), A(x)] + \theta([S(z), A(x)]). \end{aligned}$$

Furthermore,

$$[R(\theta(z)), x] = [G(z), x] + \theta([S(z), x])$$

and

$$[\theta(z), R(x)] = [\theta(z), A(x)] = \theta([z, A(x)]).$$

Consequently,

$$\begin{aligned} R([R(\theta(z)), x] + [\theta(z), R(x)]) &= \\ &= A([G(z), x] + [S(z), x] + [z, A(x)]) + \theta(F([G(z), x] + S([S(z), x] + [z, A(x)]))). \end{aligned}$$

Comparing components gives (6) and (7).

Case 4: $X = \theta(z), Y = \theta(w)$. Since $[\mathcal{I}, \mathcal{I}] = 0$, we have

$$\begin{aligned} [R(\theta(z)), R(\theta(w))] &= [G(z) + \theta(S(z)), G(w) + \theta(S(w))] \\ &= [G(z), G(w)] + \theta([S(z), G(w)]). \end{aligned}$$

Moreover, $[R(\theta(z)), \theta(w)] = 0$ and $[\theta(z), R(\theta(w))] = [\theta(z), G(w)] = \theta([z, G(w)])$. Hence

$$R(\theta([z, G(w)])) = G([z, G(w)]) + \theta(S([z, G(w)])).$$

Comparison of components yields (8) and (9).

Thus the Rota-Baxter identity is equivalent to (2)–(9). $\square$

Corollary 1. Let $R = \begin{pmatrix} A & 0 \\ \theta F & \theta S\theta^{-1} \end{pmatrix}$. Then R is a Rota-Baxter operator of weight zero on $\mathcal{L}$ if and only if

$$\begin{aligned} [A(x), A(y)] &= A([A(x), y] + [x, A(y)]), \\ [F(x), A(y)] &= F([A(x), y] + [x, A(y)]) + S([F(x), y]), \\ [S(z), A(x)] &= S([S(z), x] + [z, A(x)]) \end{aligned}$$

for all $x, y, z \in \mathfrak{sl}_2$.

Corollary 2. Let $R = \begin{pmatrix} A & 0 \\ \theta F & 0 \end{pmatrix}$. Then R is a Rota-Baxter operator of weight zero if and only if

$$[A(x), A(y)] = A([A(x), y] + [x, A(y)]),$$

$$[F(x), A(y)] = F([A(x), y] + [x, A(y)])$$

for all $x, y \in \mathfrak{sl}_2$.

In particular, A is a Rota-Baxter operator of weight zero on $\mathfrak{sl}_2$.

For a weight-zero Rota-Baxter operator $A \in \text{End}(\mathfrak{sl}_2)$, put

$$\mathcal{F}_0(A) = \{F \in \text{End}(\mathfrak{sl}_2) \mid [F(x), A(y)] = F([A(x), y] + [x, A(y)])\}.$$

Lemma 2. Let A be a Rota-Baxter operator of weight zero on $\mathfrak{sl}_2$. Then $\mathbb{C}A \subseteq \mathcal{F}_0(A)$. In other words, for every $\mu \in \mathbb{C}$, $F = \mu A$ satisfies $[F(x), A(y)] = F([A(x), y] + [x, A(y)])$.

Proof. Since A is a Rota-Baxter operator, $[A(x), A(y)] = A([A(x), y] + [x, A(y)])$. Multiplying this identity by μ gives $[\mu A(x), A(y)] = \mu A([A(x), y] + [x, A(y)])$, which is precisely the required identity for $F = \mu A$. $\square$

Corollary 3. For every linear map $F \in \text{End}(\mathfrak{sl}_2)$, the operator $R_F = \begin{pmatrix} 0 & 0 \\ \theta F & 0 \end{pmatrix}$ is a Rota-Baxter operator of weight zero on $\mathcal{L}$. Consequently, $\text{End}(\mathfrak{sl}_2) \simeq \text{Hom}(\mathfrak{sl}_2, \mathcal{I})$ gives a nine-dimensional linear family of Rota-Baxter operators on $\mathcal{L}$. Moreover, $R_F^2 = 0$ and $\text{rank } R_F = \text{rank } F \leq 3$.

Rota-Baxter operators with $A = B = 0$. We first consider Rota-Baxter operators whose $\mathfrak{sl}_2$ -component vanishes.

$$\text{Let } R = \begin{pmatrix} 0 & 0 \\ \theta F & \theta S\theta^{-1} \end{pmatrix}, F, S \in \text{End}(\mathfrak{sl}_2).$$

By Corollary 1, the Rota-Baxter identity is equivalent to

$$S([F(x), y]) = 0, \quad S([S(z), x]) = 0 \tag{10}$$

for all $x, y, z \in \mathfrak{sl}_2$.

Lemma 3. Let $0 \neq w \in \mathfrak{sl}_2$. Then $\dim[w, \mathfrak{sl}_2] = 2$ and $C_{\mathfrak{sl}_2}(w) = \{x \in \mathfrak{sl}_2 \mid [w, x] = 0\} = \mathbb{C}w$. Consequently, $[w, \mathfrak{sl}_2]^\perp = \mathbb{C}w$ with respect to the Killing form.

Proof. Since $\mathfrak{sl}_2$ has rank one, the centralizer of every nonzero element is one-dimensional. Hence $\dim C_{\mathfrak{sl}_2}(w) = 1$. By the rank-nullity theorem applied to $\text{ad}_w : \mathfrak{sl}_2 \rightarrow \mathfrak{sl}_2$, we obtain

$$\dim[w, \mathfrak{sl}_2] = 3 - \dim C_{\mathfrak{sl}_2}(w) = 2.$$

Finally, by invariance of the Killing form, $\kappa(w, [x, y]) = \kappa([w, x], y)$, and therefore $[w, \mathfrak{sl}_2]^\perp = C_{\mathfrak{sl}_2}(w) = \mathbb{C}w$. $\square$

Theorem 2. Let $R = \begin{pmatrix} 0 & 0 \\ \theta F & \theta S\theta^{-1} \end{pmatrix}$ be a linear operator on $\mathcal{L} = \mathfrak{sl}_2 \ltimes \mathcal{I}$. Then R is a Rota-Baxter operator of weight zero if and only if one of the following two alternatives holds.

(i) $S = 0$, while $F \in \text{End}(\mathfrak{sl}_2)$ is arbitrary.

(ii) There exist $0 \neq w \in \mathfrak{sl}_2$, $\lambda \in \mathbb{C}^*$, $\ell \in \mathfrak{sl}_2^*$ such that $S(x) = \lambda \kappa(w, x)w$, and $F(x) = \ell(x)w$ for all $x \in \mathfrak{sl}_2$, where κ denotes the Killing form.

Equivalently, $S = \lambda w \otimes \kappa(w, \cdot)$, $F = w \otimes \ell$.

Proof. By (10), the Rota-Baxter identity is equivalent to $S([F(x), y]) = 0$ and $S([S(z), x]) = 0$ for all $x, y, z \in \mathfrak{sl}_2$. If $S = 0$, both identities are automatically satisfied and F is arbitrary. This gives case (i).

Assume now that $S \neq 0$. Put $U = \text{Im } S$. Equation (10) implies $[U, \mathfrak{sl}_2] \subseteq \ker S$.

We first show that $\text{rank } S = 1$. Indeed, if $\text{rank } S = 3$, then $\ker S = 0$, and hence $[U, \mathfrak{sl}_2] = 0$. Since $U = \mathfrak{sl}_2$, this contradicts $Z(\mathfrak{sl}_2) = 0$. Suppose that $\text{rank } S = 2$. Then $\dim \ker S = 1$. Choose $0 \neq u \in U$. By Lemma 3, $\dim[u, \mathfrak{sl}_2] = 2$. On the other hand, $[u, \mathfrak{sl}_2] \subseteq [U, \mathfrak{sl}_2] \subseteq \ker S$, which is impossible because $\dim \ker S = 1$. Therefore $\text{rank } S = 1$.

Hence there exist $0 \neq w \in \mathfrak{sl}_2$ and $0 \neq \varphi \in \mathfrak{sl}_2^*$ such that $S(x) = \varphi(x)w$. Substituting this into (10) gives $\varphi(z)\varphi([w, x])w = 0$ for all $x, z \in \mathfrak{sl}_2$. Since $\varphi \neq 0$, we obtain $\varphi([w, x]) = 0$ for all $x \in \mathfrak{sl}_2$. Therefore $\varphi \in [w, \mathfrak{sl}_2]^\perp$.

By Lemma 3, $[w, \mathfrak{sl}_2]^\perp = \mathbb{C}w$ under the identification of $\mathfrak{sl}_2$ with its dual via the Killing form. Thus $\varphi(x) = \lambda \kappa(w, x)$ for some $\lambda \in \mathbb{C}^*$. Consequently, $S(x) = \lambda \kappa(w, x)w$.

It remains to determine F . Equation (10) becomes $\kappa(w, [F(x), y]) = 0$ for all $x, y \in \mathfrak{sl}_2$. By invariance of the Killing form, $\kappa([w, F(x)], y) = 0$. Since κ is nondegenerate, $[w, F(x)] = 0$. Hence $F(x) \in C_{\mathfrak{sl}_2}(w) = \mathbb{C}w$. Therefore there exists $\ell \in \mathfrak{sl}_2^*$ such that $F(x) = \ell(x)w$. Conversely, suppose that $S(x) = \lambda\kappa(w, x)w$, $F(x) = \ell(x)w$. Then $S([S(z), x]) = \lambda^2\kappa(w, z)\kappa(w, [w, x])w = 0$ and $S([F(x), y]) = \lambda\ell(x)\kappa(w, [w, y])w = 0$, because $\kappa(w, [w, u]) = \kappa([w, w], u) = 0$.

Thus both (10) hold, and R is a Rota-Baxter operator. $\square$

Corollary 4. Let $R = \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix}$ be a Rota-Baxter operator of weight zero on $\mathcal{L}$. Then either $D = 0$ and $C : \mathfrak{sl}_2 \rightarrow \mathcal{I}$ is arbitrary, or there exist $0 \neq w \in \mathfrak{sl}_2$, $\lambda \in \mathbb{C}^*$, $\ell \in \mathfrak{sl}_2^*$ such that $C(x) = \ell(x)\theta(w)$ and $D(\theta(x)) = \lambda\kappa(w, x)\theta(w)$.

In particular, $\text{rank } C \leq 1$, $\text{rank } D = 1$ whenever $D \neq 0$.

Remark 3. Normalize the invariant bilinear form by $\kappa(e, f) = 1$, $\kappa(h, h) = 2$.

Let $w = pe + qf + rh$. Then $\kappa(w, e) = q$, $\kappa(w, f) = p$, $\kappa(w, h) = 2r$. Therefore $S = \lambda w \otimes \kappa(w, \cdot)$ has matrix

$$S = \lambda \begin{pmatrix} pq & p^2 & 2pr \\ q^2 & pq & 2qr \\ qr & pr & 2r^2 \end{pmatrix}$$

with respect to the ordered basis $\{e, f, h\}$.

If $\ell(e) = \alpha$, $\ell(f) = \beta$, $\ell(h) = \gamma$, then $F = w \otimes \ell$ has matrix

$$F = \begin{pmatrix} p\alpha & p\beta & p\gamma \\ q\alpha & q\beta & q\gamma \\ r\alpha & r\beta & r\gamma \end{pmatrix}.$$

Corollary 5. For every nonzero operator S occurring in Theorem 2, $S^2 = \lambda\kappa(w, w)S$.

Consequently:

1. if w is nilpotent, then $\kappa(w, w) = 0$ and $S^2 = 0$;
2. if w is semisimple, then $\kappa(w, w) \neq 0$, and after multiplication by a nonzero scalar, S becomes an idempotent operator.

Rota-Baxter operators with $B = D = 0$. Let $R = \begin{pmatrix} A & 0 \\ \theta F & 0 \end{pmatrix}$, $A, F \in \text{End}(\mathfrak{sl}_2)$. By Corollary 2, the operator R is a Rota-Baxter operator of weight zero if and only if

$$[A(x), A(y)] = A([A(x), y] + [x, A(y)]), \quad (11)$$

$$[F(x), A(y)] = F([A(x), y] + [x, A(y)]) \quad (12)$$

for all $x, y \in \mathfrak{sl}_2$.

For a weight-zero Rota-Baxter operator A on $\mathfrak{sl}_2$, define

$$\mathcal{F}_0(A) = \{F \in \text{End}(\mathfrak{sl}_2) \mid [F(x), A(y)] = F([A(x), y] + [x, A(y)])\}.$$

Thus the classification of Rota-Baxter operators with $B = D = 0$ reduces to the determination of the spaces $\mathcal{F}_0(A)$.

Theorem 3. Let $R = \begin{pmatrix} A & 0 \\ \theta F & 0 \end{pmatrix}$ be a Rota-Baxter operator of weight zero on $\mathcal{L} = \mathfrak{sl}_2 \times \mathcal{I}$.

Then A is a Rota-Baxter operator of weight zero on $\mathfrak{sl}_2$. Let $A_1, \dots, A_{22}$ denote the Pei-Bai-Guo normal forms transported to the convention of the present paper as described above.

Then the corresponding spaces $\mathcal{F}_0(A_i)$ satisfy

$$\dim \mathcal{F}_0(A_i) = \begin{cases} 9, & i = 2, \\ 1, & i \in \{4, 7, 8, 10, 12, 13, 15, 16, 17, 19, 20\}, \\ 2, & i \in \{1, 3, 5, 6, 9, 11, 14, 18, 21, 22\}. \end{cases}$$

More precisely, we obtain $\mathcal{F}_0(A_2) = \text{End}(\mathfrak{sl}_2)$, whereas $\mathcal{F}_0(A_i) = \mathbb{C}A_i$ for each $i \in \{4, 7, 8, 10, 12, 13, 15, 16, 17, 19, 20\}$.

For $i \in \{1, 3, 5, 6, 9, 11, 14, 18, 21, 22\}$, there exists an operator $\widehat{Q}_i$ such that $\mathcal{F}_0(A_i) = \mathbb{C}A_i \oplus \mathbb{C}\widehat{Q}_i$. The operators $\widehat{Q}_i$ are determined by $\widehat{Q}_i = JM_iJ$, where the matrices M_i are listed in Table 1.

Proof. Since $B = D = 0$, the Rota-Baxter identity reduces to (11) and (12). Equation (11) means precisely that A is a Rota-Baxter operator of weight zero on $\mathfrak{sl}_2$. Therefore the classification of Pei-Bai-Guo applies.

Let $A = A_i$ be one of the transported normal forms. Write

$$F = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix}$$

with respect to the ordered basis $\{e, f, h\}$. For a fixed A_i , equation (12) is homogeneous and linear in the nine coefficients f_{jk} . Hence it is sufficient to substitute $(x, y) \in \{e, f, h\} \times \{e, f, h\}$. Solving the resulting finite linear systems gives

$$\dim \mathcal{F}_0(A_i) = 9, 1, \text{ or } 2,$$

according to the three cases stated in the theorem. For $A_2 = 0$, equation (12) is identically satisfied; therefore

$$\mathcal{F}_0(A_2) = \text{End}(\mathfrak{sl}_2).$$

By Lemma 2, $\mathbb{C}A_i \subseteq \mathcal{F}_0(A_i)$ for every i . For $i \in \{4, 7, 8, 10, 12, 13, 15, 16, 17, 19, 20\}$, the corresponding linear system has rank eight. Consequently, $\dim \mathcal{F}_0(A_i) = 1$, and therefore $\mathcal{F}_0(A_i) = \mathbb{C}A_i$.

For $i \in \{1, 3, 5, 6, 9, 11, 14, 18, 21, 22\}$, the system has rank seven. Hence $\dim \mathcal{F}_0(A_i) = 2$. Choosing a second generator linearly independent of A_i gives $\mathcal{F}_0(A_i) = \mathbb{C}A_i \oplus \mathbb{C}\widehat{Q}_i$, where the operators $\widehat{Q}_i$ are listed in Table 1. $\square$

Corollary 6. Let $A \neq 0$ be a weight-zero Rota-Baxter operator on $\mathfrak{sl}_2$ belonging to one of the Pei-Bai-Guo normal forms. Then, in the case $B = D = 0$, $1 \leq \dim \mathcal{F}_0(A) \leq 2$.

More precisely, $\dim \mathcal{F}_0(A) = 1$ if and only if every admissible F is proportional to A , whereas $\dim \mathcal{F}_0(A) = 2$ if and only if there exists an admissible operator F which is not proportional to A .

Remark 4. The preceding theorem shows that the extension from $\mathfrak{sl}_2$ to the simple Leibniz algebra $\mathcal{L} = \mathfrak{sl}_2 \ltimes \mathcal{I}$ introduces only a small amount of additional freedom in the case $B = D = 0$. Indeed, for every nonzero Rota-Baxter operator A on $\mathfrak{sl}_2$, the compatible block $C = \theta F$ depends on at most two independent parameters. For eleven normal forms, it is necessarily of the form $C = \lambda \theta A$. For the remaining ten nonzero normal forms, exactly one additional independent direction occurs.

Rota-Baxter operators with $B = 0$ and $D \neq 0$. Let $R = \begin{pmatrix} A & 0 \\ \theta F & \theta S\theta^{-1} \end{pmatrix}$ be a Rota-Baxter operator of weight zero on $\mathcal{L} = \mathfrak{sl}_2 \ltimes \mathcal{I}$, where $A, F, S \in \text{End}(\mathfrak{sl}_2)$, $S \neq 0$.

By Corollary 1, the Rota-Baxter identity is equivalent to

$$[A(x), A(y)] = A([A(x), y] + [x, A(y)]), \quad (13)$$

$$[S(z), A(x)] = S([S(z), x] + [z, A(x)]), \quad (14)$$

$$[F(x), A(y)] = F([A(x), y] + [x, A(y)]) + S([F(x), y]) \quad (15)$$

for all $x, y, z \in \mathfrak{sl}_2$.

Equation (13) means that A is a weight-zero Rota-Baxter operator on $\mathfrak{sl}_2$. Thus, up to the Pei-Bai-Guo classification and the change of convention fixed above, we may assume $A = A_i$, $1 \leq i \leq 22$. For a fixed pair (A, S) , define

$$\mathcal{F}(A, S) = \{F \in \text{End}(\mathfrak{sl}_2) \mid (15) \text{ holds}\}.$$

Theorem 4. Let $R = \begin{pmatrix} A & 0 \\ \theta F & \theta S\theta^{-1} \end{pmatrix}$, $A \neq 0$, $S \neq 0$, be a Rota-Baxter operator of weight zero on $\mathcal{L} = \mathfrak{sl}_2 \ltimes \mathcal{I}$. Let $A = A_i$ be one of the transported Pei-Bai-Guo normal forms. Then the structure of the solution set for S and the dimensions of the corresponding spaces $\mathcal{F}(A_i, S)$ are summarized in Table 2.

Conversely, for every pair (A_i, S) occurring in Table 2 and every $F \in \mathcal{F}(A_i, S)$, the operator $R = \begin{pmatrix} A_i & 0 \\ \theta F & \theta S \theta^{-1} \end{pmatrix}$ is a Rota-Baxter operator of weight zero on $\mathcal{L}$.

Remark 5. The case $A = A_2 = 0$ is not included in Table 2, since it was completely described in Theorem 2.

Proof (Proof of Theorem 4). For $B = 0$, the Rota-Baxter identity is equivalent to (13)–(15). Equation (13) implies that A is a weight-zero Rota-Baxter operator on $\mathfrak{sl}_2$. Therefore, we may use the transported Pei-Bai-Guo normal forms $A = A_i$.

Fix one such A_i . For each fixed A_i , substitution of the basis elements e, f, h into equation (14) yields a finite quadratic system in the entries of S . Solving this system gives the branches listed in Table 2. For every admissible S , equation (15) is a homogeneous linear system in the entries of F , whose nullity gives $\dim \mathcal{F}(A_i, S)$.

Conversely, direct substitution into equations (14) and (15) shows that every listed pair (S, F) satisfies the required identities. This completes the proof. $\square$

Proposition 1. For the family A_{22} put $\Delta = 16ab^3 - 1$. If $\Delta \neq 0$, the corresponding equation for S has four nonzero branches, and the dimensions of the associated spaces $\mathcal{F}(A_{22}, S)$ are 2, 0, 4, 2.

If $\Delta = 0$, then $S = \lambda A_{22}$, $\lambda \in \mathbb{C}^*$. Moreover, $\dim \mathcal{F}(A_{22}, \lambda A_{22}) = \begin{cases} 2, & \lambda \neq -1, \\ 4, & \lambda = -1. \end{cases}$ Thus the hypersurface $16ab^3 = 1$ is an exceptional parameter locus at which the structure of the solution space changes.

Corollary 7. Let $A \neq 0$, $B = 0$, $D \neq 0$. Then, for every admissible pair (A, S) , $\dim \mathcal{F}(A, S) \in \{0, 1, 2, 4\}$. In particular, $\dim \mathcal{F}(A, S) \leq 4$. The maximal value $\dim \mathcal{F}(A, S) = 4$ occurs only on special branches of the classification.

Remark 6. The remaining cases are $A = 0$, $B \neq 0$, and $A \neq 0$, $B \neq 0$. A complete description of Rota-Baxter operators in these cases is left as an open problem.

For $i \in \{1, 3, 5, 6, 9, 11, 14, 18, 21, 22\}$, write

$$\widehat{Q}_i = JM_iJ, \quad J = \text{diag}(1, -1, -1).$$

The matrices M_i are given in Table 1.

Table 1

i	M_i	i	M_i
1	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	11	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & a \end{pmatrix}$
3	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	14	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2a \\ a & 0 & 2a^2 \end{pmatrix}$
5	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	18	$\begin{pmatrix} -2a & 0 & 1 \\ 0 & 0 & 0 \\ -2a^2 & 0 & a \end{pmatrix}$
6	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}$	21	$\begin{pmatrix} 4a^2 & 1 & 0 \\ 16a^4 & 4a^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
9	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	22	$\begin{pmatrix} \frac{1}{-2b} & 0 & 1 \\ 2a & 0 & -4ab \\ -\frac{d}{8b^2} & 0 & \frac{d}{4b} \end{pmatrix}$

where $d = 1 + 16ab^3$.

For each normal form A_i , the admissible operators S are obtained by solving

$$[S(z), A_i(x)] = S([S(z), x] + [z, A_i(x)]), \quad x, z \in \mathfrak{sl}_2.$$

Depending on A_i , the solution set consists either of a one-parameter family or of several distinct branches. To avoid lengthy repetitions of explicit matrices, we record in Table 2 only the number of branches, their parameter dependence, and the dimensions of the corresponding spaces $\mathcal{F}(A_i, S)$.

Table 2

A_i	admissible S	$\dim \mathcal{F}(A_i, S)$
A_1	four solution branches	2, 0, 4, 2
A_3	$S = \lambda A_3, \lambda \neq 0$	2; 4 if $\lambda = -1$
A_4	$S = \lambda A_4, \lambda \neq 0$	1
A_5	$S = \lambda A_5, \lambda \neq 0$	2; 4 if $\lambda = -1$
A_6	four solution branches	0, 4, 2, 2
A_7	$S = \lambda A_7, \lambda \neq 0$	1
A_8	one-parameter family	1
A_9	four solution branches	2, 0, 4, 2
A_{10}	one-parameter family	1
A_{11}	four solution branches	0, 4, 2, 2
A_{12}	one-parameter family	1
A_{13}	$S = \lambda A_{13}, \lambda \neq 0$	1
A_{14}	four solution branches	2, 4, 0, 2
A_{15}	one-parameter family	1

A_i	admissible S	$\dim \mathcal{F}(A_i, S)$
A_{16}	one-parameter family	1
A_{17}	$S = \lambda A_{17}, \lambda \neq 0$	1
A_{18}	three solution branches	2, 0, 2
A_{19}	one-parameter family	1
A_{20}	one-parameter family	1
A_{21}	four solution branches	2, 0, 4, 2
$A_{22}, \Delta \neq 0$	four solution branches	2, 0, 4, 2
$A_{22}, \Delta = 0$	$S = \lambda A_{22}, \lambda \neq 0$	2; 4 if $\lambda = -1$

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A THEOREM ON THE EXTENSION OF A HOLOMORPHIC FUNCTION DEFINED ON AN ALGEBRAIC VARIETY TO AN ENTIRE FUNCTION

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ABSTRACT. This paper investigates the properties of the order and type of holomorphic functions defined on a regular compact set K of an algebraic variety A . Furthermore, we prove a theorem on the holomorphic extension of a function defined on K to an entire function on the whole algebraic variety A .

MSC (2020): 30F20, 30E10, 32E30.

Key words: regular compact set, holomorphic extension, entire function.

Introduction

Let $A \subset \mathbf{C}^N$ be a subset. A is an algebraic manifold of complex dimension n if for every point $a \in A$ there exists an open neighborhood $U \subset \mathbf{C}^N$ and a finite collection of polynomial functions $f_1, f_2, f_3, \dots, f_{N-n} \in \mathbf{C}[z_1, z_2, \dots, z_N]$ such that:

$$A \cap U = \{z \in U : f_1(z) = f_2(z) = f_3(z) = \dots = f_{N-n}(z) = 0\}$$

and the Jacobian matrix $\left[\frac{\partial f_j}{\partial z_k} \right]_{j=1, \dots, N-n, k=1, \dots, N}$.

Algebraic varieties' problem of extending holomorphic functions while taking into account their growth order and type was also studied by A. Zeriahi [1]. In his work, Zeriahi investigated the relationship between the rate of best polynomial approximation of holomorphic functions on affine algebraic varieties and their growth properties, and established a growth version of the Bernstein-Walsh-Siciak theorem. Using these results, he also considered the problem of extending holomorphic functions defined on an algebraic subvariety to entire functions on the ambient complex space while controlling their growth properties.

In contrast to the approach of Zeriahi [1], the present paper focuses not primarily on estimating the rate of best polynomial approximation, but rather on characterizing directly, by means of pluripotential-theoretic tools, the extension properties of holomorphic functions defined on a compact subset of an algebraic variety. In particular, the holomorphic extension problem is investigated in relation to the growth order and type of the function by using extremal functions and their associated sublevel domains. This approach makes it possible to establish a relationship between the domain of holomorphic extension, the geometric properties of the extremal function, and the growth characteristics of the holomorphic function.

Thus, while in Zeriahi [1] the relationship between the rate of polynomial approximation and the growth properties of holomorphic functions serves as one of the principal tools, in the present work the problem of extending holomorphic functions with prescribed order and type is studied directly in terms of extremal functions. This provides a distinct pluripotential-theoretic characterization of the holomorphic extension problem under consideration.

Example 1. The set $A = \{(z_1, z_2) \in \mathbf{C}^2 : z_1^2 + z_2^2 = 1\}$ is a complex algebraic manifold of complex dimension 1.

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In particular, a function f , defined on a complex manifold A is called holomorphic, $f \in \mathcal{O}(A)$, if it is holomorphic in local coordinates, i.e., the functions $f_j(t^j) = f \circ \varphi_j^{-1}(t^j) \in \mathcal{O}(B)$, $j = 1, 2, \dots$. Here, the ball B lies in the space $\mathbf{C}^n$. A function u defined on a complex manifold A is called plurisubharmonic (psh), $u \in psh(A)$, if it is (psh) in local coordinates, $\varphi_j(t^j) : U_j \rightarrow B \subset \mathbf{C}^n$, $j = 1, 2, \dots$ i.e., the functions $u_j(t^j) = u \circ \varphi_j^{-1}(t^j) \in psh(B)$, $j = 1, 2, \dots$

The algebraic manifold A is a parabolic manifold.

An Algebraic manifold is S -parabolic manifold, i.e there exists a special exhaustion function $\rho(z)$ that satisfies the conditions

- a) $\rho(z) \in psh(A)$, $\{z \in A : \rho(z) \leq c\} \subset\subset A \forall c \in \mathbf{R}$;
- b) ρ is maximal plurisubharmonic function on the out of a compact subset $K \subset\subset A$, i.e. $(dd^c \rho)^n = 0$, on $A \setminus K$.

Let us define the concept of polynomials on A algebraic manifold

Definition 1. If for a function $p(z) \in \mathcal{O}(A)$ there exist positive real numbers c, d such that for all $z \in A$ it holds inequality

$$\ln |p(z)| \leq d \cdot \rho^+(z) + c, \quad (5)$$

where $\rho^+(z) = \max\{0, \rho(z)\}$, then the function p is called ρ -polynomial of degree $\leq d$. The minimal value of d in inequality (9) is called the degree of the polynomial p .

We denote by $\mathcal{P}_\rho^d(A)$ the class of all polynomials ρ on algebraic manifold $A \subset \mathbf{C}^N$ with degree $\deg p \leq d$. Let $\mathcal{P}_\rho(A) = \bigcup_{d=1}^{\infty} \mathcal{P}_\rho^d(A)$ is the class of all polynomials on A .

Example 2. Let consider affine algebraic manifold $A \subset \mathbf{C}^N$, $\dim A = n$. By well-known W.Rudin's geometric criterion [2], we can consider (after appropriate affine transformation of coordinates)

$$A \subset \{w = (w', w'') = (w_1, \dots, w_n, w_{n+1}, \dots, w_N) : \|w''\| < C(1 + \|w'\|^k)\}$$

where, C and k are constants. If we put $\rho(w) = \ln \|w'\|$, then restriction $\rho|_A$ is a special exhaustion function for A . Obviously, ρ -polynomials on A are the restriction of ordinary polynomials $p(w', w'')$ to A . Therefore, $\mathcal{P}_\rho(A)$ are dense in $\mathcal{O}(A)$.

Definition 2. A -algebraic manifold is said to be regular if the space of ρ -polynomials $\mathcal{P}_\rho(A)$ is dense in the space of all analytic functions $\mathcal{O}(A)$.

Properties of the Order and Type

Let A be an algebraic variety. It is well known that algebraic varieties constitute a subclass of parabolic varieties. As in the classical theory, the notion of the order of growth of entire functions is introduced on regular parabolic varieties. In this setting, polynomials serve as the principal comparison objects, since among all entire functions they possess the smallest growth rate.

For this purpose, we define the pseudo-balls by

$$B_R = \{z \in A : \rho(z) < \ln R\}, \quad R > 0.$$

For an entire function $f \in \mathcal{O}(A)$, define

$$M(R, f) = \sup\{|f(z)| : z \in \partial B_R\}.$$

Definition 3. The quantity

$$\tau(f) = \limsup_{R \rightarrow \infty} \frac{\ln \ln M(R, f)}{\ln R}$$

is called the *order of growth* of the entire function f .

Clearly,

$$0 \leq \tau(f) \leq +\infty.$$

If $\tau(f) < +\infty$, then the following quantity is defined:

$$\sigma(f) = \limsup_{R \rightarrow \infty} \frac{\ln M(R, f)}{R^{\tau(f)}},$$

which is called the *type* of the entire function f .

Now let $K \subset A$ be a regular compact set. Consider the family of level sets of the extremal function:

$$D_R = \{z \in A : \Phi_A(z, K) < R\}, \quad R > 1,$$

where

$$\Phi_A(z, K) = \sup \left\{ |f(z)|^{1/\deg f} : f \in \mathcal{P}_\rho(A), \|f\|_K \leq 1 \right\}.$$

Furthermore, define

$$M_K(R, f) = \sup\{|f(z)| : z \in \partial D_R\},$$

and

$$\tau_K(f) = \limsup_{R \rightarrow \infty} \frac{\ln \ln M_K(R, f)}{\ln R}.$$

The Green function of B_r relative to A is defined by

$$V_A(z, B_r) = \sup \{u(z) : u \in \mathcal{L}(A), u|_{B_r} \leq 0\}, \quad z \in A,$$

where

$$\mathcal{L}(A) = \{u \in PSH(A) : u(z) \leq \log(1 + \|z\|) + C_u\}$$

is the Lelong class of plurisubharmonic functions on A with logarithmic growth, and C_u is a constant independent of z .

Theorem 1. Let (A, ρ) be an algebraic variety. Then, for every regular compact set $K \subset A$, we have

$$\tau_K(f) = \tau(f).$$

Proof. First, we prove that

$$\tau_K(f) \geq \tau(f).$$

Since K is compact, there exists some $z_0 \in K$ such that

$$r = \rho(z_0).$$

Then $K \subset B_r$. For every polynomial $P_d(z)$ satisfying

$$\|P_d\| \leq 1,$$

we have

$$|P_d(z)|^{1/d} \leq \|P_d\|_{B_r}^{1/d} e^{V_A(z, \overline{B_r})} \leq \sup_{\overline{B_r}} \Phi_A(z, K) e^{\max\{0, \rho(z) - \ln r\}}.$$

Hence,

$$\Phi_A(z, K) \leq L \cdot \max \left\{ 1, \frac{e^{\rho(z)}}{r} \right\}.$$

It follows that, for sufficiently large R ,

$$B_{\frac{Rr}{L}} \subset D_R,$$

and therefore

$$M\left(\frac{Rr}{L}, f\right) \leq M_K(R, f).$$

This proves that

$$\tau_K(f) \geq \tau(f).$$

Now we prove the reverse inequality

$$\tau_K(f) \leq \tau(f).$$

Since $K \subset B_r$, by the properties of the Green function,

$$V_A(z, B_r) \leq V_A(z, K).$$

Consequently,

$$\max\{0, \rho(z) - \ln r\} \leq V_A(z, K) = \ln \Phi_A(z, K).$$

Therefore, consider the set

$$D_{\frac{R}{r}} = \left\{ z \in A : \Phi_A(z, K) \leq \frac{R}{r} \right\}.$$

This set is contained in B_R , since

$$\rho(z) - \ln r \leq \ln \frac{R}{r}$$

implies

$$\rho(z) \leq \ln R.$$

Hence,

$$D_{\frac{R}{r}} \subset B_R.$$

As a result,

$$M_K\left(\frac{R}{r}, f\right) \leq M(R, f).$$

Therefore,

$$\tau_K(f) \leq \tau(f).$$

Combining both inequalities, we conclude that

$$\tau_K(f) = \tau(f).$$

Since this proves that

$$\tau_K(f) \leq \tau(f),$$

combining the two inequalities, we obtain

$$\tau_K(f) = \tau(f).$$

Main part

Definition 4. Let $f \in \mathcal{O}(A)$. If there exists a constant $C > 0$ such that

$$|f(z)| \leq Ce^{\sigma e^{\tau \rho(z)}} \quad \text{for all } z \in A.$$

then the holomorphic function f is said to be of *type* σ and *order* τ on the algebraic variety A .

$e_d(f, K)$ denotes the error of best polynomial approximation of f on K , defined by

$$e_d(f, K) = \inf_{p \in \mathcal{P}_d} \|f - p\|_K,$$

where $\mathcal{P}_d$ denotes the space of polynomials of degree at most d , and

$$\|f - p\|_K = \sup_{z \in K} |f(z) - p(z)|.$$

Theorem 2. Let $f \in \mathcal{O}(A)$ be a function defined on a regular compact set $K \subset A$. Suppose that

$$\overline{\lim}_{d \rightarrow \infty} d^{1/\tau} e_d(f, K)^{1/d} \leq (e\tau\sigma)^{1/\tau}.$$

Then f admits an extension to an entire function on A whose order does not exceed τ and whose type does not exceed σ .

Proof. In this part, we prove that a function f satisfying

$$\overline{\lim}_{d \rightarrow \infty} d^{1/\tau} e_d(f, K)^{1/d} \leq (e\tau\sigma)^{1/\tau}$$

extends to an entire function on A of order at most τ and type at most σ .

To this end, we proceed as follows. In particular, the above inequality also holds in the case

$$\lim_{d \rightarrow \infty} e_d(f, K)^{1/d} = 0.$$

Hence, by the Bernstein–Walsh theorem, the function f admits a holomorphic extension to A .

It remains to prove that

$$\lim_{R \rightarrow \infty} \frac{\ln \ln M_K(R, f)}{\ln R} \leq \tau.$$

Thus, There exists a sequence of polynomials $\{p_d\}$ such that

$$d^{1/\tau} \|f - p_d\|_K^{1/d} \leq (e^{1+\varepsilon}\tau\sigma)^{1/\tau}.$$

By the definition of the upper limit, for every $\varepsilon > 0$, there exists

$d_0 \in \mathbb{N}$ such that for all $d > d_0$,

$$d^{1/\tau} \|f - p_d\|_K^{1/d} \leq (e^{1+\varepsilon}\tau\sigma)^{1/\tau}.$$

Equivalently,

$$\|f - p_d\|_K^{1/d} \leq \left(\frac{e^{1+\varepsilon}\tau\sigma}{d} \right)^{1/\tau}.$$

From this inequality it immediately follows that

$$\|p_{d+1} - p_d\|_K^{1/d} \leq 2 \left(\frac{e^{1+\varepsilon}\tau\sigma}{d} \right)^{1/\tau}.$$

Now consider the series

$$f = p_1 + \sum_{k=1}^{\infty} (p_{k+1} - p_k).$$

Next, we estimate $M_K(R, f)$:

$$\begin{aligned} M_K(R, f) &= \|f\|_{\overline{D_R}} \\ &= \|f - p_d + p_d\|_{\overline{D_R}} \\ &\leq \|f - p_d\|_{\overline{D_R}} + \|p_d\|_{\overline{D_R}} \\ &= \left\| \sum_{k=d}^{\infty} (p_{k+1} - p_k) \right\|_{\overline{D_R}} + \|p_d\|_{\overline{D_R}} \\ &\leq \sum_{k=d}^{\infty} 2 \left(\frac{e^{1+\varepsilon}\tau\sigma R^{\tau}}{k} \right)^{k/\tau} + C R^{td}. \end{aligned}$$

Hence,

$$M_K(R, f) \leq L e^{\sigma R^{\tau}},$$

where L is a positive constant.

This inequality is equivalent to

$$\lim_{R \rightarrow \infty} \frac{\ln \ln M_K(R, f)}{\ln R} \leq \tau.$$

This completes the proof.

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**ANALYTICAL INVESTIGATION OF PURSUIT AND EVASION GAMES FOR SECOND-ORDER
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ABSTRACT. This paper considers a pursuit-evasion differential game where the dynamics of players are governed by second-order parameterized differential equations. This model substantially generalizes the problem formulation by extending the classical inertial motion to oscillating and exponentially unstable dynamic regimes. A family of parallel pursuit strategies (Π -strategy) for the pursuer is constructed, and exact necessary and sufficient conditions for guaranteed capture in a finite time interval are established alongside explicit analytical formulas for optimal capture time. Furthermore, an explicit evasion strategy for the evader is proposed, lower bounds for the distance between players are derived, and conditions for guaranteed evasion over an infinite time horizon are proven. The results obtain generalize and extend classical research on second-order controlled systems by Isaacs, Petrosyan, Pshenichnyi, Azamov, Samatov, and others.

MSC (2020): 49N79; 49N70; 91A24.

Key words: Differential game, Pursuit problem, Evasion problem, Parameterized dynamics, Parallel pursuit strategy, Π -strategy, Guaranteed capture time, Oscillatory motion, Unstable system.

1. Introduction

Differential games theory investigates conflict processes and control problems modeled by systems of differential equations. Following the development of L. S. Pontryagin's Maximum Principle, an intrinsic connection was established between optimal control theory and differential games. Essentially, differential game problems serve as a natural generalization of optimal control problems to multi-agent settings.

The foundation of differential games theory was laid by the American mathematician R. Isaacs [1], whose famous monograph provided significant momentum for the rapid development of the field. Fundamental and modern contributions in this direction have been obtained by L. S. Pontryagin [2], N. N. Krasovskii [3], L. A. Petrosyan [4], B. N. Pshenichnyi, A. A. Chikrii, A. I. Subbotin, and numerous other scholars.

In his monograph, R. Isaacs [1] formulated several special game problems that proved vital for subsequent research. For scenarios with geometric constraints on players' controls, the parallel approach strategy method was extensively analyzed by L. A. Petrosyan [4]. This approach served as a foundational step in establishing norms for multi-player differential games under complex geometric conditions. Subsequently, parallel approach strategies under geometric, integral, and mixed constraints on controls were successfully studied by A. Azamov, B. T. Samatov [5–7], and their students.

At the same time, significant progress has been made in recent years regarding pursuit-evasion and optimal control problems for systems governed by differential equations. Specifically, pursuit-evasion differential games described by first- and second-order differential equations with geometric constraints on controls, as well as optimal control problems for distributed parameter systems, have been thoroughly investigated by N. A. Mamadaliyev and Kh. Ya. Mustapokulov [8–12]. Their works deeply explore the construction of parallel

approach strategies in impulse-controlled and second-order systems, the application of the resolving functions method, and the invariance of set-valued mappings in infinite-dimensional systems.

In the works of B. T. Samatov, U. B. Soyibboev [7], and other aforementioned researchers, the dynamics of the objects were restricted to simple inertial motion regimes.

This paper addresses the general case where players' controls are subject to geometric constraints, and the dynamics of motion are governed by second-order parametric differential equations. The introduction of a dynamic parameter enables the classical inertial motion results studied by B. T. Samatov and U. B. Soyibboev [7] to be fully extended and generalized to oscillatory and exponentially unstable dynamic regimes. A parallel approach strategy that guarantees a guaranteed pursuit time for the pursuer is constructed, and conditions for guaranteed evasion over an infinite time interval along with distance estimates are established for the evader.

2. Statement of problem

Let us assume that in the space $\mathbb{R}^n$, a controlled object called the pursuer P is pursuing another object called the evader E . We denote the position of the pursuer in $\mathbb{R}^n$ by x , and the position of the evader by y .

The dynamics of the players' motion are governed by the following differential equations and initial conditions, respectively:

$$P : \ddot{x} + bx = u, \quad x(0) = x_0, \quad \dot{x}(0) = x_1, \quad (2.1)$$

$$E : \ddot{y} + by = v, \quad y(0) = y_0, \quad \dot{y}(0) = y_1, \quad (2.2)$$

where $x, y, x_0, y_0, x_1, y_1, u, v \in \mathbb{R}^n$, $b \in \mathbb{R}$, $n \geq 2$; x_0 and y_0 are the initial positions of the players, while x_1 and y_1 are their initial velocity vectors. We assume that $x_0 \neq y_0$ and $x_1 = y_1$.

Here, the time-dependent variation of u , represented as $u(\cdot) : [0, \infty) \rightarrow \mathbb{R}^n$, must be a measurable function. We impose a geometric constraint (in short, a G -constraint) on this vector-function in the following form:

$$|u(t)| \leq \alpha \quad \text{for almost all } t \geq 0, \quad (2.3)$$

where α is a given positive parameter representing the maximum acceleration value of the pursuer.

Similarly, the time-dependent variation of v , represented as $v(\cdot) : [0, \infty) \rightarrow \mathbb{R}^n$, must be a measurable function, and we impose the following G -constraint on this vector-function:

$$|v(t)| \leq \beta \quad \text{for almost all } t \geq 0, \quad (2.4)$$

where β is a given non-negative parameter representing the maximum acceleration value of the evader.

The control parameters of the players are the acceleration vectors u and v , which depend on time ($t \geq 0$). We denote the sets of all control parameters u and v satisfying conditions (2.3) and (2.4), respectively, by U and V .

Definition 1. A measurable function $u(\cdot)$ ($v(\cdot)$) satisfying condition (2.3) (condition (2.4)) is called an admissible control of the pursuer (evader) in the class U (V), where the introduced pair of admissible control classes (U, V) defines a differential game.

Integrating the differential equations (2.1) and (2.2) under the corresponding initial conditions yields the trajectories of the pursuer P and the evader E , expressed analytically depending on the value of the parameter b :

Case 1 (for $b > 0$ – harmonic oscillatory motion):

$$x(t) = x_0 \cos(\sqrt{b}t) + \frac{x_1}{\sqrt{b}} \sin(\sqrt{b}t) + \frac{1}{\sqrt{b}} \int_0^t \sin(\sqrt{b}(t-s))u(s) ds, \quad (2.5a)$$

$$y(t) = y_0 \cos(\sqrt{b}t) + \frac{y_1}{\sqrt{b}} \sin(\sqrt{b}t) + \frac{1}{\sqrt{b}} \int_0^t \sin(\sqrt{b}(t-s))v(s) ds. \quad (2.6a)$$

Case 2 (for $b = 0$ – classical inertial motion):

$$x(t) = x_0 + x_1 t + \int_0^t (t-s)u(s) ds, \quad (2.5b)$$

$$y(t) = y_0 + y_1 t + \int_0^t (t-s)v(s) ds. \quad (2.6b)$$

Case 3 (for $b < 0$ – exponentially unstable motion):

$$x(t) = x_0 \cosh(\sqrt{-b}t) + \frac{x_1}{\sqrt{-b}} \sinh(\sqrt{-b}t) + \frac{1}{\sqrt{-b}} \int_0^t \sinh(\sqrt{-b}(t-s))u(s) ds, \quad (2.5c)$$

$$y(t) = y_0 \cosh(\sqrt{-b}t) + \frac{y_1}{\sqrt{-b}} \sinh(\sqrt{-b}t) + \frac{1}{\sqrt{-b}} \int_0^t \sinh(\sqrt{-b}(t-s))v(s) ds. \quad (2.6c)$$

It is well known that control functions depending solely on the time parameter $t \geq 0$ are insufficient for the pursuer to solve the pursuit problem; appropriate forms of control must be formulated as strategies. Several methods exist for defining such concepts. Below, we introduce some notions to formalize these definitions.

First, we introduce the following notations:

$$z(t) = x(t) - y(t), \quad z_0 = x_0 - y_0, \quad \dot{z}(0) = x_1 - y_1.$$

Definition 2. A function $\mathbf{u} : \mathbb{R}^n \times S_\beta \rightarrow S_\alpha$ is called a strategy of the pursuer if the following conditions are satisfied:

- (a) The function $\mathbf{u}(z_0, v)$ is measurable with respect to $v \in V$ in the sense of Borel;
- (b) Admissibility: For every $v(\cdot) \in V$, the inclusion $\mathbf{u}(z_0, v(\cdot)) \in U$ holds on some time interval $[0, t]$. In this case, the function $\mathbf{u}(z_0, v(\cdot))$ is called a realization of the strategy $\mathbf{u}(z_0, v)$;
- (c) Volterra property: If $v_1(s) = v_2(s)$ for almost all $s \in [0, t]$ for any $v_1(\cdot), v_2(\cdot) \in V$ and $t \geq 0$, then $u_1(s) = u_2(s)$ for almost all $s \in [0, t]$, where $u_i(\cdot) = \mathbf{u}(z_0, v_i(\cdot))$, $i = 1, 2$; S_α and S_β are closed balls in $\mathbb{R}^n$ centered at the origin with radii α and β , respectively.

Definition 3. A strategy $\mathbf{u}(z_0, v)$ is called a parallel approach strategy (Π -strategy) if, for any $v(\cdot) \in V$, the solution $z(t)$ to the Cauchy problem

$$\ddot{z} + bz = \mathbf{u}(z_0, v(t)) - v(t), \quad z(0) = z_0, \quad \dot{z}(0) = 0 \quad (2.7)$$

can be represented in the form:

$$z(t) = z_0 \Lambda(t, v(\cdot)), \quad \Lambda(0, v(\cdot)) = 1,$$

where $\Lambda(t, v(\cdot))$ is a scalar function of time $t \geq 0$. Typically, this function is referred to as the approach function in pursuit problems.

Definition 4. In the pursuit problem, a Π -strategy is said to guarantee capture of the evader within the time interval $[0, t_g]$ if, for every $v(\cdot) \in V$, the following conditions hold:

- (a) There exists some time $t^* \in [0, t_g]$ such that $z(t^*) = 0$;
- (b) The inclusion $\mathbf{u}(z_0, v(\cdot)) \in U$ holds on the time interval $[0, t^*]$. Here, t_g is called the guaranteed pursuit (or capture) time.

Definition 5. A function $\mathbf{v}^* : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is called a strategy of the evader if $\mathbf{v}^*(t)$ is measurable with respect to t in the sense of Lebesgue.

Now, let us consider the (U, V) game from the evader's perspective.

Definition 6. In the evasion problem, a control $\mathbf{v}^*(\cdot) \in V$ is said to guarantee evasion if, for any $u(\cdot) \in U$, the solution $z(t)$ to the Cauchy problem

$$\ddot{z} + bz = u(t) - \mathbf{v}^*(t), \quad z(0) = z_0, \quad \dot{z}(0) = 0 \quad (2.8)$$

remains non-zero, i.e., $z(t) \neq 0$ for all $t \geq 0$.

This paper is devoted to solving the following problems under the conditions where the controls of the players are subject to constraints (2.3) and (2.4), respectively:

Problem 1. Pursuit problem: To construct a Π -strategy for the pursuer and determine the guaranteed capture time in the (U, V) game.

Problem 2. Evasion problem: To construct a strategy for the evader and estimate how the distance between the players changes over time.

3. A solution of the pursuit problem

In parametric mathematical problems, an interesting property of the final analytical results is their explicit dependence on these parameters, which are treated as constants in the solution. However, these parameters can assist in determining the conditions for the solvability (feasibility) of these problems. In this section, we present necessary and sufficient conditions for the pursuit problem to be solvable in the (U, V) game.

If the pursuer and the evader choose admissible controls $u(\cdot) \in U$ and $v(\cdot) \in V$, respectively, then by integrating the differential equation of relative motion corresponding to equations (2.7) and (2.8), the relative trajectory $z(t) = x(t) - y(t)$ is expressed in the following analytical forms depending on the value of the parameter b :

Case 1 (for $b > 0$ – harmonic oscillatory motion):

$$z(t) = z_0 \cos(\sqrt{b}t) + \frac{1}{\sqrt{b}} \int_0^t \sin(\sqrt{b}(t-s))(u(s) - v(s)) ds. \quad (3.1a)$$

Case 2 (for $b = 0$ – classical inertial motion):

$$z(t) = z_0 + \int_0^t (t-s)(u(s) - v(s)) ds. \quad (3.1b)$$

Case 3 (for $b < 0$ – exponentially unstable motion):

$$z(t) = z_0 \cosh(\sqrt{-b}t) + \frac{1}{\sqrt{-b}} \int_0^t \sinh(\sqrt{-b}(t-s))(u(s) - v(s)) ds. \quad (3.1c)$$

In terms of the newly introduced notations, the objective of the pursuer is to achieve the equality $z(t^*) = 0$ at some time $t^* > 0$. The objective of the evader is to maintain the relation $z(t) \neq 0$ for all $t \geq 0$.

For the pursuer, open-loop strategies (i.e., admissible controls depending solely on time t) are insufficient to achieve their goal. Therefore, similar to work [6], the pursuer's strategy in the present setting can be constructed based exclusively on the current values of the evader's acceleration function $v(t)$, $t \geq 0$, alongside the given constants z_0 and α .

To solve the pursuit problem, let us assume that at the current time t , the pursuer is aware of the initial parameters x_0, y_0, x_1, y_1 , the constants α, β , the current time t , and the value of the evader's control $v(t)$.

Definition 7. Suppose that $\alpha \geq \beta$. Then, in the (U, V) game, the function

$$\mathbf{u}(z_0, v) = v - \lambda(z_0, v)\xi_0 \quad (3.2)$$

is called the parallel pursuit strategy (in short, Π -strategy) of the pursuer, where

$$\lambda(z_0, v) = \langle v, \xi_0 \rangle + \sqrt{\langle v, \xi_0 \rangle^2 + \alpha^2 - |v|^2}, \quad \xi_0 = \frac{z_0}{|z_0|} \quad (3.3)$$

and $\langle v, \xi_0 \rangle$ denotes the inner product of the vectors v and ξ_0 in the space $\mathbb{R}^n$. The function $\lambda(z_0, v)$ is typically called the resolving function.

Now, we highlight several important properties of the strategy (3.2) and the resolving function (3.3).

Lemma 1. The strategy (3.2) is well-defined and continuous for all $v \in S_\beta$, and during the pursuit game, the equality $|\mathbf{u}(z_0, v)| = \alpha$ holds.

Lemma 2. The resolving function (3.3) is well-defined, continuous, and non-negative for all $v \in S_\beta$, and satisfies the following bounds:

$$\alpha - \beta \leq \lambda(z_0, v) \leq \alpha + \beta.$$

Definition 8. Depending on the value of the parameter $b \in \mathbb{R}$, let one of the following conditions hold: (i) for $b > 0$: $\alpha \geq \beta$; (ii) for $b = 0$: $\alpha > \beta$; (iii) for $b < 0$ ($k = \sqrt{-b} > 0$): $\alpha - \beta > k^2|z_0|$.

Then, in the (U, V) game, the approach function of the players $\Lambda(t, v(\cdot))$ is defined as the scalar function corresponding to the value of the parameter b as follows:

$$\Lambda(t, v(\cdot)) = \begin{cases} \cos(\sqrt{b}t) - \frac{1}{|z_0|\sqrt{b}} \int_0^t \sin(\sqrt{b}(t-s))\lambda(z_0, v(s)) ds, & b > 0, \\ 1 - \frac{1}{|z_0|} \int_0^t (t-s)\lambda(z_0, v(s)) ds, & b = 0, \\ \cosh(kt) - \frac{1}{|z_0|k} \int_0^t \sinh(k(t-s))\lambda(z_0, v(s)) ds, & b < 0, (k = \sqrt{-b}). \end{cases} \quad (3.4)$$

Lemma 3. Under the correspondig parameter conditions specified in Definition 8, the following assertions hold for all $v(\cdot) \in V$:

- (a) The function $\Lambda(t, v(\cdot))$ is monotonically decreasing with respect to t on the corresponding time intervals;
- (b) The function $\Lambda(t, v(\cdot))$ satisfies the following two-sided estimate:

$$\Lambda_1(t) \leq \Lambda(t, v(\cdot)) \leq \Lambda_2(t), \quad (3.5)$$

where the functions $\Lambda_1(t)$ and $\Lambda_2(t)$ are defined depending on the value of the parameter b as follows:

Case 1 (for $b > 0$, on the interval $t \in [0, \frac{\pi}{2\sqrt{b}}]$):

$$\Lambda_1(t) = \cos(\sqrt{b}t) - \frac{\alpha + \beta}{b|z_0|} (1 - \cos(\sqrt{b}t)), \quad \Lambda_2(t) = \cos(\sqrt{b}t) - \frac{\alpha - \beta}{b|z_0|} (1 - \cos(\sqrt{b}t)).$$

Case 2 (for $b = 0$, for all $t \geq 0$):

$$\Lambda_1(t) = 1 - \frac{\alpha + \beta}{2|z_0|} t^2, \quad \Lambda_2(t) = 1 - \frac{\alpha - \beta}{2|z_0|} t^2.$$

Case 3 (for $b < 0$, $k = \sqrt{-b} > 0$, for all $t \geq 0$):

$$\Lambda_1(t) = \cosh(kt) - \frac{\alpha + \beta}{k^2|z_0|} (\cosh(kt) - 1), \quad \Lambda_2(t) = \cosh(kt) - \frac{\alpha - \beta}{k^2|z_0|} (\cosh(kt) - 1).$$

Proof. (a) *Proof of Monotonicity:* We apply Leibniz's rule for differentiation under the integral sign. We take into account the inequality $\lambda(z_0, v(s)) \geq \alpha - \beta > 0$ from Lemma 2 and the signs of the corresponding integrand functions:

$$\textbf{Case 1} (b > 0): \Lambda(t, v(\cdot)) = \cos(\sqrt{b}t) - \frac{1}{|z_0|\sqrt{b}} \int_0^t \sin(\sqrt{b}(t-s))\lambda(z_0, v(s)) ds.$$

Taking the derivative with respect to t :

$$\frac{d\Lambda}{dt} = -\sqrt{b}\sin(\sqrt{b}t) - \frac{1}{|z_0|} \int_0^t \cos(\sqrt{b}(t-s))\lambda(z_0, v(s)) ds.$$

Since $\sin(\sqrt{b}t) > 0$ on the interval $t \in (0, \frac{\pi}{2\sqrt{b}}]$ and for all $s \in [0, t]$, $\cos(\sqrt{b}(t-s)) \geq 0$ as well as $\lambda(z_0, v(s)) > 0$, both terms of the derivative are negative. Thus, $\frac{d\Lambda}{dt} < 0$.

$$\textbf{Case 2} (b = 0): \Lambda(t, v(\cdot)) = 1 - \frac{1}{|z_0|} \int_0^t (t-s)\lambda(z_0, v(s)) ds.$$

Taking the derivative with respect to t :

$$\frac{d\Lambda}{dt} = -\frac{1}{|z_0|} \int_0^t \lambda(z_0, v(s)) ds \leq -\frac{\alpha - \beta}{|z_0|} t < 0 \quad (\forall t > 0).$$

$$\textbf{Case 3} (b < 0, k = \sqrt{-b} > 0): \Lambda(t, v(\cdot)) = \cosh(kt) - \frac{1}{|z_0|k} \int_0^t \sinh(k(t-s))\lambda(z_0, v(s)) ds.$$

Using the inequality $\lambda(z_0, v(s)) \geq \alpha - \beta$ and taking the derivative:

$$\frac{d\Lambda}{dt} = k \sinh(kt) - \frac{1}{|z_0|} \int_0^t \cosh(k(t-s)) \lambda(z_0, v(s)) ds \leq k \sinh(kt) - \frac{\alpha - \beta}{|z_0|k} \sinh(kt).$$

Since $\alpha - \beta > k^2|z_0|$ (i.e., $\frac{\alpha - \beta}{k^2|z_0|} > 1$) by condition:

$$\frac{d\Lambda}{dt} \leq k \sinh(kt) \left(1 - \frac{\alpha - \beta}{k^2|z_0|}\right) < 0 \quad (\forall t > 0).$$

Thus, in all cases, $\Lambda(t, v(\cdot))$ is strictly decreasing with respect to time t .

(b) *Proof of Two-Sided Estimate:* According to Lemma 2, the inequality $\alpha - \beta \leq \lambda(z_0, v(s)) \leq \alpha + \beta$ holds for all $s \geq 0$.

On the interval $s \in [0, t]$, the kernel terms satisfy $\sin(\sqrt{b}(t-s)) \geq 0$ for $b > 0$, $(t-s) \geq 0$ for $b = 0$, and $\sinh(k(t-s)) \geq 0$ for $b < 0$.

Integrating the inequalities and bounding the terms from above and below respectively:

$$\int_0^t K(t, s)(\alpha - \beta) ds \leq \int_0^t K(t, s)\lambda(z_0, v(s)) ds \leq \int_0^t K(t, s)(\alpha + \beta) ds$$

we obtain the corresponding relations, where $K(t, s)$ represents the respective kernel functions.

Due to the negative sign preceding the integral in $\Lambda(t, v(\cdot))$, applying the lower bound $\alpha - \beta$ yields the upper estimate $\Lambda_2(t)$, whereas applying the upper bound $\alpha + \beta$ yields the lower estimate $\Lambda_1(t)$.

Evaluating the integrals algebraically completes the proof of the two-sided estimate (3.5). $\square$

Theorem 1. Let the players' control resources satisfy $\alpha > \beta > 0$ with the initial state $z_0 \neq 0$, and in the case of $b < 0$, let the condition $\alpha - \beta > k^2|z_0|$ ($k = \sqrt{-b}$) hold. Then, the Π -strategy guarantees the capture of the evader within the time interval $[0, t_g]$, where the guaranteed capture time t_g is determined as follows:

$$t_g = \begin{cases} \frac{1}{\sqrt{b}} \arccos\left(\frac{\alpha - \beta}{\alpha - \beta + b|z_0|}\right), & b > 0, \\ \sqrt{\frac{2|z_0|}{\alpha - \beta}}, & b = 0, \\ \frac{1}{k} \ln\left(\frac{\alpha - \beta + k\sqrt{|z_0|(2(\alpha - \beta) - k^2|z_0|)}}{\alpha - \beta - k^2|z_0|}\right), & b < 0. \end{cases} \quad (3.6)$$

Proof. When the Π -strategy is applied for all admissible controls $v(\cdot) \in V$, the trajectory of relative motion has the form $z(t) = z_0\Lambda(t, v(\cdot))$.

According to Lemma 3, the upper estimate $\Lambda(t, v(\cdot)) \leq \Lambda_2(t)$ holds for all $v(\cdot) \in V$. To determine the capture time t_g , we solve the equation $\Lambda_2(t_g) = 0$ separately for each case of the parameter b :

Case 1 ($b > 0$):

$$\begin{aligned} \cos(\sqrt{b}t_g) - \frac{\alpha - \beta}{b|z_0|}(1 - \cos(\sqrt{b}t_g)) &= 0 \implies \cos(\sqrt{b}t_g) = \frac{\alpha - \beta}{\alpha - \beta + b|z_0|} \\ \implies t_g &= \frac{1}{\sqrt{b}} \arccos\left(\frac{\alpha - \beta}{\alpha - \beta + b|z_0|}\right). \end{aligned}$$

Case 2 ($b = 0$):

$$1 - \frac{\alpha - \beta}{2|z_0|}t_g^2 = 0 \implies t_g = \sqrt{\frac{2|z_0|}{\alpha - \beta}}.$$

Case 3 ($b < 0$, $k = \sqrt{-b}$):

$$\cosh(kt_g) - \frac{\alpha - \beta}{k^2|z_0|}(\cosh(kt_g) - 1) = 0 \implies \cosh(kt_g) = \frac{\alpha - \beta}{\alpha - \beta - k^2|z_0|}.$$

We apply the identity $\operatorname{arcosh}(x) = \ln(x + \sqrt{x^2 - 1})$ for the inverse hyperbolic function. Taking $x = \frac{\alpha - \beta}{\alpha - \beta - k^2|z_0|} > 1$:

$$x^2 - 1 = \frac{k^2|z_0|(2(\alpha - \beta) - k^2|z_0|)}{(\alpha - \beta - k^2|z_0|)^2}.$$

Substituting this expression into the logarithm and simplifying yields:

$$t_g = \frac{1}{k} \ln \left(\frac{\alpha - \beta + k\sqrt{|z_0|(2(\alpha - \beta) - k^2|z_0|)}}{\alpha - \beta - k^2|z_0|} \right).$$

Since initially $\Lambda(0, v(\cdot)) = 1 > 0$, and at the time instant t_g we have $\Lambda(t_g, v(\cdot)) \leq \Lambda_2(t_g) = 0$, by the continuity of the function $\Lambda(t, v(\cdot))$ with respect to time (Bolzano–Cauchy theorem), there exists some $t^* \in (0, t_g]$ such that $\Lambda(t^*, v(\cdot)) = 0$, which implies $z(t^*) = 0$ and capture is accomplished. $\square$

4. Solution of the Evasion Problem

In this section, we propose an admissible strategy for the evader that guarantees evasion (avoiding capture) in the evasion game. Using this strategy, we prove that strategy (3.2) is the optimal pursuit strategy and t_g is the optimal pursuit time.

Definition 9. In the (U, V) game, we call the following control function of the evader:

$$\mathbf{v}^*(t) = -\beta\xi_0 \quad (4.1)$$

his strategy.

Definition 10. A strategy $\mathbf{v}^*(t)$ is said to guarantee evasion on the time interval $[0, t_g]$ if for any control function $u(\cdot) \in U$ of the pursuer, the relation $z(t) \neq 0$ holds for all $t \in [0, t_g]$, where $z(t)$ is the solution of the Cauchy problem (2.8).

Theorem 2. (a) When $\alpha > \beta > 0$ (and $\alpha - \beta > k^2|z_0|$ in the case $b < 0$), strategy (4.1) guarantees evasion on the time interval $[0, t_g]$, where t_g is the optimal pursuit time given in Theorem 1.

(b) Evasion on the infinite time interval $[0, +\infty)$ ($z(t) \neq 0, \forall t \geq 0$) is guaranteed by strategy (4.1) if one of the following conditions holds: for $b = 0$ or $b < 0$: the condition $\alpha \leq \beta$ is satisfied; for $b > 0$: the condition $\beta - \alpha \geq b|z_0|$ is satisfied.

In these cases, the distance between the players is bounded from below as follows:

$$|z(t)| \geq \begin{cases} |z_0| \cos(\sqrt{b}t) + \frac{\beta - \alpha}{b}(1 - \cos(\sqrt{b}t)) \geq |z_0|, & b > 0, \\ |z_0| + \frac{\beta - \alpha}{2}t^2 \geq |z_0|, & b = 0, \\ |z_0| \cosh(kt) + \frac{\beta - \alpha}{k^2}(\cosh(kt) - 1) \geq |z_0|, & b < 0, (k = \sqrt{-b}). \end{cases} \quad (4.2)$$

Proof. (a) When the evader applies strategy (4.1) ($v^*(t) = -\beta\xi_0$), for an arbitrary control $u(\cdot) \in U$ ($|u(t)| \leq \alpha$) of the pursuer, the magnitude of the relative trajectory is bounded from below as follows:

Case 1 ($b > 0$):

$$|z(t)| \geq |z_0| \cos(\sqrt{b}t) - \frac{\alpha - \beta}{b}(1 - \cos(\sqrt{b}t)).$$

From the condition $t < t_g = \frac{1}{\sqrt{b}} \arccos\left(\frac{\alpha - \beta}{\alpha - \beta + b|z_0|}\right)$, we have $\cos(\sqrt{b}t) > \frac{\alpha - \beta}{\alpha - \beta + b|z_0|}$, which guarantees $|z(t)| > 0$.

Case 2 ($b = 0$):

$$|z(t)| \geq \left| \xi_0 \left(|z_0| + \frac{\beta t^2}{2} \right) \right| - \int_0^t (t-s) |u(s)| ds \geq |z_0| - \frac{(\alpha - \beta)t^2}{2}.$$

Since $t < t_g = \sqrt{\frac{2|z_0|}{\alpha - \beta}}$, it follows that $|z(t)| > 0$ for all $t \in [0, t_g]$.

Case 3 ($b < 0$, $k = \sqrt{-b}$):

$$|z(t)| \geq |z_0| \cosh(kt) - \frac{\alpha - \beta}{k^2} (\cosh(kt) - 1).$$

For $t < t_g$, since $\cosh(kt) < \frac{\alpha - \beta}{\alpha - \beta - k^2|z_0|}$, we obtain $|z(t)| > 0$.

(b) Next, we verify evasion on the infinite interval $[0, +\infty)$:

For $b = 0$ and $b < 0$: Since $\alpha \leq \beta \implies \beta - \alpha \geq 0$, the second terms in the estimates are non-negative. Consequently, $|z(t)| \geq |z_0| > 0$ holds for all $t \geq 0$.

For $b > 0$: We rewrite the estimate in the following form:

$$|z(t)| \geq \frac{\beta - \alpha}{b} + \left(|z_0| - \frac{\beta - \alpha}{b} \right) \cos(\sqrt{b}t).$$

From the given condition $\beta - \alpha \geq b|z_0|$, it follows that $|z_0| - \frac{\beta - \alpha}{b} \leq 0$. Taking into account that $\cos(\sqrt{b}t) \leq 1$, we get:

$$|z(t)| \geq \frac{\beta - \alpha}{b} + \left(|z_0| - \frac{\beta - \alpha}{b} \right) \cdot 1 = |z_0| > 0, \quad \forall t \in [0, +\infty).$$

Thus, under the specified conditions, $z(t) \neq 0$ holds for all $t \geq 0$, which guarantees evasion on the infinite time interval. $\square$

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ON THE SOLVABILITY OF A FORWARD NONLOCAL PROBLEM FOR A NONHOMOGENEOUS BOUSSINESQ-TYPE EQUATION

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ABSTRACT. In this paper, we study a direct nonlocal problem for a nonhomogeneous Boussinesq-type equation in a bounded multidimensional domain. The problem is considered under homogeneous boundary conditions and a weighted integral condition involving the time variable. Using the Fourier method and spectral expansion, we derive an explicit form of the solution. Sufficient conditions for the existence and uniqueness of the solution are obtained, and the convergence of the corresponding series is established.

MSC (2020): 35A02; 35M10; 35S05.

Key words: Boussinesq-type equation, nonlocal condition, existence and uniqueness, spectral method.

Introduction

Let $\Omega \subset \mathbb{R}^n$ be an arbitrary bounded domain with piecewise smooth boundary $\partial\Omega$ and $T > 0$. Consider in the cylinder $\Omega \times (0, T)$ the following equation

$$u_{tt} - \Delta u_{tt} + \nu^2 \Delta u = f(x), \quad x \in \Omega, \quad 0 < t < T, \quad (1)$$

where parameter ν is a positive number.

Let A be a positive self-adjoint extension of Laplace operator $-\Delta$: $Au = -\Delta u$, $u \in C_0^\infty(\Omega)$.

$$(Au, u) \geq \mu(u, u), \quad \mu > 0, \quad u \in D(A)$$

We assume that A has a compact inverse operator A^{-1} . In particular, this self-adjoint extension may be produced by the boundary conditions like

$$\alpha(x) \frac{\partial u}{\partial n} + \beta(x) u = 0, \quad x \in \partial\Omega$$

with some appropriate $\alpha(x)$ and $\beta(x)$.

We consider non-local conditions

$$u(x, 0) = u(x, T), \quad (2)$$

$$\int_0^T t u(x, t) dt = \varphi(x) \quad (3)$$

Equations of the form (1) are known as linearized Boussinesq-type equations (see [1, Eq. (26)]). Such equations arise in the mathematical modeling of various physical processes and therefore the study of their solvability is of particular interest. Initial and boundary value problems for Boussinesq-type equations, including nonlinear and nonhomogeneous cases, have been investigated in many works (see, for example, [2-9]).

A number of papers are devoted to the study of nonlocal problems for Boussinesq-type equations, where conditions for the existence and uniqueness of solutions were established by means of spectral and Fourier methods.

In [3], a nonlocal problem for a Boussinesq-type differential equation was studied, and the existence and uniqueness of the solution were proved by the Fourier method. The approach used in our paper is close to that work. However, the nonlocal condition considered here is different, since it is given as integral condition with respect to time and contains the factor t . Because of this, the relations for the Fourier coefficients take a different form, which in turn changes the solvability conditions of the problem.

Similar problems were considered in [7] and [10] by Khalmukhamedov A. R. and Kuchkorov E. I. Our study is also methodologically related to these works, especially in the use of spectral decomposition and the Fourier method. The main difference is that the term containing ν^2 appears with the opposite sign in the equation considered here. This change leads to a different form of the corresponding scalar equations and affects the spectral conditions that arise in the analysis of solvability.

In the present paper, we study a direct nonlocal problem for a nonhomogeneous Boussinesq-type equation subject to time-integral condition. The presence of the factor t in the integral condition, together with the changed sign of the ν^2 -term, distinguishes the present problem from those studied earlier. Due to the operator taken with the opposite sign, the form of the solution and the solution method changes. Using the Fourier method, we obtain an explicit representation of the solution and establish sufficient conditions for its existence and uniqueness.

Definition 1. We say that a function $u(x, t)$ is a solution to the non-local problem (1), (2) and (3) if the following conditions are satisfied:

- 1) $u(\cdot, t) \in D(A)$ for all $t \in (0, T)$, and the function $u(x, t)$ is continuous with respect to t in the norm of $L_2(\Omega)$;
- 2) the function $u(x, t)$ and $Au(x, t)$ are twice continuously differentiable with respect to t on the open interval $(0, T)$ in the $L_2(\Omega)$ -norm;
- 3) the function $u(x, t)$ satisfies equation (1) and the non-local conditions (2) and (3).

In order to formulate the results on existence and uniqueness we introduce some important parameters. Let λ_k and $v_k(x)$ be the eigenvalues and corresponding eigenfunctions of the self-adjoint operator A :

$$Av_k(x) = \lambda_k v_k(x), \quad x \in \Omega.$$

The spectral decomposition of any function $\varphi \in L_2(\Omega)$, $f \in L_2(\Omega)$ has the form

$$\varphi(x) = \sum_{k=1}^{\infty} \varphi_k v_k(x), \quad \varphi_k = (\varphi, v_k).$$

$$f(x) = \sum_{k=1}^{\infty} f_k v_k(x), \quad f_k = (f, v_k).$$

According to our assumptions, $\mu \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \rightarrow \infty$. Set

$$\nu_k = \nu \sqrt{\frac{\lambda_k}{1 + \lambda_k}}. \quad (4)$$

Now, we introduce the following set:

$$\Lambda_0(\nu) := \{\lambda_k \in SpA : \varphi_k + \frac{T^2}{2\nu^2 \lambda_k} f_k = 0\}.$$

Consider a linear space $L(\nu)$ formed by eigenfunctions corresponding to the eigenvalues from the $\Lambda_0(\nu)$:

$$L(\nu) = \{h | h(x) = \sum_{k=1}^{\infty} h_k v_k(x)\}.$$

In this paper, the validity of the following statement is proved.

Theorem 1. Let $\varphi \in D(A)$ and $f \in L_2(\Omega)$. Then the solution of problem (1)-(3) exists and is unique.

Preliminaries

In this section, we determine the solution to the problem using the spectral method and derive the general form of the solution.

Firstly, we can rewrite equation (1):

$$u_{tt} + Au_{tt} - \nu^2 Au = f(x), \quad x \in \Omega, \quad 0 < t < T. \quad (5)$$

Assume that $u(x, t)$ is solution of problem (5), (2) and (3). Then, we reduce problem (1)-(3) to a homogeneous problem:

$$u(x, t) = \omega(x, t) - \frac{1}{\nu^2} A^{-1} f(x).$$

In addition, the spectral decomposition of solution has the form

$$u(x, t) = \sum_{k=1}^{\infty} u_k(t) v_k(x), \quad u_k(t) = (u, v_k);$$

$$\omega(x, t) = \sum_{k=1}^{\infty} \omega_k(t) v_k(x), \quad \omega_k = (\omega, v_k).$$

The function $\omega(x, t)$ is a solution to the following equation

$$\omega_{tt} + A\omega_{tt} - \nu^2 A\omega = 0, \quad x \in \Omega, \quad 0 < t < T. \quad (6)$$

satisfying the nonlocal conditions.

Applying the $(I + A)^{-1}$ operator to both sides of equation (6) and we get

$$\omega_{tt} - \nu^2 (I + A)^{-1} A\omega = 0.$$

Recall that the operator A is self-adjoint and we can write

$$\left((I + A)^{-1} A\omega, v_k \right) = \left(\omega, A(I + A)^{-1} v_k \right) = \left(\omega, \frac{\lambda_k}{1 + \lambda_k} v_k \right) = \frac{\nu_k^2}{\nu^2} \omega_k.$$

Therefore, it follows from the above equations that

$$\begin{cases} \omega_k''(t) - \nu_k^2 \omega_k(t) = 0, & 0 < t < T, \\ \omega_k(0) = \omega_k(T), \\ \int_0^T t \omega_k(t) dt = \varphi_k + \frac{T^2}{2\nu^2 \lambda_k} f_k, \end{cases} \quad (7)$$

is valid, where $k \in N$. Hence, $\omega_k(t) = a_k \operatorname{ch} \nu_k t + b_k \operatorname{sh} \nu_k t$.

We add to this equation conditions of (7) that follow this determinant

$$\begin{cases} (1 - \operatorname{ch} \nu_k T) a_k + b_k \operatorname{sh} \nu_k T = 0 \\ a_k \operatorname{sh} \nu_k T + b_k \operatorname{ch} \nu_k T = \frac{\nu_k}{T} \phi_k \end{cases} \quad (8)$$

We note that $\begin{vmatrix} 1 - \operatorname{ch} \nu_k T & -\operatorname{sh} \nu_k T \\ \operatorname{sh} \nu_k T & \operatorname{ch} \nu_k T \end{vmatrix} = 2 \operatorname{sh}^2 \frac{\nu_k T}{2} \neq 0$, there we can determine $\phi_k = \varphi_k + \frac{T^2}{2\nu^2 \lambda_k} f_k$.

Finding the coefficients a_k and b_k from system (8), we write the form of each function $\omega_k(t)$ as follows:

$$\begin{aligned} \omega_k(t) &= a_k \operatorname{ch} \nu_k t + b_k \operatorname{sh} \nu_k t = \frac{\nu_k}{T} \phi_k \left[\frac{\operatorname{ch} \nu_k T / 2}{\operatorname{sh} \nu_k T / 2} \operatorname{ch} \nu_k t - \operatorname{sh} \nu_k t \right] = \\ &= \frac{\nu_k \phi_k}{T \operatorname{sh} \nu_k T / 2} \operatorname{ch} [\nu_k (t - T/2)]. \end{aligned}$$

Thus,

$$\omega(x, t) = \sum_{\lambda_k \notin \Lambda_0(\nu)} \frac{\nu_k}{T \operatorname{sh} \nu_k T/2} \operatorname{ch}[\nu_k(t - T/2)] \left(\varphi_k + \frac{T^2}{2\nu^2 \lambda_k} f_k \right)$$

Hence, if a solution to the problem (1)-(3) exists, then it has the following form:

$$u(x, t) = \sum_{\lambda_k \notin \Lambda_0(\nu)} \left[\frac{\nu_k}{T \operatorname{sh} \nu_k T/2} \operatorname{ch}[\nu_k(t - T/2)] \left(\varphi_k + \frac{T^2}{2\nu^2 \lambda_k} f_k \right) \right] v_k(x) - \sum_{\lambda_k \in \Lambda_0(\nu)} \frac{1}{\nu^2 \lambda_k} f_k v_k(x) \quad (9)$$

or

$$u(x, t) = \sum_{\lambda_k \notin \Lambda_0(\nu)} \left[\frac{\nu_k \operatorname{ch}[\nu_k(t - T/2)]}{T \operatorname{sh} \nu_k T/2} \varphi_k + \left(\frac{T \nu_k \operatorname{ch}[\nu_k(t - T/2)]}{2 \operatorname{sh} \nu_k T/2} \right) \frac{1}{\nu^2 \lambda_k} f_k \right] v_k(x) - \sum_{\lambda_k \in \Lambda_0(\nu)} \frac{1}{\nu^2 \lambda_k} f_k v_k(x). \quad (10)$$

Proposition 1. Let the function $u(x, t)$ be the solution of problem (1)-(3) for $\varphi, f \in L_2(\Omega)$. Then its spectral expansion has the form (9).

Auxiliary results and proof of the main Theorem

In this section, we present auxiliary lemmas and propositions which are needed for the proof of the main theorem. After establishing these auxiliary results, we proceed to the proof of the theorem itself.

For convenience, let us introduce the function

$$B(\nu_k, t) = \frac{\nu_k}{\operatorname{sh}(\nu_k T/2)} \operatorname{ch}[\nu_k(t - T/2)]. \quad (11)$$

In this regard, we can rewrite the form of the solution as follows.

$$u(x, t) = \sum_{\lambda_k \notin \Lambda_0(\nu)} \left[\frac{1}{T} B(\nu_k, t) \varphi_k + \frac{T}{2\nu^2 \lambda_k} B(\nu_k, t) f_k \right] v_k(x) - \sum_{\lambda_k \in \Lambda_0(\nu)} \frac{1}{\nu^2 \lambda_k} f_k v_k(x). \quad (12)$$

Lemma 1. There exists a constant $C > 0$ such that

$$|B(\nu_k, t)| \leq C$$

for all $k \in \mathbb{N}$ and $t \in [0, T]$.

Proof. Taking into account that $\nu_k < \nu$ and $t \in (0, T)$, we obtain

$$\operatorname{ch} \left[\nu_k \left| t - \frac{T}{2} \right| \right] \leq \operatorname{ch} \left(\frac{\nu_k T}{2} \right).$$

Therefore,

$$\begin{aligned} B(\nu_k, t) &\leq \frac{\nu_k \operatorname{ch}(\nu_k T/2)}{\operatorname{sh}(\nu_k T/2)} = \nu_k \operatorname{cth} \frac{\nu_k T}{2} = \left| \frac{\nu_k T}{2} \right| = a, \quad \operatorname{cth} = \frac{e^a + e^{-a}}{e^a - e^{-a}} = \\ &= \nu_k \left(1 + \frac{2e^{-a}}{e^a - e^{-a}} \right) = \nu_k \left(1 + \frac{2}{e^{2a} - 1} \right) \leq \nu_k \left(1 + \frac{1}{a} \right) \end{aligned}$$

Thus,

$$|B(\nu_k, t)| \leq C \quad (13)$$

where $C = \operatorname{const}$. □

Recall that for $u(x, t)$ to be a solution of the considered problem, the function itself and its second-order derivatives with respect to t must satisfy the required belong to the domain of operator A . Therefore, we prove the following statements.

Proposition 2. Let the function $u(x, t)$ be defined by (9). Then $u(\cdot, t) \in D(A)$ uniformly with respect to $t \in (0, T)$, provided that

$$\sum_{\lambda_k \notin \Lambda_0(\nu)} (\lambda_k^2 |\varphi_k|^2 + |f_k|^2) < +\infty. \quad (14)$$

In particular, if $\varphi \in D(A)$ and $f \in L_2(\Omega)$, then condition (14) is satisfied.

Proof. Applying the operator A to equation (12), we obtain

$$Au(x, t) = \sum_{\lambda_k \notin \Lambda_0(\nu)} \left[\frac{1}{T} B(\nu_k, t) \lambda_k \varphi_k + \frac{T}{2\nu^2} B(\nu_k, t) f_k \right] v_k(x) + \sum_{\lambda_k \in \Lambda_0(\nu)} \frac{1}{\nu^2} f_k v_k(x). \quad (15)$$

Since $f \in L_2(\Omega)$, the second sum in (15) is finite. It remains to estimate the first part of the expression.

Then, using (13) and Parseval's equality and the inequality $(a + b)^2 \leq 2a^2 + 2b^2$ we get

$$\begin{aligned} \|Au(\cdot, t)\|^2 &= \sum_{\lambda_k \notin \Lambda_0(\nu)} \left[\frac{1}{T} B(\nu_k, t) \lambda_k \varphi_k + \frac{T}{2\nu^2} B(\nu_k, t) f_k \right]^2 \leq \\ &\leq \sum_{\lambda_k \notin \Lambda_0(\nu)} \left(\frac{2}{T^2} B^2(\nu_k, t) \lambda_k^2 \varphi_k^2 + \frac{T^2}{2\nu^4} B^2(\nu_k, t) f_k^2 \right). \end{aligned}$$

By Lemma 1, it follows that the inequality

$$\|Au(\cdot, t)\|_{L_2(\Omega)}^2 \leq C_1 \sum_{\lambda_k \notin \Lambda_0(\nu)} (\lambda_k^2 |\varphi_k|^2 + |f_k|^2) < +\infty$$

Since condition (14) holds and $f \in L_2(\Omega)$, the right-hand side is finite.

Hence, $\|Au(\cdot, t)\|_{L_2(\Omega)} \leq \text{const.}$ uniformly with respect to $t \in (0, T)$. The proposition is proved. $\square$

Proposition 3. Let condition (14) be satisfied. Then the functions $u(x, t)$ and $Au(x, t)$ are twice continuously differentiable with respect to $t \in (0, T)$.

Proof. Differentiating with respect to t in equation (15), we obtain the following equality

$$\frac{\partial^2}{\partial t^2} Au(x, t) = \sum_{\lambda_k \notin \Lambda_0(\nu)} \left[\frac{1}{T} \nu_k^2 B(\nu_k, t) \lambda_k \varphi_k + \frac{T \nu_k^2}{2\nu^2} B(\nu_k, t) f_k \right] v_k(x).$$

By Parseval's equality, we get

$$\left\| \frac{\partial^2}{\partial t^2} Au(\cdot, t) \right\|^2 \leq \sum_{\lambda_k \notin \Lambda_0(\nu)} \left(\frac{2\nu_k^4}{T^2} B^2(\nu_k, t) \lambda_k^2 \varphi_k^2 + \frac{T^2 \nu_k^4}{2\nu^4} B^2(\nu_k, t) f_k^2 \right).$$

From definition (4), we have

$$\frac{\mu}{\mu + 1} \nu^2 \leq \nu_k^2 < \nu^2.$$

Thus ν_k^4 is uniformly bounded and together with Lemma 1, $B(\nu_k, t)$ is also uniformly bounded. Therefore the above series converges in $L_2(\Omega)$ under condition (13). Consequently,

$$\left\| \frac{\partial^2}{\partial t^2} Au(x, t) \right\|^2 \leq \text{const.} \quad 0 < t < T.$$

Moreover, $Au(x, t)$ is twice continuously differentiable with respect to t . The same argument applied directly to $u(x, t)$ shows that $u(x, t)$ is also twice continuously differentiable with respect to t . $\square$

Proof of the Main Theorem.

Let $\varphi \in D(A)$ and $f \in L_2(\Omega)$.

$$\sum_{k=1}^{\infty} \lambda_k^2 |\varphi_k|^2 < \infty, \quad \sum_{k=1}^{\infty} |f_k|^2 < \infty$$

Hence condition (14) is satisfied. Therefore by Proposition 2, the function defined by representation (12) belongs to $D(A)$ uniformly with respect to $t \in (0, T)$. By Proposition 3, the functions $u(x, t)$ and $Au(x, t)$ are twice

continuously differentiable with respect to $t \in (0, T)$. Hence all terms appearing in equation (5) are well defined in $L_2(\Omega)$.

From representation (12) we have

$$u_k(t) = \frac{1}{T} B(\nu_k, t) \varphi_k + \frac{T}{2\nu^2 \lambda_k} B(\nu_k, t) f_k - \frac{f_k}{\nu^2 \lambda_k}$$

By the definition of (11) $B(\nu_k, t)$, we have $B_{tt}(\nu_k, t) = \nu_k^2 B(\nu_k, t)$. Therefore,

$$u_k''(t) = \nu_k^2 \left[\frac{1}{T} B(\nu_k, t) \varphi_k + \frac{T}{2\nu^2 \lambda_k} B(\nu_k, t) f_k \right].$$

It follows from the above results lemma, propositions that the series defining $u(x, t)$, $Au(x, t)$, and $Au_{tt}(x, t)$ converge in $L_2(\Omega)$.

The remain series

$$- \sum_{\lambda_k \in \Lambda_0(\nu)} \frac{1}{\nu^2 \lambda_k} f_k v_k(x)$$

also converges in $L_2(\Omega)$. Indeed, since $\lambda_k \geq \lambda_1 > 0$, we have

$$\sum_{\lambda_k \in \Lambda_0(\nu)} \left| \frac{1}{\nu^2 \lambda_k} f_k \right|^2 \leq \frac{1}{\nu^4 \lambda_1^2} \sum_{\lambda_k \in \Lambda_0(\nu)} |f_k|^2 < +\infty.$$

Moreover, from representation (12), we obtain

$$|u_k(t)| \leq C_1 |\varphi_k| + \frac{C_2}{\nu^2 \lambda_k} |f_k|.$$

where C_1 and C_2 are positive constants independent of k and t . Hence,

$$|u_k(t)|^2 \leq 2C_1^2 |\varphi_k|^2 + \frac{2C_2^2}{\nu^4 \lambda_k^2} |f_k|^2.$$

Since $\lambda_k \geq \lambda_1 > 0$, we have

$$\sum_{k=1}^{\infty} |u_k(t)|^2 \leq 2C_1^2 \sum_{k=1}^{\infty} |\varphi_k|^2 + \frac{2C_2^2}{\nu^4 \lambda_1^2} \sum_{k=1}^{\infty} |f_k|^2 < \infty.$$

Therefore, the series (12) converges $L_2(\Omega)$. Consequently, the function $u(x, t)$ satisfies all the required conditions and is a solution of problem (1)-(3).

It remains to prove uniqueness and the uniqueness part of the theorem is straightforward to prove. Thus, the solution of problem (1)-(3) exists and is unique. The theorem is proved. $\square$

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AVERAGED STOCHASTIC APPROXIMATION FOR SEQUENTIAL FIXED-WIDTH CONFIDENCE INTERVALS UNDER RIGHT CENSORING

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ABSTRACT. We construct sequential fixed-width confidence intervals for a parameter of a right-censored lifetime distribution observed as a stream, where the Fisher information is unknown and must be learned online. The procedure combines the Polyak–Ruppert averaged Robbins–Monro estimator with a recursively maintained information estimate, both updated in $O(1)$ time and memory. Under standard regularity and the gain condition $cI > 1/2$, we prove asymptotic consistency and Chow–Robbins efficiency ($N_\varepsilon/n_\varepsilon \rightarrow 1$ a.s. and in mean). The analysis relies on an asymptotically linear representation of the averaged estimator, whose remainder is controlled almost surely via the martingale structure of the score. We show that stopping-time efficiency holds for every gain, while the gain condition governs coverage alone. Simulations on censored exponential and Weibull models confirm the theory and demonstrate that using the information in place of its reciprocal in the stopping rule collapses the procedure.

MSC (2020): 62L12; 62N02; 62L20; 62F12.

Key words: sequential estimation, fixed-width confidence intervals, right-censoring, stochastic approximation, Polyak–Ruppert averaging, Anscombe’s theorem.

1. Introduction

Fixed-width sequential confidence intervals, introduced by Chow and Robbins [4], determine the sample size adaptively as data accrue. The classical theory assumes i.i.d. observations and an asymptotic variance that is known or consistently estimable from a pilot. In survival and reliability analysis, however, observations are right-censored and the Fisher information depends on the unknown censoring law; it cannot be computed a priori and must be estimated *in real time*, which forces the sampling rule itself to be adaptive. For streaming data a further constraint is computational: maximum likelihood, EM, or estimating-equation solvers reprocess the sample and cost $O(n)$ or worse per update. Stochastic approximation [23] updates in $O(1)$ time and memory, and Polyak–Ruppert averaging [21, 22, 25] gives the averaged iterate the optimal asymptotic covariance I^{-1} , independently of the step size. It is thus a natural vehicle for sequential interval estimation: efficient in the Hájek–Le Cam sense, cheap, and carrying a recursively updatable information estimate that can drive a data-dependent stopping rule.

Related work. Fixed-width sequential intervals originate with [4]; coverage rests on Anscombe’s random-index theorem [1] and its nonlinear-renewal refinements [28, 8]. A recent literature treats online and streaming estimation at a *fixed* sample size: renewable regression [15], stochastic-gradient estimation for censored streams [29], and, for the Polyak–Ruppert average, variance estimation via batch-means [3], the online bootstrap [7], or random scaling [13, 26]. All fix n and report a shrinking-width interval. To our knowledge none couples a recursive information estimate with a data-driven stopping time to deliver a fully sequential *fixed-width* interval that is consistent and Chow–Robbins efficient under censoring; this is the gap we fill.

Given a stream of right-censored observations we run an averaged Robbins–Monro recursion, maintain an online estimate $\hat{I}_n$ of the Fisher information, and stop when the implied half-width reaches ε . Our contributions are: **(i)** an asymptotically consistent, Chow–Robbins efficient procedure (Corollary 1): coverage tends to γ (Theorem 2) and $N_\varepsilon/n_\varepsilon \rightarrow 1$ a.s. and in mean (Proposition 2); **(ii)** a corrected asymptotically linear representation (Lemma 2) whose frozen-score correction, not amenable to a direct Cauchy–Schwarz bound ($O(n^{-1/2})$), attains $o(n^{-1/2})$ via the martingale-difference structure of the score; **(iii)** a separation of design requirements:

stopping-time efficiency holds for *every* gain c , whereas the gain condition $cI > \frac{1}{2}$ governs coverage alone; **(iv)** a cautionary identity: the controlling scale is the variance I^{-1} , not I , and placing $\hat{I}_n$ where $\hat{I}_n^{-1}$ belongs stops a factor I^2 too early, catastrophic under heavy censoring ($I < 1$).

The paper is organized as follows. Section 2 fixes the model and assumptions; Section 3 develops the asymptotic theory, culminating in properties (H1)-(H3); Sections 4 and 5 define the procedure and prove stopping-time efficiency and coverage; Section 6 reports simulations; Section 7 discusses extensions and open problems. Longer proofs are collected in the Appendix.

2. Model, recursion, and assumptions

Censoring model and score. Let $X_1, X_2, \dots$ be i.i.d. failure times with density $f(\cdot; \theta_0)$ and c.d.f. $F(\cdot; \theta_0)$, $\theta_0 \in \Theta \subseteq \mathbb{R}^k$ open, and let $C_1, C_2, \dots$ be i.i.d. censoring times with distribution G , independent of the X_i and not depending on θ . One observes

$$Y_i = \min(X_i, C_i), \quad \delta_i = \mathbf{1}\{X_i \leq C_i\}, \quad i = 1, 2, \dots,$$

with filtration $\mathcal{F}_n = \sigma(Y_1, \delta_1, \dots, Y_n, \delta_n)$. The single-observation censored score is

$$\psi(y, \delta, \theta) = \delta \frac{\partial_\theta f(y; \theta)}{f(y; \theta)} - (1 - \delta) \frac{\partial_\theta F(y; \theta)}{1 - F(y; \theta)}, \quad (1)$$

the conditional expectation of the full-data score given (Y, δ) [19, 14]. In particular $\mathbb{E}_{\theta_0} \psi(Y, \delta, \theta_0) = 0$, and the Fisher information equals

$$I = \mathbb{E}_{\theta_0} [\psi(Y, \delta, \theta_0) \psi(Y, \delta, \theta_0)^\top].$$

Throughout this paper $k = 1$, so $I > 0$ is a scalar; the multivariate case is discussed in Remark 5.

Write $Z_i = (Y_i, \delta_i)$ for the observations, P_θ for their law, $\mathbb{P}_n = n^{-1} \sum_{i=1}^n \delta_{Z_i}$ for the empirical measure, $Pf = \int f dP_{\theta_0}$, and $\mathbb{G}_n f = \sqrt{n}(\mathbb{P}_n - P_{\theta_0})f$ for the empirical process; thus $M(\theta) = P_{\theta_0} \psi(\cdot, \theta)$.

Remark 1 (I^{-1} is the information bound). Under Assumption 2 the censored experiments are locally asymptotically normal with central sequence $\mathbb{G}_n \psi(\cdot, \theta_0)$ and information I [27, Ch. 7]; the censored score (1) is the efficient score and $\tilde{\psi}_{\theta_0} := I^{-1} \psi(\cdot, \theta_0)$ the efficient influence function. By the Hájek-Le Cam convolution theorem I^{-1} is the information bound for any regular estimator [9, 11, 12, 2]; the averaged estimator attains it (Theorem 1), so $n_\varepsilon = a^2 I^{-1} / \varepsilon^2$ is the intrinsic optimal sample size.

Averaged Robbins-Monro recursion For a compact convex $K \subset \Theta$ with $\theta_0 \in \text{int } K$ and Euclidean projection Π_K , and gain $\gamma_n = c/n$, define

$$\theta_{n+1} = \Pi_K[\theta_n + \gamma_n \psi(Y_{n+1}, \delta_{n+1}, \theta_n)], \quad \bar{\theta}_n = \frac{1}{n} \sum_{k=1}^n \theta_k. \quad (2)$$

The projection controls the boundary singularity of ψ . Since $\theta_n \xrightarrow{\text{a.s.}} \theta_0$ (proved in Theorem 1) and $\theta_0 \in \text{int } K$, the projection Π_K is eventually inactive almost surely, so it does not affect the limit theory. Write $M(\theta) = \mathbb{E}_{\theta_0} \psi(Y, \delta, \theta)$ for the drift.

Regularity conditions

We use the following standard conditions.

Assumption 1 (identifiability). θ_0 is the unique zero of M in Θ , M is C^1 near θ_0 , and $M'(\theta_0) = -I$. (The last equality is the second Bartlett identity for the censored score (1), recorded here for reference.)

Assumption 2 (smoothness). $\theta \mapsto \psi(y, \delta, \theta)$ is C^1 on a neighbourhood U_0 of θ_0 , with $\mathbb{E}_{\theta_0}[\sup_{U_0} \psi^2] < \infty$ and $\mathbb{E}_{\theta_0}[\sup_{U_0} (\partial_\theta \psi)^2] < \infty$; in particular $\theta \mapsto \mathbb{E}_{\theta_0}[\psi^2]$ is continuous on U_0 .

Assumption 3 (moments and information). $\mathbb{E}_{\theta_0} |\psi(Y, \delta, \theta_0)|^{2+\eta} < \infty$ for some $\eta > 0$ (with $\eta = 2$, i.e. a finite fourth moment, where the L^2 control in Lemma 2 requires it), and $I = \mathbb{E}_{\theta_0} [\psi(Y, \delta, \theta_0)^2] > 0$.

Assumption 4 (projection). K is compact convex, $\theta_0 \in \text{int } K$, and Π_K is the Euclidean projection onto K .

Assumption 5 (local strong monotonicity). $(\theta - \theta_0)M(\theta) \leq -\lambda(\theta - \theta_0)^2$ on a neighbourhood of θ_0 for some $\lambda > 0$. This is not independent: it follows from Assumptions 2-3 for any $0 < \lambda < I$ on a sufficiently small neighbourhood U , by the mean value theorem (the constant λ may be taken arbitrarily close to, but not equal to, I).

These conditions hold for the exponential and Weibull censored families of Section 6 and, more generally, for regular parametric lifetime models under a censoring law with sufficiently many moments. The *gain condition* $cI > \frac{1}{2}$ is invoked wherever the efficient rate is claimed; its role is delineated in Sections 3 and 5.

3. Asymptotic theory of the averaged estimator

Modulo two classical inputs – the Polyak-Juditsky averaging principle and a moment bound for harmonic-gain stochastic approximation – this section delivers the three properties used thereafter:

- (H1) asymptotic linearity $\bar{\theta}_n - \theta_0 = \mathbb{P}_n \tilde{\psi}_{\theta_0} + R_n$, $\tilde{\psi}_{\theta_0} = I^{-1}\psi(\cdot, \theta_0)$;
- (H2) negligible remainder $\sqrt{n} R_n \xrightarrow{\text{a.s.}} 0$;
- (H3) consistent scale $V_n \xrightarrow{\text{a.s.}} \sigma^2 := I^{-1}$, $0 < V_n \leq \iota^{-1}$.

Here $\mathbb{P}_n \tilde{\psi}_{\theta_0} = \frac{1}{n} \sum_i \phi_i$ with $\phi_i = I^{-1}\psi(Y_i, \delta_i, \theta_0)$ i.i.d. of mean 0 and variance $I^{-1} = \sigma^2$; (H1) states that $\bar{\theta}_n$ is asymptotically linear with the efficient influence function [27, Ch. 25].

Consistency, normality, and the non-asymptotic bias Under Assumptions 1-4, $\theta_n \xrightarrow{\text{a.s.}} \theta_0$ by the Robbins-Siegmund theorem [24], and the averaged iterate is efficient.

Theorem 1 (efficiency of the averaged estimator). Under Assumptions 1-4 and $cI > \frac{1}{2}$,

$$\sqrt{n}(\bar{\theta}_n - \theta_0) \xrightarrow{d} \mathcal{N}(0, I^{-1}).$$

Proof. The proof proceeds by linearising the drift. By Assumption 2, $M(\theta_n) = -I d_n + O(d_n^2)$, so the error $d_n = \theta_n - \theta_0$ obeys a linear recursion with drift $-cI$ and martingale noise $\xi_{n+1} = \psi(Y_{n+1}, \delta_{n+1}, \theta_n) - M(\theta_n)$. Averaging the iterates cancels the gain dependence of the asymptotic covariance, leaving $\Gamma^{-1} S \Gamma^{-\top} = I^{-1}$ with $\Gamma = -cI$ and $S = c^2 I$; the boundary condition $cI > \frac{1}{2}$ ensures the drift matrix $-cI$ is Hurwitz [6, 18], which is what makes the rate $\gamma_n = c/n$ admissible. The projection Π_K is inactive eventually, since $\theta_0 \in \text{int } K$ and $\theta_n \xrightarrow{\text{a.s.}} \theta_0$, so the unprojected limit applies [10, Thm. 5.2.1]. The full argument is that of Polyak-Juditsky [22, 10]; see the Appendix. Equivalently $\sqrt{n}(\bar{\theta}_n - \theta_0) = \mathbb{G}_n \tilde{\psi}_{\theta_0} + o_p(1)$: the averaged estimator is asymptotically linear with the efficient influence function, reproducing the MLE's limiting variance I^{-1} with no matrix inversion. $\square$

The finite-sample behaviour is governed by a non-asymptotic expansion, which explains both the coverage results and the role of cI .

Proposition 1 (non-asymptotic mean squared error). Under Assumptions 1-4 and $\gamma_n = c/n$,

$$\mathbb{E}[(\bar{\theta}_n - \theta_0)^2] = \frac{1}{In} + B_n, \quad B_n = \begin{cases} O(n^{-2cI}), & cI \neq \frac{1}{2}, \\ O(n^{-1}(\log n)^2), & cI = \frac{1}{2}, \end{cases} \quad (3)$$

so the efficient term $1/(In)$ dominates iff $cI > \frac{1}{2}$. Consequently the root-mean-square transient bias is of order n^{-cI} .

Proof idea. The proof uses the variation-of-constants (Duhamel) representation of the linearised recursion: the deterministic transient and the nonlinearity remainder contribute $O(n^{-cI})$, while the stochastic term gives the leading $1/(In)$. At $cI = \frac{1}{2}$ the logarithmic factor arises because the two competing exponents coincide. The last sentence is the fact used below: the *bias*, not its square, enters the linear representation, so the remainder there is $O(n^{-cI})$ and $\sqrt{n}$ times it is $O(n^{1/2-cI})$, which vanishes exactly when $cI > \frac{1}{2}$.

Lemma 1 (moment bound). Under Assumptions 1-4, $\gamma_n = c/n$ and $cI > \frac{1}{2}$,

$$\sup_n n \mathbb{E} d_n^2 < \infty; \quad \text{if moreover } \mathbb{E}_{\theta_0} |\psi(Y, \delta, \theta_0)|^4 < \infty, \text{ then } \sup_n n^2 \mathbb{E} d_n^4 < \infty.$$

This is the standard L^2/L^4 moment bound for harmonic-gain SA [18, Ch. 4], [17, Lemma 1]; see the harmonic-gain moment recursion.

The asymptotically linear representation

Lemma 2 (linear representation with a.s.-negligible remainder). Under Assumptions 1-4, the fourth-moment strengthening of Assumption 3, and $cI > \frac{1}{2}$, (H1)-(H2) hold. In addition $\mathbb{E}[(\sqrt{n}R_n)^2] = O(n^{-1} \log n)$.

Proof Sketch. Write $d_n = \theta_n - \theta_0$, $\xi_{n+1} = \psi(Y_{n+1}, \delta_{n+1}, \theta_n) - M(\theta_n)$ and $\xi_{n+1}^0 = \psi(Y_{n+1}, \delta_{n+1}, \theta_0)$, both martingale differences, with $\phi_{n+1} = I^{-1}\xi_{n+1}^0$.

Step 1 (Polyak-Juditsky). By Assumption 2, $M(\theta_n) = -I d_n + O(d_n^2)$; the averaging principle [22, Thm. 2] for the linear recursion with drift $-cI$ gives $\bar{\theta}_n - \theta_0 = \frac{1}{n} \sum_{k=1}^n I^{-1} \xi_k + P_n$ with $\sqrt{n}P_n \xrightarrow{\text{a.s.}} 0$. Its deterministic-transient and nonlinearity parts are $O(n^{-cI})$ pathwise (via $\pi_{n,1} = \prod_{k=1}^{n-1} (1 - cI/k) = \Theta(n^{-cI})$), hence $o(n^{-1/2})$ since $cI > \frac{1}{2}$; the stochastic averaging discrepancy is $o(n^{-1/2})$ a.s. by the refinements of [20, 16, 5], the sole external inputs beyond Step 2. (If only $\sqrt{n}P_n \xrightarrow{P} 0$ is available, (H2) holds in probability and all conclusions persist.)

Step 2 (replacing the frozen score). Put $D_n = \frac{1}{n} \sum_{k \leq n} I^{-1}(\xi_k - \xi_k^0)$, so $R_n = P_n + D_n$. The increments are martingale differences, and by the mean value theorem, using the boundedness of $\partial_\theta \psi$ on U_0 (Assumption 2), $\mathbb{E}[(\xi_k - \xi_k^0)^2 | \mathcal{F}_{k-1}] = O(d_{k-1}^2)$, so with $M_n = \sum_{k \leq n} I^{-1}(\xi_k - \xi_k^0)$ one has $\mathbb{E}[(\Delta M_k)^2] = O(\mathbb{E}d_{k-1}^2) = O(k^{-1})$ by Lemma 1. Thus $\mathbb{E}[(\sqrt{n}D_n)^2] = \frac{1}{n} \sum_{k \leq n} O(k^{-1}) = O(n^{-1} \log n)$, giving $\sqrt{n}D_n \xrightarrow{L^2} 0$; and since $\sum_k k^{-1} \mathbb{E}[(\Delta M_k)^2] < \infty$, the L^2 -bounded martingale $\sum_k k^{-1/2} \Delta M_k$ converges a.s. (by the L^2 -martingale convergence theorem [10]), so Kronecker's lemma yields $\sqrt{n}D_n \xrightarrow{\text{a.s.}} 0$. A direct Cauchy-Schwarz bound would give only $O(n^{-1/2})$; centring on the martingale difference is what recovers the sharper rate. Combining, $\sqrt{n}R_n \xrightarrow{\text{a.s.}} 0$ and $\mathbb{E}[(\sqrt{n}R_n)^2] = O(n^{-1} \log n)$. $\square$

Online variance estimate. Fix a floor $\iota \in (0, I)$ and estimate the information recursively by

$$\hat{I}_n = \frac{1}{n} \sum_{i=1}^n \psi(Y_i, \delta_i, \bar{\theta}_{i-1})^2, \quad V_n = (\hat{I}_n \vee \iota)^{-1}. \quad (4)$$

Lemma 3 (strong consistency of the variance estimate). Under Assumptions 1-5, $\hat{I}_n \xrightarrow{\text{a.s.}} I$; hence $V_n \xrightarrow{\text{a.s.}} I^{-1} = \sigma^2$ and $0 < V_n \leq \iota^{-1}$ for all n . Thus (H3) holds.

Proof. Since $\theta_n \xrightarrow{\text{a.s.}} \theta_0$, also $\bar{\theta}_{i-1} \xrightarrow{\text{a.s.}} \theta_0$. With $g(\theta) = \mathbb{E}_{\theta_0}[\psi(Y, \delta, \theta)^2]$ continuous near θ_0 and $g(\theta_0) = I$, write $\hat{I}_n - I = (a) + (b)$ where (a) $= \frac{1}{n} \sum_i [\psi(Y_i, \delta_i, \bar{\theta}_{i-1})^2 - g(\bar{\theta}_{i-1})]$ and (b) $= \frac{1}{n} \sum_i [g(\bar{\theta}_{i-1}) - I]$. Term (b) $\rightarrow 0$ a.s. by continuity and the Cesàro lemma. For term (a), on a compact sub-neighbourhood $\bar{U} \subset U_0$ of θ_0 the class $\{\psi(\cdot, \theta)^2 : \theta \in \bar{U}\}$ is P_{θ_0} -Glivenko-Cantelli—it has an integrable envelope and is continuous in θ (Assumptions 2-3), hence [27, Thm. 19.4] $\sup_{\theta \in \bar{U}} |(\mathbb{P}_n - P)\psi(\cdot, \theta)^2| \xrightarrow{\text{a.s.}} 0$; since $\bar{\theta}_{i-1} \in \bar{U}$ eventually, term (a) $\xrightarrow{\text{a.s.}} 0$. The continuous mapping theorem gives $V_n \xrightarrow{\text{a.s.}} I^{-1}$ ($I > \iota$), and the bound is immediate. $\square$

4. Sequential procedure and stopping-time efficiency

The procedure. Fix a confidence level $\gamma \in (0, 1)$ and half-width $\varepsilon > 0$, and set $a = \Phi^{-1}(\frac{1+\gamma}{2})$, so $2\Phi(a) - 1 = \gamma$. Because $\sqrt{n}(\bar{\theta}_n - \theta_0) \xrightarrow{d} \mathcal{N}(0, I^{-1})$, the error law is $\mathcal{N}(0, \sigma^2/n)$ with $\sigma^2 = I^{-1}$, and the Chow-Robbins optimal (but infeasible) nonrandom sample size is

$$n_\varepsilon = \frac{a^2 \sigma^2}{\varepsilon^2} = \frac{a^2 I^{-1}}{\varepsilon^2} = \frac{a^2}{I \varepsilon^2}. \quad (5)$$

Replacing σ^2 by V_n from (4) and choosing a warm-up $n_0 = n_0(\varepsilon)$ with $n_0 \rightarrow \infty$, $n_0 = o(n_\varepsilon)$ (e.g. $n_0 = \lceil \varepsilon^{-\alpha} \rceil$, $0 < \alpha < 2$), define

$$N_\varepsilon = \inf \left\{ n \geq n_0 : n \geq \frac{a^2 V_n}{\varepsilon^2} = \frac{a^2}{(\hat{I}_n \vee \iota) \varepsilon^2} \right\}, \quad I(N_\varepsilon) = [\bar{\theta}_{N_\varepsilon} - \varepsilon, \bar{\theta}_{N_\varepsilon} + \varepsilon]. \quad (6)$$

Both $\hat{I}_n$ and V_n update by a single rank-one addition per step; because $\hat{I}_n$ is $\mathcal{F}_n$ -measurable, $\{N_\varepsilon \leq n\} \in \mathcal{F}_n$, so N_ε is a bona fide stopping time and the whole procedure runs in $O(1)$ memory.

Remark 2 (the scale is the variance, not the information). The information I appears in the denominator of (5): a more informative model (larger I , lighter censoring) needs fewer observations. A rule must place the variance estimate $V_n \rightarrow I^{-1}$ in the numerator. Using $\hat{I}_n$ itself would target $a^2 \hat{I}_n / \varepsilon^2$, a factor I^2 of (5); under heavy censoring $I < 1$, so $I^2 \ll 1$ and the rule stops far too early. Section 6 shows the resulting collapse.

Efficiency of the stopping time

Proposition 2 (stopping-time properties). Assume only (H3). Then for every $\gamma \in (0, 1)$: (i) $N_\varepsilon \leq \max\{n_0, \lceil a^2 \iota^{-1} / \varepsilon^2 \rceil\} < \infty$ surely; (ii) $N_\varepsilon \xrightarrow{\text{a.s.}} \infty$ as $\varepsilon \rightarrow 0$; (iii) $N_\varepsilon / n_\varepsilon \xrightarrow{\text{a.s.}} 1$ as $\varepsilon \rightarrow 0$; (iv) $\mathbb{E} N_\varepsilon / n_\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow 0$.

Proof. (i) As $V_n \leq \iota^{-1}$, the threshold $a^2 V_n / \varepsilon^2 \leq a^2 \iota^{-1} / \varepsilon^2 =: m^*$ for every n , so the defining inequality holds at $n = \max\{n_0, \lceil m^* \rceil\}$.

(ii) Work on the probability-one event $\{V_n > 0 \ \forall n\} \cap \{V_n \rightarrow \sigma^2\}$. If $N_\varepsilon \leq B$ for some finite B , then $B \geq N_\varepsilon \geq a^2 V_{N_\varepsilon} / \varepsilon^2 \geq a^2 (\min_{n_0 \leq n \leq B} V_n) / \varepsilon^2$; since the minimum is strictly positive, the right side diverges as $\varepsilon \rightarrow 0$, a contradiction. Hence $N_\varepsilon \xrightarrow{\text{a.s.}} \infty$.

(iii) For ε small enough that $N_\varepsilon > n_0$, minimality gives

$$\frac{a^2 V_{N_\varepsilon}}{\varepsilon^2} \leq N_\varepsilon < \max\left(n_0, \frac{a^2 V_{N_\varepsilon-1}}{\varepsilon^2}\right) + 1.$$

Dividing by $n_\varepsilon = a^2 \sigma^2 / \varepsilon^2$, $\frac{V_{N_\varepsilon}}{\sigma^2} \leq \frac{N_\varepsilon}{n_\varepsilon} < \frac{V_{N_\varepsilon-1}}{\sigma^2} + \frac{1+n_0}{n_\varepsilon}$. As $\varepsilon \rightarrow 0$, $n_\varepsilon \rightarrow \infty$ and $n_0 / n_\varepsilon \rightarrow 0$; by (ii) $N_\varepsilon \rightarrow \infty$, so $V_{N_\varepsilon}, V_{N_\varepsilon-1} \rightarrow \sigma^2$ by the random-index principle ($v_n \rightarrow v$ and integer $N_\varepsilon \rightarrow \infty$ imply $v_{N_\varepsilon} \rightarrow v$); both bounds tend to 1.

(iv) By (i), $N_\varepsilon / n_\varepsilon \leq \max\{n_0, \lceil m^* \rceil\} / n_\varepsilon \rightarrow \iota^{-1} / \sigma^2 = I / \iota$, a deterministic constant; bounded convergence upgrades (iii) to convergence in mean. $\square$

Remark 3 (efficiency needs no gain condition). Proposition 2 uses only (H3), i.e. consistency of $\hat{I}_n$, which holds under Assumptions 1–5 *without* the gain condition. Thus $N_\varepsilon / n_\varepsilon \rightarrow 1$ for every $c > 0$. The gain condition $cI > \frac{1}{2}$ enters only in the coverage analysis below, through (H2). Efficiency and coverage are therefore governed by different requirements and must be tuned separately—a point confirmed numerically in the simulation study, where $N_\varepsilon / n_\varepsilon \approx 1$ across all gains while coverage varies with cI .

5. Consistency of the fixed-width interval

Theorem 2 (asymptotic consistency). Assume (H1)–(H3). Then for every $\gamma \in (0, 1)$, $\lim_{\varepsilon \rightarrow 0} \mathbb{P}\{\theta_0 \in I(N_\varepsilon)\} = \gamma$.

Proof. Let $S_m = \sum_{i=1}^m \phi_i$, i.i.d. with mean 0 and variance $\sigma^2 = I^{-1}$. By (H1), $\sqrt{m}(\bar{\theta}_m - \theta_0) = S_m / \sqrt{m} + \sqrt{m}R_m$.

Leading term. By Proposition 2(ii), $N_\varepsilon / n_\varepsilon \xrightarrow{\text{P}} 1$ with $n_\varepsilon \rightarrow \infty$ nonrandom. Anscombe’s random-sum central limit theorem for i.i.d. finite-variance summands [1], whose uniform-continuity-in-probability condition holds automatically for such partial sums [8, Thm. 3.1], gives $S_{N_\varepsilon} / \sqrt{N_\varepsilon} \xrightarrow{d} \mathcal{N}(0, \sigma^2)$.

Remainder. By (H2), $\sqrt{m}R_m \xrightarrow{\text{a.s.}} 0$, and $N_\varepsilon \xrightarrow{\text{a.s.}} \infty$ (Proposition 2(ii)); the random-index principle gives $\sqrt{N_\varepsilon}R_{N_\varepsilon} \xrightarrow{\text{a.s.}} 0$, hence $\xrightarrow{\text{P}} 0$. (In the in-probability form of (H2), the same follows from uniform negligibility of $\sqrt{m}R_m$ on windows $[(1 \mp \eta)n_\varepsilon]$ via Doob’s L^2 maximal inequality and the finite-time bounds of [16, 5].)

Combination. By Slutsky’s lemma, $\sqrt{N_\varepsilon}(\bar{\theta}_{N_\varepsilon} - \theta_0) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$, equivalently $T_\varepsilon := \sqrt{N_\varepsilon}I(\bar{\theta}_{N_\varepsilon} - \theta_0) \xrightarrow{d} \mathcal{N}(0, 1)$. By Proposition 2(iii) and $\varepsilon^2 n_\varepsilon I = a^2$, $\varepsilon^2 N_\varepsilon I = a^2 (N_\varepsilon / n_\varepsilon) \xrightarrow{\text{a.s.}} a^2$, so $b_\varepsilon := \varepsilon \sqrt{N_\varepsilon}I \xrightarrow{\text{a.s.}} a$. Then

$$\mathbb{P}\{\theta_0 \in I(N_\varepsilon)\} = \mathbb{P}\{|\bar{\theta}_{N_\varepsilon} - \theta_0| \leq \varepsilon\} = \mathbb{P}\{|T_\varepsilon| \leq b_\varepsilon\}.$$

Since $(T_\varepsilon, b_\varepsilon) \xrightarrow{d} (Z, a)$ with $Z \sim \mathcal{N}(0, 1)$ and $\mathbb{P}\{|Z| = a\} = 0$, the portmanteau theorem gives $\mathbb{P}\{\theta_0 \in I(N_\varepsilon)\} \rightarrow \mathbb{P}\{|Z| \leq a\} = 2\Phi(a) - 1 = \gamma$. $\square$

Corollary 1 (efficient consistent procedure). Under Assumptions 1–4, the fourth-moment strengthening of Assumption 3, and $cI > \frac{1}{2}$, the family $\{I(N_\varepsilon)\}_{\varepsilon > 0}$ of (6) is asymptotically consistent and asymptotically efficient in the Chow-Robbins sense ($N_\varepsilon / n_\varepsilon \rightarrow 1$ a.s. and in mean).

Proof. Lemmas 2-3 give (H1)-(H3); combine Proposition 2 and Theorem 2. $\square$

Remark 4 (the gain condition and coverage). The gain condition enters only through (H2). If $cI \leq \frac{1}{2}$, the transient bias is $O(n^{-cI})$ (Proposition 1), so $\sqrt{n}R_n = O(n^{1/2-cI}) \not\rightarrow 0$; the Anscombe step fails and coverage falls below γ . (*This is the corrected order: the bias is n^{-cI} , not n^{-2cI} ; the latter is the order of its square B_n .*) Quantifying the deficit as $P\{\theta_0 \in I(N_\varepsilon)\} = \gamma + O(\varepsilon^\beta)$ requires the nonlinear renewal / Edgeworth machinery of Woodroffe [28] applied to the rates of $\hat{I}_n^{-1} \rightarrow I^{-1}$ and of $\sqrt{n}(\bar{\theta}_n - \theta_0)$, and is left to future work.

Remark 5 (multivariate case). For $k > 1$, replace the interval by a fixed maximal-axis confidence ellipsoid. With $\hat{I}_n = n^{-1} \sum_i \psi(Y_i, \delta_i, \theta_{i-1}) \psi(\cdot)^\top$ and $E_n = \{\theta : n(\bar{\theta}_n - \theta)^\top \hat{I}_n (\bar{\theta}_n - \theta) \leq \chi_{k,\gamma}^2\}$, the largest semi-axis is $\{\chi_{k,\gamma}^2 \lambda_{\max}(\hat{I}_n^{-1})/n\}^{1/2}$; requiring it $\leq \varepsilon$ gives $N_\varepsilon = \inf\{n \geq n_0 : n \geq \chi_{k,\gamma}^2 \lambda_{\max}(\hat{I}_n^{-1})/\varepsilon^2\}$. The vector analogue of Theorem 2 holds under $c \lambda_{\min}(I) > \frac{1}{2}$; the least-informative direction, through $\lambda_{\max}(\hat{I}_n^{-1}) = \lambda_{\min}(\hat{I}_n)^{-1}$, governs the sample size. Preconditioning by $\hat{I}_{n-1}^{-1}$ (an $O(k^2)$ Sherman-Morrison update) reduces the gain condition to $c > \frac{1}{2}$ regardless of the conditioning of I .

6. Simulation study and practical guidance

We use the exponential censored model: $X \sim \text{Exp}(\lambda_0)$, $C \sim \text{Exp}(\lambda_C)$ with $\lambda_C = p_c \lambda_0 / (1 - p_c)$, so the censoring fraction is p_c and $I = (1 - p_c) / \lambda_0^2$; the score is $\psi(y, \delta, \lambda) = \delta / \lambda - y$. We take $\lambda_0 = 1$, $\gamma = 0.95$ ($a = 1.96$), gain $\gamma_n = c / (n + 50)$, floor $\iota = 10^{-6}$, projection radius $R = 100$, a 200-observation pilot MLE start, and $M = 2500$ Monte Carlo replications. All quantities are computed in a single streaming pass per replication.

Table 2: Table 1. Effect of the gain at $p_c = 0.5$ ($I = 0.5$), $\varepsilon = 0.03$, $n_\varepsilon = 8537$, $M = 2500$ replications. Efficiency $N_\varepsilon / n_\varepsilon$ equals 1.00 to two decimals throughout (observed range 0.999–1.004); coverage is governed by cI . Monte Carlo standard errors are ≈ 0.006 – 0.010 for coverage and $\lesssim 0.003$ for the ratio.

cI	0.30	0.50	0.75	1.00	1.50	2.00	3.00	4.00	5.00
$N_\varepsilon / n_\varepsilon$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
coverage	0.46	0.61	0.75	0.83	0.88	0.90	0.92	0.93	0.93

Stopping-time efficiency holds for every gain. Table 1 sweeps the gain at $p_c = 0.5$ ($I = 0.5$), $\varepsilon = 0.03$. Across the entire range—including $cI \leq \frac{1}{2}$ —the ratio $N_\varepsilon / n_\varepsilon$ stays at 1.00 within Monte Carlo error, confirming Remark 3: efficiency of the stopping time does not require the gain condition. Coverage, by contrast, rises monotonically through the threshold $cI = \frac{1}{2}$ and approaches—but at these sample sizes does not reach—the nominal 0.95. The residual gap is the finite-sample efficiency deficit of the averaged estimator quantified by B_n in Proposition 1 (the averaged MSE sits at a constant factor above the Cramér–Rao bound that shrinks only slowly in n); it narrows as c increases (coverage $0.90 \rightarrow 0.93$ from $cI = 2$ to $cI = 5$).

Table 3: Table 2. Correct versus naive stopping scale, $\varepsilon = 0.03$, $cI \geq 3$, $M = 2500$; coverage in brackets (Monte Carlo standard error ≈ 0.006 – 0.010). The naive rule stops at $\approx I^2$ of the correct size.

	$p_c = 0.3$ ($I=0.7$)	$p_c = 0.5$ ($I=0.5$)	$p_c = 0.7$ ($I=0.3$)
correct: $N_\varepsilon / n_\varepsilon$ [cov]	1.02 [0.93]	1.00 [0.93]	0.98 [0.93]
naive: $N_\varepsilon / n_\varepsilon$ [cov]	0.47 [0.78]	0.25 [0.64]	0.11 [0.44]

The correct scale is essential. Table 2 contrasts the correct rule ($V_n = \hat{I}_n^{-1}$) with the naive one ($\hat{I}_n$) at $\varepsilon = 0.03$ and gain tuned to $cI \geq 3$. The correct rule attains $N_\varepsilon / n_\varepsilon \approx 1$ and coverage near 0.93 at all censoring levels. The naive rule stops at a fraction $N_\varepsilon / n_\varepsilon \approx I^2$ of the required size—0.47, 0.25, 0.11 for $p_c = 0.3, 0.5, 0.7$ (against $I^2 = 0.49, 0.25, 0.09$; agreement to within Monte Carlo error ≈ 0.02) and its coverage collapses to 0.78, 0.64, 0.44. The error is largest exactly where censoring is heaviest and sequential methods are most needed.

A two-parameter check: censored Weibull. To go beyond $k = 1$ we check the ellipsoidal rule of Remark 5 on $X \sim \text{Weibull}(\lambda, \gamma)$ with $\theta = (\lambda, \gamma)$ unknown and independent exponential censoring at a 25% fraction. With

$W = (\lambda y)^\gamma$, $e = \delta - W$, the censored score (1) is $\psi_\lambda = \frac{\gamma}{\lambda}e$, $\psi_\gamma = \frac{\delta}{\gamma} + e \ln(\lambda y)$, verified against the Bartlett identity $-\mathbb{E}[\partial_\theta \psi] = I$ by finite differences at $(\lambda_0, \gamma_0) = (1, 2)$; the information is ill-conditioned, $\lambda_{\max}(I)/\lambda_{\min}(I) \approx 9.3$. We run (2) from a 300-observation pilot, gain $c = 15$ ($c\lambda_{\min}(I) \approx 4.9$), floor $\iota = 10^{-6}$, box projection, and the ellipsoidal rule with $\gamma = 0.95$ ($\chi_{2,0.95}^2 = 5.99$), $M = 1500$.

Table 4: Table 3. Two-parameter censored Weibull ($\lambda_0 = 1$, $\gamma_0 = 2$, 25% censoring, $c\lambda_{\min}(I) \approx 4.9$, $M = 1500$); $n_\varepsilon = \chi_{2,0.95}^2 \lambda_{\max}(I^{-1})/\varepsilon^2$.

ε	n_ε	mean N_ε	$N_\varepsilon/n_\varepsilon$	coverage
0.05	7381	7007	0.95	0.68
0.04	11533	11150	0.97	0.76
0.03	20502	20132	0.98	0.81
0.02	46130	45791	0.99	0.89

As in the scalar case $N_\varepsilon/n_\varepsilon \rightarrow 1$ regardless of coverage, confirming Remark 3 in the multivariate setting. Coverage approaches 0.95 more slowly, consistent with the larger finite-sample bias under a correlated, ill-conditioned I ; it worsens for larger c (e.g. $c = 30, 45$ gave 0.37, 0.02), since a scalar gain tuned to $\lambda_{\min}(I)$ over-drives the best-conditioned direction. The preconditioned recursion of Remark 5 is designed to remove this; a full study is left to future work.

Practical guidance. Choose $c = \min\{\kappa/\hat{I}_0, c_{\max}\}$ with $\kappa \geq 3$ from the pilot $\hat{I}_0$ ($c_{\max}$ guarding the boundary singularity of ψ), floor $\iota = 10^{-6}$, and warm-up $n_0 = \lceil \varepsilon^{-\alpha} \rceil$, $\alpha \in (1, 2)$. The gain may be self-tuned by $c_n = \kappa/(\hat{I}_{n-1} \vee \iota)$, so $c_n I \rightarrow \kappa$ without knowing I . Where accurate coverage at moderate n is required, a larger gain or a bias correction to $\bar{\theta}_n$ is advisable (Remark 4).

7. Discussion and open problems

The procedure enjoys asymptotic coverage consistency, first-order Chow-Robbins efficiency, and $O(1)$ per-observation cost, under conditions verifiable for regular parametric lifetime models. A few points merit discussion. Efficiency and coverage are governed by different requirements (the simulation study), so a two-stage design—a pilot estimate of I then a refined c with $cI \gg \frac{1}{2}$ —is natural. The projection (Assumption 4) is asymptotically inactive since $\theta_n \rightarrow \theta_0 \in \text{int } K$; a generous K avoids boundary effects. The fourth-moment strengthening of Assumption 3 is needed only for the L^2 control of P_n ; the martingale part D_n and consistency of $\hat{I}_n$ need only finite variance and a $2 + \eta$ moment, all finite for exponential and Weibull models with censoring bounded away from one.

Sensitivity to model misspecification. The procedure relies on a correctly specified family $f(\cdot; \theta)$ in (1). If the true law $P \notin \{P_\theta\}$ but $M(\theta) = \mathbb{E}_P \psi(Y, \delta, \theta)$ still has a unique zero θ^* , then $\theta_n \xrightarrow{\text{a.s.}} \theta^*$, the Kullback-Leibler projection of P ; but $\hat{I}_n \rightarrow I^* = \mathbb{E}_P[\psi(\cdot, \theta^*)^2] \neq -M'(\theta^*)$, so the correct variance is the sandwich $\Sigma = M'(\theta^*)^{-1} I^* M'(\theta^*)^{-\top}$, not I^{*-1} , and coverage is no longer protected by Theorem 2. Mitigations: enlarge the family, or use a bounded-influence (Huber-type) score—to which the martingale argument of Lemma 2 applies essentially unchanged. In practice, a goodness-of-fit test on the pilot sample can detect severe misspecification before the sequential phase begins, at which point the family can be enlarged or the sandwich variance Σ estimated online.

Open problems. (i) The exact coverage rate $\gamma + O(\varepsilon^\beta)$ via nonlinear renewal theory [28], licensing bias-corrected intervals at moderate n ; (ii) the multivariate ellipsoidal procedure of Remark 5, with control of the least-informative direction; (iii) accelerated-failure-time and Cox regression on streams, where I may be ill-conditioned; (iv) a bounded-influence score variant; (v) a fully nonparametric online information estimate for unknown censoring.

8. Conclusion

We proposed and analysed a sequential fixed-width confidence procedure for right-censored data based on the averaged Robbins-Monro estimator and an online Fisher-information estimate. Under standard regularity and $cI > \frac{1}{2}$ it is asymptotically consistent and Chow-Robbins efficient at $O(1)$ per-observation cost. The

analysis isolates three transferable properties (H1)-(H3), corrects the linearisation step by a martingale-difference argument, and separates the two design requirements: stopping-time efficiency holds for every gain, while the gain condition controls coverage alone. Using the information rather than its reciprocal as the stopping scale is shown to collapse the procedure under heavy censoring.

Appendix: Proof notes

Monotonicity (Assumption 5). Since $M'(\theta_0) = -I < 0$ and M' is continuous, $M' \leq -\lambda$ near θ_0 for any $\lambda < I$, and the fundamental theorem of calculus gives $(\theta - \theta_0)M(\theta) \leq -\lambda(\theta - \theta_0)^2$.

Efficiency (Theorem 1). Consistency is Robbins-Siegmund applied to $(\theta_n - \theta_0)^2$; the projection binds finitely often since $\theta_0 \in \text{int } K$ [10, Thm. 5.2.1]. The averaged CLT is Polyak-Juditsky [22, Thm. 2]/[10, Thm. 10.1.1] with drift $\Gamma = -cI$, noise $S = c^2I$, limiting covariance $\Gamma^{-1}S\Gamma^{-1} = I^{-1}$; the rate $\gamma_n = c/n$ needs $cI > \frac{1}{2}$ [6, 18].

Non-asymptotic MSE (Proposition 1). With $\pi_{n,j} = \prod_{k=j}^{n-1} (1 - cI/k) = (j/n)^{cI} (1 + O(1/j))$, the variation-of-constants form of the linearised recursion has a deterministic transient contributing $O(n^{-cI})$, a stochastic term $1/(In) + o(1/n)$, and a nonlinearity remainder $O(n^{-cI})$; squaring and summing gives (3), with a $(\log n)^2$ factor at $cI = \frac{1}{2}$.

Moment bound (Lemma 1). With $V_n = d_n^2$, $\mathbb{E}[V_{n+1} \mid \mathcal{F}_n] \leq (1 - 2\lambda c/n)V_n + Cc^2/n^2(1 + V_n)$; for $2\lambda c > 1$ the substitution $Z_n = nV_n$ is contractive in expectation and Robbins-Siegmund gives $\sup_n \mathbb{E}Z_n < \infty$, the L^4 bound analogously [18, Ch. 4].

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TRAJECTORY ROUTES AND VELOCITY-SIGNATURE CLASSIFICATION OF TRANSITIVE
AND MIXED LOTKA-VOLTERRA TOURNAMENT OPERATORS ON S^3

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ABSTRACT. We give a complete velocity-signature classification of two discrete Lotka–Volterra operators generated by fully oriented tournaments on the three-dimensional simplex S^3 : the transitive tournament T_1 and the mixed tournament T_2 , in which one vertex is dominated by all others while the remaining three vertices form a directed 3-cycle. The two matrices coincide in their first row and column, so any difference in the resulting dynamics is attributable solely to the internal order of the three subordinate strategies. Using the identity $x'_k - x_k = x_k(Ax)_k$, we convert the sixteen formal sign patterns of Ax into systems of linear inequalities and classify all admissible ones for both models, obtaining $N(T_1) = 3$ and $N(T_2) = 7$. For each operator we give the explicit convex-hull domains (where tractable), the complete Jacobian spectrum at every vertex, a Lyapunov-type argument for global convergence, the fixed-point set, and the evolutionary-game reading. Although both operators eliminate the same dominated strategy, T_1 converges to a single dominant vertex, whereas the restriction of V_2 to the complementary invariant face coincides with the classical non-convergent three-cycle Volterra operator.

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Key words: Lotka–Volterra operator, quadratic stochastic operator, tournament, simplex, velocity signature, convex polyhedron, evolutionary game theory, Lyapunov function.

Introduction

Quadratic stochastic operators form an important class of nonlinear mappings acting on finite-dimensional simplexes, originating in mathematical genetics and heredity theory [1, 2, 3] and later developed as discrete dynamical systems with associated fixed points, invariant faces, and Lyapunov functions [4]. Among them, *Lotka–Volterra operators* are governed by a skew-symmetric interaction matrix, combining algebraic, geometric, and combinatorial methods [5, 6, 7, 8], in the tradition of the classical continuous Lotka–Volterra equations [13, 14].

Let

$$S^{m-1} = \left\{ x = (x_1, \dots, x_m) \in R^m : x_i \geq 0, \sum_{i=1}^m x_i = 1 \right\}.$$

A discrete-time Lotka–Volterra operator $V : S^{m-1} \rightarrow S^{m-1}$ is defined by

$$x'_k = x_k \left(1 + \sum_{i=1}^m a_{ki} x_i \right), \quad A = (a_{ki}), \quad a_{ki} = -a_{ik}, \quad |a_{ki}| \leq 1.$$

Skew-symmetry gives $x^T A x = 0$, so mass is preserved, and $|a_{ki}| \leq 1$ keeps all coordinates non-negative, so V maps S^{m-1} into itself. In the *tournament* case $a_{ki} \in \{-1, 1\}$ ($k \neq i$), every pair of components has a strict

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dominance relation, with the standard convention $i \rightarrow j \iff a_{ij} = 1$ [11], giving a natural correspondence between Lotka–Volterra dynamics and zero-sum evolutionary games [12].

This paper gives a complete classification of two non-isomorphic tournaments on $m = 4$ vertices that share a common structural feature: one vertex is dominated by all three others, while the relative order among those three differs. In T_1 the dominance hierarchy is fully transitive ($4 \rightarrow 3 \rightarrow 2 \rightarrow 1$); in T_2 the same hub vertex is present (vertex 1, dominated by all others), but the remaining three vertices 2, 3, 4 form a directed 3-cycle $2 \rightarrow 3 \rightarrow 4 \rightarrow 2$ rather than a linear order. Since A_1 and A_2 coincide in their first row and column, every difference between the two classifications below is attributable to this single change in the subordinate order.

The main tool is the sign structure of the velocity vector $Vx - x$. Since $x'_k - x_k = x_k(Ax)_k$, the sign of $(Ax)_k$ decides growth or decline of strategy k in $\text{int}(S^3)$. Each $\sigma \in \{+, -\}^4$ defines a *velocity signature*; the admissible signatures determine a decomposition of S^3 into convex polyhedra [9, 10]. We give a complete classification for both T_1 and T_2 at the same level of detail: explicit domains, full Jacobian spectra at every vertex, the Lyapunov-type method of Ganikhodzhaev [5] for global convergence, the fixed-point set, and the evolutionary-game reading.

Our main results are

$$N(T_1) = 3, \quad N(T_2) = 7,$$

and although both operators eliminate the same hub strategy, the cyclic subtournament in T_2 more than doubles the number of admissible domains and replaces T_1 's convergence to a single vertex with the non-convergent dynamics of the classical 3-cycle Volterra operator studied by Zakharevich [3].

Preliminaries

Write

$$S^3 = \left\{ x = (x_1, x_2, x_3, x_4) \in R^4 : x_i \geq 0, \sum_{i=1}^4 x_i = 1 \right\},$$

with vertices e_1, e_2, e_3, e_4 . Its edge midpoints, face barycenters, and overall barycenter are, respectively,

$$M_{ij} = \frac{e_i + e_j}{2}, \quad N_{ijk} = \frac{e_i + e_j + e_k}{3}, \quad K_{1234} = \frac{1}{4} \sum_{i=1}^4 e_i.$$

For V as above with $a_{ki} \in \{-1, 1\}$ ($k \neq i$), every face of S^3 is V -invariant, and the set of fixed points is denoted

$$\text{Fix}(V) = \{x \in S^3 : Vx = x\}.$$

Since $x'_k - x_k = x_k(Ax)_k$, the sign of the k -th coordinate of $Vx - x$ in the interior of the simplex is determined by the sign of $(Ax)_k$; this observation underlies the following definitions.

Definition 1. (Speed of the mapping) [5]. For V as above, the vector $Vx - x$ is called the *speed of the mapping at the point* $x \in S^3$.

Definition 2. (Velocity signature) [5, 9]. A *velocity signature* $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in \{+, -\}^4$ is assigned to a point $x \in \text{int}(S^3)$ according to the signs of $(Ax)_k$. For the classification of admissible signatures, we consider the strict systems

$$\sigma_k(Ax)_k > 0, \quad k = 1, \dots, 4.$$

Definition 3. (Admissible signature). A signature σ is *admissible* if the strict system $\sigma_k(Ax)_k > 0$ ($k = 1, \dots, 4$) has a solution in $\text{int } S^3$. The corresponding non-strict system $\sigma_k(Ax)_k \geq 0$ ($k = 1, \dots, 4$), together with the simplex constraints, then determines the associated convex polyhedron F_σ .

Definition 4. (Trajectory route) [9]. The sequence of admissible domains F_σ through which a trajectory $\{x^{(n)}\}$ passes, in order, is called its *trajectory route*.

For a skew-symmetric matrix A we also use the standard sets

$$P = \{x \in S^3 : Ax \geq 0\}, \quad Q = \{x \in S^3 : Ax \leq 0\},$$

which are closely related to the fixed points of the operator and describe possible repelling and attracting configurations [5]. The local character of a fixed point $x^* \in \text{Fix}(V)$ is studied by means of the Jacobian matrix

$$J(V)(x) = \left(\frac{\partial V_k(x)}{\partial x_i} \right)_{k,i=1}^4,$$

evaluated at $x = x^*$, together with its eigenvalues.

Two Tournament Models on S^3

Both tournaments below share the same hub vertex, which is dominated by all three others; they differ only in the internal orientation of the subtournament on the remaining three vertices, as shown in Figure 1.

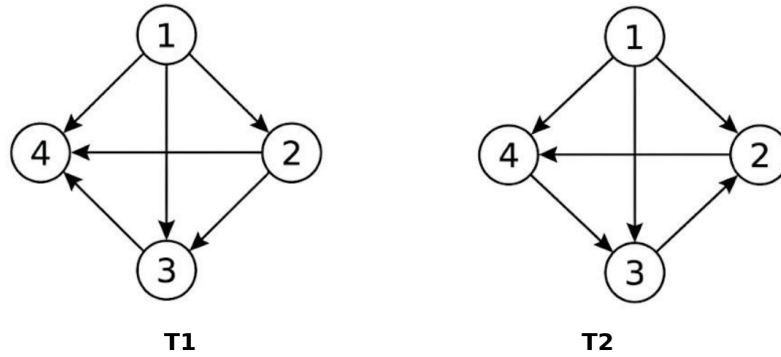


Figure 1: The two tournament representatives on four vertices. T_1 is transitive, with dominance order $4 \rightarrow 3 \rightarrow 2 \rightarrow 1$. T_2 has the same hub structure (vertex 1 dominated by all others) but the subtournament on $\{2, 3, 4\}$ is a directed 3-cycle $2 \rightarrow 3 \rightarrow 4 \rightarrow 2$ instead of a linear order.

The Transitive Tournament T_1 . Let T_1 be the transitive tournament with ordering $4 \rightarrow 3 \rightarrow 2 \rightarrow 1$. The corresponding matrix and operator are

$$A_1 = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad V_1 : \begin{cases} x'_1 = x_1(1 - x_2 - x_3 - x_4) \\ x'_2 = x_2(1 + x_1 - x_3 - x_4) \\ x'_3 = x_3(1 + x_1 + x_2 - x_4) \\ x'_4 = x_4(1 + x_1 + x_2 + x_3) \end{cases}$$

$$(A_1x)_1 = -x_2 - x_3 - x_4, \quad (A_1x)_2 = x_1 - x_3 - x_4, \quad (A_1x)_3 = x_1 + x_2 - x_4, \quad (A_1x)_4 = x_1 + x_2 + x_3.$$

For every $x \in \text{int}(S^3)$,

$$(A_1x)_1 < 0 \quad \text{and} \quad (A_1x)_4 > 0.$$

The Mixed Tournament T_2 . Let T_2 be the mixed tournament in which vertex 1 is dominated by all other vertices, while 2, 3, 4 form the directed cycle $2 \rightarrow 3 \rightarrow 4 \rightarrow 2$. The corresponding matrix and operator are

$$A_2 = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}, \quad V_2 : \begin{cases} x'_1 = x_1(1 - x_2 - x_3 - x_4) \\ x'_2 = x_2(1 + x_1 + x_3 - x_4) \\ x'_3 = x_3(1 + x_1 - x_2 + x_4) \\ x'_4 = x_4(1 + x_1 + x_2 - x_3) \end{cases}$$

$$(A_2x)_1 = -x_2 - x_3 - x_4, \quad (A_2x)_2 = x_1 + x_3 - x_4, \quad (A_2x)_3 = x_1 - x_2 + x_4, \quad (A_2x)_4 = x_1 + x_2 - x_3.$$

For every $x \in \text{int}(S^3)$, $(A_2x)_1 < 0$ – identical to T_1 's first coordinate, since A_1 and A_2 share their first row and column exactly. Unlike T_1 , none of $(A_2x)_2, (A_2x)_3, (A_2x)_4$ has a fixed sign, since the cyclic subtournament on $\{2, 3, 4\}$ admits no internal dominance order.

General Signature Decomposition Method

For V_r generated by A_r ($r = 1, 2$), $x'_k - x_k = x_k(A_r x)_k$, so on $\text{int}(S^3)$ the sign of $x'_k - x_k$ equals that of $(A_r x)_k$.

Proposition 1. *For every tournament T_r , each admissible set $F_\sigma^{(r)}$ is a convex polyhedron in S^3 .*

Proof. The simplex S^3 is a convex polytope. For a fixed signature σ , the conditions defining $F_\sigma^{(r)}$ are linear inequalities of the form $(A_r x)_k \geq 0$ or $(A_r x)_k \leq 0$. Therefore $F_\sigma^{(r)}$ is the intersection of S^3 with finitely many closed half-spaces. $\square$

Algorithm 1 (Velocity-signature feasibility verification). For a given tournament matrix A_r : (1) list all $\sigma \in \{+, -\}^4$; (2) for each σ , test the strict system $\sigma_k(A_r x)_k > 0$ ($k = 1, \dots, 4$) together with $x_i > 0$, $\sum_i x_i = 1$; (3) if feasible, σ is admissible, defining $F_\sigma^{(r)}$ by the corresponding non-strict system; (4) otherwise discard σ ; (5) enumerate the surviving domains $F_1^{(r)}, \dots, F_{N_r}^{(r)}$. For the present four-dimensional case, all feasibility tests are verified analytically (Theorems 1 and 3); the finite enumeration above is included only as a systematic organizational procedure.

Lemma 1. *Algorithm 1 terminates after finitely many steps and returns exactly the set of admissible signatures.*

Proof. The algorithm tests each of the $2^4 = 16$ elements of $\{+, -\}^4$ exactly once, so every possible sign pattern of $A_r x$ in the interior of the simplex is tested exactly once; feasible systems give admissible signatures, while infeasible systems are excluded. $\square$

Complete Analysis of the Transitive Tournament T_1

Admissible signatures are determined by the signs of $(A_1 x)_2 = x_1 - x_3 - x_4$ and $(A_1 x)_3 = x_1 + x_2 - x_4$. Moreover,

$$(A_1 x)_3 - (A_1 x)_2 = x_2 + x_3 > 0,$$

so $(A_1 x)_2 > 0$ together with $(A_1 x)_3 < 0$ is impossible.

Theorem 1. *A signature σ is admissible for T_1 if and only if*

$$\sigma \in \{(-, -, -, +), (-, -, +, +), (-, +, +, +)\}.$$

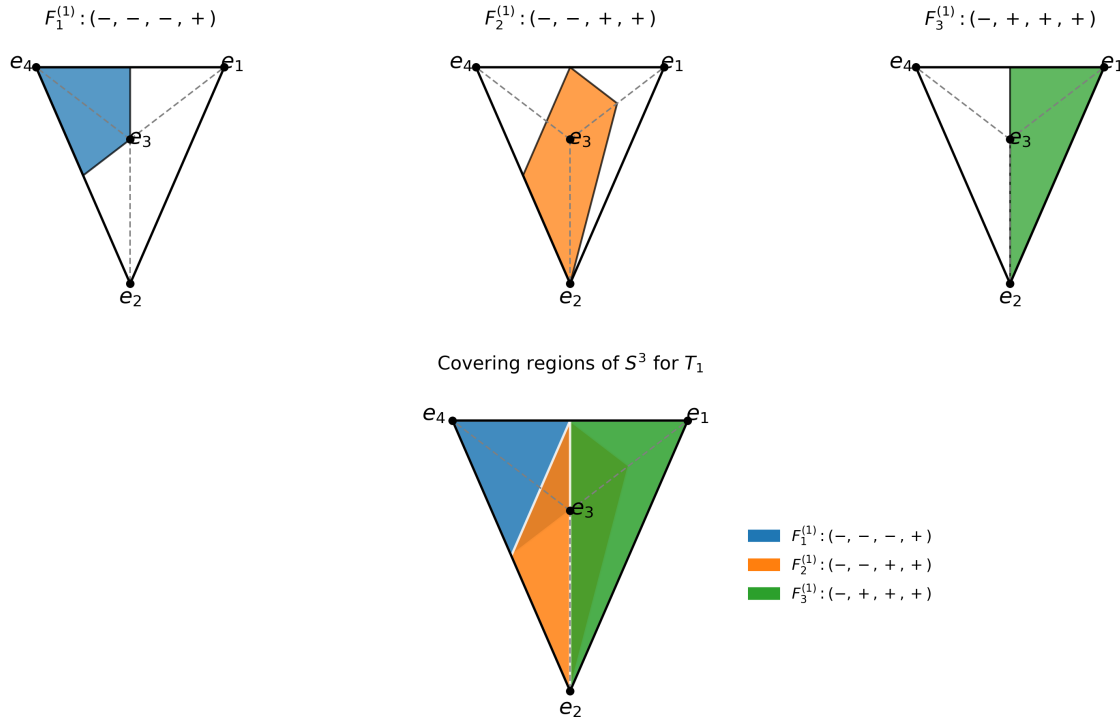
Consequently $N(T_1) = 3$.

Proof. Only σ_2, σ_3 can vary; as noted above, $(A_1 x)_3 - (A_1 x)_2 = x_2 + x_3 > 0$ throughout $\text{int}(S^3)$, so the pattern $\sigma_2 = +, \sigma_3 = -$ is excluded, leaving exactly the three signatures above. $\square$

The corresponding domains and their convex hulls are

$$F_1^{(1)} = \text{co}\{e_4, e_3, M_{24}, M_{14}\}, \quad F_2^{(1)} = \text{co}\{e_3, e_2, M_{24}, M_{14}, M_{13}\}, \quad F_3^{(1)} = \text{co}\{e_2, e_1, M_{14}, M_{13}\},$$

so that $S^3 = F_1^{(1)} \cup F_2^{(1)} \cup F_3^{(1)}$ (Figure 2).

Velocity-signature decomposition of S^3 for T_1 ($N = 3$)

 Figure 2: Velocity-signature decomposition of S^3 for T_1 : the three admissible domains $F_1^{(1)}$, $F_2^{(1)}$, $F_3^{(1)}$ and their union.

The sets are $P_1 = \{e_1\}$, $Q_1 = \{e_4\}$. The Jacobian of V_1 at each vertex has the following spectrum on S^3 (Table 1).

 Table 1: Jacobian spectra of V_1 at the four vertices.

Vertex	Eigenvalues	Character
e_1	1, 2, 2, 2	repelling
e_2	0, 1, 2, 2	saddle
e_3	0, 0, 1, 2	saddle
e_4	0, 0, 0, 1	attracting

The global attracting property is obtained by the Lyapunov-function method of Ganikhodzhaev [5]. Ganikhodzhaev's Theorem 1 associates a monomial Lyapunov function

$$\varphi(x) = x_1^{p_1} x_2^{p_2} x_3^{p_3} x_4^{p_4}$$

with every solution $p = (p_1, p_2, p_3, p_4)$ of $A_1 y \geq 0$ in S^3 , i.e., with every $p \in P_1$; since $P_1 = \{e_1\}$, taking $p = e_1$ gives the degenerate monomial $\varphi(x) = x_1$. This is complemented by the block-sum construction of the same paper (§3), applied to $Q_1 = \{e_4\}$: the sum of the coordinates *outside* Q_1 ,

$$\psi(x) = x_1 + x_2 + x_3 = 1 - x_4.$$

Lemma 2. (Lyapunov functions for T_1) [5]. *For every $x \in S^3$,*

$$\varphi(V_1 x) = \varphi(x)^2, \quad \psi(V_1 x) = \psi(x)^2.$$

Consequently $0 \leq \varphi(V_1x) \leq \varphi(x) \leq 1$ and $0 \leq \psi(V_1x) \leq \psi(x) \leq 1$, with strict inequality on $\text{int}(S^3)$, and

$$\lim_{n \rightarrow \infty} \varphi(x^{(n)}) = \lim_{n \rightarrow \infty} \psi(x^{(n)}) = 0 \quad \text{for every } x \in \text{int}(S^3).$$

Proof. Since $x_1 + x_2 + x_3 + x_4 = 1$,

$$\varphi(V_1x) = x'_1 = x_1(1 - x_2 - x_3 - x_4) = x_1(1 - (1 - x_1)) = x_1^2 = \varphi(x)^2,$$

$$\psi(V_1x) = 1 - x'_4 = 1 - x_4(1 + x_1 + x_2 + x_3) = 1 - x_4(2 - x_4) = (1 - x_4)^2 = \psi(x)^2.$$

Since $0 \leq \varphi(x) \leq 1$ and $0 \leq \psi(x) \leq 1$ on S^3 , squaring gives $\varphi(V_1x) \leq \varphi(x)$ and $\psi(V_1x) \leq \psi(x)$, with equality only at the respective endpoints $\{0, 1\}$; both inequalities are strict on $\text{int}(S^3)$. Iterating, $\varphi(x^{(n)}) = \varphi(x)^{2^n} \rightarrow 0$ and $\psi(x^{(n)}) = \psi(x)^{2^n} \rightarrow 0$. $\square$

Thus $x_1 \rightarrow 0$ and $x_4 \rightarrow 1$ strictly monotonically along every interior trajectory.

Theorem 2. For every $x \in \text{int}(S^3)$, $\lim_{n \rightarrow \infty} V_1^n x = e_4$.

Proof. By Lemma 2, $\psi(x^{(n)}) = 1 - x_4^{(n)} \rightarrow 0$, so $x_4^{(n)} \rightarrow 1$. Since $\sum_{k=1}^4 x_k^{(n)} = 1$ with all coordinates non-negative, the remaining coordinates tend to 0. $\square$

The trajectory route (Definition 4) is $P_1 \rightarrow F_3^{(1)} \rightarrow F_2^{(1)} \rightarrow F_1^{(1)} \rightarrow Q_1$, from the weakest to the dominant pure strategy.

Complete Analysis of the Mixed Tournament T_2

Define

$$g_2(x) = (A_2x)_2 = x_1 + x_3 - x_4,$$

$$g_3(x) = (A_2x)_3 = x_1 - x_2 + x_4,$$

$$g_4(x) = (A_2x)_4 = x_1 + x_2 - x_3.$$

As noted above, $(A_2x)_1 < 0$ throughout $\text{int}(S^3)$, and $g_2 + g_3 + g_4 = 3x_1 > 0$, so the all-negative pattern for (g_2, g_3, g_4) is impossible.

Theorem 3. A signature σ is admissible for T_2 if and only if

$$\sigma_1 = - \quad \text{and} \quad (\sigma_2, \sigma_3, \sigma_4) \neq (-, -, -).$$

Consequently, $N(T_2) = 7$.

Proof. By Definition 2, $\sigma_k = \text{sign}((A_2x)_k)$ for $k = 1, 2, 3, 4$; in particular $\sigma_k = \text{sign}(g_k(x))$ for $k = 2, 3, 4$. As shown above, $(A_2x)_1 < 0$ throughout $\text{int}(S^3)$, so

$$\sigma_1 = \text{sign}((A_2x)_1) = -$$

for every admissible signature. Among the eight sign patterns of $(\sigma_2, \sigma_3, \sigma_4) \in \{+, -\}^3$, exactly one is infeasible:

$$(\sigma_2, \sigma_3, \sigma_4) = (-, -, -) \implies g_2, g_3, g_4 < 0 \implies g_2 + g_3 + g_4 < 0,$$

which contradicts the identity

$$g_2 + g_3 + g_4 = 3x_1 > 0.$$

It remains to show that each of the other seven patterns is genuinely realized by a point of $\text{int}(S^3)$; this is verified directly from the formulas for g_2, g_3, g_4 at the points below.

$(\sigma_2, \sigma_3, \sigma_4)$	$x = (x_1, x_2, x_3, x_4)$
$(-, -, +)$	$(0.1, 0.5, 0.1, 0.3)$
$(-, +, -)$	$(0.1, 0.1, 0.3, 0.5)$
$(-, +, +)$	$(1/6, 1/6, 1/6, 1/2)$
$(+, -, -)$	$(0.1, 0.3, 0.5, 0.1)$
$(+, -, +)$	$(1/6, 1/2, 1/6, 1/6)$
$(+, +, -)$	$(1/6, 1/6, 1/2, 1/6)$
$(+, +, +)$	$(1/4, 1/4, 1/4, 1/4)$

Each point has strictly positive coordinates summing to 1, and a direct substitution into $g_2 = x_1 + x_3 - x_4$, $g_3 = x_1 - x_2 + x_4$, $g_4 = x_1 + x_2 - x_3$ confirms the stated signs in every row. Hence all seven patterns are admissible, and $N(T_2) = 7$. $\square$

The seven domains are listed in Table 2, together with their explicit vertex sets. Unlike T_1 's three domains, most of T_2 's domains are polytopes with vertices beyond the standard points e_i, M_{ij}, N_{ijk} ; each vertex set below is obtained by the same half-space intersection method.

Table 2: The seven admissible domains of T_2 : signatures, defining inequalities, and vertices.

Domain	Signature	Conditions	Vertices
$F_1^{(2)}$	$(-, -, -, +)$	$g_2 \leq 0, g_3 \leq 0, g_4 \geq 0$	$\text{co}\{N_{234}, M_{24}, e_2, (\frac{1}{4}, \frac{1}{2}, 0, \frac{1}{4})\}$
$F_2^{(2)}$	$(-, -, +, -)$	$g_2 \leq 0, g_3 \geq 0, g_4 \leq 0$	$\text{co}\{e_4, M_{34}, (\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}), N_{234}\}$
$F_3^{(2)}$	$(-, -, +, +)$	$g_2 \leq 0, g_3 \geq 0, g_4 \geq 0$	$\text{co}\{M_{14}, (\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}), e_4, M_{24}, (\frac{1}{4}, \frac{1}{2}, 0, \frac{1}{4}), N_{234}\}$
$F_4^{(2)}$	$(-, +, -, -)$	$g_2 \geq 0, g_3 \leq 0, g_4 \leq 0$	$\text{co}\{e_3, N_{234}, M_{23}, (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)\}$
$F_5^{(2)}$	$(-, +, -, +)$	$g_2 \geq 0, g_3 \leq 0, g_4 \geq 0$	$\text{co}\{M_{12}, (\frac{1}{4}, \frac{1}{2}, 0, \frac{1}{4}), e_2, M_{23}, (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0), N_{234}\}$
$F_6^{(2)}$	$(-, +, +, -)$	$g_2 \geq 0, g_3 \geq 0, g_4 \leq 0$	$\text{co}\{M_{13}, (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0), e_3, M_{34}, (\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}), N_{234}\}$
$F_7^{(2)}$	$(-, +, +, +)$	$g_2 \geq 0, g_3 \geq 0, g_4 \geq 0$	$\text{co}\{e_1, M_{12}, M_{14}, (\frac{1}{4}, \frac{1}{2}, 0, \frac{1}{4}), (\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}), N_{234}, M_{13}, (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)\}$

Each vertex set was obtained by intersecting the corresponding half-spaces $\sigma_k g_k \geq 0$ with S^3 and computing the extreme points of the resulting polytope. The barycenter K_{1234} lies in the strict part of $F_7^{(2)}$, since

$$g_2(K_{1234}) = g_3(K_{1234}) = g_4(K_{1234}) = \frac{1}{4}.$$

Since each $F_\sigma^{(2)}$ is a full three-dimensional sub-polytope of S^3 , no single planar picture can display all seven domains without some overlap once they are projected onto the page; Figure 3 therefore shows each domain on its own set of axes.

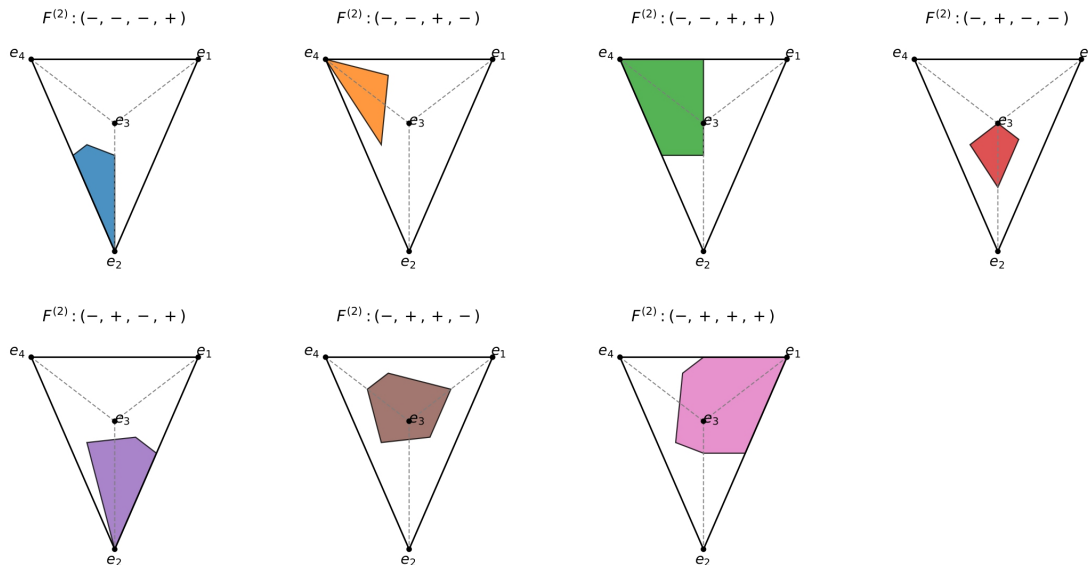


Figure 3: The seven admissible domains of T_2 , shown individually within the simplex S^3 .

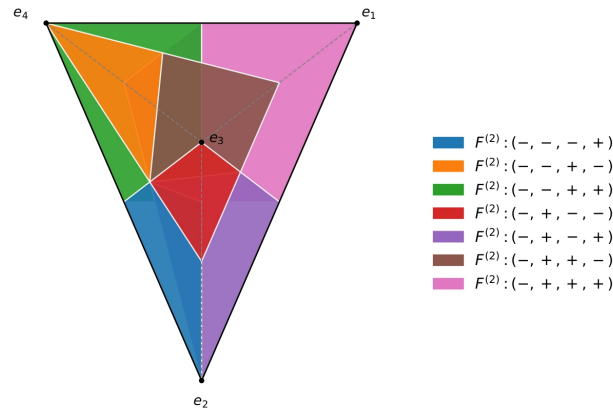


Figure 4: Velocity-signature decomposition of S^3 for T_2 : the seven domains of Figure 3 superimposed. Several projected domains overlap in this two-dimensional visualization, although they remain distinct subsets of the three-dimensional simplex.

The sets are $P_2 = \{e_1\}$, $Q_2 = \{N_{234}\}$, where $N_{234} = (0, 1/3, 1/3, 1/3)$.

Table 3: Jacobian spectra of V_2 at the four vertices.

Vertex	Eigenvalues	Character
e_1	1, 2, 2, 2	repelling
e_2, e_3, e_4	0, 0, 1, 2	saddle

Unlike T_1 , where the four vertices have four distinct spectra, in T_2 the three subordinate vertices e_2, e_3, e_4 share a single spectrum, a direct consequence of the cyclic symmetry among them.

Proposition 2. $\text{Fix}(V_2) = \{e_1, e_2, e_3, e_4\} \cup \{N_{234}\}$, and V_2 has no fixed point in $\text{int}(S^3)$.

Proof. Applying the support-based fixed-point method of [6] to A_2 : fixed points with $|S| \leq 2$ reduce to vertices; the unique 3-cycle $\{2, 3, 4\}$ gives N_{234} ; and $|S| = 4$ is excluded since $\det A_2 = 9 \neq 0$. $\square$

Theorem 4. For every $x \in \text{int}(S^3)$, $x_1^{(n)} \rightarrow 0$ and $V_2^n x$ approaches the face $\text{co}\{e_2, e_3, e_4\}$.

Proof. Since $(A_2 x)_1 = -x_2 - x_3 - x_4 = -(1 - x_1)$ on S^3 , we get $x'_1 = x_1(1 - (1 - x_1)) = x_1^2$, so $x_1^{(n)} = (x_1^{(0)})^{2^n} \rightarrow 0$. $\square$

Unlike T_1 , whose three domains are totally ordered by a single inequality and therefore admit the linear trajectory route $P_1 \rightarrow F_3^{(1)} \rightarrow F_2^{(1)} \rightarrow F_1^{(1)} \rightarrow Q_1$, the seven domains of T_2 carry no such total order: (g_2, g_3, g_4) vary independently subject only to $g_2 + g_3 + g_4 = 3x_1$, so a trajectory's passage among $F_1^{(2)}, \dots, F_7^{(2)}$ depends on the initial point, and the only structural invariant along every route is the monotone elimination $x_1^{(n)} \rightarrow 0$.

Remark 1. Restricted to the invariant face $x_1 = 0$, V_2 reduces exactly to the classical 3-cycle Volterra operator on $\{e_2, e_3, e_4\}$ studied by Zakharevich [3], whose trajectories exhibit non-convergent (“historic”) behavior: the time spent near each vertex grows without bound, and no interior trajectory converges. Thus, although $x_1 \rightarrow 0$ always, the limiting behavior of (x_2, x_3, x_4) is generally *not* convergence to N_{234} or to any vertex.

Comparative Discussion

Table 4 summarizes the classification. Both operators share the same repelling vertex e_1 and the same first-coordinate identity $(A_r x)_1 < 0$; the difference in the number of admissible domains traces to the internal cyclic structure on $\{2, 3, 4\}$ alone.

Table 4: Comparative summary of T_1 and T_2 .

	T_1 (transitive)	T_2 (mixed)
Domains N	3	7
P_r	$\{e_1\}$	$\{e_1\}$
Q_r	$\{e_4\}$	$\{N_{234}\}$
Domain ordering	Total order (single route)	No total order; route depends on initial data
Asymptotics	$V_1^n x \rightarrow e_4$ for all $x \in \text{int}(S^3)$	$x_1 \rightarrow 0$; residual dynamics on $\text{co}\{e_2, e_3, e_4\}$
Spectra at subordinate vertices	Four distinct saddle types	Single saddle type, repeated 3 times

In particular, $N(T_1) \neq N(T_2)$: the velocity-signature count distinguishes the transitive and mixed structures even though both tournaments share an identical hub vertex and an identical first coordinate $(A_r x)_1$.

Evolutionary Game-Theoretic Interpretation

Both A_1, A_2 are payoff matrices of zero-sum games on four pure strategies [12]. In T_1 , the strict hierarchy $4 \rightarrow 3 \rightarrow 2 \rightarrow 1$ makes e_4 the globally attracting pure strategy for interior initial conditions: Theorem 2 shows every interior population converges to it, consistent with the evolutionary-game reading of e_4 as the unique stable outcome. In T_2 , the hub strategy 1 is still eliminated (Theorem 4), but the cyclic subgame on $\{2, 3, 4\}$ has no such stable pure outcome: by Remark 1, the residual dynamics is the classical non-convergent rock-paper-scissors-type cycle, so the population perpetually cycles among e_2, e_3, e_4 rather than settling on any one of them. This is the qualitative dynamical consequence of replacing a single transitive edge-order among the three subordinate strategies with a 3-cycle, while leaving the dominance relation to the hub vertex unchanged.

Conclusion

We gave a complete velocity-signature classification of two Lotka–Volterra tournament operators on S^3 sharing a common dominance hub: $N(T_1) = 3$ for the fully transitive case and $N(T_2) = 7$ for the mixed case with a cyclic subordinate triple. For T_1 , explicit convex-hull domains, full Jacobian spectra, and a Lyapunov-type argument establish global convergence to e_4 . For T_2 , the analogous machinery shows elimination of the same hub strategy, but the residual dynamics has the classical non-convergent behavior of the Zakharevich three-cycle Volterra operator [3], which does not converge to any pure strategy. The two classifications differ only through the internal order of the three subordinate vertices, so this order alone controls both the size of the velocity-signature decomposition and whether the population ultimately settles on a pure strategy or exhibits this non-convergent cyclic behavior.

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CONSISTENCY OF WILD BOOTSTRAP FOR RANDOM VARIABLES WITH VALUES IN $L_p[0; 1]$

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ABSTRACT. This paper investigates the consistency of the wild bootstrap for independent identically distributed random elements in the separable Banach space $L_p[0; 1]$, $p > 2$. It is shown that, conditionally on the original sample, the normalized bootstrap mean converges weakly in probability to the same limiting distribution as the normalized sample mean. The proof is based on a central limit theorem for triangular arrays of random elements in $L_p[0; 1]$.

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Key words: wild bootstrap, L_p -space, bootstrap consistency, central limit theorem, triangular arrays.

Introduction

The bootstrap, introduced in [3], is one of the most widely used resampling methods in modern statistics. Its main idea is to approximate the unknown distribution of a statistic by the conditional distribution of the corresponding statistic computed from a bootstrap sample. One of the fundamental theoretical problems in this context is to establish the consistency of the bootstrap procedure.

Fundamental results on bootstrap consistency were obtained in [1], as well as in [4]. For dependent observations, classical bootstrap procedures generally fail to preserve the dependence structure of the original data, which has led to the development of block bootstrap methods and other resampling techniques for dependent sequences; see, for example, [6, 12].

The wild bootstrap originates from the ideas of [14] and was further developed in [8] and [9]. Consistency of wild bootstrap for dependent random variables with values in Hilbert space were considered in [13]. In [2] the authors demonstrate the validity of the wild bootstrap in $C[0; 1]$.

In the present paper, we establish the consistency of the Gaussian wild bootstrap for independent and identically distributed random elements taking values in a $L_p[0; 1]$ space. Using independent standard Gaussian multipliers and a central limit theorem for triangular arrays [10], we verify a Lindeberg-type condition, and convergence of the covariance structure. As a result, we prove the conditional weak convergence of the normalized bootstrap mean to the same centered Gaussian limit as that of the normalized mean of the original observations.

Main part

In this section, we establish the conditional weak convergence of the Gaussian multiplier wild bootstrap in the separable Banach space $L_p[0, 1]$, $p > 2$. The space $L_p[0, 1]$ consists of all equivalence classes of measurable functions $f : [0, 1] \rightarrow \mathbb{R}$ such that $\int_0^1 |f(t)|^p \mu(dt) < \infty$ with the norm $\|f\|_p = \left(\int_0^1 |f(t)|^p \mu(dt) \right)^{1/p}$. In what follows we will use $\|\cdot\|$ for the norm of all spaces under consideration, hopefully that this will not lead to confusion.

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Definition.[7] A separable Banach space B is called type 2 Banach space if for any finite collection of B -valued independent random variables with $\mathbb{E}X_i = 0$, $\mathbb{E}\|X_i\|^2 < \infty$, $i = \overline{1, n}$ the following inequality holds

$$\mathbb{E} \left\| \sum_{i=1}^n X_i \right\|^2 \leq C \sum_{i=1}^n \mathbb{E}\|X_i\|,$$

where a constant C depends on B only.

Theorem 1.[7] Let $\{X_i(t)\}_{i \geq 1}$ be a sequence of independent identically distributed mean-zero random elements taking values in a separable Banach space B of type 2, and suppose that $\mathbb{E}\|X_1\|^{2+\delta} < \infty$. Then the following weak convergence holds as $n \rightarrow \infty$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i(t) \Rightarrow Y(t) \quad \text{in } B \text{ and } K(s, t) = \mathbb{E}Y(s)Y(t) = \mathbb{E}X_1(s)X_1(t) \quad s, t \in [0, 1],$$

where $Y(t)$ is a centered Gaussian random element. Conversely, if every sequence of independent identically distributed mean-zero B -valued random elements satisfying $\mathbb{E}\|X_1\|^2 < \infty$ satisfies the central limit theorem, then B is of type 2.

This theorem is applicable in our setting, as $L_p[0, 1]$, $p \geq 2$ is type 2 Banach Space.

Let $X_1(t), X_2(t), \dots$ be independent identically distributed $L_p[0, 1]$ -valued random elements. Denote

$$\bar{X}_n(t) = \frac{1}{n} \sum_{i=1}^n X_i(t).$$

Let $W_1, W_2, \dots$ be independent standard Gaussian random variables, independent of the original sample. Define the wild bootstrap observations by

$$X_i^*(t) = \bar{X}_n(t) + (X_i(t) - \bar{X}_n(t))W_i, \quad i = 1, \dots, n,$$

and define the corresponding bootstrap sample mean by

$$\bar{X}_n^*(t) = \frac{1}{n} \sum_{i=1}^n X_i^*(t).$$

Then

$$\sqrt{n}(\bar{X}_n^*(t) - \bar{X}_n(t)) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i^*(t) - \bar{X}_n(t)) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i(t) - \bar{X}_n(t))W_i.$$

Thus, the validity of the wild bootstrap is reduced to establishing the conditional weak convergence of the normalized centered bootstrap sample mean. The following theorem gives the corresponding result in $L_p[0, 1]$.

Theorem 2. Let $X_1(t), X_2(t), \dots$ be independent identically distributed $L_p[0, 1]$ -valued random elements such that

$$\mathbb{E}X_1(t) = 0, \quad \mathbb{E}\|X_1(t)\|^{2+\delta} < \infty \text{ for some } \delta > 0.$$

Then the following weak convergence holds as $n \rightarrow \infty$

$$\sqrt{n}(\bar{X}_n^*(t) - \bar{X}_n(t)) \Rightarrow Y(t) \text{ in Probability,}$$

where $Y(t)$ is a centered Gaussian random element in $L_p[0, 1]$ with mean zero and covariance function $K(s, t) = \mathbb{E}Y(s)Y(t) = \mathbb{E}X_1(s)X_1(t)$, $s, t \in [0, 1]$.

Thus theorems 1 and 2 imply consistency of the wild bootstrap.

Proof. The proof of our theorem is based on the central limit theorem for triangular arrays of independent random elements in $L_p[0, 1]$, $p > 2$, obtained in [10]. We first recall this result. For convenience, in what follows we omit the argument t and write X_i , $\bar{X}_n$, and X_i^* instead of $X_i(t)$, $\bar{X}_n(t)$, and $X_i^*(t)$, respectively. We set $S = [0, 1]$, let $\mathcal{S}$ be the Lebesgue σ -algebra on $[0, 1]$, and let μ be the Lebesgue measure.

Theorem 3.[10] Let $(X_{nk}, k = 1, \dots, r_n; n \in \mathbb{N})$ be a triangular array of mean zero independent random elements in the separable space $L_p(S, \mathcal{S}, \mu; \mathbb{R})$, $p > 2$. Assume that the following conditions are satisfied.

(a)

$$\sup_{n \in \mathbb{N}} \sum_{k=1}^{r_n} \mathbb{E} \|X_{nk}\|^2 < \infty.$$

(b) For every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{r_n} \mathbb{E} [\|X_{nk}\|^2 \mathbf{1}_{\{\|X_{nk}\| > \varepsilon\}}] = 0.$$

(c) There is a mean zero Gaussian random element $Y \in L_p(S, \mathcal{S}, \mu; \mathbb{R})$ such that

$$\sum_{k=1}^{r_n} \mathbb{E}(X_{nk} \otimes X_{nk}) \longrightarrow \Gamma := \mathbb{E}(Y \otimes Y) \text{ in } L_p(S^2, \mathcal{S} \otimes \mathcal{S}, \mu \otimes \mu; \mathbb{R}),$$

where $(X_{nk} \otimes X_{nk})(s, t) = X_{nk}(s)X_{nk}(t)$, $(s, t) \in S^2$. Since the random elements are centered, $\mathbb{E}(X_{nk} \otimes X_{nk})$ represents their covariance function.

(d) Denoting by σ_n and σ the non-negative elements $L_p(S, \mathcal{S}, \mu; \mathbb{R})$ defined by $\sigma_n^2(s) := \sum_{k=1}^{r_n} \mathbb{E} X_{nk}(s)^2$ and $\sigma^2(s) := \mathbb{E} Y(s)^2$, μ -a.e. on S ,

$$\int_S \sigma_n^p, d\mu \xrightarrow{n \rightarrow \infty} \int_S \sigma^p, d\mu.$$

Then

$$\sum_{k=1}^{r_n} X_{nk} \xrightarrow{n \rightarrow \infty} Y \text{ in distribution.}$$

Theorem 4.[7] Let $X_1, X_2, \dots$ be independent identically distributed random elements with values in a separable Banach space B . Then $\frac{1}{n} \sum_{i=1}^n X_i \rightarrow 0$ a.s. if and only if $E|X_1| < \infty$ and $EX_1 = 0$.

A fixed realization of the original sample $X_1, \dots, X_n$, we denote by $x_1, \dots, x_n$, and set $\bar{x}_n = \frac{1}{n} \sum_{i=1}^n x_i$. Then $X_i^* = \frac{1}{\sqrt{n}}(x_i - \bar{x}_n)W_i$, $i = 1, \dots, n$.

Thus, it remains to verify that the wild bootstrap random elements satisfy the conditions of Theorem 2. We check these conditions successively.

a) By definition,

$$X_i^* = \frac{1}{\sqrt{n}}(x_i - \bar{x}_n)W_i, \quad i = 1, \dots, n.$$

Hence,

$$\begin{aligned} \sup_{n \in \mathbb{N}} \sum_{i=1}^n \mathbb{E} \left\| \frac{1}{\sqrt{n}}(x_i - \bar{x}_n)W_i \right\|^2 &= \sup_{n \in \mathbb{N}} \sum_{i=1}^n \mathbb{E} \left[\frac{1}{n} W_i^2 \|x_i - \bar{x}_n\|^2 \right] \\ &= \sup_{n \in \mathbb{N}} \sum_{i=1}^n \frac{1}{n} \|x_i - \bar{x}_n\|^2. \end{aligned}$$

Let $M > 0$. Then, by Markov's inequality,

$$\begin{aligned} \mathbb{P} \left(\sum_{i=1}^n \frac{1}{n} \|X_i - \bar{X}_n\|^2 > M \right) &= \mathbb{P} \left(\frac{1}{n} \sum_{i=1}^n \|X_i - \bar{X}_n\|^2 > M \right) \\ &\leq \frac{\mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|X_i - \bar{X}_n\|^2 \right]}{M} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n (\|X_i\|^2 + \|\bar{X}_n\|^2) \right]}{M} \\
&\leq \frac{\mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|X_i\|^2 + \frac{1}{n} \sum_{i=1}^n \|\bar{X}_n\|^2 \right]}{M} \\
&= \frac{\mathbb{E}\|X_i\|^2 + \mathbb{E}\|\bar{X}_n\|^2}{M}.
\end{aligned}$$

From the definition of type 2 Banach spaces we have

$$\begin{aligned}
&= \frac{\mathbb{E}\|X_i\|^2 + \mathbb{E}\|\bar{X}_n\|^2}{M} \leq \frac{\mathbb{E}\|X_i\|^2}{M} + \frac{\mathbb{E}\|\sum_{i=1}^n X_i\|^2}{n^2 M} \leq \\
&\leq \frac{\mathbb{E}\|X_i\|^2}{M} + \frac{C \sum_{i=1}^n \mathbb{E}\|X_i\|^2}{n^2 M} = \frac{\mathbb{E}\|X_i\|^2}{M} + \frac{C \mathbb{E}\|X_i\|^2}{nM} \rightarrow 0 \text{ as } M \rightarrow \infty.
\end{aligned}$$

Consequently,

$$\sup_{n \in \mathbb{N}} \sum_{i=1}^n \mathbb{E}\|X_i^*\|^2 < \infty$$

is bounded in Probability and thus condition (a) is satisfied.

b) We now verify the Lindeberg condition. Using $X_i^* = \frac{1}{\sqrt{n}}(x_i - \bar{x}_n)W_i$ and the inequality $\mathbf{1}_{\{x > \varepsilon\}} \leq \frac{x^\delta}{\varepsilon^\delta}$, for $x \geq 0$, we obtain

$$\begin{aligned}
&\sum_{i=1}^n \mathbb{E} \left[\left\| \frac{1}{\sqrt{n}}(x_i - \bar{x}_n)W_i \right\|^2 \mathbf{1}_{\left\{ \left\| \frac{1}{\sqrt{n}}(x_i - \bar{x}_n)W_i \right\| > \varepsilon \right\}} \right] \\
&\leq \frac{1}{\varepsilon^\delta} \sum_{i=1}^n \mathbb{E} \left[\left\| \frac{1}{\sqrt{n}}(x_i - \bar{x}_n)W_i \right\|^{2+\delta} \right] \\
&= \frac{\mathbb{E}|W_1|^{2+\delta}}{\varepsilon^\delta n^{1+\delta/2}} \sum_{i=1}^n \|x_i - \bar{x}_n\|^{2+\delta} \\
&= \frac{\mathbb{E}|W_1|^{2+\delta}}{\varepsilon^\delta n^{\delta/2}} \left(\frac{1}{n} \sum_{i=1}^n \|x_i - \bar{x}_n\|^{2+\delta} \right).
\end{aligned}$$

The convergence $\frac{1}{n} \sum_{i=1}^n \|X_i - \bar{X}_n\|^{2+\delta} \rightarrow \mathbb{E}\|X_1 - \mathbb{E}X_1\|^{2+\delta} = \mathbb{E}\|X_1\|^{2+\delta} < \infty$ a.s. implies $\frac{\mathbb{E}|W_1|^{2+\delta}}{\varepsilon^\delta n^{\delta/2}} \left(\frac{1}{n} \sum_{i=1}^n \|X_i - \bar{X}_n\|^{2+\delta} \right) \rightarrow 0$ a.s..

Therefore, condition **(b)** is satisfied.

c) Now we will verify condition (c).

It is sufficient to prove that $\sum_{i=1}^n \text{Cov}(X_i^*(s), X_i^*(t)) \rightarrow \text{Cov}(Y(s), Y(t))$ in probability in $L_p([0, 1]^2)$.

By the definition of covariance function

$$K(s, t) = \mathbb{E}[(X(s) - \mathbb{E}X(s))(X(t) - \mathbb{E}X(t))] = \mathbb{E}[X(s)X(t)] - \mathbb{E}[X(s)]\mathbb{E}[X(t)].$$

Now we calculate the covariance of the bootstrap variables. Since

$$\mathbb{E}[X_i^*(s)] = \mathbb{E}[X_i^*(t)] = 0,$$

we have

$$\begin{aligned}\sum_{i=1}^n \text{Cov}(X_i^*(s), X_i^*(t)) &= \sum_{i=1}^n (\mathbb{E}[X_i^*(s)X_i^*(t)] - \mathbb{E}[X_i^*(s)]\mathbb{E}[X_i^*(t)]) \\ &= \sum_{i=1}^n \mathbb{E}[X_i^*(s)X_i^*(t)].\end{aligned}$$

Using the definition

$$X_i^* = \frac{1}{\sqrt{n}}(x_i - \bar{x}_n)W_i,$$

we obtain

$$\begin{aligned}\sum_{i=1}^n \mathbb{E}[X_i^*(s)X_i^*(t)] &= \sum_{i=1}^n \mathbb{E}\left[\frac{1}{\sqrt{n}}(x_i(s) - \bar{x}_n(s))W_i \frac{1}{\sqrt{n}}(x_i(t) - \bar{x}_n(t))W_i\right] \\ &= \frac{1}{n} \sum_{i=1}^n (x_i(s) - \bar{x}_n(s))(x_i(t) - \bar{x}_n(t))\mathbb{E}W_i^2.\end{aligned}$$

Since $\mathbb{E}W_i^2 = 1$, it follows that

$$\sum_{i=1}^n \text{Cov}(X_i^*(s), X_i^*(t)) = \frac{1}{n} \sum_{i=1}^n (x_i(s) - \bar{x}_n(s))(x_i(t) - \bar{x}_n(t)).$$

Expanding the product, we get

$$\begin{aligned}&\frac{1}{n} \sum_{i=1}^n (x_i(s) - \bar{x}_n(s))(x_i(t) - \bar{x}_n(t)) \\ &= \frac{1}{n} \sum_{i=1}^n (x_i(s)x_i(t) - x_i(s)\bar{x}_n(t) - \bar{x}_n(s)x_i(t) + \bar{x}_n(s)\bar{x}_n(t)).\end{aligned}$$

Using

$$\frac{1}{n} \sum_{i=1}^n x_i(s) = \bar{x}_n(s), \quad \frac{1}{n} \sum_{i=1}^n x_i(t) = \bar{x}_n(t),$$

we obtain

$$\begin{aligned}&\frac{1}{n} \sum_{i=1}^n x_i(s)x_i(t) - \bar{x}_n(s)\bar{x}_n(t) - \bar{x}_n(t)\bar{x}_n(s) + \bar{x}_n(s)\bar{x}_n(t) \\ &= \frac{1}{n} \sum_{i=1}^n x_i(s)x_i(t) - \bar{x}_n(s)\bar{x}_n(t).\end{aligned}$$

By Theorem 4, we have

$$\frac{1}{n} \sum_{i=1}^n X_i(s)X_i(t) - \bar{X}_n(s)\bar{X}_n(t) \rightarrow \mathbb{E}[X_1(s)X_1(t)] - \mathbb{E}[X_1(s)]\mathbb{E}[X_1(t)] = \text{Cov}(X_1(s), X_1(t)) = \text{Cov}(Y(s), Y(t)),$$

where the convergence holds almost surely.

Hence, the required covariance structure coincides with the covariance structure of the limiting Gaussian element, and condition (c) is satisfied.

d) Consider

$$\sigma_n^{*2}(s) = \sum_{i=1}^n \mathbb{E} [(X_i^*(s))^2] .$$

and obtain

$$\begin{aligned} \sigma_n^{*2}(s) &= \sum_{i=1}^n \mathbb{E} \left[\frac{1}{n} (x_i(s) - \bar{x}_n(s))^2 W_i^2 \right] \\ &= \frac{1}{n} \sum_{i=1}^n (x_i(s) - \bar{x}_n(s))^2 \mathbb{E} W_i^2 . \end{aligned}$$

Since $\mathbb{E} W_i^2 = 1$, it follows that

$$\sigma_n^{*2}(s) = \frac{1}{n} \sum_{i=1}^n (x_i(s) - \bar{x}_n(s))^2 .$$

Expanding the square, we get

$$\begin{aligned} \sigma_n^{*2}(s) &= \frac{1}{n} \sum_{i=1}^n (x_i^2(s) - 2x_i(s)\bar{x}_n(s) + \bar{x}_n^2(s)) \\ &= \frac{1}{n} \sum_{i=1}^n x_i^2(s) - \frac{2}{n} \sum_{i=1}^n x_i(s)\bar{x}_n(s) + \frac{1}{n} \sum_{i=1}^n \bar{x}_n^2(s) . \end{aligned}$$

Since

$$\frac{1}{n} \sum_{i=1}^n x_i(s) = \bar{x}_n(s) ,$$

we obtain

$$\begin{aligned} \sigma_n^{*2}(s) &= \frac{1}{n} \sum_{i=1}^n x_i^2(s) - 2\bar{x}_n^2(s) + \bar{x}_n^2(s) \\ &= \frac{1}{n} \sum_{i=1}^n x_i^2(s) - \bar{x}_n^2(s) . \end{aligned}$$

By Theorem 4

$$\frac{1}{n} \sum_{i=1}^n X_i^2(s) \longrightarrow \mathbb{E} X_1^2(s) \text{ a.s.}$$

and

$$\bar{X}_n(s) \longrightarrow \mathbb{E} X_1(s) = \mathbb{E} Y(s) \implies \bar{X}_n^2(s) \longrightarrow (\mathbb{E} X_1(s))^2 = \mathbb{E} Y(s)^2 \text{ a.s.},$$

we get $\sigma_n^{*2}(s) \rightarrow \mathbb{E} X_1^2(s) - (\mathbb{E} X_1(s))^2 = \sigma^2(s)$ a.s..

Thus, condition **(d)** is satisfied.

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A CLASS OF ONE-DIMENSIONAL SOLVABLE EXTENSIONS OF THE NATURALLY GRADED FILIFORM 3-LIE ALGEBRA $NGF_{5,3}$

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ABSTRACT. In this paper, we study one-dimensional solvable extensions of the naturally graded filiform 3-Lie algebra $NGF_{5,3}$. Using the description of derivations of the algebra $NGF_{m,3}$, we determine the general form of the multiplication table of such extensions and derive the restrictions on its parameters from the fundamental identity. The analysis naturally leads to two cases according to the coefficients occurring in the product $[x, e_1, e_2]$. In the present paper, we consider in detail the case $\alpha = \beta = 0$. By applying suitable adapted changes of basis, the corresponding multiplication tables are reduced to simpler forms and the remaining parameters are analyzed. The case $(\alpha, \beta) \neq (0, 0)$ is left for a separate investigation.

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Key words: 3-Lie algebra, naturally graded filiform algebra, solvable extension, hypo-nilpotent ideal, derivation.

Introduction

The concept of an n -Lie algebra was introduced by Filippov as a natural multilinear generalization of a Lie algebra [1]. In contrast to the binary case, the structure theory of nilpotent and solvable n -Lie algebras contains a number of phenomena which have no direct analogue in the theory of Lie algebras. In particular, the notions of nilpotence as a subalgebra and nilpotence as an ideal need not coincide. This leads naturally to the notion of a hypo-nilpotent ideal, which plays an important role in the description of solvable n -Lie algebras; see [2-3].

Naturally graded filiform n -Lie algebras and their solvable extensions were studied in [4]. Among the naturally graded filiform algebras, the authors distinguished a model algebra with the simplest multiplication table, denoted by $NGF_{m,n}$. It is an n -ary analogue of the model filiform Lie algebra and is defined, in a suitable basis $\{e_1, \dots, e_m\}$, by

$$[e_1, e_2, \dots, e_{n-1}, e_j] = e_{j+1}, \quad n \leq j \leq m-1, \quad (6)$$

all other products being zero or obtained from skew-symmetry. This model is the starting point of the present paper.

The specialization $n = 3, m = 5$ gives the five-dimensional algebra $NGF_{5,3}$ used below. Its low dimension allows us to display the derivations and compatibility relations of a codimension-one extension explicitly.

The model used here is not a new nilpotent algebra. In [3, p. 44], the authors recall the simplest filiform model of [2], with products $[f_1, f_2, f_j] = f_{j-1}$ for $4 \leq j \leq m$, and the earlier description of its codimension-one solvable extensions. For $m = 5$, the identification

$$f_1 = e_1, \quad f_2 = e_2, \quad f_3 = e_5, \quad f_4 = e_4, \quad f_5 = e_3$$

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identifies that model with $NGF_{5,3}$. Thus the extension problem below lies within the scope of [2]. The calculations are presented as an explicit low-dimensional treatment of this known problem, rather than as a new classification. The second model in [3, equation (3.1)] has additional products for general m and should be distinguished from the simplest model of [2].

We give a direct derivation calculation, a normalized parameter table, and explicit adapted basis reductions for $\alpha = \beta = 0$. We establish reduction to three displayed normal forms, without asserting that they represent three distinct abstract isomorphism classes. The embedded ideal N is preserved in the reductions; no assertion that every abstract isomorphism preserves this ideal is needed. A precise family-by-family comparison with [2] is required before any independent originality claim can be made.

Throughout the paper the ground field is $\mathbb{C}$.

Preliminaries

We recall the notions used below. For subspaces A, B, C of a 3-Lie algebra L , by $[A, B, C]$ we mean the linear span of all products $[a, b, c]$, where $a \in A$, $b \in B$, and $c \in C$.

Definition 1. A complex vector space L endowed with a skew-symmetric trilinear operation $[\cdot, \cdot, \cdot]$ is called a 3-Lie algebra if

$$[a, b, [u, v, w]] = [[a, b, u], v, w] + [u, [a, b, v], w] + [u, v, [a, b, w]] \quad (7)$$

for all $a, b, u, v, w \in L$. Identity (7) is called the *fundamental identity* [1].

Definition 2. A linear mapping $d : L \rightarrow L$ is called a *derivation* if

$$d([u, v, w]) = [d(u), v, w] + [u, d(v), w] + [u, v, d(w)] \quad (8)$$

for all $u, v, w \in L$. The set of all derivations is denoted by $\text{Der}(L)$. By the fundamental identity, for arbitrary fixed $a, b \in L$, the mapping

$$\text{ad}(a, b)(z) = [a, b, z]$$

is a derivation.

Definition 3. Let I be an ideal of a 3-Lie algebra L . Put

$$I^0 = I, \quad I^{r+1} = [I^r, I, L], \quad r \geq 0.$$

The ideal I is nilpotent if $I^s = 0$ for some s . On the other hand, regarding I as a 3-Lie subalgebra, put

$$J^0 = I, \quad J^{r+1} = [J^r, I, I].$$

The ideal I is called *hypo-nilpotent* if the latter sequence reaches zero while the former does not. A hypo-nilpotent ideal is called maximal if it is not properly contained in another hypo-nilpotent ideal [3].

Definition 4. We use the derived series $L^{(0)} = L$ and $L^{(r+1)} = [L^{(r)}, L^{(r)}, L]$. The algebra is solvable when this series terminates at zero.

Following [4], consider the model naturally graded filiform n -Lie algebra $NGF_{m,n}$ defined by (6). Specializing first to $n = 3$ and then to $m = 5$, we obtain the five-dimensional 3-Lie algebra

$$N = NGF_{5,3} = \text{Span}\{e_1, e_2, e_3, e_4, e_5\}, \quad (9)$$

whose multiplication table is

$$\begin{cases} [e_1, e_2, e_3] = e_4, \\ [e_1, e_2, e_4] = e_5. \end{cases} \quad (10)$$

All products not displayed in (10) are zero or follow from skew-symmetry. In particular, if

$$U = \text{Span}\{e_3, e_4, e_5\},$$

then

$$[U, U, N] = 0. \quad (11)$$

Relation (11) will be used repeatedly below.

We now describe the derivation algebra of N . The statement is written independently of the extension problem and will subsequently be applied to the inner derivations arising from the complementary element.

Proposition 1. Relative to the ordered basis $(e_1, \dots, e_5)$, with columns representing images of basis vectors, all derivations of N have the form

$$D = \begin{pmatrix} a_{11} & a_{21} & 0 & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 & 0 \\ a_{13} & a_{23} & a_{33} & 0 & 0 \\ a_{14} & a_{24} & a_{34} & a_{11} + a_{22} + a_{33} & 0 \\ a_{15} & a_{25} & a_{35} & a_{34} & 2a_{11} + 2a_{22} + a_{33} \end{pmatrix},$$

where the thirteen displayed independent coefficients are arbitrary. This is the five-dimensional specialization of the derivation calculation for the model algebra; see [4] and the related conference contribution [5]. We verify it directly to make the present calculation self-contained.

Proof. Write $D(e_i) = \sum_k a_{ik}e_k$. Applying the derivation identity to $[e_1, e_2, e_3] = e_4$ gives

$$D(e_4) = (a_{11} + a_{22} + a_{33})e_4 + a_{34}e_5.$$

Applying it to $[e_1, e_2, e_4] = e_5$ gives $D(e_5) = (2a_{11} + 2a_{22} + a_{33})e_5$. The zero products $[e_1, e_3, e_4] = 0$ and $[e_2, e_3, e_4] = 0$ imply $a_{32} = a_{31} = 0$, respectively. The images of e_1, e_2 are unrestricted. Conversely, the displayed matrix satisfies the identity on both nonzero triples. On each of the other eight increasing triples, substitution gives zero on both sides. Skew-symmetry and multilinearity complete the verification. $\square$

One-dimensional solvable extensions

Let $R = N \dot{+} \text{Span}\{x\}$ be a six-dimensional solvable 3-Lie algebra in which $N = \text{NGF}_{5,3}$ is a maximal hypo-nilpotent ideal. Since N is an ideal, every product $[x, e_i, e_j]$ belongs to N .

Theorem 1. Let R be a six-dimensional solvable 3-Lie algebra with maximal hypo-nilpotent ideal $\text{NGF}_{5,3}$. Then there exists a basis $\{x, e_1, e_2, e_3, e_4, e_5\}$ of R such that the multiplication table of R with respect to this basis has the following form:

$$\begin{cases} [x, e_1, e_2] = \alpha e_1 + \beta e_2 + \gamma e_3, \\ [x, e_1, e_3] = p_3 e_3 + p_5 e_5, \\ [x, e_1, e_4] = (\beta + p_3) e_4, \\ [x, e_1, e_5] = (2\beta + p_3) e_5, \\ [x, e_2, e_3] = q_3 e_3 + q_5 e_5, \\ [x, e_2, e_4] = (-\alpha + q_3) e_4, \\ [x, e_2, e_5] = (-2\alpha + q_3) e_5, \\ [x, e_i, e_j] = 0, \quad 3 \leq i, j \leq 5. \end{cases} \quad (12)$$

Moreover, the parameters satisfy

$$\alpha p_3 + \beta q_3 = 0, \quad \alpha p_5 + \beta q_5 = 0. \quad (13)$$

Conversely, the tables (10) and (12) define a 3-Lie algebra precisely when (13) holds. Every such algebra is solvable. This converse alone does not assert that N is hypo-nilpotent: in the case $\alpha = \beta = 0$, that additional condition is equivalent to $(p_3, q_3) \neq (0, 0)$.

Proof. Let

$$R = \text{NGF}_{5,3} \dot{+} \langle x \rangle.$$

Since $\text{NGF}_{5,3}$ is an ideal of R , all products involving x and two elements of $\text{NGF}_{5,3}$ belong to $\text{NGF}_{5,3}$.

First, for each $i = 1, \dots, 5$, the mapping

$$\text{ad}(x, e_i) : \text{NGF}_{5,3} \longrightarrow \text{NGF}_{5,3}, \quad \text{ad}(x, e_i)(z) = [x, e_i, z],$$

is a derivation of $NGF_{5,3}$. Applying Proposition 1 to these derivations and taking into account the skew-symmetry of the bracket, we obtain

$$[x, e_1, e_2] = \alpha e_1 + \beta e_2 + \gamma_3 e_3 + \gamma_4 e_4 + \gamma_5 e_5,$$

and

$$\begin{cases} [x, e_1, e_3] = p_3 e_3 + p_4 e_4 + p_5 e_5, \\ [x, e_1, e_4] = (\beta + p_3) e_4 + p_4 e_5, \\ [x, e_1, e_5] = (2\beta + p_3) e_5. \end{cases}$$

whereas

$$\begin{cases} [x, e_2, e_3] = q_3 e_3 + q_4 e_4 + q_5 e_5, \\ [x, e_2, e_4] = (-\alpha + q_3) e_4 + q_4 e_5, \\ [x, e_2, e_5] = (-2\alpha + q_3) e_5. \end{cases}$$

Similarly, applying Proposition 1 to $\text{ad}(x, e_i)$, $3 \leq i \leq 5$, yields

$$[x, e_i, e_j] = 0, \quad 3 \leq i, j \leq 5.$$

Next, replacing x by

$$x' = x + p_4 e_2 - q_4 e_1 - \gamma_4 e_3 - \gamma_5 e_4,$$

and then omitting the prime, we may assume

$$p_4 = q_4 = \gamma_4 = \gamma_5 = 0.$$

Putting $\gamma = \gamma_3$, we obtain the table (12).

Finally, applying the fundamental identity to

$$[x, e_1, [x, e_2, e_3]]$$

and using the multiplication table obtained above, we have

$$[x, e_1, [x, e_2, e_3]] = [x, [x, e_1, e_2], e_3] + [x, e_2, [x, e_1, e_3]].$$

Comparing the coefficients of e_3 and e_5 on both sides gives, respectively,

$$\alpha p_3 + \beta q_3 = 0$$

and

$$\alpha p_5 + \beta q_5 = 0.$$

Thus all the asserted relations are obtained.

For the converse, define the residual

$$F(a, b; u, v, w) = [a, b, [u, v, w]] - [[a, b, u], v, w] - [u, [a, b, v], w] - [u, v, [a, b, w]].$$

It is alternating in a, b and separately in u, v, w . Hence it suffices to check the $\binom{6}{2}\binom{6}{3} = 300$ choices of an increasing basis pair and an increasing basis triple in the order $(x, e_1, \dots, e_5)$. Put $A = \alpha p_3 + \beta q_3$ and $B = \alpha p_5 + \beta q_5$. Substitution into the sparse table gives all potentially nonzero residuals as follows:

(a, b)	(u, v, w)	$F(a, b; u, v, w)$
(x, e_1)	(x, e_2, e_3)	$-Ae_3 + Be_5$
(x, e_1)	(x, e_2, e_4)	$-Ae_4$
(x, e_1)	(x, e_2, e_5)	$-Ae_5$
(x, e_2)	(x, e_1, e_3)	$Ae_3 - Be_5$
(x, e_2)	(x, e_1, e_4)	Ae_4
(x, e_2)	(x, e_1, e_5)	Ae_5
(x, e_3)	(x, e_1, e_2)	$-Ae_3 + Be_5$
(x, e_4)	(x, e_1, e_2)	$-Ae_4$
(x, e_5)	(x, e_1, e_2)	$-Ae_5$

The other 291 residuals vanish identically. This enumeration follows by using $[U, U, R] = 0$, the two internal nonzero products, and the displayed actions on U ; it does not assume any parameter constraints. It proves sufficiency as well as the absence of additional relations.

Finally, put $v = \alpha e_1 + \beta e_2$. The table gives $R^{(1)} \subseteq \mathbb{C}v + U$ and $[\mathbb{C}v + U, \mathbb{C}v + U, R] \subseteq U$. Since $[U, U, R] = 0$, we have $R^{(3)} = 0$. Thus every parameter table satisfying the fundamental identity is solvable. If all parameters vanish, N is a nilpotent ideal, illustrating why the hypo-nilpotent condition must be imposed separately.

We now proceed to simplify the multiplication table obtained above. The further analysis essentially depends on the coefficients α and β occurring in the product $[x, e_1, e_2]$. In view of (13), there are two essentially different cases:

$$\alpha = \beta = 0 \quad \text{and} \quad (\alpha, \beta) \neq (0, 0).$$

In the present paper, we restrict our attention to the first case, that is, $\alpha = \beta = 0$, and investigate the corresponding one-dimensional solvable extensions of $NGF_{5,3}$. The case $(\alpha, \beta) \neq (0, 0)$ will be considered separately.

Case 1. $\alpha = \beta = 0$.

In this case, the multiplication table obtained above takes the form

$$\left\{ \begin{array}{l} [x, e_1, e_2] = \gamma e_3, \\ [x, e_1, e_3] = p_3 e_3 + p_5 e_5, \\ [x, e_1, e_4] = p_3 e_4, \\ [x, e_1, e_5] = p_3 e_5, \\ [x, e_2, e_3] = q_3 e_3 + q_5 e_5, \\ [x, e_2, e_4] = q_3 e_4, \\ [x, e_2, e_5] = q_3 e_5. \end{array} \right. \quad (14)$$

We first observe that

$$(p_3, q_3) \neq (0, 0).$$

Indeed, if $p_3 = q_3 = 0$, then the ideal series satisfies

$$[N, N, R] \subseteq \text{Span}\{e_3, e_4, e_5\}, \quad [[N, N, R], N, R] \subseteq \text{Span}\{e_4, e_5\},$$

and its next terms are contained in $\text{Span}\{e_5\}$ and then vanish. Thus N would be nilpotent as an ideal of R , contrary to the hypo-nilpotent assumption.

For completeness, if $(p_3, q_3) \neq (0, 0)$, one of $\text{ad}(x, e_1)|_U$ and $\text{ad}(x, e_2)|_U$ is invertible. Since $[N, N, R] \subseteq U$, this gives $[N, N, R] = U$ and $[U, N, R] = U$. Thus N is not nilpotent as an ideal, although its internal lower central series terminates. Maximality follows from codimension one, since R cannot be hypo-nilpotent as an ideal of itself.

We now specify the normalization completely. If $p_3 \neq 0$, put

$$x' = x, \quad e'_1 = p_3^{-1} e_1, \quad e'_2 = -q_3 e_1 + p_3 e_2, \quad e'_3 = e_3, \quad e'_4 = e_4, \quad e'_5 = e_5.$$

If $p_3 = 0$ and $q_3 \neq 0$, put

$$x' = x, \quad e'_1 = q_3^{-1} e_2, \quad e'_2 = -q_3 e_1, \quad e'_3 = e_3, \quad e'_4 = e_4, \quad e'_5 = e_5.$$

In both cases the determinant on $\text{Span}\{e_1, e_2\}$ is 1, so both defining products of N are preserved. The new coefficients satisfy $p'_3 = 1$, $q'_3 = 0$. The transformed products still have no e_4 component in $[x', e'_i, e'_j]$ and no e_4, e_5 component in $[x', e'_1, e'_2]$. Omitting primes gives the desired normalization.

Hence (14) reduces to

$$\begin{cases} [x, e_1, e_2] = \gamma e_3, \\ [x, e_1, e_3] = e_3 + p_5 e_5, \\ [x, e_1, e_4] = e_4, \\ [x, e_1, e_5] = e_5, \\ [x, e_2, e_3] = q_5 e_5, \\ [x, e_2, e_4] = 0, \\ [x, e_2, e_5] = 0. \end{cases} \quad (15)$$

We next eliminate γ by the adapted change

$$e'_2 = e_2 - \gamma e_3 + \gamma p_5 e_5.$$

Indeed, $[x, e_1, e'_2] = 0$, while the defining products of N and the remaining products in (15) retain their form. Hence we may assume $[x, e_1, e_2] = 0$.

Consequently, the multiplication table can be written as

$$\begin{cases} [x, e_1, e_2] = 0, \\ [x, e_1, e_3] = e_3 + c e_5, \\ [x, e_1, e_4] = e_4, \\ [x, e_1, e_5] = e_5, \\ [x, e_2, e_3] = d e_5, \\ [x, e_2, e_4] = 0, \\ [x, e_2, e_5] = 0, \end{cases} \quad (16)$$

where $c, d \in \mathbb{C}$.

The remaining parameters can be reduced as follows. If $d \neq 0$, the change $e'_1 = e_1 - (c/d)e_2$ eliminates c . The rescaling

$$x' = x, \quad e'_1 = e_1, \quad e'_2 = d e_2, \quad e'_3 = e_3, \quad e'_4 = d e_4, \quad e'_5 = d^2 e_5$$

then normalizes d to 1: $[x', e'_2, e'_3] = d^2 e_5 = e'_5$. The preceding shear has determinant 1 and leaves e_3, e_4, e_5, x fixed; the rescaling is applied to that sheared basis. If $d = 0$ and $c \neq 0$, choose $s \in \mathbb{C}$ with $s^2 = c$ and put

$$x' = s^{-1} x, \quad e'_1 = s e_1, \quad e'_2 = e_2, \quad e'_3 = e_3, \quad e'_4 = s e_4, \quad e'_5 = s^2 e_5.$$

This normalizes c to 1, since $[x', e'_1, e'_3] = e'_3 + (c/s^2)e'_5 = e'_3 + e'_5$. All previously vanishing products remain zero. Each change preserves the defining products of N . We therefore obtain the following three tables, corresponding respectively to $c = d = 0$, $d = 0, c \neq 0$, and $d \neq 0$.

$$R_{5,3}^{(1)} : \begin{cases} [e_1, e_2, e_3] = e_4, \\ [e_1, e_2, e_4] = e_5, \\ [x, e_1, e_3] = e_3, \\ [x, e_1, e_4] = e_4, \\ [x, e_1, e_5] = e_5. \end{cases} \quad (17)$$

$$R_{5,3}^{(2)} : \begin{cases} [e_1, e_2, e_3] = e_4, \\ [e_1, e_2, e_4] = e_5, \\ [x, e_1, e_3] = e_3 + e_5, \\ [x, e_1, e_4] = e_4, \\ [x, e_1, e_5] = e_5. \end{cases} \quad (18)$$

$$R_{5,3}^{(3)} : \begin{cases} [e_1, e_2, e_3] = e_4, \\ [e_1, e_2, e_4] = e_5, \\ [x, e_1, e_3] = e_3, \\ [x, e_1, e_4] = e_4, \\ [x, e_1, e_5] = e_5, \\ [x, e_2, e_3] = e_5. \end{cases} \quad (19)$$

All omitted products in the above multiplication tables are zero or follow from skew-symmetry.

It should be noted that the multiplication tables $R_{5,3}^{(1)}$, $R_{5,3}^{(2)}$, and $R_{5,3}^{(3)}$ define 3-Lie algebras. Indeed, the parameter relations obtained from the fundamental identity are satisfied in the case $\alpha = \beta = 0$, and all subsequent reductions are performed by changes of basis preserving the 3-Lie algebra structure.

We summarize the above considerations in the following statement.

Proposition 2. Let R be a six-dimensional solvable 3-Lie algebra with maximal hypo-nilpotent ideal $NGF_{5,3}$, and suppose that $\alpha = \beta = 0$ in the product

$$[x, e_1, e_2] = \alpha e_1 + \beta e_2 + \gamma e_3.$$

Then, by suitable adapted changes of basis, the multiplication table of R can be reduced to one of the forms

$$R_{5,3}^{(1)}, \quad R_{5,3}^{(2)}, \quad R_{5,3}^{(3)}$$

given in (17)–(19), respectively.

Proof. The preceding explicit reductions yield the three tables, whose parameters satisfy (13). In each case,

$$[R, R, R] = U, \quad [U, U, R] = 0, \quad [U, N, R] = U.$$

Thus R is solvable, while N is not nilpotent as an ideal. Internally, $[N, N, N] = \text{Span}\{e_4, e_5\}$, its next lower central term is $\mathbb{C}e_5$, and the following term is zero. Hence N is hypo-nilpotent. Codimension one proves maximality. These statements concern existence and reduction to normal forms, not pairwise non-isomorphism.

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INTEGRATION OF THE EXTENDED FIFTH-ORDER KORTEWEG-DE VRIES EQUATION WITH
TIME-DEPENDENT COEFFICIENTS AND A SELF-CONSISTENT SOURCE IN THE CLASS OF RAPIDLY
DECREASING FUNCTIONS

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ABSTRACT. In this article, the problem of integrating the extended Korteweg-de Vries equation with time-dependent coefficients and a self-consistent source in the class of rapidly decreasing functions using the inverse scattering method for the self-adjoint Sturm-Liouville operator is studied. Furthermore, an integration algorithm based on this method is developed, and an illustrative example is solved using the proposed algorithm.

MSC (2020): 34L25, 35P25, 47A40, 37K15.

Key words: Korteweg-de Vries equation, Jost solutions, scattering data, Gelfand-Levitan-Marchenko integral equation.

Introduction

The inverse scattering method was first studied in the work of Gardner, Green, Kruskal, and Miura[1] in 1967. They succeeded in finding a global solution of the Cauchy problem for the following Korteweg-de Vries(KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0$$

using the method of inverse problems of scattering theory for the self-adjoint Sturm-Liouville operator on the class of rapidly decreasing functions. In doing so, they used the method of solving the proper and inverse problem of scattering theory, studied by L.D. Faddeev [2], for a given Sturm-Liouville operator on the entire axis. After that, P. Lax [3] showed that the method used by these scientists was universal and found a higher-order analogue of the nonlinear Korteweg-de Vries equation. In the work of V.K. Melnikov[4], integration of the Korteweg-de Vries equation with a self-consistent source in the class of rapidly decreasing functions is achieved. Apart from the Korteweg-de Vries equation, using the method of inverse problems of scattering theory, equations such as Hirota [5] equation, Modified Korteweg-de Vries equation [6, 7], Sine-Gordon [8], and mKdF-Sine-Gordon equation have been integrated [9].

The classical third-order KdV model has shortcomings in representing several important properties of nonlinear waves observed in laboratory experiments, such as the non-monotonic relationship between the velocity and amplitude of a solitary wave or the table-top shape of large-amplitude solitary waves [10]. Therefore, there is a need to study the fifth-order Korteweg-de Vries equation and its extended form.

In [11], the dependence of the wave amplitude, width, and velocity of the solution to the fifth-order Korteweg-de Vries equation on the equation parameters is studied. Some forms of the fifth-order KdV equation, with some coefficients set to zero, have also been used to study plasma waves [12] electromagnetic waves in discrete transmission lines [13], gravity-capillary water waves [14,15] and waves in floating ice sheets.

Several methods are used to integrate the Korteweg-de Vries equation and its higher-order analogues, in addition to the inverse scattering transform method. Examples include the Hirota method, the tanh-coth method.

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The application of Hirota's method to KdV and higher-order KdV equations was studied in [16]. In [17], N-soliton solutions for the fifth-order KdV equation were studied under the Bargmann constraint via the Hirota method and the Wronsky technique, respectively. In [18], the solution of the fifth-order Korteweg-de Vries (Lax, Sawada-Kotera, Kaup-Kuperschmidt) equation in several forms was found using the tanh method.

Let us consider the following extended fifth-order Korteweg-de Vries equation with time-dependent coefficients and a self-consistent source:

$$\begin{cases} u_t + \alpha(t)(u_{xxxxx} - 20u_x u_{xx} - 10u u_{xxx} + 30u^2 u_x) + \beta(t)(u_{xxx} - 6u u_x) + \\ + \gamma(t)u_x = 4 \sum_{m=1}^N \omega_m(t) \frac{\partial}{\partial x} |\varphi_m|^2, \\ -\varphi_m'' + u(x, t)\varphi_m = k_m^2 \varphi_m. \end{cases} \quad (1)$$

Where $\alpha(t), \beta(t), \gamma(t)$ and $\omega_m(t)$, $m = \overline{1, N}$ are the previously given continuously differentiable functions. In the case under consideration, the functions $\varphi_m = \varphi_m(x, t)$, $m = \overline{1, N}$ are the eigenfunctions corresponding to the eigenvalues of the self-adjoint Sturm-Liouville operator

$$L(t) := -\frac{d^2}{dx^2} + u(x, t), \quad t > 0$$

and satisfy the following conditions

$$\int_{-\infty}^{+\infty} |\varphi_m(x, t)|^2 dx = A_m(t), \quad m = \overline{1, N}. \quad (2)$$

Where $A_m(t)$, $m = \overline{1, N}$ are positive functions given. We study the given equation (1) with the following initial condition

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}. \quad (3)$$

Where $u_0(x)$ is a real function with the following properties:

1)

$$\int_{-\infty}^{\infty} (1 + |x|) |u_0(x)| dx < \infty \quad (4)$$

2) The operator $L(0) = -\frac{d^2}{dx^2} + u_0(x)$, $x \in \mathbb{R}$ has N eigenvalues $\lambda_1(0), \lambda_2(0), \dots, \lambda_N(0)$ and does not possess spectral singularities.

Let us assume that the function $u(x, t)$ has the required smoothness and rather quickly tends to its limits at $x \rightarrow \pm\infty$, i.e

$$\int_{-\infty}^{\infty} \left[(1 + |x|) |u(x, t)| + \sum_{j=1}^5 \left| \frac{\partial^j u(x, t)}{\partial x^j} \right| \right] dx < \infty, \quad \forall t \geq 0. \quad (5)$$

The main goal of the work is to obtain representations for the solution problem $u(x, t)$, $\varphi_m(x, t)$, $m = \overline{1, N}$ (1)-(5) in the framework of the method of the inverse scattering problem for the operator $L(t)$.

Preliminaries

Consider the Sturm-Liouville operator

$$L(0)y := -y'' + u_0(x)y = k^2 y, \quad x \in \mathbb{R} \quad (6)$$

where the potential $u_0(x)$ satisfies condition (3). In this section, information necessary for further presentation concerning the direct and inverse scattering problems for equation (1). Denote by $f(x, k)$ and $g(x, k)$ the Jost solutions of equation (1) with asymptotics

$$\begin{aligned} f(x, k) &= e^{ikx} + \bar{o}(1), \quad x \rightarrow +\infty, \quad f'(x, k) = ike^{ikx} + \bar{o}(1), \quad x \rightarrow +\infty (\operatorname{Im} k = 0); \\ g(x, k) &= e^{-ikx} + \bar{o}(1), \quad x \rightarrow -\infty, \quad g'(x, k) = -ike^{-ikx} + \bar{o}(1), \quad x \rightarrow -\infty (\operatorname{Im} k = 0) \end{aligned} \quad (7)$$

Under the condition (3), such solutions exist and are uniquely determined by the asymptotics (7). Solutions $f(x, k), g(x, k)$ have the representations

$$f(x, k) = e^{ikx} + \int_x^\infty K^+(x, t) e^{ikt} dt, \quad g(x, k) = e^{-ikx} + \int_{-\infty}^x K^-(x, t) e^{-ikt} dt \quad (8)$$

where kernels $K^+(x, t), K^-(x, t)$ are real functions related to the potential $u_0(x)$ as follows:

$$u_0(x) = \mp 2 \frac{dK^\pm(x, x)}{dx}$$

For real k , pairs of functions $\{f(x, k), f(x, -k)\}$ and $\{g(x, k), g(x, -k)\}$ are pairs of linearly independent solutions to the equation (6), therefore

$$f(x, k) = a(k)g(x, -k) + b(k)g(x, k),$$

$$g(x, k) = -b(-k)f(x, k) + a(k)f(x, -k). \quad (9)$$

The functions $r^+(k) = -\frac{b(-k)}{a(k)}$ and $r^-(k) = \frac{b(k)}{a(k)}$ are called reflection coefficients (right and left, respectively). The coefficients $a(k), b(k)$ and $r^\pm(k)$ have the following properties (see [19], p.121):

Defenition. The following set $\{r^+(k, 0), \chi_1(0), \chi_2(0), \dots, \chi_N(0), B_1(0), B_2(0), \dots, B_N(0)\}$ is called the data of the scattering theory of the operator $L(0)$.

(8) The kernel $K^+(x, y)$ in the Levin representation consists of the unique global solution of this Gelfand-Levitan-Marchenko integral equation

$$K^+(x, y) + F^+(x + y) + \int_x^\infty K^+(x, z) F^+(z + y) dz = 0, \quad (y > x). \quad (10)$$

Where

$$F^+(x) = \sum_{j=1}^N \alpha_j^+(0) e^{-\chi_j(0)x} + \frac{1}{2\pi} \int_{-\infty}^\infty r^+(k, 0) e^{ikx} dk, \quad \alpha_j^+(0) = \frac{iB_j(0)}{i \frac{da(z)}{dz} \Big|_{z=i\chi_j}}.$$

A. For real $k \neq 0$

$$\overline{a(k)} = a(-k), \quad \overline{b(k)} = b(-k), \quad |a(k)|^2 = 1 + |b(k)|^2, \quad a(k) = -\frac{1}{2ik} W\{f(x, k), g(x, k)\},$$

$$b(k) = \frac{1}{2ik} W\{f(x, k), g(x, -k)\}; \quad a(k) = 1 + O\left(\frac{1}{k}\right), \quad b(k) = O\left(\frac{1}{k}\right), \quad k \rightarrow \pm\infty;$$

where $W\{f(x, k), g(x, k)\} \equiv f'(x, k)g(x, k) - f(x, k)g'(x, k)$;

B. The function $a(k)$ extends analytically to the half-plane $\operatorname{Im} k > 0$ and there it has a finite number of zeros $k_m = i\chi_m$, ($\chi_n > 0$), $m = 1, 2, \dots, N$, these zeros are simple, and $\lambda_m = -\chi_m^2$, $m = 1, 2, \dots, N$ are the eigenvalues of the operator $L(0)$. In addition, the following relation holds

$$g(x, i\chi_j) = B_j f(x, i\chi_j), \quad j = 1, 2, \dots, N.$$

C. For real $k \neq 0$, the function $r^+(k)$ is continuous,

$$\overline{r^+(k)} = r^+(-k), \quad |r^+(k)| < 1, \quad r^+(k) = o(k^{-1}), \quad |k| \rightarrow \infty, \quad k^2 \left[1 - |r^+(k)|^2\right]^{-1} = O(1), \quad |k| \rightarrow 0;$$

D. This function $k(a(k) - 1)$ is continuously bounded in $\text{Im } k > 0$, and the following relations hold:

$$a^{-1}(k) = O(1), \quad |k| \rightarrow 0, \quad \text{Im } k \geq 0, \quad \lim_{k \rightarrow 0} ka(k) (r^+(k) + 1) = 0, \quad \text{Im } k = 0,$$

where the function $a(k)$ is defined by the following formula

$$a(k) = \prod_{j=1}^N \frac{k - i\chi_j}{k + i\chi_j} \exp \left\{ -\frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{\ln(1 - |r^+(\xi)|^2)}{\xi - k} d\xi \right\}, \quad \text{Im } k > 0.$$

E. These

$$R^{\pm}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} r^{\pm}(k) e^{\pm ikx} dk$$

functions satisfy the following conditions for each $a > -\infty$:

$$(1 + |x|) |R^{\pm}(\pm x)| \in L^1(a, \infty)$$

Lemma 1. Let the functions $y(x, \lambda)$ and $z(x, \mu)$ be solutions of the following equations $Ly(x, \lambda) = \lambda y(x, \lambda)$, $Lz(x, \mu) = \mu z(x, \mu)$ respectively. In this case, the following equalities are true

$$\frac{d}{dx} W\{y(x, \lambda), z(x, \mu)\} = (\lambda - \mu)y(x, \lambda)z(x, \mu).$$

Theorem 1 [(see [19, p. 121])]. The potential of the Sturm-Liouville operator $u_0(x)$ is defined as a single value by the data of its scattering theory.

Evolution of scattering data

Let us consider the following equation

$$u_t + \alpha(t) (u_{xxxxx} - 20u_x u_{xx} - 10u u_{xxx} + 30u^2 u_x) + \beta(t) (u_{xxx} - 6u u_x) = G(x, t). \quad (11)$$

Where we introduce the following notation $G(x, t) = -\gamma(t)u_x + 4 \sum_{m=1}^N \omega_m(t) \frac{\partial}{\partial x} |\varphi_m|^2$.

For equation (11), the Lax pair is in the following form:

$$\begin{aligned} -\phi''_{xx}(x, k, t) + (u(x, t) - k^2) \phi(x, k, t) &= 0, \\ \frac{\partial \phi(x, k, t)}{\partial t} &= 8\alpha(t) \left(2\phi_{xxxxx} - 5u\phi_{xxx} + \frac{15}{4}uu_x\phi - \frac{25}{4}u_{xx}\phi_x - \frac{15}{8}\phi u_{xxx} + \frac{15}{4}u^2\phi_x \right) + \\ &+ 60k^2\alpha(t)u_x\phi + 16ik^5\alpha(t)\phi + \beta(t) (-u_x + 4ik^3)\phi + \beta(t) (2u + 4k^2)\phi_x + F(x, t). \end{aligned} \quad (12)$$

Now, based on relation $\phi_{txx} = \phi_{xxt}$ and using equalities (11)-(12), we get the following equation

$$-F_{xx} + (u - k^2) F = -G(x, t)\phi. \quad (13)$$

Where $\phi(x, k, t) = g(x, k, t)$. To solve this equation, we use the method of variation of constants. That is, if we put expression

$$F(x, t) = B(x, t)g(x, k, t) + C(x, t)g(x, -k, t) \quad (14)$$

into equation (13), we get this system of equations

$$\begin{cases} B'_x(x, t)g(x, k, t) + C'_x(x, t)g(x, -k, t) = 0 \\ B'_x(x, t)g'_x(x, k, t) + C'_x(x, t)g'_x(x, -k, t) = G(x, t)g(x, k, t). \end{cases} \quad (15)$$

If we solve the last system of equations with respect to the unknowns $B(x, t)$ and $C(x, t)$, then we get the following equalities

$$B(x, t) = -\frac{1}{2ik} \int_{-\infty}^x G(s, t)g(s, k, t)g(s, -k, t)ds, C(x, t) = \frac{1}{2ik} \int_{-\infty}^x G(s, t)g^2(s, k, t)ds.$$

Then, by substituting the obtained $B(x, t)$ and $C(x, t)$ into equality (14) and using relation (12), we obtain the following expression:

$$\begin{aligned} \frac{\partial g(x, k, t)}{\partial t} = & 8\alpha(t) \left(2g_{xxxxx} - 5ug_{xxx} + \frac{15}{4}uu_xg - \frac{25}{4}u_{xx}g_x - \frac{15}{8}gu_{xxx} + \frac{15}{4}u^2g_x \right) + 60k^2\alpha(t)u_xg + \\ & + 16ik^5\alpha(t)g + \beta(t)(-u_x + 4ik^3)g + \beta(t)(2u + 4k^2)g_x - \\ & - \frac{g(x, k, t)}{2ik} \int_{-\infty}^x G(s, t)g(s, k, t)g(s, -k, t)ds + \frac{g(x, -k, t)}{2ik} \int_{-\infty}^x G(s, t)g^2(s, k, t)ds \end{aligned} \quad (16)$$

If we use (5),(7),(9) and the last equality, then pass to the limit $x \rightarrow +\infty$, we obtain the following equalities:

$$\begin{aligned} \frac{db(-k, t)}{dt} = & \left(32ik^5\alpha(t) + 8ik^3\beta(t) - \frac{1}{2ik} \int_{-\infty}^{+\infty} G(s, t)g(s, k, t)g(s, -k, t)ds \right) b(-k, t) - \\ & - \frac{a(-k, t)}{2ik} \int_{-\infty}^{+\infty} G(s, t)g^2(s, k, t)ds, \\ \frac{da(k, t)}{dt} = & -\frac{a(k, t)}{2ik} \int_{-\infty}^{+\infty} G(s, t)g(s, k, t)g(s, -k, t)ds - \frac{b(k, t)}{2ik} \int_{-\infty}^{+\infty} G(s, t)g^2(s, k, t)ds \end{aligned} \quad (17)$$

Using the last equations, we find the time evolution of the scattering function $r^+(k, t)$. That is, as follows:

$$\frac{dr^+(k, t)}{dt} = (32ik^5\alpha(t) + 8ik^3\beta(t))r^+(k, t) + \frac{1}{2ika^2(k, t)} \int_{-\infty}^{+\infty} G(s, t)g^2(s, k, t)ds. \quad (18)$$

Lemma 2. If functions $g(x, k, t)$ and $g(x, -k, t)$ are solutions of equation

$$-\phi''_{xx}(x, k, t) + (u(x, t) - k^2)\phi(x, k, t) = 0,$$

then the following properties

$$\int_{-\infty}^{+\infty} G(s, t)g^2(s, k, t)ds = -4k^2a(k, t)b(-k, t), \quad (19)$$

$$\int_{-\infty}^{+\infty} G(s, t)g(s, k, t)g(s, -k, t)ds = 4k^2b(k, t)b(-k, t) \quad (20)$$

hold.

Proof. To prove this lemma, we first calculate this integral using the relations (5), (7) and (9):

$$\int_{-\infty}^{+\infty} g^2(s, k, t)u'_s(s, t)ds = g^2(s, k, t)u(s, t) \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} 2g(s, k, t)g'_s(s, k, t)u(s, t)ds =$$

$$\begin{aligned} &= -2 \int_{-\infty}^{+\infty} (g''_{ss}(s, k, t) + k^2 g(s, k, t)) g'_s(s, k, t) ds = \int_{-\infty}^{+\infty} \left(-k^2 (g^2(s, k, t))'_s - (g'^2_s(s, k, t))'_s \right) ds = \\ &= \left(-k^2 g^2(s, k, t) - (g'_s(s, k, t))^2 \right) \Big|_{-\infty}^{+\infty} = 4k^2 a(k, t) b(-k, t). \end{aligned}$$

Therefore, we have obtained the following equality

$$\int_{-\infty}^{+\infty} g^2(s, k, t) u'_s(s, t) ds = 4k^2 a(k, t) b(-k, t). \quad (21)$$

Now we calculate the following integral using lemma 1.

$$\begin{aligned} &4 \int_{-\infty}^{+\infty} \sum_{m=1}^N \omega_m(t) \frac{\partial}{\partial s} \left(|\varphi_m(s, t)|^2 \right) g^2(s, k, t) ds = 2 \sum_{m=1}^N \omega_m(t) |\varphi_m(s, t)|^2 g^2(s, k, t) \Big|_{-\infty}^{+\infty} - \\ &-4 \int_{-\infty}^{+\infty} \sum_{m=1}^N \omega_m(t) g(s, k, t) g'_s(s, k, t) |\varphi_m(s, t)|^2 ds + 2 \int_{-\infty}^{+\infty} \sum_{m=1}^N \omega_m(t) g^2(s, k, t) \frac{\partial \left(|\varphi_m(s, t)|^2 \right)}{\partial s} ds = \\ &= 2 \int_{-\infty}^{+\infty} \sum_{m=1}^N \omega_m(t) [g \bar{\varphi}_m W \{g, \varphi_m\} + g \varphi_m W \{g, \bar{\varphi}_m\}] ds = \\ &= 2 \int_{-\infty}^{+\infty} \sum_{m=1}^N \frac{\omega_m(t)}{\lambda - \lambda_m} \left[\frac{dW \{g, \bar{\varphi}_m\}}{ds} W \{g, \varphi_m\} + W \{g, \bar{\varphi}_m\} \frac{dW \{g, \varphi_m\}}{ds} \right] ds = \\ &= 2 \int_{-\infty}^{+\infty} \sum_{m=1}^N \frac{\omega_m(t)}{\lambda - \lambda_m} \frac{d}{ds} \{W \{g, \bar{\varphi}_m\} W \{g, \varphi_m\}\} ds = \sum_{m=1}^N \frac{2\omega_m(t)}{\lambda - \lambda_m} W \{g, \bar{\varphi}_m\} W \{g, \varphi_m\} \Big|_{-\infty}^{+\infty} = 0. \end{aligned}$$

Thus, based on the last equality and equality (21), we were able to derive the following relationship:

$$\int_{-\infty}^{+\infty} G(s, t) g^2(s, k, t) ds = -4k^2 a(k, t) b(-k, t) \gamma(t).$$

Now, we derive the equality (20) in Lemma 2. To do this, we evaluate the following integral using the relations (5), (7) and (9):

$$\begin{aligned} &\int_{-\infty}^{+\infty} G(s, t) g(s, k, t) g(s, -k, t) ds = -\gamma(t) \int_{-\infty}^{+\infty} u'_s g(s, k, t) g(s, -k, t) ds + \\ &+ 4 \int_{-\infty}^{+\infty} \sum_{m=1}^N \alpha_m(t) \frac{\partial |\varphi_m(s, t)|^2}{\partial s} g(s, k, t) g(s, -k, t) ds = \gamma(t) I_1 + \sum_{m=1}^N \alpha_m(t) I_m. \quad (22) \\ &I_1 = - \int_{-\infty}^{+\infty} g(s, k, t) g(s, -k, t) u'_s(s, t) ds = \int_{-\infty}^{+\infty} g'_s(s, k, t) g(s, -k, t) u(s, t) ds + \\ &+ \int_{-\infty}^{+\infty} g(s, k, t) g'_s(s, -k, t) u(s, t) ds = \int_{-\infty}^{+\infty} (g''_{ss}(s, -k, t) + k^2 g(s, -k, t)) g'_s(s, k, t) ds + \end{aligned}$$

$$+ \int_{-\infty}^{+\infty} (g''_{ss}(s, k, t) + k^2 g(s, k, t)) g'_s(s, -k, t) ds = \frac{\partial g(s, k, t)}{\partial s} \cdot \frac{\partial g(s, -k, t)}{\partial s} \Big|_{-\infty}^{+\infty} +$$

$$+ k^2 g(s, k, t) g(s, -k, t) \Big|_{-\infty}^{+\infty} = 4k^2 b(k, t) b(-k, t).$$

We can also derive this expression using lemma 1.

$$I_m = 4 \int_{-\infty}^{+\infty} \frac{\partial}{\partial s} (|\varphi_m(s, t)|^2) g(s, k, t) g(s, -k, t) ds = 0.$$

If we substitute the found I_1 and I_m into expression (22), we obtain equality (20). **The lemma is proved.**

According to equality (18) and lemma 2, we form the following equality:

$$\frac{dr^+(k, t)}{dt} = (32ik^5 \alpha(t) + 8ik^3 \beta(t) - 2ik\gamma(t)) r^+(k, t). \quad (23)$$

Furthmore, based on equality (17) and lemma 2, the following relation $a_t(k, t) = 0$ hold. According to last equality, $a(k, t) = a(k, 0)$, $k_m(t) = k_m(0)$ is valid. That is

$$\frac{d\lambda_m(t)}{dt} = 0. \quad (24)$$

Now, we calculate the time evolution of the normalizing constants corresponding to the eigenvalues. Therefore, using relation (16), we obtain the following equality:

$$\frac{\partial g(x, k, t)}{\partial t} = 8\alpha(t) \left(2g_{xxxxx} - 5ug_{xxx} + \frac{15}{4}uu_xg - \frac{25}{4}u_{xx}g_x - \frac{15}{8}gu_{xxx} + \frac{15}{4}u^2g_x \right) + 60k^2\alpha(t)u_xg +$$

$$+ 16ik^5\alpha(t)g + \beta(t) (-u_x + 4ik^3)g + \beta(t) (2u + 4k^2)g_x - \frac{g(x, k, t)}{2ik} \int_{-\infty}^x G(s, t)g(s, k, t)g(s, -k, t)ds +$$

$$+ \frac{g(x, -k, t)}{2ik} \int_{-\infty}^x G(s, t)g^2(s, k, t)ds = -ikg(x, k, t) - g'_x(x, k, t) +$$

$$+ 4 \sum_{m=1}^N \alpha_m(t) \left[g(x, -k, t) \int_{-\infty}^x \frac{\partial |\varphi_m(s, t)|^2}{\partial s} g^2(s, k, t)ds + g(x, k, t) \int_{-\infty}^x \frac{\partial |\varphi_m(s, t)|^2}{\partial s} g(s, k, t)g(s, -k, t)ds \right].$$

If we use the following equality $g(x, k_m(t), t) = B_m(t)f_m(x, k_m(t), t)$ in the last expression and take the limit in $x \rightarrow +\infty$ and equate the terms before $e^{ik_n x}$, then we get the following equality

$$\frac{dB_m(t)}{dt} = (32ik_m^5 \alpha(t) + 8ik_m^3 \beta(t) - 2ik_m \gamma(t) + 2\omega_m(t)A_m(t)) B_m(t), \quad m = \overline{1, N}. \quad (25)$$

Finally, we were able to prove this theorem.

Theorem 2. If functions $u(x, t)$, $\varphi_m(x, t)$, $m = \overline{1, N}$, $x \in \mathbb{R}$, $t > 0$ are solutions to problem (1)-(5), then the potential $u(x, t)$ and the operator $L(t)$ given by the scattering theory $\{r^+(k, t), \lambda_m(t), B_m(t), m = \overline{1, N}\}$ satisfy the following differential equations (23),(24),(25).

Let's present an algorithm for solving problems (1)-(5):

- First of all, we find the givens of the scattering theory of the Sturm-Liouville operator whose potential $u_0(x)$ satisfies the condition (3), that is, the following summation

$$\left\{ r^+(k, 0) = -\frac{b(-k, 0)}{a(k, 0)}, \quad \lambda_m(0), \quad B_m(0), \quad m = 1, 2, \dots, N \right\}.$$

- Using the solutions of the differential equations in Theorem 2, we find the scattering data of the operator $L(t)$, $t > 0$:

$$\left\{ r^+(k, t) = -\frac{b(-k, t)}{a(k, t)}, \quad \lambda_m(t), \quad B_m(t), \quad m = 1, 2, \dots, N \right\}.$$

- Using the obtained scattering data for $t > 0$, the Gelfand-Levitan-Marchenko integral equation is solved, resulting in the determination of the unique solution $u(x, t)$, as established in theorem 1.
- Then, we solve the correct problem for the family of operators $L(t)$, $t > 0$ with potential $u(x, t)$ and find its Yost solutions and the functions $\varphi_m = \varphi_m(x, t)$, $m = 1, N$ satisfying condition (2).

Remark. The derived relations completely determine the evolution of the scattering data, thereby enabling the application of the inverse scattering transform method to solve the initial-value problem (1)-(5).

Example 1. Solve the following Cauchy problem

$$\begin{cases} u_t + t(u_{xxxxx} - 20u_x u_{xx} - 10uu_{xxx} + 30u^2 u_x) + 3t^2(u_{xxx} - 6uu_x) + \frac{1}{1+t^2}u_x = \\ = e^{2 \arctg t} (32t - 24t^2) \frac{\partial}{\partial x} |\varphi_1(x, t)|^2, \\ -\varphi_1'' + u(x, t)\varphi_1 = -\varphi_1 \end{cases}$$

$$u|_{t=0} = -\frac{2}{ch^2 x}, \quad \int_{-\infty}^{+\infty} |\varphi_1(x, t)|^2 dx = \frac{1}{2} e^{-2 \arctg t}.$$

The solution to this Cauchy problem is as follows:

$$u(x, t) = \frac{-2}{ch^2(x - \arctg t)}, \quad \varphi_1(x, t) = \frac{e^{-x}}{1 + e^{-2x+2 \arctg t}}.$$

Integration of the loaded and extended fifth-order Korteweg-de Vries equation with a self-consistent source

The most general definition of loaded equations and a detailed classification of various loaded equations are given in the works of A.M. Nakhushev [20]. Among the works devoted to loaded Korteweg-de Vries equations, the works [21-24] should be especially noted.

Let us now turn to the consideration of a special case of equation (1), namely, we will consider the loaded Korteweg-de Vries equation with a source:

$$\begin{aligned} u_t + P(u(x_0, t)) (u_{xxxxx} - 20u_x u_{xx} - 10uu_{xxx} + 30u^2 u_x) + Q(u(x_1, t)) (u_{xxx} - 6uu_x) + \\ + Z(u(x_2, t)) u_x = 4 \sum_{m=1}^N F_m(u(x_3, t)) \frac{\partial}{\partial x} |\varphi_m|^2. \end{aligned} \quad (26)$$

Where $P(y)$, $Q(z)$, $Z(s)$ and $F_m(t)$, $m = \overline{1, N}$ are polynomials from y , z , s and t and $x_0, x_1, x_2, x_3 \in \mathbb{R}$ respectively. Equation (26) is not a special case of equation (1), since in equation (26) the coefficients depend on the value of the solution on a manifold of lower dimension. If in problem (26)+(2)-(5) instead of equation (1) we consider equation (26), then the following theorem is valid.

Theorem 3. If the functions $u(x, t)$, $\varphi_m(x, t)$, $m = \overline{1, N}$, $x \in \mathbb{R}$, $t > 0$ are a solution to the problem (26)+(2)-(5), then the scattering data $\{r^+(k, t), \lambda_m(t), B_m(t), m = \overline{1, N}\}$ of the operator $L(t)$ with potential $u(x, t)$ satisfy the following differential equations:

$$\frac{\partial r^+(k, t)}{\partial t} = (32ik^5 P(u(x_0, t)) + 8ik^3 Q(u(x_1, t)) - 2ikZ(u(x_2, t))) r^+(k, t), \quad \frac{\partial \lambda_m(t)}{\partial t} = 0, \quad m = \overline{1, N},$$

$$\frac{dB_m(t)}{dt} = (32ik_m^5 P(u(x_0, t)) + 8ik_m^3 Q(u(x_1, t))) B_m(t) +$$

$$+ (-2ik_m Z(u(x_2, t)) + 2F_m(u(x_3, t)) A_m(t)) B_m(t), \quad m = \overline{1, N}.$$

Example 2. Solve the following Cauchy problem

$$\begin{cases} u_t + \alpha(t)u(0, t) (u_{xxxx} - 20u_x u_{xx} - 10u u_{xxx} + 30u^2 u_x) + \beta(t)u(\ln 2, t) (u_{xxx} - 6u u_x) + \\ + \gamma(t)u(1, t)u_x = (t+1)u(2, t) \frac{\partial}{\partial x} |\varphi_1(x, t)|^2, \\ -\varphi_1'' + u(x, t)\varphi_1 = -\varphi_1. \end{cases}$$

$$u|_{t=0} = -\frac{2}{ch^2 x}, \quad \int_{-\infty}^{+\infty} |\varphi_1(x, t)|^2 dx = \frac{1}{2(t+1)^2}.$$

Where

$$\alpha(t) = \frac{(-24t^3 - 23t^2 + t + 2)(t^2 + 2t + 2)^2}{256(t+1)^3}, \quad \beta(t) = \frac{-3t^2(t^2 + 2t + 5)^2}{32(t+1)^2},$$

$$\gamma(t) = \frac{e^2 \left(t(e^2 + e^{-2}(t+1)^2)^2 - 8(t+1) \right) (e^2 + (t+1)^2)^2}{16(t+1)^2 (e^4 + (t+1)^2)^2}.$$

The solution to this Cauchy problem is as follows:

$$u(x, t) = \frac{-2}{ch^2(x - \ln(t+1))}, \quad \varphi_1(x, t) = \frac{e^{-x}}{1 + (t+1)^2 e^{-2x}}.$$

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STATISTICAL ANALYSIS OF AGREEMENT BETWEEN DBSCAN AND HDBSCAN CLUSTERING RESULTS

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ABSTRACT. We propose a statistical approach to comparing the DBSCAN and HDBSCAN clustering algorithms. The approach examines transitions in object status (core, border, or noise) and the association between clustering results. The neighborhood radius ε for DBSCAN is selected subject to a constraint defined by the proportion of objects labeled as noise by HDBSCAN. The resulting assignments are analyzed using contingency tables, Pearson's χ^2 statistic, Cramér's V , and Tschuprow's T . A computational experiment is conducted on data with clusters of differing densities. The two algorithms identify nearly identical numbers of noise objects. Cramér's V increases with the number of nearest neighbors k , whereas Tschuprow's T shows an overall increase but is not monotonic. The results support the use of HDBSCAN-derived information for calibrating DBSCAN and for a statistical comparison of density-based clustering results.

MSC (2020): 62H30; 62H17; 62H20.

Key words: comparison methodology, hybrid algorithm, object status, optimal radius selection.

Introduction

A major difficulty in comparing clustering algorithms is the absence of a universal comparison criterion. Theoretical limitations of a unified clustering framework have been discussed in [7]. This difficulty limits the methodological basis for applying density-based clustering to data containing nonspherical clusters. In addition to differences in distance measures and density-estimation parameters, algorithms may differ substantially in both the number and membership of the clusters they identify. External and internal validity criteria for nonspherical clusters formed through object connectivity have been considered in [1].

This study compares two density-based clustering methods, DBSCAN and HDBSCAN, in the context of related clustering approaches [2–5]. Although the two methods belong to the same broad class, they approach clustering differently. Their principal difference concerns the treatment of clusters with varying densities.

A major limitation of DBSCAN [6] is its sensitivity to the neighborhood radius ε . The proposed modification addresses this limitation through the following steps:

- determining an admissible range of ε from the data and a specified number of nearest neighbors k ;
- selecting ε by solving an optimization problem subject to a constraint on the proportion of objects labeled as noise;
- using the objects labeled as noise at the selected ε to subsequently cluster lower-density local regions that do not overlap previously identified clusters.

An important advantage of HDBSCAN is its ability to analyze data with heterogeneous density. The proposed radius-selection approach broadens the applicability of DBSCAN and provides a more systematic basis for comparing it with other density-based methods. Under these conditions, the frequently stated expectation

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that HDBSCAN generally produces better clustering because it automatically analyzes the density structure of the data requires further examination.

The comparison is motivated by two observations:

- HDBSCAN identifies noise objects while accounting for local differences in data density;
- the proportion of noise objects identified by HDBSCAN can serve as a regularization parameter when selecting an optimal radius ε for DBSCAN.

Conventional comparisons typically rely on aggregate clustering-quality indices. Here, we instead examine statistical patterns in object-status transitions between the results of DBSCAN and HDBSCAN. This makes it possible to investigate how differences between the algorithms arise, assess the stability of object statuses as ε changes, determine the role of border objects in cluster formation, and relate the HDBSCAN noise proportion to the selected DBSCAN radius.

Mathematical model

Let $E_0 = \{S_1, \dots, S_m\}$ be a set of objects. Each object is described by n heterogeneous features, $X(n) = (x_1, \dots, x_n)$, of which ξ are measured on interval scales and the remaining $n - \xi$ on nominal scales. The metric $\rho(x, y)$ is used to quantify proximity between objects.

The interpretability of DBSCAN and confidence in its clustering results can be improved by constructing a hybrid model that combines cluster analysis, classification, and fuzzy logic. During clustering, each object receives one of three statuses: core, border, or noise. These statuses are treated as additional nominal features. A constraint on the proportion of objects assigned noise status is used to select an optimal neighborhood radius ε .

The hybrid procedure for selecting an optimal DBSCAN radius ε is implemented as follows [8]. Admissible values of ε are examined sequentially to construct a new data table. Each column represents a nominal feature with either three or two categories corresponding to object statuses.

The problem is thus reduced to optimizing a feature-stability measure for either three categories or two categories, with core and border statuses combined in the latter case. The two-category formulation is equivalent to a two-class recognition problem. The first class, K_1 , represents the DBSCAN clustering results; the second class, K_2 , is an artificially constructed opposing class. The distribution of object categories in the opposing class is initially modeled by a Dirichlet distribution. The proportion of noise objects acts as a regularization parameter.

Within the theory of estimating dependencies from empirical data described in [9], the search for ε can be interpreted as a problem of recovering information from indirect experiments. If the target function $f(t)$ cannot be measured at any point t , another measurable function $F(x)$ may be available that is related to $f(t)$ through the operator equation $Af(t) = F(x)$.

Concepts from fuzzy set theory [10] are used to interpret such indirect experimental results. Let the admissible radius values lie in $[\varepsilon_{\min}, \varepsilon_{\max}]$, where $\varepsilon_{\min}$ corresponds to an assignment in which all objects are labeled as noise, and $\varepsilon_{\max}$ to an assignment in which no object is labeled as noise.

Let category $\mu = 1$ combine core and border statuses, and let $\mu = 2$ denote noise status. Denote by $d(\mu)$ the proportion of objects in class K_1 belonging to category μ , and by $\theta(\mu)$ the corresponding proportion in class K_2 .

The membership function for class K_1 is defined by

$$f(\mu) = \frac{d(\mu)}{d(\mu) + \theta(\mu)}. \quad (1)$$

Using (1), the stability measure is defined as

$$g = \frac{1}{2} \sum_{\mu=1}^2 \begin{cases} (d(\mu) + \theta(\mu)) \cdot f(\mu), & f(\mu) \geq 0.5, \\ (d(\mu) + \theta(\mu)) \cdot (1 - f(\mu)), & f(\mu) < 0.5. \end{cases} \quad (2)$$

For the two-category assignments of the objects in E_0 , the stability values in (2) are evaluated over the admissible interval $[\varepsilon_{\min}, \varepsilon_{\max}]$. The optimal radius ε is obtained by solving the following unimodal minimization problem:

$$g(\varepsilon) - 0.5 \longrightarrow \min_{\varepsilon \in [\varepsilon_{\min}, \varepsilon_{\max}]}.$$

The curve in Fig. 1 has a single pronounced minimum, which is consistent with the assumed unimodal form of the objective.

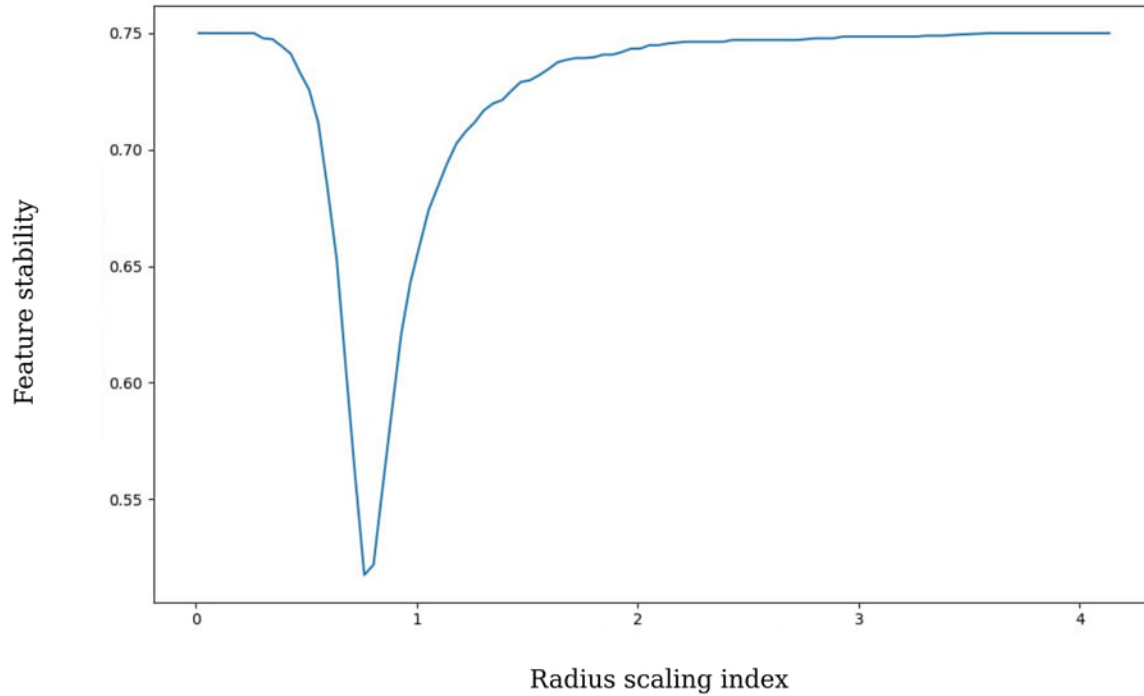


Fig. 1. Stability measure (2) over the admissible radius interval $[\varepsilon_{\min}, \varepsilon_{\max}]$.

Differences in cluster-density structure are reflected in how border objects are assigned to neighboring clusters. This motivates testing the hypothesis that some objects labeled as noise by HDBSCAN become core or border objects under DBSCAN.

The present study reverses the earlier formulation. Previously, the noise proportion was specified as a constraint when selecting ε for DBSCAN. Here, HDBSCAN determines that proportion. The resulting value is then used to select an optimal radius ε and to cluster the data with DBSCAN.

DBSCAN and HDBSCAN share the following properties:

- the ability to identify clusters of arbitrary shape;
- the use of proximity relations in a metric space and nearest-neighbor analysis;
- no requirement to specify the number of clusters in advance;
- the identification of objects that are not assigned to any cluster, referred to here as noise objects.

The aim of the study is to identify structural patterns that reflect differences in the mechanisms of DBSCAN and HDBSCAN. Its contribution is a statistical comparison procedure based on object-status transitions and the use of the HDBSCAN noise proportion as a regularization parameter for selecting the DBSCAN radius. To address this aim, the following hypotheses are proposed.

Hypothesis 1. Object status transitions

Objects labeled as noise by HDBSCAN are hypothesized to form a heterogeneous group. At the selected optimal radius, some of these objects may become border or core objects in DBSCAN. Support for this hypothesis would suggest that an HDBSCAN noise label can reflect a local density deficit rather than the absence of membership in a cluster under an alternative density criterion.

Hypothesis 2. Clustering mechanisms and regularization

Differences between DBSCAN and HDBSCAN are hypothesized to arise primarily from the treatment of cluster boundaries and objects with low local density. The dense interiors of the clusters identified by the two algorithms are expected to overlap substantially, while the main differences involve the reassignment of border and noise objects. The HDBSCAN noise proportion may therefore serve as a regularization parameter when selecting an optimal DBSCAN radius. Support for this hypothesis would indicate a relationship between the local density structure identified by HDBSCAN and the parameters selected for DBSCAN.

Hypothesis 3. Stability of object statuses

The stability of object statuses under changes in ε is hypothesized to differ across categories. Core objects are expected to be highly stable, whereas border and noise objects may exhibit substantially greater variability. This would allow the reliability of each object's category assignment to be assessed quantitatively.

Hypothesis 4. Reassignment of border objects

Changes in ε are hypothesized to affect primarily the classification of border objects. If parameter changes predominantly reassign objects in this category, this would suggest that the main difference between the algorithms lies in how competing neighboring clusters assign objects located in boundary regions.

Hypothesis 5. Association between results and algorithm selection

The association between the algorithms' results is examined by cross-tabulating HDBSCAN assignments with DBSCAN object statuses and cluster memberships. Statistical dependence is assessed using contingency tables and Pearson's χ^2 statistic, together with Cramér's V and Tschuprow's T . Cramér's V is treated as the principal measure of association, while Tschuprow's T provides a supplementary measure. Pearson's χ^2 test of independence can be used to assess the statistical significance of an observed association when its assumptions are satisfied.

The strength of association is proposed as a possible guide to algorithm selection. High values would support considering DBSCAN as a potentially simpler computational alternative, whereas low values would indicate substantial differences and motivate closer examination of the density structure captured by HDBSCAN. Validation of this hypothesis would clarify whether association measures can help characterize the conditions under which each algorithm is applicable.

Computational experiment

The experiment used the Breast Cancer Wisconsin (Diagnostic) dataset [11]. Its features were computed from digitized images of fine-needle aspirates of breast masses. Cluster analysis was performed on 569 objects described by 30 quantitative features. Before clustering, the data were mapped to the interval $[-1, 1]$ using a fractional-linear transformation, after which Euclidean distance was used. For HDBSCAN, the minimum cluster size parameter, `min_cluster_size`, was set to 2.

Table 1 presents DBSCAN results obtained by selecting ε subject to a constraint determined by the HDBSCAN noise proportion. The values $k \in \{3, 7, 10, 15\}$ were chosen to examine changes in the clustering results.

As shown in Table 1, the numbers of objects assigned to clusters by HDBSCAN and DBSCAN differ only slightly. At $k = 15$, HDBSCAN assigns 253 objects to clusters, whereas DBSCAN assigns 254.

This result is consistent with calibrating the DBSCAN radius to the HDBSCAN noise proportion. The two algorithms classify nearly the same number of objects as noise, providing a basis for comparing their cluster assignments. Matching these totals does not imply that the same individual objects receive the same labels.

Table 2 relates the HDBSCAN assignments to the object statuses determined by DBSCAN.

Of the objects labeled as noise by HDBSCAN, 16 are border objects in DBSCAN, and none are core objects (Table 2).

The correspondence between the two algorithms' cluster assignments is shown in Table 3.

Table 1. Clustering results for different numbers of nearest neighbors k .

k	HDBSCAN			DBSCAN	
	Cluster sizes	Clustered objects	Noise (%)	Cluster sizes	Clustered objects
3	2, 3, 2, 2, 304	313	45.0	19, 248, 14, 4, 8, 3, 3, 3, 5, 3, 3	313
7	9, 2, 3, 147	161	71.7	154, 7	161
10	9, 252	261	54.1	14, 247	261
15	7, 246	253	55.5	254	254

Table 2. DBSCAN object statuses at $k = 10$ and an HDBSCAN noise proportion of 54.1%.

HDBSCAN assignment	DBSCAN status		
	Core	Border	Noise
Noise	0	16	292
Cluster 1 (9 objects)	3	5	1
Cluster 2 (252 objects)	164	73	15

Table 3. Cross-tabulation of DBSCAN and HDBSCAN assignments at $k = 10$.

DBSCAN assignment	HDBSCAN assignment		
	Noise	Cluster 1	Cluster 2
Noise	292	1	15
Cluster 1 (14 objects)	6	8	0
Cluster 2 (247 objects)	10	0	237

Tables 2 and 3 show that, among the 16 objects labeled as noise by HDBSCAN but as border objects by DBSCAN, six are assigned to DBSCAN cluster 1 and the remaining ten to DBSCAN cluster 2.

Table 4 reports the contingency-table statistics and measures of association.

Table 4. Contingency-table statistics and measures of association.

Statistic or coefficient	$k = 3$	$k = 7$	$k = 10$	$k = 15$
Pearson's χ^2 statistic	914.1066	634.7085	752.2534	461.4917
Information-based association measure	277.1457	228.1456	313.5447	280.2493
Pearson's contingency coefficient C	0.7851	0.7262	0.7546	0.6690
Tschuprow's T	0.4464	0.6280	0.8130	0.7573
Cram Γ @r's V	0.5669	0.7468	0.8130	0.9006

The statistics and association measures in Table 4 serve different purposes, so their numerical values should not be compared directly. Cram Γ @r's V and Tschuprow's T are particularly useful for describing the strength of association between the clustering results. Figure 2 shows their variation with k , together with Pearson's contingency coefficient C .

The results obtained for the different values of k show the following patterns:

- Cram Γ @r's V increases from 0.567 at $k = 3$ to 0.901 at $k = 15$, indicating stronger association between

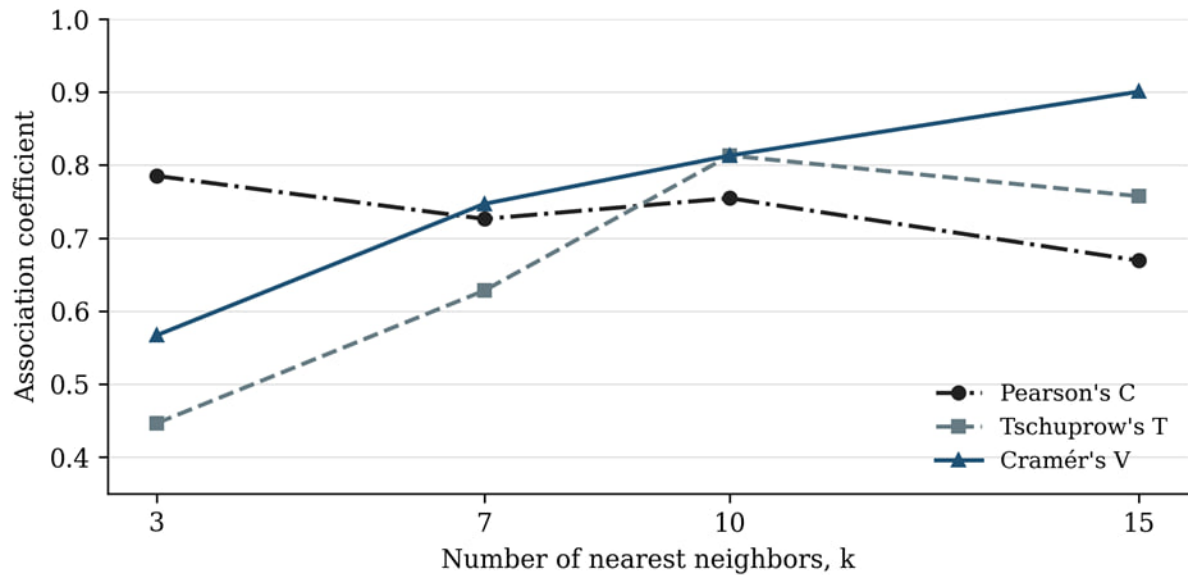


Fig. 2. Association coefficients for different numbers of nearest neighbors k .

the assignments produced by the two algorithms.

- Tschuprow's T also increases overall, from 0.446 at $k = 3$ to 0.757 at $k = 15$, but reaches its maximum of 0.813 at $k = 10$ and then decreases.
- Pearson's contingency coefficient remains relatively high (0.669–0.785), but cannot be compared directly with Cramér's V because its maximum depends on the dimensions of the contingency table.

The χ^2 statistic and the information-based measure depend on the contingency-table dimensions and the number of categories. Their raw values are therefore less suitable for comparing results across different values of k . Cramér's V and Tschuprow's T provide normalized measures of association, although the table structure must still be considered when interpreting them.

For the settings examined, Cramér's V indicates increasing association as k increases, reaching 0.901 at $k = 15$. Tschuprow's T supports an overall increase relative to $k = 3$, but its decrease between $k = 10$ and $k = 15$ shows that the trend is not uniformly increasing across both measures. Thus, the experiment demonstrates a general tendency toward stronger association between DBSCAN and HDBSCAN assignments as k increases, rather than a monotonic increase in every coefficient.

Conclusion

We have proposed a statistical procedure for comparing DBSCAN and HDBSCAN based on object-status transitions and contingency-table analysis of clustering results. In the reported experiment, using the HDBSCAN noise proportion as a constraint on the DBSCAN radius produced nearly identical numbers of noise objects and substantial association between the resulting assignments. The approach may support parameter selection and comparative assessment of density-based clustering algorithms for data with heterogeneous structure. DBSCAN may be considered as a computationally simpler candidate in suitable settings, but any efficiency advantage requires separate validation.

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DYNAMICS OF RANDOM PRODUCTS OF POSITIVE CONTRACTIONS ON L_1

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ABSTRACT. We study a trajectorywise analogue of the classical zero–two phenomenon for products of positive contractions on L^1 -spaces. Motivated by the strong zero–two law for a single positive contraction, we ask whether a uniform bound strictly below 2 for the limiting successive differences along a fixed realization forces these differences to converge to zero. We show that this is false in general. More precisely, for every $c \in (0, 2)$ we construct two rank-one positive contractions on $L^1([0, 1])$ and an alternating realization ω such that

$$\|T_\omega^{n+1}f - T_\omega^n f\|_1 = c$$

for every $n \geq 1$ and every nonnegative f with $\|f\|_1 = 1$. Hence the hypothesis is satisfied for a suitable $\varepsilon > 0$, while the conclusion fails. In fact, the same equality holds in operator norm. Thus a direct pathwise extension of the deterministic zero–two criterion to non-autonomous, and hence random, products is impossible without additional assumptions.

Key words: positive contractions, zero–two law, random products, L^1 -spaces, rank-one operators, counterexample.

Introduction

The asymptotic behaviour of iterates of linear operators on Banach spaces is a classical topic in functional analysis, with important connections to ergodic theory and Markov processes (see, for example, [4–6]). For positive contractions on L^1 -spaces, a fundamental question concerns the decay of the successive differences

$$\|T^{n+1}f - T^n f\|_1.$$

A classical result of Ornstein and Sucheston is the strong zero–two law; related forms and refinements were subsequently obtained by several authors (see [8–10, 12]). One useful formulation is that if the limiting successive difference for every positive unit vector is bounded uniformly away from 2, then the limiting successive difference must in fact be zero for every vector in L^1 .

The purpose of the present note is to examine whether an analogous statement remains valid for products of different positive contractions. Let $\{T_\alpha\}_{\alpha \in A}$ be a family of positive contractions and, for a realization $\omega = (\alpha_1, \alpha_2, \dots)$, define

$$T_\omega^n = T_{\alpha_n} \circ \dots \circ T_{\alpha_1}.$$

The question considered here is trajectorywise: for a fixed realization ω , does a uniform strict bound below 2 for

$$\lim_{n \rightarrow \infty} \|T_\omega^{n+1}f - T_\omega^n f\|_1$$

over all positive unit vectors force this limit to be zero?

We show that the answer is negative, even in a particularly simple situation. For every prescribed $c \in (0, 2)$ we construct two rank-one Markov contractions on the standard probability space $[0, 1]$ and an

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alternating realization for which the successive difference is identically equal to c for every positive unit vector. Thus the obstruction is not caused by a specially chosen initial function; it holds uniformly over the whole positive unit sphere. Moreover, the operator-norm successive difference is also equal to c .

The paper is organized as follows. In the main part we recall the deterministic zero-two criterion, formulate its pathwise analogue for products of positive contractions, and then give the counterexample.

Main part

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $L^1 = L^1(X, \mathcal{A}, \mu)$ be the real Banach space of integrable functions with norm $\|\cdot\|_1$. Denote by L^1_+ the cone of nonnegative elements of L^1 . A linear operator $T : L^1 \rightarrow L^1$ is called *positive* if $T(L^1_+) \subset L^1_+$, and it is called a *contraction* if $\|T\| \leq 1$. As usual, equalities between L^1 -functions are understood modulo sets of measure zero.

For a contraction T and $f \in L^1$, the sequence $\|T^n(f - Tf)\|_1 = \|T^n f - T^{n+1} f\|_1$ is non-increasing and therefore has a limit.

Theorem 1. (see [10].) Let T be a positive linear contraction on L^1 . Assume that there exists $\varepsilon > 0$ such that, for every $f \in L^1_+$ with $\|f\|_1 = 1$,

$$\lim_{n \rightarrow \infty} \|T^n f - T^{n+1} f\|_1 < 2(1 - \varepsilon).$$

Then, for every $f \in L^1$,

$$\lim_{n \rightarrow \infty} \|T^n f - T^{n+1} f\|_1 = 0.$$

We now formulate the corresponding question for products of different positive contractions. Let $(A, \mathcal{P}, \lambda)$ be a probability space, so that $\lambda(A) = 1$, and let $\{T_\alpha : L^1(X) \rightarrow L^1(X)\}_{\alpha \in A}$ be a family of positive linear contractions. Set $\Omega = A^{\mathbb{N}}$, $\omega = (\alpha_1, \alpha_2, \dots) \in \Omega$, and, for $n \geq 1$, define $T_\omega^n := T_{\alpha_n} \circ T_{\alpha_{n-1}} \circ \dots \circ T_{\alpha_1}$. When needed, Ω may be equipped with the product probability measure $\mathbb{P} = \lambda^{\mathbb{N}}$. The question below, however, is pathwise and concerns a fixed realization ω .

Question 1. Fix $\omega \in \Omega$. Assume that there exists a constant $\varepsilon > 0$, independent of f , such that for every $f \in L^1_+$ with $\|f\|_1 = 1$ the limit exists and

$$\lim_{n \rightarrow \infty} \|T_\omega^{n+1} f - T_\omega^n f\|_1 < 2(1 - \varepsilon). \quad (27)$$

Does it follow that, for every $f \in L^1$,

$$\lim_{n \rightarrow \infty} \|T_\omega^{n+1} f - T_\omega^n f\|_1 = 0? \quad (28)$$

The answer is negative. In fact, the next construction shows that every value strictly between 0 and 2 can occur as a constant successive difference, uniformly over all positive unit vectors.

Theorem 2. For every $c \in (0, 2)$ there exist two positive linear contractions T_1, T_2 on $L^1([0, 1])$ and a realization $\omega = (1, 2, 1, 2, \dots)$ such that, for every $f \in L^1_+$ with $\|f\|_1 = 1$ and every $n \geq 1$,

$$\|T_\omega^{n+1} f - T_\omega^n f\|_1 = c. \quad (29)$$

Moreover,

$$\|T_\omega^{n+1} - T_\omega^n\| = c, \quad n \geq 1. \quad (30)$$

Consequently, Question 1 has a negative answer.

Proof. Fix $c \in (0, 2)$ and put $a = 1 - \frac{c}{2} \in (0, 1)$. On $X = [0, 1]$ with Lebesgue measure define

$$u(x) = 1, \quad v(x) = \frac{1}{a} \mathbf{1}_{[0, a]}(x).$$

Then $u, v \in L^1_+$ and $\|u\|_1 = \|v\|_1 = 1$. Furthermore,

$$\|u - v\|_1 = \int_0^a \left(\frac{1}{a} - 1 \right) dx + \int_a^1 1 dx = (1 - a) + (1 - a) = 2(1 - a) = c.$$

Define rank-one operators $T_1, T_2 : L^1([0, 1]) \rightarrow L^1([0, 1])$ by

$$T_1 h = \left(\int_0^1 h(t) dt \right) u, \quad T_2 h = \left(\int_0^1 h(t) dt \right) v. \quad (31)$$

Both operators are linear and positive. Moreover, for every $h \in L^1([0, 1])$,

$$\|T_i h\|_1 = \left| \int_0^1 h(t) dt \right| \|u_i\|_1 \leq \|h\|_1,$$

where $u_1 = u$ and $u_2 = v$. Hence $\|T_i\| \leq 1$. Since equality is attained on any nonnegative unit vector, we have $\|T_1\| = \|T_2\| = 1$. Because $\int_0^1 u dt = \int_0^1 v dt = 1$, for $i, j \in \{1, 2\}$ we obtain

$$T_i T_j h = \left(\int_0^1 T_j h(t) dt \right) u_i = \left(\int_0^1 h(t) dt \right) u_i = T_i h.$$

Thus

$$T_i T_j = T_i, \quad i, j \in \{1, 2\}. \quad (32)$$

For the alternating realization $\omega = (1, 2, 1, 2, \dots)$, formula (32) gives

$$T_\omega^n = \begin{cases} T_1, & n \text{ odd}, \\ T_2, & n \text{ even}. \end{cases}$$

Now let $f \in L_+^1$ with $\|f\|_1 = 1$. Since $f \geq 0$, $\int_0^1 f(t) dt = 1$, and therefore $T_1 f = u$, $T_2 f = v$. Hence for every $n \geq 1$,

$$\|T_\omega^{n+1} f - T_\omega^n f\|_1 = \|u - v\|_1 = c,$$

which proves (29). The same calculation gives the operator-norm statement. Indeed,

$$(T_2 - T_1)h = \left(\int_0^1 h(t) dt \right) (v - u),$$

so $\|(T_2 - T_1)h\|_1 \leq c\|h\|_1$. Taking a nonnegative h with $\|h\|_1 = 1$ gives equality. Thus $\|T_2 - T_1\| = c$, and (30) follows from the alternating form of T_ω^n .

Finally, choose any $0 < \varepsilon < 1 - \frac{c}{2}$. Then $c < 2(1 - \varepsilon)$, so the hypothesis (27) holds for every positive unit vector f , whereas by (29)

$$\lim_{n \rightarrow \infty} \|T_\omega^{n+1} f - T_\omega^n f\|_1 = c > 0.$$

Therefore (28) fails. This proves that Question 1 has a negative answer. $\square$

Remark 1. The operators in Theorem 2 preserve integrals:

$$\int_0^1 T_i h dx = \int_0^1 h dx, \quad i = 1, 2.$$

Thus the counterexample already occurs for rank-one Markov operators. It is also uniform in the initial positive unit vector and, by (30), persists at the operator-norm level.

In this note we have shown that the deterministic zero-two criterion for a single positive contraction does not admit a direct trajectorywise extension to products of positive contractions. The counterexample is elementary but strong: for any prescribed $c \in (0, 2)$ there exist two rank-one Markov contractions and an alternating realization for which

$$\|T_\omega^{n+1} f - T_\omega^n f\|_1 = c$$

for every n and every positive unit vector f . Hence the limiting successive differences may stay uniformly and strictly below 2 without converging to zero.

This shows that additional structural assumptions are indispensable if one seeks a zero-two type theorem for random or non-autonomous products. Possible assumptions may concern the admissible realizations, common invariant structures, mixing properties, or probabilistic hypotheses on the driving process. The present result isolates the basic obstruction: once the operator is allowed to vary with time, the deterministic dichotomy can fail even for very simple positive contractions on a standard L^1 -space.

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GEOMETRIC APPLICATION OF FUNCTORS OF PROBABILITY MEASURES P AND IDEMPOTENT PROBABILITY MEASURES I

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ABSTRACT. In the work defines the geometric properties of the space of idempotent probability measures. It also considers the construction of one-dimensional, two-dimensional, three-dimensional,..., $(n-1)$ -dimensional and infinite-dimensional simplices in the space of idempotent probability measures. In particular, for a compact of X it is established that if the space of $I_3(X) \setminus X$ is hereditary normally, then X is metrized; some subsets of space of idempotent probability measures which are, respectively, Z - sets, are allocated to max - plus - convex subsets.

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Key words: simplices, an idempotent measure, a compact, Z - a set, max - plus - a convex set.

The theory of idempotent measures belongs to idempotent mathematics, i.e. the fields of the mathematics based on replacement of usual arithmetic operations idempotent (as, for example, $x \oplus y = \max\{x, y\}$). The idempotent mathematics intensively develops now (see, for example, [3], the review article [4] and the bibliography in it). Her communication with traditional mathematics is described by the informal principle [4] according to which there is a heuristic compliance between important, interesting and useful designs the last and similar results of idempotent mathematics.

In the present article we investigate a functor of I idempotent probability measures in category compact hausdorff spaces. In traditional mathematics to him there corresponds the functor of P probability measures. The concept of an idempotent measure (Maslov's measure) finds numerous applications in various fields of mathematics, mathematical physics and economy. In particular, such measures arise in problems of dynamic optimization [4]; analogy between Maslov's integration and optimization is noted also in [4].

Unlike a case of probability measures to which consideration extensive literature is devoted (see [5]) geometrical and topological properties of spaces of idempotent measures were practically not investigated.

We will consider the set $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ with two algebraic operations: addition $\oplus$ and multiplication $\odot$ determined as follows $u \oplus v = \max\{u, v\}$ and $u \odot v = u + v$ where $\mathbb{R}$ a set of real numbers.

Let X be a Hausdorff compact space ($\equiv$ a compact), $C(X)$ be the algebra of continuous functions on X with the usual algebraic operations. For $C(X)$ operations $\oplus$ and $\odot$ we define as follows: $\varphi \oplus \psi = \max\{\varphi, \psi\}$ and $\varphi \odot \psi = \varphi + \psi$, $\varphi, \psi \in C(X)$.

The problem of constructing one-dimensional, two-dimensional, three-dimensional,..., $(n-1)$ - dimensional and infinite-dimensional simplices in the space of idempotent probability measures: Suppose we are given $\alpha, \beta \in \mathbb{R}_{\max}$ and $x, y \in L$. Using these, we can write their max -plus linear combination as follows $\alpha \odot x \oplus \beta \odot y$.

If $\alpha \oplus \beta = 1$, ($\max\{\alpha, \beta\} = 0$), and $-\infty \leq \alpha, \beta \leq 0$, then we call the expression $\alpha \odot x \oplus \beta \odot y$ is a max -plus-affine combination.

$$\sigma_{\oplus}^1 = [x, y]_{\oplus} = \{\alpha \odot x \oplus \beta \odot y : \max\{\alpha, \beta\} = 0, -\infty \leq \alpha, \beta \leq 0\}$$

is a one-dimensional max -plus simplex.

$$\sigma_{\oplus}^2 = [x, y, z]_{\oplus} = \{\alpha \odot x \oplus \beta \odot y \oplus \gamma \odot z : \max\{\alpha, \beta, \gamma\} = 0, -\infty \leq \alpha, \beta, \gamma \leq 0\}$$

is a two-dimensional max-plus simplex.

Similarly,

$$\sigma_{\oplus}^{(n-1)} = [x_1, x_2, \dots, x_n]_{\oplus} = \left\{ \bigoplus_{i=1}^n \alpha_i \odot x_i : -\infty \leq \alpha_i \leq 0, i = \overline{1, n}, \bigoplus_{i=1}^n \alpha_i = 0 \right\}$$

– $(n-1)$ is a dimensional max-plus simplex.

In this case $I(2) \simeq \sigma_{\oplus}^1, I(3) \simeq \sigma_{\oplus}^2, \dots, I(n) \simeq \sigma_{\oplus}^{(n-1)}$ s are appropriate.

If X is a topological space and $|X| = \aleph_0$, then a set of idempotent probability measures $I(X)$ is defined. This set is an infinite-dimensional max-plus simplex:

$$\sigma_{\oplus}^{\infty} = [x_1, x_2, \dots, x_n, \dots]_{\oplus} = \left\{ \bigoplus_{i=1}^{\infty} \alpha_i \odot x_i : -\infty \leq \alpha_i \leq 0, i = \overline{1, \infty}, \bigoplus_{i=1}^{\infty} \alpha_i = 0 \right\}$$

Definition 1. A functional $\mu: C(X) \rightarrow \mathbb{R}$ called an idempotent probability measure on X , if it satisfies the following conditions:

- (I1) μ is normed, i.e. $\mu(\lambda_X) = \lambda$ for all $\lambda \in \mathbb{R}$,
- (I2) μ is max-plus-homogeneous, i.e. $\mu(\lambda \odot \varphi) = \lambda \odot \mu(\varphi)$ for all $\lambda \in \mathbb{R}$ and $\varphi \in C(X)$,
- (I3) μ is max-plus-additive, i.e. $\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$ for all $\varphi, \psi \in C(X)$.

The set of all idempotent probability measures on X denotes by $I(X)$ [1]. Consider $I(X)$ as a subspace of $\mathbb{R}^{C(X)}$.

For given compacts X, Y and a continuous mapping $f: X \rightarrow Y$ we can verify that the natural map $I(f): I(X) \rightarrow I(Y)$, defined by the formula $I(f)(\mu)(\psi) = \mu(\psi \circ f)$, is continuous. Moreover, the construction I is a normal functor.

Therefore, for an arbitrary idempotent probability measure $\mu \in I(X)$ one can define the concept of support:

$$\text{supp} \mu = \bigcap \{A \subset X : \overline{A} = A, \mu \in I(A)\}.$$

For a positive integer n , we define the following set

$$I_n(X) = \{\mu \in I(X) : |\text{supp} \mu| \leq n\}.$$

Set

$$I_{\omega}(X) = \bigcup_{n=1}^{\infty} I_n(X).$$

The set $I_{\omega}(X)$ is everywhere dense [1] in $I(X)$.

Idempotent probability measure $\mu \in I_{\omega}(X)$ is called an idempotent probability measure with finite support. Note that if μ is an idempotent probability measure with finite support $\text{supp} \mu = \{x_1, x_2, \dots, x_k\}$, then μ can be represented as $\mu = \lambda_1 \odot \delta_{x_1} \oplus \lambda_2 \odot \delta_{x_2} \oplus \dots \oplus \lambda_k \odot \delta_{x_k}$ the only way, where $-\infty < \lambda_i \leq 0, i = 1, 2, \dots, k, \lambda_1 \oplus \lambda_2 \oplus \dots \oplus \lambda_k = 0$. Here, as usual, for $x \in X$ by δ_x is denoted a functional on $C(X)$, defined by the formula $\delta_x(\varphi) = \varphi(x), \varphi \in C(X)$, and called the Dirac measure. It is concentrated at the point x .

Definition 2. A functional $\mu: C(X) \rightarrow \mathbb{R}$ called a probability measure on X , if it satisfies the following conditions:

- (P1) μ is normed, i.e. $\mu(\lambda_X) = \lambda$, here λ_X is a constant function such that $\lambda_X(x) = \lambda, \lambda \in \mathbb{R}$.
- (P2) μ is homogeneous, i.e. $\mu(\lambda \varphi) = \lambda \mu(\varphi)$ for all $\lambda \in \mathbb{R}$ and $\varphi \in C(X)$,
- (P3) μ is additive, i.e. $\mu(\varphi + \psi) = \mu(\varphi) + \mu(\psi)$ for all $\varphi, \psi \in C(X)$.

The set of all probability measures on X denotes by $P(X)$ [3]. Consider $P(X)$ as subspaces of $\mathbb{R}^{C(X)}$. In the induced topology the sets of the view

$$\langle \mu; \varphi_1, \dots, \varphi_k; \varepsilon \rangle = \{v \in F(X) : |\mu(\varphi_i) - \nu(\varphi_i)| < \varepsilon, i = 1, \dots, k\}$$

form a base of neighbourhoods of $\mu \in F(X)$ where $\varphi_i \in C(X), i = 1, \dots, k, \varepsilon > 0$ and $F = P$ or $F = I$.

The topology generated by this base coincide with the pointwise convergence topology on $F(X)$. For a compact Hausdorff space X the topological spaces $I(X)$ and $P(X)$ are compact Hausdorff spaces.

To show the difference between sets $P(X)$ and $I(X)$, consider a finite set X . Let X be a set consisting of 3 different points, and equipped with the discrete topology. Consider the following sets:

$$P(X) = \left\{ \sum_{i=1}^3 \mu_i \delta_{x_i} : \mu_i \geq 0, i = 1, 2, 3, \sum_{i=1}^3 \mu_i = 1 \right\},$$

$$I(X) = \left\{ \bigoplus_{i=1}^3 \mu_i \odot \delta_{x_i} : \mu_i \geq -\infty, i = 1, 2, 3, \bigoplus_{i=1}^3 \mu_i = 0 \right\}.$$

It is well-known that $P(X)$ represents the set of all probability measures on X , and $I(X)$ represents the set of all idempotent probability measures on X . Moreover, $P(X) \subset \mathbb{R}^n$ and $I(X) \subset \mathbb{R}_{\max}^n$.

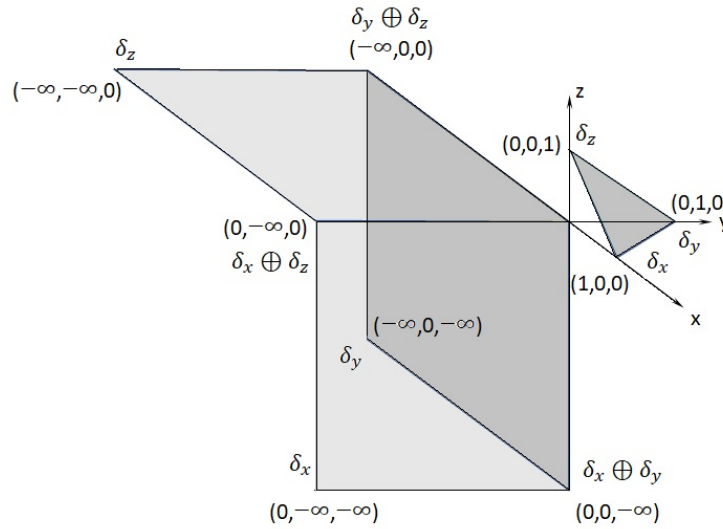


Figure 5: The images $I(X)$ (trihedron) and $P(X)$ (triangle) when X consists of three points.

For a given map $f: X \rightarrow Y$ of compact hausdorff spaces formula

$$F(f)(\mu)(\varphi) = \mu(\varphi \circ f), \quad \mu \in F(X),$$

defines a map $F(f): F(X) \rightarrow F(Y)$ where $\varphi \in C(Y)$, $F = P$ or $F = I$. In such cases they say that f induces $F(f)$.

If X consists of n points then $P(X)$ is a compact convex subset of $\mathbb{R}^n$, and $I(X)$ is a compact max-plus-convex subset of $\mathbb{R}_{\max}^n$

Theorem 1. For an arbitrary compact space X , the spaces $P_\omega(X)$ and $I_\omega(X)$ are homeomorphic.

The sets $P(X)$ of all probability measures and $I(X)$ of all idempotent probability measures are endowed with the topology of pointwise convergence. The topological spaces $P(X)$ and $I(X)$ are compact Hausdorff spaces.

Theorem 2. For an arbitrary finite compact set X , the spaces $P(X)$ and $I(X)$ are homeomorphic.

Proof. Consider the mapping

$$z_I^P : P(X) \longrightarrow I(X),$$

given by equality

$$z_I^P \left(\sum_{i=1}^n \alpha_i \delta_{x_i} \right) = \bigoplus_{i=1}^n \left(\left(\ln \alpha_i - \bigoplus_{j=1}^n \ln \alpha_j \right) \odot \delta_{x_i} \right),$$

and

$$z_P^I : I(X) \longrightarrow P(X),$$

rule defined

$$z_P^I \left(\bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \right) = \sum_{i=1}^n \frac{e^{\lambda_i}}{\sum_{j=1}^n e^{\lambda_j}} \cdot \delta_{x_i}.$$

We show that z_I^P and z_P^I – are mutually inverse mappings.

1) For each probability measure $\sum_{i=1}^n \alpha_i \delta_{x_i} \in P(X)$ the following equalities hold

$$\begin{aligned} z_P^I \left(z_I^P \left(\sum_{i=1}^n \alpha_i \delta_{x_i} \right) \right) &= z_P^I \left(\bigoplus_{i=1}^n \left(\ln \alpha_i - \bigoplus_{j=1}^n \ln \alpha_j \right) \odot \delta_{x_i} \right) = \\ &= \sum_{i=1}^n \frac{e^{\ln \alpha_i - \bigoplus_{j=1}^n \ln \alpha_j}}{\sum_{l=1}^n e^{\ln \alpha_l - \bigoplus_{j=1}^n \ln \alpha_j}} \cdot \delta_{x_i} = \sum_{i=1}^n \frac{e^{\ln \alpha_i} : e^{\bigoplus_{j=1}^n \ln \alpha_j}}{\left(\sum_{l=1}^n e^{\ln \alpha_l} \right) : e^{\bigoplus_{j=1}^n \ln \alpha_j}} \cdot \delta_{x_i} = \\ &= \sum_{i=1}^n \frac{\alpha_i \delta_{x_i}}{\sum_{l=1}^n \alpha_l} = \sum_{i=1}^n \alpha_i \delta_{x_i}; \end{aligned}$$

2) For each idempotent probability measure $\bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \in I(X)$ we have

$$\begin{aligned} z_I^P \left(z_P^I \left(\bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \right) \right) &= z_I^P \left(\sum_{i=1}^n \frac{e^{\lambda_i}}{\sum_{j=1}^n e^{\lambda_j}} \delta_{x_i} \right) = \\ &= \bigoplus_{i=1}^n \left(\ln \frac{e^{\lambda_i}}{\sum_{j=1}^n e^{\lambda_j}} - \bigoplus_{l=1}^n \ln \frac{e^{\lambda_l}}{\sum_{j=1}^n e^{\lambda_j}} \right) \odot \delta_{x_i} = \\ &= \bigoplus_{i=1}^n \left(\ln e^{\lambda_i} - \ln \sum_{j=1}^n e^{\lambda_j} - \bigoplus_{l=1}^n \left(\ln e^{\lambda_l} - \ln \sum_{j=1}^n e^{\lambda_j} \right) \right) \odot \delta_{x_i} = \\ &= \bigoplus_{i=1}^n \left(\lambda_i - \ln \sum_{j=1}^n e^{\lambda_j} - \bigoplus_{l=1}^n \lambda_l + \ln \sum_{j=1}^n e^{\lambda_j} \right) \odot \delta_{x_i} = \bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i}. \end{aligned}$$

Consequently, $z_I^P z_P^I : I(X) \rightarrow I(X)$ and $z_P^I z_I^P : P(X) \rightarrow P(X)$ – identical mappings.

We now show that the mapping z_I^P and z_P^I are continuous. Since they are mutually inverse mappings between compacta, it suffices to show the continuity of only one of them.

We show that the mapping $z_I^P : P(X) \rightarrow I(X)$ is continuous. Let $\mu_0 \in P(X)$ – be a probability measure, $\{\mu_t\}_{t=1}^\infty \subset P(X)$ – sequence converging to μ_0 in topology of pointwise convergence (symbolically $\lim_{t \rightarrow \infty} \mu_t = \mu_0$).

Since X is finite, without loss of generality, we can assume that $\text{supp} \mu_t = \text{supp} \mu_0 = \{x_1, x_2, \dots, x_n\}$, $n \in \mathbb{N}$, for all $t = 1, 2, \dots$

Then we have $\lim_{t \rightarrow \infty} \alpha_t^i = \alpha_0^i$, где $\mu_t = \sum_{i=1}^n \alpha_t^i \delta_{x_i}$, $t = 0, 1, 2, \dots$. On the other hand, according to the continuity of $\ln$ and the operation *oplus*, the equality $\lim_{t \rightarrow \infty} \ln \alpha_t^i = \ln \alpha_0^i$ follows. It attracts her t

$$\lim_{t \rightarrow \infty} \left(\ln \alpha_t^i - \bigoplus_{j=1}^n \ln \alpha_t^j \right) = \ln \alpha_0^i - \bigoplus_{j=1}^n \ln \alpha_0^j.$$

Therefore, $\lim_{t \rightarrow \infty} z_I^P(\mu_t) = z_I^P(\mu_0)$, i.e. the mapping z_I^P is continuous.

Theorem 1 is proved.

Corollary 1. For an arbitrary metrizable compact set X , the spaces $P(X)$ and $I(X)$ are homomorphic. The proof of the following assertion is established using the max-plus variant of the Hahn-Banach theorem [9].

We call a subset L of the space $C(X)$ max-plus-a linear subspace in $C(X)$, if:

- 1) $c_X \in L$ for each $c \in \mathbb{R}$;
- 2) $\lambda \odot \varphi \in L$, for any $\lambda \in \mathbb{R}$ and $\varphi \in L$;
- 3) $\varphi \oplus \psi \in L$, for each $\varphi, \psi \in L$.

Lemma 1 [7]. Let L – max-plus- be a linear subspace in $C(X)$. Let $\mu : L \rightarrow \mathbb{R}$ – be a functional satisfying the conditions of normalization, homogeneity and additivity (with $C(X)$ replaced by L). For an arbitrary $\varphi_0 \in C(X) \setminus L$ there exists a continuation of the mapping μ satisfying the conditions of normalization, homogeneity and additivity for the minimal max-plus- is a linear subspace of L' , containing $L \cup \{\varphi_0\}$ (with $C(X)$ replaced by L').

Consider the following subset in $C(X \times Y)$:

$$C_0 = \left\{ \bigoplus_{i=1}^n \varphi_i \odot \psi_i : \varphi_i \in C(X) \text{ and } \psi_i \in C(Y), i = 1, 2, \dots, n, n \in \mathbb{N} \right\}.$$

It is obvious that C_0 – max-plus-is a linear subspace in $C(X)$.

For every pair $(\mu, \nu) \in I(X) \times I(Y)$ we set

$$(\mu \widetilde{\otimes} \nu) \left(\bigoplus_{i=1}^n \varphi_i \odot \psi_i \right) = \bigoplus_{i=1}^n \mu(\varphi_i) \odot \nu(\psi_i).$$

Proposition 1. $\mu \widetilde{\otimes} \nu$ is an idempotent probability measure on C_0 .

Proof. Each $c \in \mathbb{R}$ can be represented as $c_{X \times Y} = a_X \odot b_Y$, where $a, b \in \mathbb{R}$ and $a + b = c$. Therefore, $(\mu \widetilde{\otimes} \nu)(c_{X \times Y}) = (\mu \otimes \nu)(a_X \odot b_Y) = \mu(a) \odot \nu(b) = a \odot b = c$.

Let $\lambda \in \mathbb{R}_{\max}$ and $\bigoplus_{i=1}^n \varphi_i \odot \psi_i \in C_0$. Then $(\mu \widetilde{\otimes} \nu)(\lambda \odot \bigoplus_{i=1}^n \varphi_i \odot \psi_i) = (\mu \widetilde{\otimes} \nu) \left(\bigoplus_{i=1}^n (\lambda \odot \varphi_i) \odot \psi_i \right) = \bigoplus_{i=1}^n \mu(\lambda \odot \varphi_i) \odot \nu(\psi_i) = \bigoplus_{i=1}^n \lambda \odot \mu(\varphi_i) \odot \nu(\psi_i) = \lambda \odot \bigoplus_{i=1}^n \mu(\varphi_i) \odot \nu(\psi_i) = \lambda \odot (\mu \widetilde{\otimes} \nu) \left(\bigoplus_{i=1}^n \varphi_i \odot \psi_i \right)$.

Finally, let $\bigoplus_{i=1}^n \varphi_{1i} \odot \psi_{1i} \in C_0$ and $\bigoplus_{j=1}^m \varphi_{2j} \odot \psi_{2j} \in C_0$. Then fair $(\mu \widetilde{\otimes} \nu) \left(\bigoplus_{i=1}^n \varphi_{1i} \odot \psi_{1i} \oplus \bigoplus_{j=1}^m \varphi_{2j} \odot \psi_{2j} \right) = (\mu \widetilde{\otimes} \nu) \left(\bigoplus_{k=1}^n \varphi_{kl} \odot \psi_{kl} \right) = \bigoplus_{k=1}^n \mu(\varphi_{kl}) \odot \nu(\psi_{kl}) = \bigoplus_{i=1}^n \mu(\varphi_{1i}) \odot \nu(\psi_{1i}) \oplus \bigoplus_{j=1}^m \mu(\varphi_{2j}) \odot \nu(\psi_{2j}) = (\mu \widetilde{\otimes} \nu) \left(\bigoplus_{i=1}^n \varphi_{1i} \odot \psi_{1i} \right) \oplus (\mu \widetilde{\otimes} \nu) \left(\bigoplus_{j=1}^m \varphi_{2j} \odot \psi_{2j} \right)$. Proposition 1 is proved.

Since C_0 – max-plus-is a linear subspace in $C(X \times Y)$, according to Lemma 1, there is the extension of the idempotent probability measure $\mu \widetilde{\otimes} \nu$ на $C(X \times Y)$.

Set

$$\mu \otimes \nu = \bigoplus \{ \xi \in I(X \times Y) : \xi|_{C_0} = \mu \widetilde{\otimes} \nu \}.$$

Thus, we have proved the following max-plus-a variant of the Fubini theorem.

Theorem 3. For every pair $(\mu, \nu) \in I(X) \times I(Y)$ there exists a unique idempotent probability measure $\mu \otimes \nu \in I(X \times Y)$ such that $(\mu \otimes \nu)(\varphi \odot \psi) = \mu(\varphi) \odot \nu(\psi)$, $\varphi \in C(X)$, $\psi \in C(Y)$.

Although the spaces $I(X)$ and $P(X)$ are homeomorphic for an arbitrary metrizable compact set X , however, as the following example shows, the P and I constructions form different functors from each other.

To construct a corresponding example in which the functors P and I are not isomorphic, we need some concepts from the categorical algebra.

Let $F, G : \mathfrak{R}_1 \rightarrow \mathfrak{R}_2$ – functors from the category $\mathfrak{R}_1$ to the category $\mathfrak{R}_2$.

The natural transformation $\alpha : F \rightarrow G$ of the functor F into the functor G is called [6] such a function $\alpha : ob\mathfrak{R}_1 \rightarrow Mor\mathfrak{R}_2$, such that $\alpha_A := \alpha(A) : F(A) \rightarrow G(A)$, (α_A , which means the value of the function α in the object A , is a morphism of the category $\mathfrak{R}_2$ from the object $F(A)$ to $G(A)$) and for any $\varphi : A \rightarrow B$, (from the object A to the object B) $B \in ob\mathfrak{R}_1$, in the category $\mathfrak{R}_2$ a commutative diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ F(\varphi) \downarrow & & \downarrow G(\varphi) \\ F(B) & \xrightarrow{\alpha_B} & G(B), \end{array}$$

i.e.

$$G(\varphi) \circ \alpha_A = \alpha_B \circ F(\varphi)$$

The morphisms, $\alpha_A, A \in ob\mathfrak{R}_1$ are called the components of the natural transformation α .

Thus, $\alpha = \{\alpha_A : A \in ob\mathfrak{R}_1\}$. The natural transformation $\varepsilon : F \rightarrow G$ of the functors $F, G : \mathfrak{R}_1 \rightarrow \mathfrak{R}_2$ is called [6] (natural) isomorphism if $\varepsilon_A : F(A) \rightarrow G(A)$ – an isomorphism of the category $\mathfrak{R}_2$, that is, in $\mathfrak{R}_2$ there exists a morphism $\eta_A : G(A) \rightarrow F(A)$, such that $\eta_A \circ \varepsilon_A = id_{F(A)}$ and $\varepsilon_A \circ \eta_A = id_{G(A)}$ for any object $A \in ob\mathfrak{R}_1$.

We now give the construction of an example in which the functors P and I are not isomorphic.

Example 1. Consider the sets $X = \{a, b, c\}$, $Y = \{a, b\}$, $Z = \{a, c\}$, where a, b, c – are different points (these sets are supplied with discrete topologies). Define the following mappings:

$$\begin{aligned} f : X &\rightarrow Y, & f(a) &= f(c) = a, & f(b) &= b \\ g : X &\rightarrow Z, & g(a) &= g(b) = a, & g(c) &= c. \end{aligned}$$

We show that there is no natural transformation of the functor P to the functor I and vice versa. For this purpose, consider the objects $X, Y \times Z$ from the category $Comp$ and the morphism $(f, g) : X \rightarrow Y \times Z$ from the category $Comp$.

It suffices to show that the mapping $P((f, g))$ has a property that the mapping $I((f, g))$ does not possess.

The mapping

$$X \xrightarrow{(f, g)} Y \times Z$$

under the influence of the functor P goes to the map

$$P(X) \xrightarrow{P((f, g))} P(Y \times Z),$$

and under the influence of the I functor –

$$I(X) \xrightarrow{I((f, g))} I(Y \times Z).$$

The morphisms $P((f, g))$ and $I((f, g))$ are determined pointwise [6], i.e.

$$P((f, g)) = (P(f), P(g))$$

and

$$I((f, g)) = (I(f), I(g)).$$

Note that, by the Fubini theorem [5], the correspondence $(P(Y) \times P(Z)) \ni (\mu, \nu) \mapsto \mu \otimes \nu \in P(Y \times Z)$ sets up an attachment

$$P(Y) \times P(Z) \subset \rightarrow P(Y \times Z).$$

By max-plus to the variant of the Fubini theorem (Theorem 2), $(I(Y) \times I(Z)) \ni (\mu, \nu) \mapsto \mu \otimes \nu \in I(Y \times Z)$ sets the embedding

$$I(Y) \times I(Z) \hookrightarrow I(Y \times Z).$$

The morphisms $(P(f), P(g))$ and $(I(f), I(g))$ are determined by the formulas

$$(P(f), P(g))(\mu)(\varphi \cdot \psi) = P(f)(\mu)(\varphi) \cdot P(g)(\mu)(\psi)$$

and

$$(I(f), I(g))(\mu)(\varphi \odot \psi) = I(f)(\mu)(\varphi) \odot I(g)(\mu)(\psi),$$

where $\varphi \in C(Y)$, $\psi \in C(Z)$, $\mu \in P(X)$ or $\mu \in I(X)$.

We show that the mapping $(P(f), P(g)) : P(X) \rightarrow P(Y) \times P(Z)$ – is an embedding.

In fact, for any

$$\mu = \alpha_1 \delta_a + \alpha_2 \delta_b + \alpha_3 \delta_c,$$

$$\nu = \beta_1 \delta_a + \beta_2 \delta_b + \beta_3 \delta_c,$$

with positive $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$, $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $\beta_1 + \beta_2 + \beta_3 = 1$, equality takes place

$$P(f)(\mu) = (\alpha_1 + \alpha_3) \delta_a + \alpha_2 \delta_b,$$

$$P(g)(\mu) = (\alpha_1 + \alpha_2) \delta_a + \alpha_3 \delta_c,$$

$$P(f)(\nu) = (\beta_1 + \beta_3) \delta_a + \beta_2 \delta_b,$$

$$P(g)(\nu) = (\beta_1 + \beta_2) \delta_a + \beta_3 \delta_c.$$

Therefore, $(P(f), P(g))(\mu) = (P(f), P(g))(\nu)$ if and only if

$$\begin{cases} \alpha_1 + \alpha_3 = \beta_1 + \beta_3, \\ \alpha_2 = \beta_2, \\ \alpha_1 + \alpha_2 = \beta_1 + \beta_2, \\ \alpha_3 = \beta_3. \end{cases}$$

This system has a unique solution $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \alpha_3 = \beta_3$. Hence, $\mu = \nu$. Thus, $(P(f), P(g))(\mu) = (P(f), P(g))(\nu)$ if and only if $\mu = \nu$, i.e., $(P(f), P(g)) : P(X) \rightarrow P(Y) \times P(Z)$ – is an embedding.

We show that the mapping $(I(f), I(g)) : I(X) \rightarrow I(Y) \times I(Z)$ is not an embedding. In fact, for idempotent probability measures

$$\mu = \lambda_1 \odot \delta_a \oplus \lambda_2 \odot \delta_b \oplus \lambda_3 \odot \delta_c,$$

$$\nu = \gamma_1 \odot \delta_a \oplus \gamma_2 \odot \delta_b \oplus \gamma_3 \odot \delta_c,$$

to $-\infty < \lambda_1, \lambda_2, \lambda_3, \gamma_1, \gamma_2, \gamma_3 \leq 0$ и $\lambda_1 \oplus \lambda_2 \oplus \lambda_3 = \gamma_1 \oplus \gamma_2 \oplus \gamma_3 = 0$, there are equalities

$$I(f)(\mu) = (\lambda_1 \oplus \lambda_3) \odot \delta_a \oplus \lambda_2 \odot \delta_b,$$

$$I(g)(\mu) = (\lambda_1 \oplus \lambda_2) \odot \delta_a \oplus \lambda_3 \odot \delta_c,$$

$$I(f)(\nu) = (\gamma_1 \oplus \gamma_3) \odot \delta_a \oplus \gamma_2 \odot \delta_b,$$

$$I(g)(\nu) = (\gamma_1 \oplus \gamma_2) \odot \delta_a \oplus \gamma_3 \odot \delta_c.$$

The equality $(I(f), I(g))(\mu) = (I(f), I(g))(\nu)$ holds if and only if

$$\begin{cases} \lambda_1 \oplus \lambda_3 = \gamma_1 \oplus \gamma_3, \\ \lambda_2 = \gamma_2, \\ \lambda_1 \oplus \lambda_2 = \gamma_1 \oplus \gamma_2, \\ \lambda_3 = \gamma_3. \end{cases}$$

This, the system has a number of solutions. For example, with $-\infty < \lambda_1 \leq 0$, $-\infty < \gamma_1 \leq 0$ set $(\lambda_1, \gamma_1, 0, 0, 0, 0)$ is her solution.

The equality $(I(f), I(g))(\mu) = (I(f), I(g))(\nu)$ is true, although $\lambda_1 \neq \gamma_1$ for this set. This means that the mapping $(I(f), I(g))$ is not an embedding. Thus, there is no natural transformation of the functor P to the functor I , since the morphism $I((f, g)) = (I(f), I(g))$ is not an embedding.

In other words, there is no natural transformation $\alpha = \{\alpha_X : X \in \text{Comp}\}$ such that the diagram

$$\begin{array}{ccc} I(X) & \xrightarrow{I((f,g))} & I(Y \times Z) \\ \alpha_X \downarrow & & \downarrow \alpha_{Y \times Z} \\ P(X) & \xrightarrow{P((f,g))} & P(Y \times Z) \end{array}$$

would be commutative.

The set of all nonempty closed subsets of the topological space X is denoted by $\exp X$. For the subsets $U_1, \dots, U_n \subset X$ we set

$$\begin{aligned} O\langle U_1, \dots, U_n \rangle &= \{F : F \in \exp X, F \subset \bigcup_{i=1}^n U_n, F \cap U_1 \neq \emptyset, \dots, F \cap U_n \neq \emptyset\} = \\ &= \{F : F \in \exp X, F \subset \bigcup_{i=1}^n U_n\} \cap (\bigcap_{i=1}^n F : F \in \exp X, F \cap U_i \neq \emptyset) \end{aligned}$$

If the sets $U_1, \dots, U_n$ are open, then the sets

$$\{F : F \in \exp X, F \subset \bigcup_{i=1}^n U_n\} = \exp(\bigcup_{i=1}^n U_i, X)$$

$$\{F : F \in \exp X, F \cap U_i \neq \emptyset\} = \exp X \setminus \exp(X \setminus U_i, X)$$

open in the space of closed subsets by definition of the Vietoris topology. Therefore, the set $O\langle U_1, \dots, U_n \rangle$ is open. On the other hand, this kind They also have elements of the prebase of the Vietoris topology: $\exp(U, X) = O\langle U \rangle$, $\exp X \setminus \exp(X \setminus U, X) = O\langle U, X \rangle$. Thus, the family of all sets of the form $O\langle U_1, \dots, U_n \rangle$, where the sets $U_1, \dots, U_n$ are open in the space X , is the base of Vietoris topology. The resulting topological space $\exp X$ is called a hyperspace of X . For the compact X , the hyperspace $\exp X$ is compact.

For the compact X , the natural number n , the functor F we put

$$F_n = \{a \in F(X) : |\text{supp} a| \leq n\},$$

$$F_n^0 = F_n(X) \setminus F_{n-1}(X).$$

Proposition 2. If τ – is an uncountable cardinal, then the space $\exp_3^0 \alpha N_\tau$ is not normal.

Proof. Since the cardinal τ is infinite and the set N_τ is infinite, there are F_1 and F_2 spaces of N_τ , such that F_1 is uncountable, F_2 is countable and $F_1 \cap F_2 = \emptyset$. Take the point $x_0 \in F_1 \cup F_2$. Choose the subsets A_1 and A_2 of the space $\exp_3^0 \alpha N_\tau$, assuming

$$A_1 = \{\{p, x, x_0\} : x \in F_1 \setminus \{x_0\}\}, \quad A_2 = \{\{p, x', x_0\} : x' \in F_2 \setminus \{x_0\}\},$$

where $p \in \alpha N_\tau \setminus N_\tau$. Obviously, $A_1 \cap A_2 = \emptyset$.

Let $F = \{x_1, x_2, x_3\} \in \exp_3^0 \alpha N_\tau \setminus A_1$. The set $O(\{x_1\}, \{x_2\}, \{x_3\})$ is an open neighborhood of the set F , that does not intersect A_1 . Therefore, the set A_1 is closed in $\exp_3^0(\alpha N_\tau)$. The closure of the set A_2 is proved in a similar way.

For each $x \in N_\tau$ we put $U_x = O(\alpha N_\tau \setminus \{x_0, x\}, \{x_0\}, \{x\}) \cap \exp_3^0(\alpha N_\tau)$. It is easy to see that the smallest neighborhoods by inclusion of the sets A_1 and A_2 in $\exp_3^0(\alpha N_\tau)$ are the sets $OA_1 = \bigcup_{x \in F_1} U_x$ and $OA_2 = \bigcup_{x \in F_2} U_x$, respectively. For the set $\{a, b, x_0\}$, where $a \in F_1$, $b \in F_2$, we have

$$\{a, b, x_0\} \in O(\alpha N_\tau \setminus \{x_0, a\}, \{x_0\}, \{a\}) \subset OA_1,$$

$$\{a, b, x_0\} \in O(\alpha N_\tau \setminus \{x_0, b\}, \{x_0\}, \{b\}) \subset OA_2.$$

This means, $OA_1 \cap OA_2 \neq \emptyset$. Proposition 2 is proved.

Proposition 3. If τ is an uncountable cardinal, then the $I_3^0(\alpha N_\tau)$ space is not normal.

Proof. Since $\exp_3^0(X)$ is closed in $I_3^0(X)$ for the compact X , and since normality is a hereditary property for closed subsets of the space, it follows that $I_3^0(\alpha N_\tau)$ is not normal. Proposition 3 is proven.

Theorem 4. Let X – be a compact. If the space $I_3(X) \setminus X$ is hereditarily normal, then X is metrizable.

Proof. Suppose that the compact, X is non-metrizable. If X has one nonisolated point, then X is homeomorphic to $\alpha \mathbb{N}_\tau$ для $\tau = |X| > \omega$. Proposition 3 implies that $I_3^0(X)$ is not normal. But, according to the condition $I_3^0(X)$ must be normal as a subset of the hereditarily normal space $I_3(X) \setminus X$. Got a contradiction.

Now let a and b – be different nonisolated points of the compact X . There are open neighborhoods of U and V of points a and b , such that $\overline{U} \cap \overline{V} = \emptyset$.

Consider the set $Z = \overline{U} \times \exp_2 \overline{V}$, and the formula $\lambda(x_0, x, y) = 0 \odot \delta_x \oplus 0 \odot \delta_y \oplus 0 \odot \delta_z$ we define the topological embedding $\lambda_0 : Z \rightarrow I_3(X) \setminus X$. The result of M. Katetov [11] (which asserts the perfect normality of the factor X under the condition of hereditary normality of the product of $X \times Y$ compact sets, $|Y| \geq \omega$) implies that the factor $\exp_2 \overline{V}$ product $Z = \overline{U} \times \exp_2 \overline{V}$ is perfectly normal. Further, applying the result of V. V. Fedorchuk [9] (which asserts the metrizable of the compact X , if for a normal functor F of degree at least 2 the space $F(X)$ is completely normal) we conclude that $\overline{V}$ is metrizable. Similarly, it can be shown that $\overline{U}$ is metrizable. Therefore, each nonisolated point of the compact X has a metrizable closed neighborhood. Hence, the compact X is locally metrizable. Therefore, it is metrizable. Theorem 4 is proved.

Corollary 2. Let X be a compact set and $n \geq 3$. If the space $I_n X \setminus X$ is hereditarily normal, then the compact X is metrizable.

A closed set A of a space X is called a Z -set in X , if the identity map id_X can be arbitrarily closely approximated by $f : X \rightarrow X \setminus A$. A countable union of Z -sets in X is called a σ - Z -set in X .

Theorem 5. For an arbitrary non-empty closed subset A of a compact X , where $A \neq X$, the subspace $I(A)$ is a Z -set in $I(X)$.

Proof. Fix $x_0 \in X \setminus A$, and for an arbitrary $\varepsilon > 0$, $\varepsilon \leq 1$ we define a continuous mapping $f_\varepsilon : I(X) \rightarrow I(X)$, assuming

$$f_\varepsilon(\mu) = (\ln(1 - \varepsilon) - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \mu \oplus (\ln \varepsilon - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \delta_{x_0}, \mu \in I(X).$$

The mapping is defined correctly. In addition, $f_\varepsilon(\mu) \notin I(A)$ for any $\mu \in I(X)$, because $x_0 \in \text{supp} f_\varepsilon(\mu)$, i.e. $(X \setminus A) \cap \text{supp} f_\varepsilon(\mu) \neq \emptyset$. Therefore, the mapping f_ε acts from $I(X)$ to $I(X) \setminus I(A)$. The distance between points x, y max-plus-the segment $[-\infty, 0]$ is determined by the rule $\rho(x, y) = |e^y - e^x|$. The distance between the points $\alpha_1 \odot x \oplus \beta_1 \odot y$ and $\alpha_2 \odot x \oplus \beta_2 \odot y$ max-plus-the segment $[x, y] = \{\alpha \odot x \oplus \beta \odot y : -\infty \leq \alpha \leq 0, -\infty \leq \beta \leq 0, \alpha \oplus \beta = 0\}$ we define by the formula

$$\rho(\alpha_1 \odot x \oplus \beta_1 \odot y, \alpha_2 \odot x \oplus \beta_2 \odot y) = |e^{\alpha_2} - e^{\alpha_1}| + |e^{\beta_2} - e^{\beta_1}|.$$

Consider the max-plus-segment

$$[\mu, \delta_{x_0}] = \{(\ln(1 - \varepsilon) - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \mu \oplus (\ln \varepsilon - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \delta_{x_0} : 0 \leq \varepsilon \leq 1\}.$$

Calculate the distance between the points $\mu = 0 \odot \mu \oplus (-\infty) \odot \delta_{x_0}$ and $f_\varepsilon(\mu) = (\ln(1 - \varepsilon) - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \mu \oplus (\ln \varepsilon - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \delta_{x_0}$ max-plus-segment $[\mu, \delta_{x_0}]$:

$$\begin{aligned} \rho(\mu, f_\varepsilon(\mu)) &= |e^{\ln(1-\varepsilon)-\ln(1-\varepsilon)\oplus\ln\varepsilon} - e^0| + |e^{\ln\varepsilon-\ln(1-\varepsilon)\oplus\ln\varepsilon} - e^{-\infty}| = \\ &= |e^{\ln(1-\varepsilon)-\ln(1-\varepsilon)\oplus\ln\varepsilon} - 1| + e^{\ln\varepsilon-\ln(1-\varepsilon)\oplus\ln\varepsilon}. \end{aligned}$$

Two cases are possible.

1-case. $0 < \varepsilon \leq \frac{1}{2}$. Then

$$|e^{\ln(1-\varepsilon)-\ln(1-\varepsilon)\oplus\ln\varepsilon} - 1| + e^{\ln\varepsilon-\ln(1-\varepsilon)\oplus\ln\varepsilon} = 0 + e^{\ln\frac{\varepsilon}{1-\varepsilon}} = \frac{\varepsilon}{1-\varepsilon}.$$

2-case. $\frac{1}{2} < \varepsilon \leq 1$. Then

$$|e^{\ln(1-\varepsilon)-\ln(1-\varepsilon)\oplus\ln\varepsilon} - 1| + e^{\ln\varepsilon-\ln(1-\varepsilon)\oplus\ln\varepsilon} = e^{\ln\frac{1-\varepsilon}{\varepsilon}} + 1 = \frac{1-\varepsilon}{\varepsilon} + 1 = \frac{1}{\varepsilon}.$$

So, for $0 < \varepsilon \leq \frac{1}{2}$ the equality $\rho(\mu, f_\varepsilon(\mu)) = \frac{\varepsilon}{1-\varepsilon}$ and idempotent probabilistic measures of $f_\varepsilon(\mu)$ approach μ in the metric of ρ with $\varepsilon \rightarrow 0$. This, in particular, implies that for $\varepsilon \rightarrow 0$ the measures $f_\varepsilon(\mu)$ converge to μ with respect to the topology of the space $I(X)$. Indeed, for an arbitrary function $\varphi \in C(X)$ with $0 < \varepsilon < \frac{1}{2}$ we have $f_\varepsilon(\mu)(\varphi) = 0 \odot \mu(\varphi) \oplus \ln \frac{\varepsilon}{1-\varepsilon} \odot \delta_{x_0}(\varphi) = \mu(\varphi) \oplus (\ln \frac{\varepsilon}{1-\varepsilon} + \varphi(x_0))$. For $0 < \varepsilon < \frac{1}{1+e^{2\|\varphi\|}}$ we have $f_\varepsilon(\mu)(\varphi) = \mu(\varphi)$. Therefore, $0 = |f_\varepsilon(\mu)(\varphi) - \mu(\varphi)| < \varepsilon$. In other words, the identity mapping $id_{I(X)}$ is approximated by the mappings $f_\varepsilon : I(X) \rightarrow I(X) \setminus I(A)$. Theorem 5 is proved.

Theorem 6. For an arbitrary compact X and every natural $n \in \mathbb{N}$, $n < |X|$, the subspace $I_n(X)$ is Z -set in $I(X)$. Therefore, $I_\omega(X)$ is a σ - Z -set.

Proof. Fix $\nu \in I(X) \setminus I_n(X)$. Then $|\text{supp}\nu| \geq n + 1$. For an arbitrary $\varepsilon > 0$, $\varepsilon \leq 1$ we define the mapping $f_\varepsilon : I(X) \rightarrow I(X)$ by the equality

$$f_\varepsilon(\mu) = (\ln(1 - \varepsilon) - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \mu \oplus (\ln \varepsilon - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \nu, \quad \mu \in I(X).$$

The mapping f_ε is defined correctly, and it is continuous. In addition, $f_\varepsilon(\mu) \notin I_n(X)$ for any $\mu \in I(X)$, because $\text{supp}f_\varepsilon(\mu) \supset \text{supp}\mu \cup \text{supp}\nu$, therefore, $|\text{supp}f_\varepsilon(\mu)| \geq n + 1$. Therefore, the mapping f_ε acts from $I(X)$ to $I(X) \setminus I_n(X)$.

Consider the max-plus-segment

$$[\mu, \nu] = \{(\ln(1 - \varepsilon) - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \mu \oplus (\ln \varepsilon - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \nu : 0 \leq \varepsilon \leq 1\}.$$

Calculate the distance between the points $\mu = 0 \odot \mu \oplus (-\infty) \odot \nu$ and $f_\varepsilon(\mu) = (\ln(1 - \varepsilon) - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \mu \oplus (\ln \varepsilon - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \nu$ max-plus-segment $[\mu, \nu]$:

$$\rho(\mu, f_\varepsilon(\mu)) = |e^{\ln(1-\varepsilon)-\ln(1-\varepsilon)\oplus\ln\varepsilon} - 1| + e^{\ln\varepsilon-\ln(1-\varepsilon)\oplus\ln\varepsilon}.$$

For $0 < \varepsilon \leq \frac{1}{2}$ we have

$$\rho(\mu, f_\varepsilon(\mu)) = \frac{\varepsilon}{1-\varepsilon}.$$

When $\frac{1}{2} < \varepsilon \leq 1$ holds

$$\rho(\mu, f_\varepsilon(\mu)) = \frac{1}{\varepsilon}.$$

So, for $0 < \varepsilon \leq \frac{1}{2}$ the equality $\rho(\mu, f_\varepsilon(\mu)) = \frac{\varepsilon}{1-\varepsilon}$ and idempotent probabilistic measures of $f_\varepsilon(\mu)$ approach μ in the metric of ρ with $\varepsilon \rightarrow 0$. This, in particular, implies that for $\varepsilon \rightarrow 0$ the measures $f_\varepsilon(\mu)$ converge to μ with respect to the topology of the space $I(X)$. Indeed, for an arbitrary function $\varphi \in C(X)$ with $0 < \varepsilon < \frac{1}{2}$ we have $f_\varepsilon(\mu)(\varphi) = 0 \odot \mu(\varphi) \oplus \ln \frac{\varepsilon}{1-\varepsilon} \odot \nu(\varphi) = \mu(\varphi) \oplus (\ln \frac{\varepsilon}{1-\varepsilon} + \nu(\varphi))$. For $0 < \varepsilon < \frac{1}{1+e^{2\|\varphi\|}}$ we have $f_\varepsilon(\mu)(\varphi) = \mu(\varphi)$. Therefore $0 = |f_\varepsilon(\mu)(\varphi) - \mu(\varphi)| < \varepsilon$. In other words, the identity mapping $id_{I(X)}$ is approximated by the

mappings $f_\varepsilon : I(X) \rightarrow I(X) \setminus I_n(X)$. Therefore, $I_n(X)$ is a Z -set in $I(X)$. Since $I_\omega(X) = \bigcup_{i=1}^\infty I_n(X)$, it therefore becomes a σ - Z -set. Theorem 6 is proved.

For the functor I , the compact X , the nonempty set $A \subset X$ we set

$$S_I(A) = \{\mu \in I(X) : \text{supp } \mu \cap A \neq \emptyset\}.$$

It is clear that $S_I(\emptyset) = \emptyset$, $S_I(X) = I(X)$. By construction, the inclusion of $A \subset B$ implies $S_I(A) \subset S_I(B)$.

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NUMERICAL MODELING OF THE IMPACT OF ONLINE REVIEWS AND NEGATIVE INFORMATION ON TOURIST VISIT INTENTION

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ABSTRACT. This article develops a nonlinear dynamic model for assessing the joint effect of online reviews, ratings, communication activity, and negative information on tourist visit intention. The model contains five dimensionless state indices: destination image, visit intention, communication influence, online-rating influence, and negative-information pressure. Numerical solutions are obtained using the classical fourth-order Runge–Kutta method. Five scenarios are compared over sixty conditional time units: baseline, stronger communication, improved online ratings, growth of negative information, and integrated management. Using a single consistent parameter set, the final visit-intention index increases by about 19.5% under stronger communication, by 33.8% when the rating channel improves, and by 58.3% under integrated management, whereas stronger negative information reduces it by about 23.0%. The results support the use of the model as a scenario-analysis tool for digital destination-image management.

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Key words: online reviews; eWOM; destination image; tourist visit intention; negative information; nonlinear dynamic model; Runge–Kutta method; scenario analysis.

Introduction

The digital information environment has fundamentally transformed the way tourists choose destinations. Before planning a trip, users form perceptions of a destination through search engines, social networks, booking platforms, and reviews on map services. Consequently, tourism attractiveness is now determined not only by natural and cultural resources or service infrastructure, but also by destination image, ratings, and reputational signals in the digital environment. Uzbekistan's tourism policy likewise identifies the active promotion of national tourism products in international markets, targeted advertising, and the use of modern marketing tools as priority tasks [1]. Under these conditions, quantitatively assessing the impact of online information flows on visit intention can strengthen the scientific basis of managerial decision-making.

Destination image is a multifactorial and time-varying system. Positive reviews and high ratings may increase trust in a destination and raise the likelihood of a visit, whereas negative reviews, complaints about service quality, and unfavorable information concerning safety or prices may have the opposite effect. In addition, a communication campaign may simultaneously affect destination image, review activity, and the spread of negative information. Therefore, the relationships among these variables should be examined not only through static regression coefficients but also through a dynamic system that explicitly accounts for time.

The purpose of this study is to represent the impact of online reviews and negative information on tourist visit intention using a system of nonlinear differential equations, solve the model numerically by the fourth-order Runge–Kutta method, and compare the outcomes of different management scenarios. The practical

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objectives are to formalize the influence mechanisms among the main digital and reputational factors, provide a reproducible computational algorithm based on a single parameter set, and evaluate how management measures targeting communication, ratings, and negative information affect final visit intention. Until the model is calibrated using observed data for a real destination, it should be interpreted not as a forecasting instrument but as a conceptual and numerical platform for scenario and sensitivity analysis.

Literature review and positioning of the study

Classical studies of destination image distinguish its functional and psychological components as well as holistic and individual attributes [2]. Baloglu and McCleary linked destination-image formation to information sources and personal factors [3], while Beerli and Martin systematized the interacting roles of information sources, previous experience, and tourist characteristics [4]. These approaches provide a theoretical basis for introducing digital information sources into the model as an independent driver of image dynamics.

The Internet has sharply increased the influence of electronic word of mouth (eWOM) in the tourism market. Litvin, Goldsmith, and Pan showed that online opinion exchange significantly affects consumer decisions in hospitality and tourism [5]. Xiang and Gretzel demonstrated that social media has become an important source in the travel-information search process [6]. Jalilvand et al. empirically examined the structural relationships among eWOM, destination image, tourist attitude toward destination and travel intention, documenting both a direct effect of eWOM on visit intention and an indirect effect mediated by destination image [7].

Subsequent research has confirmed the robustness of these relationships using broader empirical evidence. In a meta-analysis of 87 studies, Afshardoost and Eshaghi showed that destination image significantly predicts tourists' behavioral intentions [8]. Setiawan et al. analyzed the mediating roles of destination image, trust, and satisfaction in the effect of eWOM on visit intention [9]. Guo and Pesonen showed that the rating and content of online travel reviews can affect tourists' cognitive, affective, and conative image components differently; in particular, low-rated negative reviews may cause substantial damage to affective destination image [10].

Reputational risk and negative information also constitute independent determinants of behavioral intentions. Siyamiyan Gorji, Almeida Garcia, and Mercade-Mele found that risk- and crisis-related information can weaken destination image and subsequent behavioral intentions [11]. Ko, combining large-scale online-review and survey data, showed that destination image is associated with the perceived usefulness and reception of reviews [12]. Thus, although the empirical literature has extensively examined the relationships among eWOM, image, ratings, and visit intention, scenario-based analysis of feedback links and saturation mechanisms among these factors within a single nonlinear dynamic system remains relatively uncommon. The present study addresses this gap at the conceptual and numerical level.

Model and numerical strategy

The study is based on a deductive modeling approach. Five state indicators of the tourist decision process are selected and their interactions are represented by differential equations in the form of positive and negative flows. The variables are interpreted as dimensionless indices of relative intensity and are not restricted to the interval $[0, 1]$. This interpretation preserves the possibility of calibrating the model using different data sources.

Table 1. Model variables

Symbol	Meaning	Interpretation
$I(t)$	Destination image	Intensity of the overall positive perception of the destination
$B(t)$	Visit intention	Relative index of propensity to visit or book
$C(t)$	Communication influence	Flow of advertising, media activity, and positive content
$R(t)$	Online-rating influence	Positive reputational signal from reviews and ratings
$N(t)$	Negative-information pressure	Intensity of negative reviews and reputational risk

Source: Authors' calculations.

The following nonlinear system is adopted for the state variables:

$$\begin{aligned}\frac{dI}{dt} &= u_I + a_1C + a_2R - a_3NI - a_4I - \mu_1I^2, \\ \frac{dB}{dt} &= b_1I + b_2R - b_3NB - b_4B - \mu_2B^2, \\ \frac{dC}{dt} &= u_C + c_1B - c_2C - \mu_3C^2, \\ \frac{dR}{dt} &= u_R + r_1B + r_2I - r_3R - \mu_4R^2, \\ \frac{dN}{dt} &= d_1B - d_2CN - d_3N - \mu_5N^2.\end{aligned}$$

Here, all coefficients a_i , b_i , c_i , r_i , d_i , and μ_i are positive, while u_I , u_C , and u_R are nonnegative parameters representing source inflows. In the first equation, communication and ratings increase destination image, whereas negative information damages image through the product NI . The second equation represents the positive response of visit intention to destination image and ratings and its negative response to adverse information. The quadratic terms $-\mu_iX^2$ act as internal saturation mechanisms that prevent unbounded growth. The term $-d_2CN$ in the $N(t)$ equation represents the possibility that active communication can mitigate negative-information pressure.

For the numerical experiment, the initial conditions are

$$I(0) = 0.40, \quad B(0) = 0.30, \quad C(0) = 0.35, \quad R(0) = 0.32, \quad N(0) = 0.25.$$

The computational horizon is $T = 60$ conditional time units with step size $h = 0.1$. The baseline parameters are

$$\begin{aligned}u_I &= 0.02, & u_C &= 0.03, & u_R &= 0.02, \\ a_1 &= 0.15, & a_2 &= 0.12, & a_3 &= 0.06, & a_4 &= 0.10, & \mu_1 &= 0.02, \\ b_1 &= 0.13, & b_2 &= 0.10, & b_3 &= 0.05, & b_4 &= 0.09, & \mu_2 &= 0.018, \\ c_1 &= 0.11, & c_2 &= 0.12, & \mu_3 &= 0.015, \\ r_1 &= 0.08, & r_2 &= 0.09, & r_3 &= 0.11, & \mu_4 &= 0.014, \\ d_1 &= 0.06, & d_2 &= 0.07, & d_3 &= 0.10, & \mu_5 &= 0.016.\end{aligned}$$

Because a closed-form analytical solution is generally unavailable, the classical fourth-order Runge–Kutta scheme is applied [13,14,15]. Let

$$X(t) = (I(t), B(t), C(t), R(t), N(t))^T, \quad \frac{dX}{dt} = F(X).$$

At each step,

$$\begin{aligned}K_1 &= F(X_n), \\ K_2 &= F\left(X_n + \frac{h}{2}K_1\right), \\ K_3 &= F\left(X_n + \frac{h}{2}K_2\right), \\ K_4 &= F(X_n + hK_3), \\ X_{n+1} &= X_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4).\end{aligned}$$

All scenarios are computed from the same initial state and with the same values of T and h . The parameter modifications are summarized in Table 2.

Table 2. Parameter changes across scenarios

Scenario	Modification
Baseline	All parameters at baseline values
Communication +30%	u_C, a_1, c_1 multiplied by 1.30
Rating +25%	a_2, b_2, r_1, r_2 multiplied by 1.25
Negative information +40%	a_3, b_3, d_1 multiplied by 1.40
Integrated management	Communication +30%; rating +25%; d_2 multiplied by 1.30

Source: Authors' calculations.

Verification of the first RK4 step.

To demonstrate computational reproducibility, the first step of the baseline scenario, from $t_0 = 0$ to $t_1 = 0.1$, is presented numerically. The initial vector is

$$X_0 = (0.40, 0.30, 0.35, 0.32, 0.25).$$

The four RK4 increments are

$$\begin{aligned} K_1 &= (0.061700, 0.051630, 0.019163, 0.043366, -0.014125), \\ K_2 &= (0.061717, 0.051966, 0.019321, 0.043593, -0.013893), \\ K_3 &= (0.061719, 0.051965, 0.019322, 0.043593, -0.013894), \\ K_4 &= (0.061738, 0.052300, 0.019482, 0.043819, -0.013663). \end{aligned}$$

Therefore,

$$X_1 = X_0 + \frac{0.1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = (0.406172, 0.305197, 0.351932, 0.324359, 0.248611).$$

Thus, after the first step,

$$\begin{aligned} I(0.1) &= 0.406172, & B(0.1) &= 0.305197, & C(0.1) &= 0.351932, \\ R(0.1) &= 0.324359, & N(0.1) &= 0.248611. \end{aligned}$$

The remaining 599 steps are carried out using exactly the same algorithm.

Results

Table 3 presents the final values obtained at $T = 60$ from the same RK4 algorithm and a consistent parameter set for all scenarios.

Table 3. Final scenario results at $T = 60$

Scenario	$I(60)$	$B(60)$	$C(60)$	$R(60)$	$N(60)$
Baseline	4.300	4.616	3.155	4.434	0.827
Communication +30%	5.731	5.518	4.404	5.264	0.786
Rating +25%	5.739	6.179	3.929	6.389	0.949
Negative information +40%	3.357	3.554	2.598	3.685	0.999
Integrated management	7.429	7.306	5.376	7.398	0.729

Source: Authors' calculations.

In the baseline scenario, visit intention reaches $B(60) = 4.616$. When the communication channel is strengthened, $B(60) = 5.518$, representing a 19.5% increase relative to the baseline. When the influence of online ratings is strengthened, $B(60) = 6.179$, corresponding to a 33.8% increase. This result indicates that the rating signal affects visit intention directly through the term b_2R and indirectly through mutual reinforcement between destination image and ratings. When the negative-information parameters are increased by 40%, $B(60)$ falls to 3.554, which is 23.0% below the baseline. Integrated management produces the largest effect: $B(60) = 7.306$, or 58.3% above the baseline level.

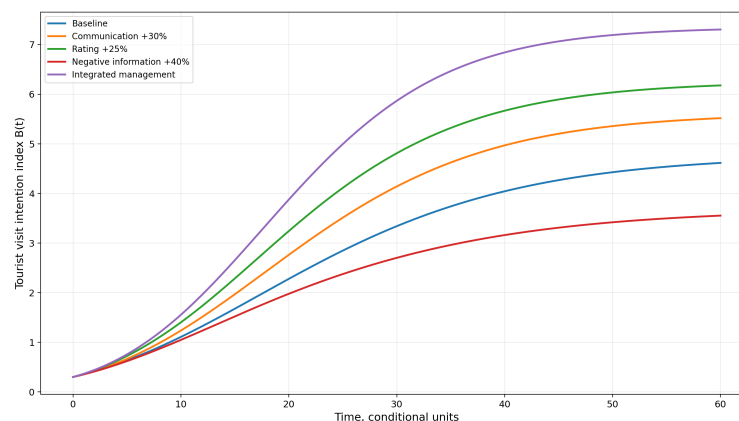


Figure 1. Dynamics of tourist visit intention $B(t)$ under different scenarios

Source: Authors' calculations.

Figure 1 shows how differences among the scenarios develop over time. Initially, all trajectories start from the same point, but when positive reputational channels are strengthened, the gap increases cumulatively. Under the negative-information scenario, the growth rate begins to slow at an early stage. The advantage of integrated management is explained not by a change in a single coefficient, but by the simultaneous strengthening of feedback links among communication, destination image, and ratings.

An important trade-off is observed in the dynamics of negative information. Strengthening only the rating channel increases visit intention and activity, but the rise in $B(t)$ creates additional pressure on $N(t)$ through the term $d_1 B$. As a result, under the rating scenario $N(60) = 0.949$, which is 14.8% above the baseline. Direct strengthening of the negative-information parameters leads to $N(60) = 0.999$. Under integrated management, however, the increase in d_2 strengthens the channel through which communication mitigates negative information and reduces $N(60)$ to 0.729, which is 11.8% below the baseline level.

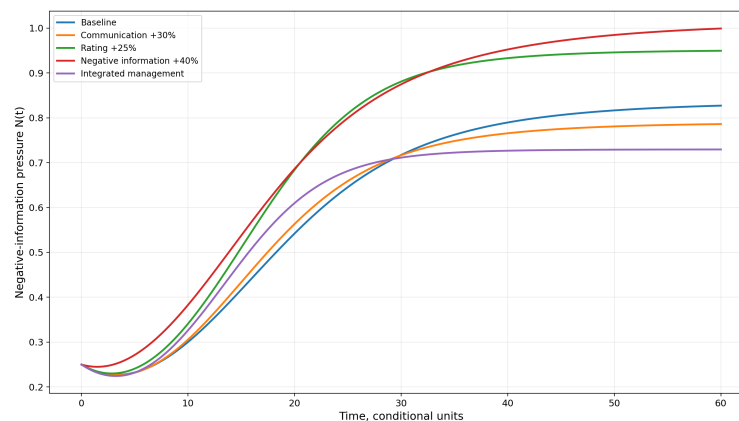


Figure 2. Dynamics of negative-information pressure $N(t)$ under different scenarios

Source: Authors' calculations.

Discussion and practical implications

From a managerial perspective, the results yield three conclusions. First, improving online ratings and destination image increases visit intention, but stimulating demand alone does not automatically reduce reputational risk. Second, a mitigation mechanism—such as responding to negative reviews, correcting service deficiencies, and maintaining active communication—is required as an independent management channel. Third, the highest visit intention is achieved under multichannel management. Therefore, tourism organizations should monitor not only advertising and rating indicators, but also review valence, response speed, complaint topics, and reputational-risk indicators within a unified system.

The model also highlights the distinction between demand stimulation and reputational stabilization. The rating scenario strongly increases $B(t)$, but it also produces a higher value of $N(60)$ than the baseline. By contrast, integrated management combines communication, stronger rating effects, and an explicit negative-information mitigation mechanism. This produces the highest final destination image and visit intention while keeping negative-information pressure relatively low.

The current coefficients are hypothetical and have not been estimated using observational data for a specific destination. Therefore, the numerical values do not represent actual tourist numbers or exact probabilities; rather, they show relative dynamics across scenarios. At the next stage, the coefficients can be calibrated using Google Reviews, TripAdvisor, Booking.com, social-media posts, search trends, booking data, and visit statistics. Parameter-estimation methods such as nonlinear least squares, Bayesian calibration, or state-space approaches could transform the model into a forecasting tool.

Additional graphical results

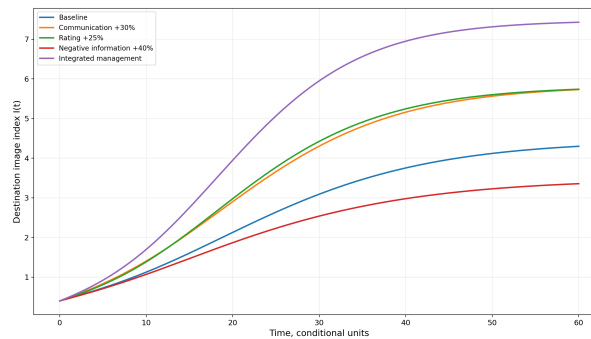


Figure 3. Dynamics of destination image $I(t)$ under different scenarios

Source: Authors' calculations.

Figure 3 shows that strengthening the communication and rating channels shifts the destination-image trajectory above the baseline, whereas stronger negative information substantially constrains the growth of $I(t)$. Under integrated management, the combined effect of the positive channels produces the highest final destination-image index.

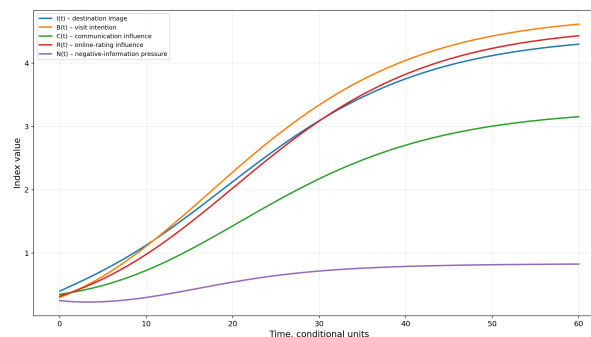


Figure 4. Dynamics of $I(t)$, $B(t)$, $C(t)$, $R(t)$, and $N(t)$ in the baseline scenario

Source: Authors' calculations.

Figure 4 illustrates the simultaneous evolution of the five state variables in the baseline scenario. $I(t)$, $B(t)$, $C(t)$, and $R(t)$ initially increase more rapidly and then slow under the effect of the quadratic saturation terms, whereas $N(t)$ stabilizes at a lower level.

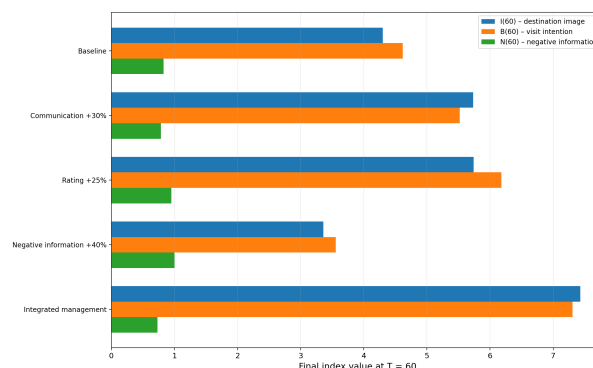


Figure 5. Comparison of final $I(60)$, $B(60)$, and $N(60)$ values across scenarios at $T = 60$

Source: Authors' calculations.

Figure 5 compares the final states in a single graph. Integrated management yields the highest values of $I(60)$ and $B(60)$ and one of the lowest values of $N(60)$. This result indicates that positive communication, online ratings, and measures to mitigate negative information are more effective when applied jointly rather than separately.

Conclusion

This study developed a five-variable nonlinear dynamic model describing the interactions among online reviews, ratings, communication influence, negative information, and tourist visit intention. A unified parameter set and the fourth-order Runge–Kutta method were used to ensure consistent and reproducible scenario calculations.

In the numerical experiment, stronger communication increased final visit intention by 19.5%, improvement of the rating channel by 33.8%, and integrated management by 58.3%. Stronger negative information reduced visit intention by 23.0%. At the same time, increasing ratings alone was found not to eliminate negative-information pressure completely. The most favorable outcome was observed under the integrated scenario, which actively manages the mitigation of negative information while simultaneously strengthening positive communication and ratings.

From a practical perspective, four directions are recommended for destination management: continuous monitoring of online ratings and reviews; rapid and substantive responses to negative reviews; coordination of positive media and social-media communication with service quality; and integration of digital reputation indicators with actual visit and booking data. Future research should calibrate the model using real data for a specific tourist destination in Uzbekistan, analyze parameter sensitivity and the stability of equilibrium states, and introduce seasonality and external shocks as separate variables.

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MODELLING TOURIST FLOWS TO UZBEKISTAN: A GRAVITY APPROACH

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ABSTRACT. This paper estimates the determinants of inbound tourist flows to Uzbekistan using a 2023 cross-section of 18 origin countries. Nested log-linear OLS specifications, HC3 robust standard errors and Poisson pseudo-maximum likelihood are employed. The contribution does not lie in applying the gravity model itself, which is well established in tourism research, but in identifying an Uzbekistan-specific regional concentration effect: the largest origin markets are simultaneously geographically close, contiguous, visa-free and culturally connected. Consequently, the baseline distance elasticity of -2.2 captures a composite spatial-regional effect and becomes unstable once controls are introduced. Of the six hypotheses, the negative distance effect is supported only in the baseline model, while cultural and social proximity is positively significant in PPML. Income, common border, visa-free access and direct flights are not statistically supported because of multicollinearity and insufficient variation. Comparison with prior tourism studies suggests that these deviations reflect the structure of the Uzbekistan sample rather than a rejection of general tourism-demand mechanisms. A testable digital-proximity extension and a two-stage framework linking predicted arrivals to tourism receipts and GDP effects are proposed.

Key words: inbound tourism; gravity model; Uzbekistan; distance; cultural proximity; PPML; multicollinearity; digital proximity.

JEL: C21, L83, Z32.

1. Introduction

Tourism matters for Uzbekistan not only as a source of foreign-currency earnings but also as a cross-sectoral driver of transport, the hotel business, food service, trade, cultural industries and regional employment. According to official data, 2.52 million guests were registered at hotels and similar accommodation establishments in 2023; 46.6% of them were citizens of CIS countries and other foreign states. The largest purposes of travel were leisure and recreation (984,400 stays) and business and professional trips (962,600) [15]. The strategic role of the sector is also enshrined in state policy: by 2030 the targets are to raise tourism's share of GDP from 3.5% to 7% and to double the annual inflow of foreign tourists from 10 to 20 million people [16].

For a landlocked country situated in the center of Eurasia, the structure of inbound flows is especially dependent on spatial accessibility. At the same time, Uzbekistan is linked to its neighboring states by a common border, shared history, language contacts, and migration and family networks. These features act in the same direction and are concentrated mainly within a single regional group of countries. This complicates the standard task of estimating individual coefficients: distance, contiguity, visa regime and cultural proximity partly measure one and the same latent factor of regional proximity.

The aim of the study is to determine which economic, spatial, institutional, transport and cultural-social factors are statistically associated with inbound flows to Uzbekistan, and how reliably their individual effects

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can be separated in a small cross-section. Six hypotheses are tested: H_1 — the income of the origin country is positively associated with the flow; H_2 — distance is negatively associated with it; H_3 — a common border increases the flow; H_4 — a visa-free regime increases the flow; H_5 — direct air service increases the flow; H_6 — cultural-social proximity increases the flow.

Applying the gravity model to tourism is not in itself a scientific novelty. The empirical novelty of this study lies in establishing that, for the Uzbekistan sample, the high baseline distance elasticity is not a stable “pure” transport effect, but reflects the combined action of distance, a common border and cultural-regional proximity. The second result is the identification of cultural-social proximity as the only positive factor that retains statistical significance in the PPML specification. The third result is a substantive interpretation of statistically insignificant coefficients through the sample’s saturation with visa-free regimes and direct flights, rather than a mechanical claim that the corresponding effects are absent.

2. Literature review and positioning of the study

The gravity model was originally proposed for the analysis of international trade [1], and its modern structural interpretation accounts for multilateral resistance [2]. In tourism the approach has been used for no less than fifteen years: K. Keum’s panel study demonstrated the applicability of gravity logic to international tourism flows [3], while C. Morley, J. RossellFi and M. Santana-Gallego derived a tourism-demand model from consumer behaviour [4]. The review by J. RossellFi and M. Santana-Gallego confirms that gravity specifications have become a stable direction in the analysis of international tourism [6]. Consequently, the research gap should be sought not in a repeated application of the method, but in a new data structure, a new identification problem, or a new substantive finding.

In broader panel samples, the origin country’s income usually has a positive effect, while distance has a negative one. For example, F. Yerdelen Tatoglu and H. Gul, using flow data from 30 countries to 14 major tourist destinations, established the significance of income and a negative dependence of arrivals on distance [7]. H_1 and H_2 therefore reproduce the findings of the core literature, but their confirmation for a single destination country and a single year is not automatic.

Institutional barriers have also been shown to matter. L. A. Cantillo, using OLS and PPML estimates, showed that a traditional visa regime reduced international arrivals by approximately 21%, while more flexible visa forms had a smaller negative effect [8]. At the same time, estimating the visa effect requires a sufficient number of observations under both restrictive and liberal regimes. If nearly all countries in the sample share the same regime, the coefficient is determined by a handful of exceptions and becomes statistically unstable.

Cultural and social ties reduce the informational and psychological costs of travel. U. Khalid, L. Okafor and O. Sanusi showed that a common language and linguistic proximity increase international tourism flows [9]. L. Okafor, U. Khalid and K. Burzynska established the positive role of migration and language networks [10]. These findings provide the theoretical basis for H_6 and make it possible to interpret diaspora, shared history and linguistic proximity as channels that reduce intangible barriers.

Direct air service is usually regarded as an element of transport accessibility and of a route’s potential market. A counterfactual analysis based on an augmented gravity model shows that the presence of direct service can unlock additional demand, although the route’s sustainability depends on market size and other characteristics [17]. The positive sign expected for H_5 is therefore economically plausible, but its empirical estimation is not possible when there is almost no variation in the feature.

Digitalisation extends the traditional gravity model by lowering search, quality-verification and booking costs. E. Lopez-Cordova showed that internet penetration in origin countries and the development of digital tools in the destination country weaken spatial and informational frictions [11]. A. Guedes and co-authors distinguish digital adoption from virtual proximity; their results for Southern Europe show that virtual proximity, measured by search interest, can be a more stable factor than the overall level of internet use [12]. This matters for the present study: simply expanding the list of factors does not guarantee validity — a digital variable must have a clear economic meaning and sufficient variation.

The positioning of this study consists in comparing the general conclusions of the literature with the specific structure of flows to Uzbekistan. Unlike multi-country panels, this study considers a single destination country, a single year, and a small number of origin markets. This makes it possible to reveal the national configuration of factors, but at the same time limits the possibility of causal interpretation and of separating closely related effects.

3. Data and econometric strategy

The estimation is carried out on a cross-section of 18 origin countries for 2023. The unit of observation is the registered flow of visits from country i to Uzbekistan. If the official source counts trips rather than unique persons, the dependent variable is interpreted as the number of arrivals (visits) rather than the number of individual tourists.

$$\text{Tourists}_i = A \cdot \text{GDPpc}_i^\alpha \cdot \text{Distance}_i^{-\gamma} \cdot \exp(\eta_i)$$

$$\eta_i = \theta_1 \text{Border}_i + \theta_2 \text{VisaFree}_i + \theta_3 \text{DirectFlight}_i + \theta_4 \text{CulturalIndex}_i$$

The following equation is estimated in log-linear form:

$$\ln \text{Tourists}_i = \beta_0 + \alpha \ln \text{GDPpc}_i - \gamma \ln \text{Distance}_i + \eta_i + \varepsilon_i$$

Table 1. Model variables and expected signs

Variable	Notation	Measurement	Expected sign
Tourist flow	Tourists_i	number of visits	dependent
GDP per capita	GDPpc_i	US dollars per person	+
Distance	Distance_i	km	-
Common border	Border_i	1/0	+
Visa-free regime	VisaFree_i	1/0	+
Direct flight	DirectFlight_i	1/0	+
Cultural-social index	CulturalIndex_i	0–3	+

GDPpc_i characterises the purchasing power of the origin market and is taken from the World Development Indicators [14]. Distance_i is the great-circle (orthodromic) distance from Tashkent to the capital of the corresponding country, calculated from CEPII GeoDist data and geographic coordinates [13]. Border_i , VisaFree_i and DirectFlight_i take the value 1 if there is a common land border, a visa-free regime and regular direct service, respectively, in 2023. The exact percentage effect of a binary variable with coefficient θ equals $100[\exp(\theta) - 1]\%$.

The cultural-social index is formed as the sum of three binary components — linguistic proximity, a shared historical-institutional connection, and the presence of a notable diaspora network: $\text{CulturalIndex}_i = \text{Language}_i + \text{History}_i + \text{Diaspora}_i$. The index has an ordinal scale from 0 to 3. Its use requires a transparent coding protocol; the result is therefore treated as a preliminary association rather than a causal effect.

Five specifications are estimated. Model 1 contains only income and distance. Model 2 adds border, visa-free regime and direct service. Model 3 includes the cultural index. Model 4 repeats the full OLS specification with HC3-robust standard errors, which are preferable for a small sample. Model 5 estimates the conditional mean in multiplicative form using PPML:

$$E(\text{Tourists}_i | X_i) = \exp(X_i' \beta)$$

PPML does not require the dependent variable to follow a Poisson distribution; the method is used as a pseudo-maximum-likelihood estimator and is robust to certain forms of heteroskedasticity [5]. The ordinary R^2 of OLS and the pseudo- R^2 of PPML are not directly comparable.

Digitalisation is not mechanically included in the current estimation. Uzbekistan's national level of digitalisation is identical for all 18 observations of a single year and is fully absorbed by the constant. An econometrically meaningful digital variable must vary either across origin markets or over time — for example, internet penetration in the origin country, search interest in Uzbekistan, the availability of an e-visa, the share of online bookings, or the reach of digital advertising in that country.

4. Results

Table 2. Descriptive statistics

Variable	N	Mean	St. dev.	Minimum	Maximum
Tourists _i	18	359,011	674,334	12,400	2,155,200
ln Tourists _i	18	11.10	1.79	9.43	14.58
GDPpc _i	18	24,589	21,942	1,150	82,000
ln GDPpc _i	18	9.59	1.22	7.05	11.31
Distance _i	18	3,658	2,534	308	10,464
ln Distance _i	18	7.89	0.94	5.73	9.26
Border _i	18	0.22	0.43	0	1
VisaFree _i	18	0.83	0.38	0	1
DirectFlight _i	18	0.94	0.24	0	1
CulturalIndex _i	18	1.00	1.28	0	3

Flows are extremely heterogeneous: the maximum value is roughly 174 times the minimum. At the same time, the binary features show weak variation. A common border is present for 4 of the 18 countries, a visa-free regime for 15, and direct air service for 17. Consequently, the coefficients on VisaFree_i and especially DirectFlight_i are effectively identified by a handful of exceptions and cannot be estimated with high precision.

4.1. Multicollinearity

In the full model, the variance inflation factors are approximately $VIF_{\ln \text{Distance}} \approx 33$, $VIF_{\ln \text{GDPpc}} \approx 15$ and $VIF_{\text{Border}} \approx 8.2$. Multicollinearity means that the explanatory variables carry similar information, so the model cannot reliably separate their individual contributions. A high VIF does not necessarily reduce the model's joint explanatory power, but it inflates standard errors, makes coefficients sensitive to the composition of the sample, and can change their signs. In this case, about 97% of the variation in log distance and 93% of the variation in log income is reproduced by the other regressors of the full specification.

The loss of significance of distance after the controls are included therefore does not mean that space has ceased to affect travel. The more accurate conclusion is that, in the Uzbekistan sample, distance is difficult to separate from neighbourhood, cultural proximity and the regional structure of flows.

4.2. Model estimates

Table 3. OLS and PPML estimation results

Variable	Model 1 OLS	Model 2 OLS	Model 3 OLS	Model 4 OLS- HC3	Model 5 PPML
ln GDPpc _i	0.56 (0.40)	0.20 (0.76)	0.21 (0.96)	0.21 (1.6)	-0.10 (0.64)
ln Distance _i	-2.2*** (0.57)	-1.2 (1.7)	-0.86 (1.9)	-0.86 (3.2)	-0.50 (1.2)
Border _i	—	1.4 (2.3)	0.74 (3.5)	0.74 (5.7)	-0.60 (1.5)
VisaFree _i	—	-0.21 (0.61)	-0.54 (0.73)	-0.54 (1.2)	-0.32 (0.57)
DirectFlight _i	—	0.34 (0.49)	0.16 (0.62)	0.16 (13)	0.15 (0.39)
CulturalIndex _i	—	—	0.55 (0.89)	0.55 (1.3)	1.2** (0.57)
Constant	23*** (1.7)	18** (7.8)	15* (8.1)	15 (19)	15*** (4.9)
N	18	18	18	18	18
R ² / pseudo-R ²	0.66	0.68	0.71	0.71	0.83

*Note: standard errors are shown in parentheses; *, **, *** denote statistical significance at the 10%, 5% and 1% levels, respectively. HC3-robust standard errors are reported for Model 4. The PPML pseudo-R² is not directly comparable with the OLS R².*

The baseline estimate is $\ln \text{Tourists}_i = 23 + 0.56 \ln \text{GDPpc}_i - 2.2 \ln \text{Distance}_i$. The distance coefficient is negative and significant at the 1% level. All else equal, a 1% increase in distance is associated with a decrease in flow of approximately 2.2%. However, after institutional and cultural variables are added, the magnitude of the coefficient falls to 0.86 in OLS and to 0.50 in PPML, and its statistical significance disappears. This pattern confirms that the baseline coefficient captures not only transport costs but also regional proximity.

Income has a positive sign in OLS but is insignificant in all specifications; in PPML the sign becomes negative. This result should not be interpreted as a negative effect of prosperity. In the sample, large flows are formed above all by neighbouring countries, whereas the wealthiest markets are often distant. The strong association of income with distance

and regional affiliation makes it impossible to isolate an independent income elasticity.

The coefficients on common border, visa-free regime and direct service are statistically insignificant. For the visa regime and direct flights, the main reason is the absence of an adequate control group: the visa-free regime applies to 15 countries, and direct service to 17. The negative estimates for Border_i and VisaFree_i in PPML therefore have no substantive causal interpretation.

The cultural-social index is positive in every model in which it is included, and becomes significant in PPML: the coefficient is 1.2 with a standard error of 0.57 ($p \approx 0.038$). Formally, $\exp(1.2) \approx 3.3$, meaning that a one-point increase in the index is associated with a conditional mean flow approximately 3.3 times higher. Because of the ordinal scale, the small N , and the correlation with neighbourhood, this result should be understood as a strong statistical association rather than a precise causal magnitude.

5. Hypothesis testing and comparison with prior studies

Table 4. Hypothesis status and comparative interpretation

Hypothesis	Result in this study	Comparison and interpretation
H ₁ : income (+)	Not supported	In panel models, income is usually significant [7]. For Uzbekistan the effect is masked by a combination of lower incomes and larger flows from neighbouring countries; the VIF for income is 15.
H ₂ : distance (-)	Partially supported	Consistent with [3, 7] in the baseline model. The loss of significance after the controls are added shows a composite regional, rather than a purely transport-related, distance effect.
H ₃ : common border (+)	Not supported	The sign is unstable. Four neighbouring countries are simultaneously close both culturally and geographically; a separate border effect is not identified.
H ₄ : visa-free regime (+)	Not statistically supported	Visa restrictions have previously been shown to reduce flows [8]. In the current sample 15 of 18 countries are visa-free, so there is insufficient variation to test the effect.
H ₅ : direct flight (+)	Not statistically supported	The transport literature points to a positive potential from direct connectivity [17], but 17 of 18 markets already have a direct flight; a control group is virtually absent.
H ₆ : cultural proximity (+)	Partially supported	A positive and significant coefficient in PPML is consistent with research on language, migration and diaspora [9, 10], but it is insignificant in OLS and is not causal.

The comparison shows that the “non-confirmation” of H₁, H₃, H₄ and H₅ is not a new economic law for Uzbekistan. It means that the available cross-section does not contain sufficient independent variation to separate the corresponding effects. The national specificity is expressed in the overlap of several types of proximity: geographical proximity is almost automatically accompanied by a common border, cultural ties and a large flow. As a result, the standard regressors compete to explain one and the same cross-country variation.

Two findings are, in fact, confirmed. First, distance materially constrains the flow in the baseline model, but this estimate is not robust to the inclusion of regional controls. Second, cultural-social proximity is positively associated with the flow in PPML. The remaining hypotheses should be regarded not as refuted, but as unidentified given the available data.

6. Scientific novelty and national specificity

The scientific novelty consists in identifying, for Uzbekistan, an effect of combined regional proximity. The baseline distance coefficient of -2.2 cannot be interpreted as an isolated elasticity of transport costs: once border and the cultural index are included, it falls by more than half and loses significance. This demonstrates that large flows from Central Asia are shaped not by a single factor, but by a cluster of spatial, cultural and institutional ties.

The second new finding is an asymmetry in the identifiability of factors. The cultural-social index retains positive significance in PPML, whereas the visa-free regime and direct-service indicators cannot be estimated because they cover almost the entire sample. This allows a more precise formulation of the national specificity: liberalised access and transport connectivity are already typical characteristics of Uzbekistan's principal markets and therefore, in a single cross-section, do not explain the differences between them. The differences are more strongly related to the depth of cultural-social networks and to overall regional position.

The third result is methodologically interpretative in nature: statistical insignificance is distinguished from the absence of an economic effect. This conclusion matters for studies with a small number of countries, because a mechanical expansion of the list of regressors can reduce precision and even undermine the model's substantive validity.

7. Practical relevance and forecasting

The results make it possible to segment origin markets. Neighbouring countries form the core of short-stay, family, medical, event-related and cross-border tourism; for them, convenient overland routes, repeat visits and regional tourism packages are priorities. For culturally connected but more distant markets, pilgrimage, educational and historical-cultural products are promising. For distant, high-income markets, product quality, international promotion, connecting and direct transport accessibility, and the reduction of digital information barriers are decisive.

The gravity model can be used for conditional forecasting of flows by origin country under scenario changes in the visa regime, flight frequency, digital promotion and other variables. However, the current model cannot directly forecast GDP: its dependent variable is the number of visits. Assessing additional economic growth requires a second stage linking the forecast of arrivals to tourist spending and a sectoral multiplier:

$$\Delta \text{Receipts} = \sum_i \left(\hat{T}_i^{\text{scen}} - \hat{T}_i^{\text{base}} \right) \cdot \text{Spend}_i$$

$$\Delta \text{GDP}_{\text{tour}} = m \cdot \Delta \text{Receipts}$$

where $\hat{T}_i^{\text{scen}}$ and $\hat{T}_i^{\text{base}}$ are the scenario and baseline forecasts of the flow, Spend_i is the average spending of one visitor from country i , and m is a multiplier obtained from a tourism satellite account or an input-output model. Without such data, a claim about a specific increase in GDP would be methodologically unfounded. Nevertheless, the proposed two-stage scheme shows how the results of the gravity model can be built into national tourism and infrastructure planning.

8. Digital extension and limitations of the study

For further analysis it is proposed to move to an "origin country – year" panel and to include a bilateral digital-proximity term:

$$E(\text{Tourists}_{it} | X_{it}) = \exp \left(X'_{it} \beta + \delta \text{DigitalProximity}_{it} + \mu_i + \lambda_t \right)$$

$\text{DigitalProximity}_{it}$ could combine: the share of internet users in the origin country; search interest in Uzbekistan; the reach of advertising for the national tourism brand; the use of e-visas and online booking; and the availability of official content in the language of the relevant market. Unlike a general "digitalisation index," such an indicator has a genuinely bilateral meaning, varies both across countries and over time, and directly reflects the reduction in information costs. Prior research shows that digital tools can weaken the role of distance and language barriers [11], but the effect of virtual proximity and of overall internet penetration is not always the same [12].

The main limitations of the present estimation are: $N = 18$; a single year of observations; possible mixing of recreational, business, family and other travel purposes; an approximate cultural index; the endogeneity of direct flights and visa policy; weak variation in the binary features; and strong multicollinearity. The results therefore characterise the structure of associations in 2023, rather than the causal effects of government policy.

Robustness checks should include sequential exclusion of countries (leave-one-out), models without Border_i and without CulturalIndex_i , decomposition of the cultural index into its components, alternative distance measures, robust regression, and comparison of out-of-sample OLS and PPML errors. When the database is expanded, it would be worthwhile to account for the exchange rate, relative prices, population, flight frequency and fares, hotel infrastructure, and external shocks.

9. Conclusion

The study has shown that the structure of inbound flows to Uzbekistan is consistent with general gravity logic, but is distinguished by a high concentration of interrelated factors. In the baseline OLS model, distance has a strong negative coefficient of -2.2; however, after common border, visa regime, direct service and cultural proximity are added, its magnitude shrinks and loses statistical significance. The initial estimate therefore reflects a composite effect of spatial and regional proximity.

Of the six hypotheses, H_2 is partially confirmed, since the negative distance effect is robust only in the baseline model. H_6 is also partially confirmed: cultural-social proximity is positively significant in PPML but not in OLS. H_1 , H_3 , H_4 and H_5 are not statistically supported. Unlike larger panel studies, this reflects not a refutation of the role of income, border, visas and air connectivity, but the small sample, multicollinearity and weak variation of the features.

Uzbekistan's national specificity lies in the fact that its largest markets are simultaneously close, contiguous, visa-free and culturally connected. The scientific contribution of the paper consists in identifying this identification configuration and in providing a more accurate interpretation of insignificant coefficients. In practice, the model can be used for market segmentation and scenario forecasting of arrivals, but assessing the impact on GDP requires a separate stage using spending data and sectoral multipliers. The model's prospective development lies in panel data and bilateral digital proximity, rather than in a formal expansion of the number of factors.

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DENSITY-TYPE PROPERTIES OF TOPOLOGICAL GROUPS

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ABSTRACT. In this paper, we study the density-type properties of topological groups. In particular, proved that for a topological group G , which the identity element 1 is closed in G , and there is a countable basis of neighborhoods for 1 in G the following relations hold: 1) $d(G) = wd(G)$; 2) $ld(G) = lwd(G)$. Besides, shown that for a topological group G , which G is a second countable locally compact (Hausdorff) space, the following relations hold: 1) $d(G) = wd(G)$; 2) $ld(G) = lwd(G)$. Finally, shown that for a topological group G , which G is Lie groups and locally Euclidean, the following relations hold: 1) $d(G) = wd(G)$; 2) $ld(G) = lwd(G)$.

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Key words: density, weak density, topological group, Lie groups.

Introduction

A fundamental chapter of topological spaces that has attracted the interest of researchers are undoubtedly the density-type properties of topological spaces.

Topological groups arise when a topological space carries binary operations with the elements of the topological space satisfying all conditions of groups. In recent years, such structures, especially with Lie groups, have attracted considerable attention in mathematics and mathematical physics.

One of the central problems in the field is the cardinal invariants of these topological groups. A major step in this direction has been the density-type invariants of the topological groups. This approach gives us the opportunity to evaluate geometric properties of topological groups.

In recent years many authors have investigated influence of different functors to various density-type properties. In particular, such studies have been done for the functor of permutation degree, finite hyperspace, and finite product (see, for instance, [3, 4, 5, 6, 7]). For example, most important density-type properties of the space of permutation degree of a topological space were considered in [5].

The papers [4] and [6] contain several similar interesting results on various density-type properties, and the paper [7] presents several results about cardinal invariants in the space of permutation degree, finite hyperspace, and finite product.

Throughout this paper, all spaces are assumed to be an infinite T_1 -spaces. Topological terminology and notation follow [1, 2].

Preliminaries

In this section we recall the notions and elementary facts that will be used below.

A topological group G is a topological space equipped with a group structure such that the multiplication

$$\cdot : G \times G \longrightarrow G, \quad (x, y) \longmapsto xy,$$

and the inversion

$$^{-1} : G \longrightarrow G, \quad x \longmapsto x^{-1},$$

are continuous. The product $G \times G$ is endowed with the product topology. A topology satisfying these requirements is called a group topology [1].

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Continuity of multiplication can equivalently be expressed as follows: for arbitrary $x, y \in G$ and every neighborhood W of xy , there exist neighborhoods U of x and V of y such that $UV \subseteq W$, where $UV = \{uv : u \in U, v \in V\}$. Similarly, continuity of inversion means that for every $x \in G$ and every neighborhood V of x^{-1} there is a neighborhood U of x with $U^{-1} \subseteq V$, where $U^{-1} = \{u^{-1} : u \in U\}$.

It is enough to require continuity of the single map

$$G \times G \longrightarrow G, \quad (x, y) \longmapsto xy^{-1}.$$

In neighborhood form, for every $x, y \in G$ and every neighborhood W of xy^{-1} , there are neighborhoods U of x and V of y such that $UV^{-1} \subseteq W$.

A subset $A \subset X$ is called dense in X when $\overline{A} = X$. The density of X , denoted by $d(X)$, is the least cardinality of a dense subset of X . Thus

$$d(X) = \min\{|A| : A \subset X, \overline{A} = X\}.$$

A space with $d(X) \leq \aleph_0$ is called separable [2].

The local density of X at x is the least cardinal τ for which x possesses a neighborhood of density τ in X . We denote this cardinal by $ld(x)$. The global local density is then

$$ld(X) = \sup\{ld(x) : x \in X\}.$$

It is standard that

$$ld(X) \leq d(X)$$

for every topological space X .

Example. Consider the real line R endowed with the discrete topology τ_d . For every $x \in R$, the singleton $\{x\}$ is an open neighborhood and has density one. Hence

$$ld(R, \tau_d) = 1.$$

On the other hand, a discrete space has no proper dense subset, so its density equals its cardinality. Therefore

$$d(R, \tau_d) = |R| = c,$$

and consequently

$$1 = ld(R, \tau_d) < d(R, \tau_d) = c.$$

The weak density of a topological space is the least cardinal $\tau \geq \aleph_0$ for which there exists a π -base B that can be represented as

$$B = \bigcup_{\alpha \in A} B_\alpha, \quad |A| = \tau,$$

where each B_α is a centered family of open sets. We denote this cardinal invariant by $wd(X)$ [4].

A topological space X is said to be locally weakly τ -dense at x if τ is the least cardinal for which x has a neighborhood whose weak density is τ . The corresponding pointwise invariant is denoted by $lwd(x)$, and the global local weak density is

$$lwd(X) = \sup\{lwd(x) : x \in X\}.$$

Main part

A real Lie group is a group that is also a finite-dimensional real smooth manifold, in which the group operations of multiplication and inversion are smooth maps. Smoothness of the group multiplication:

$$\mu : G \times G \rightarrow G, \quad \mu(x, y) = xy$$

means that μ is a smooth mapping of the product manifold $G \times G$ into G . The two requirements can be combined to the single requirement that the mapping:

$$(x, y) \mapsto x^{-1}y$$

be a smooth mapping of the product manifold into G [1].

Theorem 1. Let a topological group G , the identity element 1 is closed in G , and there is a countable basis of neighborhoods for 1 in G . Then the following relations hold:

- 1) $d(G) = wd(G)$;
- 2) $ld(G) = lwd(G)$.

Proof. We first compare the global invariants and subsequently establish the corresponding local equality.

Because $\{1\}$ is closed and left translations are homeomorphisms, every singleton is closed. Indeed, for each $x \in G$ the map

$$L_x : G \longrightarrow G, \quad L_x(y) = xy,$$

is a homeomorphism and therefore

$$\{x\} = L_x(\{1\})$$

is closed. Thus G is a T_1 -space. A T_1 topological group is Hausdorff. To see this directly, take distinct $x, y \in G$. Then $x^{-1}y \neq 1$. Since $\{x^{-1}y\}$ is closed,

$$G \setminus \{x^{-1}y\}$$

is a neighborhood of 1. By continuity of the group operations, one can choose a symmetric neighborhood V of 1 such that

$$V^{-1}V = V^2 \subseteq G \setminus \{x^{-1}y\}.$$

If xV and yV had a common point, then $xu = yv$ for some $u, v \in V$, which would imply

$$x^{-1}y = uv^{-1} \in V^2,$$

a contradiction. Hence xV and yV are disjoint, and G is Hausdorff.

The countable local base at 1 can be transported by translations. Hence every point of G has a countable neighborhood base. Thus G is first countable and Hausdorff. By the Birkhoff–Kakutani metrization theorem, G admits a compatible metric.

We use the standard fact that for every metrizable T_1 -space X ,

$$d(X) = wd(X).$$

For completeness, one direction follows immediately from the definitions. If D is dense and $|D| = d(X)$, then, using countable local bases at the points of D , the union of these bases is a π -base that is a union of $d(X)$ centered families. Consequently,

$$wd(X) \leq d(X).$$

The reverse inequality for metrizable spaces follows from the density–weak-density theorem for stratifiable spaces; metrizable spaces are stratifiable. Hence the two invariants agree.

Applying this result to the metrizable group G gives

$$\boxed{d(G) = wd(G)}.$$

It remains to prove the local equality. Fix $x \in G$ and let $\mathcal{N}(x)$ denote the collection of neighborhoods of x . Every neighborhood $U \in \mathcal{N}(x)$, with the subspace topology, is metrizable. Therefore

$$d(U) = wd(U)$$

for each such U . By the definitions,

$$ld(x) = \min\{d(U) : U \in \mathcal{N}(x)\}, \quad lwd(x) = \min\{wd(U) : U \in \mathcal{N}(x)\}.$$

Since the two quantities agree for every neighborhood involved, we obtain

$$ld(x) = lwd(x).$$

As x was arbitrary,

$$\begin{aligned} ld(G) &= \sup\{ld(x) : x \in G\} \\ &= \sup\{lwd(x) : x \in G\} \\ &= lwd(G). \end{aligned}$$

Therefore

$$\boxed{ld(G) = lwd(G)}.$$

Both assertions follow.

Theorem 2. Let a topological group G is a second countable locally compact (Hausdorff) space. Then the following relations hold:

- 1) $d(G) = wd(G)$;
- 2) $ld(G) = lwd(G)$.

Proof. We again treat the global and local invariants separately.

Part 1: Proof that $d(G) = wd(G)$.

Choose a countable base

$$\mathcal{B} = \{B_n : n \in \mathbb{N}\}$$

for G . For each n select $x_n \in B_n$ and define

$$D = \{x_n : n \in \mathbb{N}\}.$$

If O is a nonempty open subset of G , some basic set B_n satisfies

$$\emptyset \neq B_n \subseteq O.$$

The selected point x_n therefore belongs to O , and so $D \cap O \neq \emptyset$. Thus D is dense and

$$d(G) \leq \aleph_0.$$

Since G is infinite,

$$d(G) = \aleph_0.$$

The same countable base is a π -base. For every $n \in \mathbb{N}$ put

$$\mathcal{B}_n = \{B_n\}.$$

Each $\mathcal{B}_n$ is centered, and

$$\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n.$$

Hence $\mathcal{B}$ witnesses

$$wd(G) \leq \aleph_0.$$

By the definition of weak density, $wd(G) \geq \aleph_0$, so

$$wd(G) = \aleph_0.$$

Consequently,

$$\boxed{d(G) = wd(G)}.$$

Alternatively, second countability makes G first countable; since G is Hausdorff, the Birkhoff–Kakutani theorem shows that it is metrizable, and the standard metrizable-space equality $d(G) = wd(G)$ gives the same conclusion. The local compactness assumption is retained as part of the class specified in the theorem.

Part 2: Proof that $ld(G) = lwd(G)$.

Let $x \in G$ and let $\mathcal{N}(x)$ be the neighborhood family at x . Every subspace of a second countable space is second countable. Thus, for each $U \in \mathcal{N}(x)$,

$$\mathcal{B}|_U = \{B_n \cap U : B_n \cap U \neq \emptyset\}$$

is a countable base for U . Since G is Hausdorff, U is Hausdorff as well.

Repeating the argument from Part 1 for the space U gives

$$d(U) = wd(U).$$

Therefore

$$ld(x) = \min\{d(U) : U \in \mathcal{N}(x)\}$$

and

$$lwd(x) = \min\{wd(U) : U \in \mathcal{N}(x)\}$$

have the same value. Hence

$$ld(x) = lwd(x)$$

for every $x \in G$. Taking suprema yields

$$\begin{aligned} ld(G) &= \sup\{ld(x) : x \in G\} \\ &= \sup\{lwd(x) : x \in G\} \\ &= lwd(G). \end{aligned}$$

Thus

$$\boxed{ld(G) = lwd(G)}.$$

Combining the two parts, we have

$$\boxed{d(G) = wd(G)} \quad \text{and} \quad \boxed{ld(G) = lwd(G)}.$$

This completes the proof.

Theorem 3. Let G is Lie groups, which are locally Euclidean. Then the following relations hold:

- 1) $d(G) = wd(G)$;
- 2) $ld(G) = lwd(G)$.

Proof. A Lie group is, in particular, a finite-dimensional smooth manifold. We exploit its local Euclidean structure together with the homogeneity supplied by translations.

Part 1: Proof that $d(G) = wd(G)$.

For each $x \in G$ there are an open neighborhood U_x of x , an integer $n \geq 0$, and a homeomorphism

$$\varphi_x : U_x \longrightarrow V_x,$$

where V_x is an open subset of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is second countable and metrizable, the same is true of V_x and hence of U_x .

Choose such a coordinate neighborhood U at the identity element 1, together with a homeomorphism

$$\varphi : U \longrightarrow V \subseteq \mathbb{R}^n.$$

Let

$$\mathcal{B} = \{B_k : k \in \mathbb{N}\}$$

be a countable base for U . Select $u_k \in B_k$ and put

$$D_U = \{u_k : k \in \mathbb{N}\}.$$

Given any nonempty open subset O of U , some B_k satisfies

$$\emptyset \neq B_k \subseteq O.$$

Hence $u_k \in O$, showing that D_U is dense in U .

For each $g \in G$, left translation

$$L_g : G \longrightarrow G, \quad L_g(y) = gy,$$

is a homeomorphism. It follows that gU is a metrizable neighborhood of g and

$$gD_U = L_g(D_U)$$

is dense in gU . Thus the local Euclidean pieces of the group all have the same qualitative density behavior.

More importantly, a finite-dimensional manifold is first countable. Hence the Lie group G is first countable at 1 and, by translation, at every point. Since a Lie group is Hausdorff, the Birkhoff–Kakutani metrization theorem applies and shows that G is metrizable.

For metrizable T_1 -spaces the density and weak density coincide. Therefore

$$\boxed{d(G) = wd(G)}.$$

Part 2: Proof that $ld(G) = lwd(G)$.

Fix $x \in G$. A coordinate neighborhood U_x of x is metrizable. More generally, whenever V is a neighborhood of x contained in a coordinate neighborhood, V is metrizable and consequently

$$d(V) = wd(V).$$

Thus the two density notions agree on the neighborhood system generated by the local Euclidean structure.

The group structure gives an equivalent description at every point. Namely,

$$L_x : G \longrightarrow G, \quad L_x(y) = xy,$$

is a homeomorphism and maps neighborhoods of 1 bijectively onto neighborhoods of x . Therefore the local invariants are translation invariant:

$$ld(x) = ld(1), \quad lwd(x) = lwd(1).$$

At the identity, a Euclidean neighborhood is metrizable and has equal density and weak density. Hence

$$ld(1) = lwd(1),$$

and consequently

$$ld(x) = lwd(x)$$

for every $x \in G$.

Finally,

$$\begin{aligned} ld(G) &= \sup\{ld(x) : x \in G\} \\ &= \sup\{lwd(x) : x \in G\} \\ &= lwd(G). \end{aligned}$$

Thus

$$\boxed{ld(G) = lwd(G)}.$$

We have proved both required relations:

$$\boxed{d(G) = wd(G)} \quad \text{and} \quad \boxed{ld(G) = lwd(G)}.$$

This completes the proof of Theorem 3.

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Regular parameters and local pluriregularity

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Abstract. We study weighted extremal functions on a compact subset of $\mathbb{C}^n$ when the weight is fixed and the logarithmic growth parameter varies. Concavity and two-sided estimates show that the regular parameters at a point form either the empty set, the smallest contact parameter alone, or the whole contact ray. At a regular parameter, we estimate the right derivative and characterize both persistence of regularity and equality of this derivative with the unweighted Green function. A local estimate involving dd^c shows that a global admissible extension that is strictly plurisubharmonic near the point makes weighted regularity equivalent to local pluriregularity. We also describe failure of local pluriregularity by removing a negligible subset. This gives a characterization of the points at which a compact set is non-plurithin but not locally pluriregular, and identifies its largest relatively plurifine open negligible subset. The paper presents a focused account of these results from the author's longer preprint.

MSC (2020): 32U05; 32U15; 32U35.

Key words: weighted extremal function, local pluriregularity, plurithinness, negligible set, plurifine topology.

1. Introduction

Weighted extremal functions connect polynomial approximation with pluripotential theory. Their regularity depends on both the compact set and the weight. A further parameter is the allowed logarithmic growth at infinity. We keep the compact set and the weight fixed and ask how regularity changes when this parameter increases.

The notion of weighted regularity used here was studied by Sadullaev [6], Alan [1], and Narzillaev [4]. It requires equality between the regularized extremal function and the weight at a contact point. This is stronger than continuity of the extremal function alone; see [1, Example 2.3]. Alan asked whether regularity is preserved as the growth parameter increases when the weight has a global plurisubharmonic extension of the prescribed growth [1, Open Problem 3.7].

The present paper gives a short, self-contained account of selected results from [5]. We first show that regularity can be lost only at the smallest contact parameter. The proof uses concavity in the growth parameter. We then study the right derivative at a regular parameter and obtain a local criterion for persistence under strict plurisubharmonicity. Finally, we use negligible sets and the plurifine topology to distinguish local pluriregularity from non-plurithinness. The measure representation of the derivative and the detailed counterexamples in [5] are outside the scope of this short account.

The results have two complementary parts. Parameter estimates describe the possible changes in the weighted envelope. Local regularity and negligible sets explain the geometry of the points at which those changes can occur. In particular, strict plurisubharmonicity is needed only near the point under consideration, provided that a global extension with the required growth is available.

2. Preliminaries

Let $\text{PSH}(\Omega)$ denote the plurisubharmonic functions on an open set $\Omega \subset \mathbb{C}^n$. We use the Euclidean norm and write $\log^+ t = \max\{\log t, 0\}$, with $\log^+ 0 = 0$. For $\delta > 0$, put

$$\mathcal{L}_\delta = \{u \in \text{PSH}(\mathbb{C}^n) : u(z) \leq \delta \log^+ \|z\| + C_u\}.$$

The constant C_u may depend on u . The subclass $\mathcal{L}_\delta^+$ consists of the functions that also satisfy $u(z) \geq \delta \log^+ \|z\| - C$ for some constant C . Thus $\mathcal{L}_\delta = \delta \mathcal{L}_1$.

Throughout the paper, $K \subset \mathbb{C}^n$ is a non-pluripolar compact set and ψ is a bounded real-valued function on K , unless stronger assumptions are stated. The weighted extremal function and its regularization are

$$V_{\delta,K,\psi}(z) = \sup\{u(z) : u \in \mathcal{L}_\delta, u \leq \psi \text{ on } K\}, \quad F_\delta = V_{\delta,K,\psi}^*. \quad (1)$$

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Here $f^*(z) = \limsup_{\zeta \rightarrow z} f(\zeta)$ denotes upper semicontinuous regularization. For $\delta = 1$ and $\psi = 0$, (1) gives the classical Siciak–Zahariuta extremal function [7],

$$V_K := V_{1,K,0}.$$

Its regularization V_K^* is the pluricomplex Green function with pole at infinity. It is finite and belongs to $\mathcal{L}_1^+$. We have $V_K = 0$ on K , whereas V_K^* may be positive at some points of K .

The contact parameter set and the global regular parameter set are

$$\begin{aligned}\Lambda(K, \psi) &= \{\delta > 0 : V_{\delta,K,\psi}|_K = \psi\}, \\ \Lambda_{\text{reg}}(K, \psi) &= \{\delta \in \Lambda(K, \psi) : F_\delta|_K = \psi\}.\end{aligned}$$

These sets were introduced in [4]. If $\Lambda(K, \psi)$ is nonempty, then it is $(0, \infty)$ or $[\lambda, \infty)$ for some $\lambda > 0$ [4, Section 4, Proposition 1]. We also use the pointwise notation from [5],

$$\Lambda_{\text{reg}}(a; K, \psi) = \{\delta \in \Lambda(K, \psi) : F_\delta(a) = \psi(a)\}, \quad a \in K.$$

Thus $\Lambda_{\text{reg}}(K, \psi)$ is the intersection of these pointwise sets. At a contact parameter δ , we say that K is (δ, ψ) -regular at a if $F_\delta(a) = \psi(a)$. If $\psi = \Psi|_K$ for some $\Psi \in \mathcal{L}_\tau$, then $[\tau, \infty) \subset \Lambda(K, \psi)$, since Ψ itself is admissible at every such parameter.

Put $m = \inf_K \psi$ and $M = \sup_K \psi$. Scaling and comparison of the obstacles in (1) give

$$V_{\delta,K,\psi} = \delta V_{1,K,\psi/\delta}, \quad V_{\delta,K,\psi+c} = V_{\delta,K,\psi} + c,$$

and

$$\delta V_K^* + m \leq F_\delta \leq \delta V_K^* + M. \quad (2)$$

These inequalities hold on all of $\mathbb{C}^n$.

A set is pluripolar if locally it is contained in the $-\infty$ locus of a plurisubharmonic function that is not identically $-\infty$. A set is negligible if locally it is contained in a set $\{v < v^*\}$, where v is the supremum of a locally uniformly bounded above family of plurisubharmonic functions. Negligible sets and pluripolar sets coincide [2; 3, Section 4.7]. In particular, countable unions of negligible sets are negligible. An extremal envelope agrees with its regularization outside such a set.

For a bounded set E , not necessarily compact, define V_E by the same unweighted supremum. Pluripolar removal gives

$$V_{E \cup P}^* = V_E^* \quad \text{if } P \text{ is pluripolar and } E \cup P \text{ is bounded.} \quad (3)$$

We will use this identity for non-pluripolar E ; see [3, Theorem 5.2.4 and Corollary 5.2.5]. The regularization in (3) is essential.

We need one more consequence of negligibility. If U and W are envelopes of locally uniformly bounded above families of plurisubharmonic functions and have finite regularizations, then, for $s, t \geq 0$,

$$(sU + tW)^* = sU^* + tW^*. \quad (4)$$

Indeed, the two sums agree outside a pluripolar set. A plurisubharmonic function is recovered by upper regularization of its values off a pluripolar set. Applying this to $sU^* + tW^*$ proves (4). Terms with zero coefficient are omitted.

We write $B(a, r)$ for the open Euclidean ball and $\overline{B}(a, r)$ for the closed ball. The compact set K is locally pluriregular at $a \in K$ if

$$V_{K \cap \overline{B}(a,r)}^*(a) = 0 \quad \text{for every } r > 0.$$

It is non-pluripolar at every point if $K \cap B(a, r)$ is non-pluripolar for every $a \in K$ and $r > 0$. The latter assumption will be imposed in Section 5. Global pluriregularity, namely $V_K^*|_K = 0$, is a weaker condition than local pluriregularity at every point.

For an open ball B and a compact set $E \subset B$, the relative extremal function is

$$h_{E,B} = \sup\{u \in \text{PSH}(B) : -1 \leq u \leq 0, u|_E \leq -1\}. \quad (5)$$

The standard local characterization says that K is locally pluriregular at a if and only if

$$h_{K \cap \overline{B}(a,r), B(a,R)}^*(a) = -1 \quad (0 < r < R)$$

for sufficiently small concentric balls [3, Chapter 4].

For the differential operator we use the normalization

$$d^c = \frac{\partial - \bar{\partial}}{4i}, \quad dd^c = \frac{i}{2} \partial \bar{\partial}, \quad \omega_0 = dd^c \|z\|^2.$$

For a real-valued C^2 function u , plurisubharmonicity means $dd^c u \geq 0$, and strict plurisubharmonicity means $dd^c u > 0$, as $(1, 1)$ -forms. For continuous functions, these inequalities are understood in the sense of currents. In particular, $dd^c \Psi \geq c\omega_0$ near a means that $\Psi - c\|z - a\|^2$ is plurisubharmonic there.

3. Regular parameters

We begin with estimates that hold without any continuity assumption on the weight. They imply both the classification of regular parameters and the bounds for the right derivative.

Lemma 1. For $\alpha, \beta > 0$ and $0 \leq t \leq 1$,

$$F_{(1-t)\alpha+t\beta} \geq (1-t)F_\alpha + tF_\beta.$$

Moreover, if $0 < \alpha < \beta$, then on $\mathbb{C}^n$,

$$(\beta - \alpha)V_K^* \leq F_\beta - F_\alpha \leq \frac{\beta - \alpha}{\alpha}(F_\alpha - m). \quad (6)$$

Proof. If $u \in \mathcal{L}_\alpha$ and $v \in \mathcal{L}_\beta$ are at most ψ on K , then $(1-t)u + tv$ is admissible at parameter $(1-t)\alpha + t\beta$. Taking independent suprema first proves concavity for the unregularized envelopes. Formula (4) then proves the stated inequality.

For the lower bound in (6), take $u \in \mathcal{L}_\alpha$ with $u \leq \psi$ on K , and $v \in \mathcal{L}_1$ with $v \leq 0$ on K . The function $u + (\beta - \alpha)v$ is admissible at parameter β . Hence

$$V_{\beta, K, \psi} \geq V_{\alpha, K, \psi} + (\beta - \alpha)V_K.$$

Regularization and (4) give the lower bound. For the upper bound, set $q = \psi - m \geq 0$ on K . Scaling and monotonicity in the obstacle yield

$$V_{\beta, K, q} = \beta V_{1, K, q/\beta} \leq \beta V_{1, K, q/\alpha} = \frac{\beta}{\alpha} V_{\alpha, K, q}.$$

Adding back m and regularizing proves the upper bound in (6). $\square$

Since $V_K^* \geq 0$, Lemma 1 shows that $F_\delta(z)$ is nondecreasing in δ . For each fixed z , it is also finite and concave, hence continuous for $\delta > 0$. The signs in (6) are useful: increasing the growth parameter always increases the admissible envelope, and the unweighted Green function gives a lower bound for the increase.

Theorem 1. Suppose that $\Lambda = \Lambda(K, \psi)$ is nonempty. If $\Lambda = [\lambda, \infty)$ with $\lambda > 0$, then, for every $a \in K$,

$$\Lambda_{\text{reg}}(a; K, \psi) \in \{\emptyset, \{\lambda\}, [\lambda, \infty)\}.$$

The same alternatives hold for $\Lambda_{\text{reg}}(K, \psi)$. If $\Lambda = (0, \infty)$, each of these regular parameter sets is either empty or equal to $(0, \infty)$.

Proof. For $a \in K$ and $\delta \in \Lambda$, put

$$d_\delta(a) = F_\delta(a) - \psi(a).$$

Contact makes this function nonnegative. By Lemma 1, it is concave and nondecreasing in δ . Suppose $d_\alpha(a) = 0$ and that some $\eta \in \Lambda$ satisfies $\eta < \alpha$. For any $\beta > \alpha$, write

$$\alpha = (1-t)\eta + t\beta, \quad t = \frac{\alpha - \eta}{\beta - \eta} \in (0, 1).$$

Then

$$0 = d_\alpha(a) \geq (1-t)d_\eta(a) + td_\beta(a) \geq 0.$$

Thus $d_\beta(a) = 0$. For smaller parameters, monotonicity gives $0 \leq d_\delta(a) \leq d_\alpha(a) = 0$. Therefore, a zero at any parameter other than the smallest one forces vanishing on the whole contact ray.

When $\Lambda = [\lambda, \infty)$, the only remaining possibilities are no zero or a zero only at λ . When $\Lambda = (0, \infty)$, each parameter has a smaller parameter in Λ , so any zero forces vanishing everywhere. Intersecting the pointwise possibilities over $a \in K$ gives the global assertion. $\square$

Theorem 1 rules out nondegenerate bounded intervals of regular parameters among the possibilities left by the interval description in [4]. It also locates any loss of regularity in Alan's question. If regularity holds at τ and fails at some larger parameter, then necessarily

$$\Lambda(K, \psi) = [\tau, \infty), \quad \Lambda_{\text{reg}}(a; K, \psi) = \{\tau\}.$$

The constructions in [5] realize this alternative even with a smooth extending weight and a polynomially convex compact set that is non-pluripolar and non-plurithin at every point.

There are two immediate special cases. If ψ is constant, then $F_\delta = \delta V_K^* + \psi$, and regularity at a either holds for every $\delta > 0$ or fails for every $\delta > 0$, according as $V_K^*(a) = 0$ or $V_K^*(a) > 0$. More generally, (6) shows that $V_K^*(a) > 0$ permits at most one regular parameter, since $F_\delta(a)$ is then strictly increasing.

We next measure the change immediately after a regular parameter. The following proposition also separates the initial slope from the slope for large parameters.

Proposition 1. Let $\tau \in \Lambda_{\text{reg}}(a; K, \psi)$. The right derivative

$$D_\tau(a) = \lim_{h \downarrow 0} \frac{F_{\tau+h}(a) - \psi(a)}{h} \quad (7)$$

exists and is finite. The quotient is nonincreasing as a function of $h > 0$, and

$$V_K^*(a) \leq \frac{F_{\tau+h}(a) - \psi(a)}{h} \leq D_\tau(a) \leq \frac{\psi(a) - m}{\tau}. \quad (8)$$

Furthermore,

$$0 \leq \frac{F_{\tau+h}(a) - \psi(a)}{h} - V_K^*(a) \leq \frac{M - m}{h}. \quad (9)$$

Equality $D_\tau(a) = V_K^*(a)$ holds if and only if

$$F_\delta(a) = \psi(a) + (\delta - \tau)V_K^*(a) \quad \text{for every } \delta \geq \tau.$$

Regularity persists for every $\delta > \tau$ if and only if $D_\tau(a) = 0$. If $D_\tau(a) > 0$, regularity fails at every $\delta > \tau$.

Proof. Since $d_\tau(a) = 0$, concavity gives, for $0 < h_1 < h_2$,

$$d_{\tau+h_1}(a) \geq \frac{h_1}{h_2} d_{\tau+h_2}(a).$$

This proves monotonicity of the quotient. Inequality (6), applied with $\alpha = \tau$ and $\beta = \tau + h$, bounds it between $V_K^*(a)$ and $(\psi(a) - m)/\tau$. Thus the limit in (7) exists, is finite, and satisfies (8).

For (9), combine (2) and (6) to obtain

$$\begin{aligned} 0 &\leq F_{\tau+h}(a) - \psi(a) - hV_K^*(a) \\ &\leq M + \tau V_K^*(a) - \psi(a) \leq M - m. \end{aligned}$$

The last inequality uses $\psi(a) = F_\tau(a) \geq \tau V_K^*(a) + m$. Dividing by h proves (9).

If $D_\tau(a) = V_K^*(a)$, the lower and upper bounds for every quotient in (8) agree. This gives the affine formula. Conversely, that formula immediately gives equality of the derivatives. If $D_\tau(a) = 0$, (8) forces every defect to vanish. If $D_\tau(a) > 0$, the defect is positive for all sufficiently small $h > 0$, and its monotonicity makes it positive for every $h > 0$. $\square$

It follows from (9) that $V_K^*(a)$ is the limiting slope as $h \rightarrow \infty$, whereas $D_\tau(a)$ is the initial slope. Vanishing of the former does not by itself force vanishing of the latter. In fact, [5] contains a globally pluriregular example with $D_\tau(a) > 0$. A condition that does force both slopes to vanish is given in the next section.

4. Local regularity

We first recall how local pluriregularity controls a weighted envelope. Suppose ψ is bounded and upper semicontinuous on K , and $\delta \in \Lambda(K, \psi)$. If a is locally pluriregular, set $K_r = K \cap \overline{B}(a, r)$ and $M_r = \sup_{K_r} \psi$. Monotonicity in the compact set and (2) give

$$\psi(a) \leq F_\delta(a) \leq V_{\delta, K_r, \psi}^*(a) \leq M_r + \delta V_{K_r}^*(a) = M_r.$$

As $r \downarrow 0$, upper semicontinuity gives $M_r \downarrow \psi(a)$. Hence $F_\delta(a) = \psi(a)$. The same argument applies on each smaller compact K_s : contact is inherited because the envelope on K_s lies between the envelope on K and the common obstacle. This also proves local weighted regularity at a . For continuous weights, equivalence of local weighted and unweighted regularity is established in [4, Theorem 4.1].

The converse for global weighted regularity needs more information about the weight. The following estimate uses a global admissible extension but requires strict plurisubharmonicity only near a .

Theorem 2. Let $a \in K$, let $\Psi \in \mathcal{L}_\tau \cap C(\mathbb{C}^n)$, and put $\psi = \Psi|_K$. Suppose that for some $R > 0$ and $c > 0$,

$$dd^c \Psi \geq c\omega_0 \quad \text{on } B(a, R).$$

Then, for $0 < r < R$ and every $\delta \geq \tau$,

$$F_\delta(a) - \Psi(a) \geq cr^2(1 + h_{K \cap \overline{B}(a, r), B(a, R)}^*(a)). \quad (10)$$

Consequently, (δ, ψ) -regularity at a for one $\delta \geq \tau$ is equivalent to local pluriregularity at a . If it holds at one such parameter, it holds at every $\delta \geq \tau$.

Proof. Put $B = B(a, R)$ and $E = K \cap \overline{B}(a, r)$. For a competitor u in (5), define on B

$$H_u(z) = \Psi(z) - c\|z - a\|^2 + cr^2(u(z) + 1).$$

This function is plurisubharmonic, since

$$dd^c H_u = dd^c \Psi - c\omega_0 + cr^2 dd^c u \geq 0.$$

The bound $u \leq 0$ gives $H_u \leq \Psi$ whenever $\|z - a\| \geq r$. Therefore the function

$$W_u(z) = \begin{cases} \max\{\Psi(z), H_u(z)\}, & z \in B, \\ \Psi(z), & z \notin B \end{cases}$$

is globally plurisubharmonic. It equals Ψ outside $B(a, r)$, so it belongs to $\mathcal{L}_\tau$. On E , we have $u = -1$ and $H_u \leq \Psi$. On $K \setminus E$, the latter inequality follows from $\|z - a\| \geq r$. Thus $W_u = \psi$ on K , and W_u is admissible in (1) for every $\delta \geq \tau$.

Taking the supremum over u near a gives

$$V_{\delta, K, \psi}(z) \geq \Psi(z) - c\|z - a\|^2 + cr^2(h_{E, B}(z) + 1), \quad z \in B.$$

Regularize at a . Continuity of Ψ proves (10). If $F_\delta(a) = \Psi(a)$, then (10) and $h_{E, B}^* \geq -1$ imply $h_{E, B}^*(a) = -1$ for every $0 < r < R$. The relative-extremal characterization in Section 2 gives local pluriregularity. The converse and persistence follow from the preceding observation, since $[\tau, \infty) \subset \Lambda(K, \psi)$. $\square$

Under the assumptions of Theorem 2, regularity at τ therefore implies

$$D_\tau(a) = V_K^*(a) = 0.$$

Indeed, persistence makes the derivative zero, and local pluriregularity makes the Green function zero. If a C^2 plurisubharmonic extension gives regularity at τ but this regularity is lost at a larger parameter, then necessarily $(dd^c \Psi)^n(a) = 0$. Otherwise, positivity and continuity of $dd^c \Psi$ would give the strict lower bound required in Theorem 2 on a small ball.

Sadullaev's global-to-local theorem uses a globally strictly plurisubharmonic extension [6, Theorem 2.4]. Alan assumes strictness on a ball containing the compact set [1, Theorem 3.14]. Estimate (10) isolates the local condition needed at the point. It does not dispense with the global extension in $\mathcal{L}_\tau$: that extension is what allows W_u to be a competitor with the prescribed growth.

5. Plurithinness and negligible sets

In this section, assume that K is non-pluripolar at every point. We describe the points that fail to be locally pluriregular, and then distinguish those at which K is plurithin from those at which it is not.

A set E is plurithin at a if a is not an accumulation point of $E \setminus \{a\}$, or if there is a plurisubharmonic function u near a such that

$$\limsup_{E \setminus \{a\} \ni z \rightarrow a} u(z) < u(a).$$

The puncture is part of the definition. Finite unions of sets plurithin at a are plurithin there. We also use the following form of the thinness theorem: if E is plurithin at an accumulation point a , there is $u \in \mathcal{L}_1$ such that

$$u(a) > -\infty, \quad \lim_{E \setminus \{a\} \ni z \rightarrow a} u(z) = -\infty. \quad (11)$$

See [3, Proposition 4.8.2 and Corollary 4.8.4].

Negligibility and plurithinness at a point are different notions. For example, let $\overline{\mathbb{D}} = \{\zeta \in \mathbb{C} : |\zeta| \leq 1\}$. The set $\{0\} \times \overline{\mathbb{D}} \subset \mathbb{C}^2$ is negligible because it is contained in $\{\log |z_1| = -\infty\}$. It is not plurithin at $(0, 0)$: restriction to the complex line $z_1 = 0$ and the submean inequality exclude a strict jump at the center of the disk. If that restriction is identically $-\infty$, a strict jump is also impossible. Conversely, being plurithin at one point does not make a whole set negligible. For instance, $\overline{\mathbb{D}} \cup \{2\}$ is thin at its isolated point 2, but contains a nonpolar disk.

The plurifine topology is the smallest topology in which all plurisubharmonic functions are continuous. Finite intersections of sets $B \cap \{u > c\}$, where B is a Euclidean ball and $u \in \text{PSH}(B)$, form a basis. A set E is plurithin at a if and only if some plurifine neighborhood of a misses $E \setminus \{a\}$ [3, Section 4.8]. We use this equivalence below; no general theory of plurifinely plurisubharmonic functions is needed.

Following [1], decompose K into

$$\begin{aligned} R &= \{a \in K : K \text{ is locally pluriregular at } a\}, \\ T &= \{a \in K : K \text{ is plurithin at } a\}, \\ S &= K \setminus (R \cup T). \end{aligned}$$

Local pluriregularity implies non-plurithinness, so these sets are disjoint. The set S is the part not detected by either local regularity or plurithinness. Alan asked for its characterization in [1, Open Problem 3.17].

Lemma 2. For $a \in K$, the following conditions are equivalent:

- (i) $a \notin R$;
- (ii) there is a negligible subset $N \subset K$ containing a such that $K \setminus N$ is plurithin at a ;
- (iii) there is a plurifine open neighborhood U of a such that $K \cap U$ is negligible.

Proof. Suppose $a \notin R$. Choose $r > 0$ with

$$g = V_{K \cap \overline{B}(a, r)}^*, \quad c = g(a) > 0.$$

Local non-pluripolarity makes g finite and plurisubharmonic. Set

$$U = B(a, r) \cap \{g > c/2\}, \quad N = K \cap U.$$

The set U is a plurifine open neighborhood of a . Since the unregularized extremal function is zero on $K \cap \overline{B}(a, r)$,

$$N \subset \{V_{K \cap \overline{B}(a, r)}^* < V_{K \cap \overline{B}(a, r)}^*\}.$$

Thus N is negligible. Near a , the function g is at most $c/2$ on $K \setminus N$, whereas $g(a) = c$. This proves (ii) and (iii).

Assume (ii) and set $E = K \setminus N$. The point a is an accumulation point of E . Otherwise a small neighborhood of a would meet K only in the pluripolar set N , contrary to local non-pluripolarity. Apply (11) and choose $\rho > 0$ small enough that

$$u \leq u(a) - 1 \quad \text{on } E \cap \overline{B}(a, \rho).$$

Then $v = u - u(a) + 1 \in \mathcal{L}_1$ is nonpositive on $K \cap \overline{B}(a, \rho) \setminus N$, while $v(a) = 1$. By (3),

$$V_{K \cap \overline{B}(a, \rho)}^*(a) = V_{K \cap \overline{B}(a, \rho) \setminus N}^*(a) \geq 1.$$

Hence $a \notin R$. Finally, (iii) implies (ii) with $N = K \cap U$, because the complement of a plurifine neighborhood of a is plurithin at a . $\square$

Theorem 3. A point $a \in K$ belongs to S if and only if there is a negligible subset $N \subset K$ such that $a \in N$, N is non-plurithin at a , and $K \setminus N$ is plurithin at a .

Proof. If $a \in S$, Lemma 2 gives a negligible set N containing a for which $K \setminus N$ is plurithin at a . If N were plurithin there, their union K would also be plurithin, contrary to $a \notin T$. Conversely, if such an N exists, its non-plurithinness makes K non-plurithin at a . Lemma 2 gives $a \notin R$. Thus $a \in S$. $\square$

The theorem describes S without a local extremal function in its statement. Lemma 2 also gives a criterion for regular points: $a \in R$ if and only if $K \setminus N$ is non-plurithin at a for every negligible subset $N \subset K$. Indeed, replacing N by $N \cup \{a\}$ changes neither negligibility nor thinness at a .

There is a canonical choice of negligible subset. Define

$$N_K = \bigcup \{K \cap U : U \text{ is plurifine open and } K \cap U \text{ is negligible}\}. \quad (12)$$

For a set A , write $A'_{\mathcal{F}}$ for its plurifine derived set: $a \in A'_{\mathcal{F}}$ if every plurifine neighborhood of a meets $A \setminus \{a\}$.

Proposition 2. The set N_K is the largest relatively plurifine open negligible subset of K . Moreover,

$$\begin{aligned} R &= K \setminus N_K, \\ T &= \{a \in K : a \text{ is isolated in the relative plurifine topology}\}, \\ S &= N_K \cap (N_K)'_{\mathcal{F}}. \end{aligned}$$

Proof. Lemma 2 gives $N_K = K \setminus R$. We must still prove negligibility, since an arbitrary union of negligible sets need not be negligible. Let B_j run through the balls with rational centers and positive rational radii for which $K \cap B_j \neq \emptyset$. Put

$$A_j = K \cap \overline{B}_j, \quad N_j = K \cap B_j \cap \{V_{A_j}^* > 0\}.$$

Every A_j is non-pluripolar, and each N_j is negligible. We claim that $K \setminus R = \bigcup_{j \geq 1} N_j$.

If $a \notin R$, choose r with $V_{K \cap \overline{B}(a,r)}^*(a) > 0$ and a rational ball B_j such that $a \in B_j$ and $\overline{B}_j \subset B(a, r)$. Monotonicity in the set gives $V_{A_j}^*(a) > 0$, hence $a \in N_j$. Conversely, if $a \in N_j$, choose $\rho > 0$ with $\overline{B}(a, \rho) \subset B_j$. Then

$$V_{K \cap \overline{B}(a,\rho)}^*(a) \geq V_{A_j}^*(a) > 0,$$

so $a \notin R$. This proves the claim and shows that N_K is a countable union of negligible sets.

Relative plurifine openness and maximality follow from (12). The thinness criterion gives the description of T . Each point of T belongs to N_K , since its singleton is relatively plurifine open in K and is pluripolar. At a point of N_K , isolation in N_K is equivalent to isolation in K , because N_K is relatively plurifine open. Removing these isolated points from $N_K = K \setminus R$ gives $S = N_K \cap (N_K)'_{\mathcal{F}}$. $\square$

In one complex variable, polar sets are thin at each of their points. Theorem 3 therefore gives $S = \emptyset$. In several complex variables, the negligible disk considered above shows why this argument no longer applies. The product construction in [5] realizes an entire nonempty exceptional polydisc as S while the compact set remains non-pluripolar at every point. Thus the distinction captured by Theorem 3 is specific to the local geometry in several complex variables.

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LOTKA-VOLTERRA DYNAMICS ON S^3 WITH A PARTIALLY ORIENTED INTERACTION GRAPH

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ABSTRACT. We study a five-parameter Lotka–Volterra quadratic stochastic operator on the three-dimensional simplex S^3 in the degenerate case $ce = ad$. The corresponding skew-symmetric matrix has exactly one zero interaction coefficient, and its partially oriented interaction graph contains two strong three-vertex subtournaments. On the fixed labelled vertex set $\{1, 2, 3, 4\}$, the isomorphism class of this graph contains exactly twelve labelled realizations. The degeneracy reduces the four coordinate growth forms to two independent linear forms. We determine the complete fixed-point set, prove that the kernel intersects the simplex in a segment of fixed points, construct the six non-empty signature polytopes and their adjacency cycle, and describe the admissible forward and backward crossing directions. We also prove convergence of global negative trajectories contained in the relative interior to unique points of the fixed segment and compute the Jacobian spectrum along that segment, including explicit eigenvalues and eigenvectors at its endpoints.

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Key words: Lotka–Volterra operator, simplex, partially oriented graph, strong tournament, fixed point, signature, convex polytope, trajectory, Jacobian spectrum.

Introduction

Quadratic stochastic operators and their Lotka–Volterra subclasses provide discrete models in population genetics, ecology, epidemiology, and related nonlinear dynamical systems. Their connection with Lyapunov functions, tournaments, and positive/negative trajectories is classical in the theory of Volterra operators [1, 2, 10, 11]. A useful structural feature of a Lotka–Volterra operator is its connection with a skew-symmetric interaction matrix and, in the non-degenerate case, with a tournament. When some interaction coefficients vanish, the associated object becomes a partially oriented graph, as in recent studies of degenerate Lotka–Volterra systems [3]; this is precisely the setting considered here.

For broader background on quadratic stochastic operators, trajectory and fixed-point problems, and their population-dynamical interpretation, see [4–8]. A recent graph-based study of discrete Lotka–Volterra operators associated with weighted oriented bigraphs is [9].

The operator studied in this article has one vanishing interaction, namely $a_{14} = 0$, while the remaining five interaction strengths are positive. Its associated partially oriented graph contains the two directed 3-cycles on the vertex sets $\{1, 2, 3\}$ and $\{2, 3, 4\}$. Thus, it has exactly two strong three-vertex subtournaments. Up to graph isomorphism this is one partially oriented graph type; however, on a fixed set of four labels its isomorphism class contains exactly twelve labelled graphs. Our principal degeneracy condition is $ce = ad$. It is equivalent to $\det A = 0$ for the skew-symmetric interaction matrix. Under this condition, the four growth forms are determined by only two linear forms. This reduction produces a common fixed segment, six strict signature domains, six closed signature polytopes meeting face-to-face, and a cyclic admissible order of transitions between neighbouring regions.

The paper is organized as follows. We first recall the basic notions used throughout, including fixed points, positive and negative trajectories, ω - and α -limit sets, signatures, convex hulls, and adjacent polytopes. After establishing the operator and its degeneracy, we determine the kernel segment and the entire fixed-point set. The six signature regions and their convex-hull descriptions are derived next, followed by their adjacency structure and trajectory directions. Finally, we analyze the Jacobian spectrum.

Preliminaries

Let $S^3 = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_i \geq 0, x_1 + x_2 + x_3 + x_4 = 1\}$ be the standard three-dimensional simplex, with vertices

$$e_1 = (1, 0, 0, 0), \quad e_2 = (0, 1, 0, 0), \quad e_3 = (0, 0, 1, 0), \quad e_4 = (0, 0, 0, 1).$$

For $I \subset \{1, 2, 3, 4\}$, we write $\Gamma_I = \{x \in S^3 : x_j = 0 \text{ for every } j \notin I\}$ for the corresponding coordinate face.

Definition 1. Let $V : S^3 \rightarrow S^3$. A point $x \in S^3$ is called a fixed point of V if $V(x) = x$. The set of all fixed points is denoted by $\text{Fix}(V) = \{x \in S^3 : V(x) = x\}$.

Definition 2. Let $x^{(0)} \in S^3$. The positive trajectory of $x^{(0)}$ is $O^+(x^{(0)}) = \{x^{(n)} : n \geq 0\}$, $x^{(n)} = V^n(x^{(0)})$. A negative trajectory of $x^{(0)}$ is a sequence $O^-(x^{(0)}) = \{x^{(-n)} : n \geq 0\}$ with $x^{(0)}$ as its terminal point and $V(x^{(-n-1)}) = x^{(-n)}$ for $n \geq 0$. If V is invertible, then the negative trajectory is unique and $x^{(-n)} = V^{-n}(x^{(0)})$. Without invertibility, a negative trajectory can be non-unique or may fail to exist globally.

Definition 3. The ω -limit set of the positive trajectory is

$$\omega(x^{(0)}) = \{y \in S^3 : \exists n_k \rightarrow \infty \text{ with } x^{(n_k)} \rightarrow y\}.$$

Whenever a negative trajectory is fixed, its α -limit set is

$$\alpha(x^{(0)}) = \{y \in S^3 : \exists n_k \rightarrow \infty \text{ with } x^{(-n_k)} \rightarrow y\}.$$

Thus $O^+(x^{(0)})$ and $O^-(x^{(0)})$ denote the trajectories themselves, whereas $\omega(x^{(0)})$ and $\alpha(x^{(0)})$ denote their sets of accumulation points.

Definition 4. Let $B \subset \mathbb{R}^n$. The convex hull of B , denoted by $\text{co}(B)$ or $\text{conv}(B)$, is the smallest convex set containing B . Equivalently,

$$\text{co}(B) = \left\{ \sum_{i=1}^m \lambda_i x_i : x_i \in B, \lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1, m \in \mathbb{N} \right\}.$$

Definition 5. A collection of signs $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, where $\sigma_i \in \{+, -\}$, is called a signature. For a given signature σ , we associate the set of all $x \in S^3$ satisfying

$$x'_k > x_k \quad \text{if } \sigma_k = +, \quad x'_k < x_k \quad \text{if } \sigma_k = -, \quad k = 1, 2, 3, 4.$$

The set of points corresponding to a given signature is relatively open and convex in S^3 (possibly empty). We denote by F_σ its closure in S^3 . Thus, F_σ is a convex polyhedron determined by a system of linear inequalities corresponding to the signature σ .

Definition 6. (Adjacent polytopes) Let P and Q be full-dimensional convex polytopes in the same d -dimensional affine space. They are called adjacent if $P \cap Q$ is a common facet of both polytopes, that is, $\dim(P \cap Q) = d - 1$. In the present three-dimensional simplex, two full-dimensional signature polytopes are adjacent precisely when they share a two-dimensional face.

For a skew-symmetric matrix $A = (a_{ij})$, we use the following convention. If $a_{ij} > 0$, the edge between i and j is oriented $j \rightarrow i$; if $a_{ij} < 0$, it is oriented $i \rightarrow j$; and if $a_{ij} = 0$, the edge is left unoriented. A tournament is called strong if every vertex can be reached from every other vertex by a directed path; see [12, 13] for standard graph-theoretic background.

Definition 7. Let d be the Euclidean distance on S^3 . An invariant set $Y \subset S^3$ is called attracting if there exists a neighbourhood U of Y such that $\text{dist}(V^n(x), Y) \rightarrow 0$ ($n \rightarrow \infty$) for every $x \in U$. Its basin of attraction is $B(Y) = \{x \in S^3 : \text{dist}(V^n(x), Y) \rightarrow 0\}$. A fixed point p is attracting when $Y = \{p\}$. A fixed point is a local repeller if every sufficiently close point different from it eventually leaves a prescribed neighbourhood of it.

Main part

Consider the Lotka–Volterra quadratic stochastic operator $V : S^3 \rightarrow S^3$ defined by

$$\begin{cases} x'_1 = x_1(1 + ax_2 - cx_3), \\ x'_2 = x_2(1 - ax_1 + bx_3 - ex_4), \\ x'_3 = x_3(1 + cx_1 - bx_2 + dx_4), \\ x'_4 = x_4(1 + ex_2 - dx_3), \end{cases} \quad (1)$$

or, equivalently, $V_i(x) = x_i(1 + (Ax)_i)$, $i = 1, 2, 3, 4$, where

$$A = \begin{pmatrix} 0 & a & -c & 0 \\ -a & 0 & b & -e \\ c & -b & 0 & d \\ 0 & e & -d & 0 \end{pmatrix}.$$

Throughout the article we assume

$$0 < a, b, c, d, e \leq 1, \quad ce = ad. \quad (2)$$

Define the coordinate growth forms

$$\begin{aligned} h_1(x) &= (Ax)_1 = ax_2 - cx_3, \\ h_2(x) &= (Ax)_2 = -ax_1 + bx_3 - ex_4, \\ h_3(x) &= (Ax)_3 = cx_1 - bx_2 + dx_4, \\ h_4(x) &= (Ax)_4 = ex_2 - dx_3. \end{aligned}$$

Then

$$x'_i - x_i = x_i h_i(x), \quad i = 1, 2, 3, 4. \quad (3)$$

Thus, in the relative interior of S^3 , the sign of $x'_i - x_i$ is exactly the sign of $h_i(x)$. In particular, the signature conditions in Definition 5 are equivalent to $h_k(x) > 0$ when $\sigma_k = +$ and $h_k(x) < 0$ when $\sigma_k = -$. For the above matrix A , the graph has edges $2 \rightarrow 1$, $1 \rightarrow 3$, $3 \rightarrow 2$, $2 \rightarrow 4$, $4 \rightarrow 3$, and the single unoriented edge $1 - 4$. Consequently, the induced subtournaments on $\{1, 2, 3\}$ and $\{2, 3, 4\}$ are directed 3-cycles and hence are strong.

The unique unoriented edge is $\{1, 4\}$. The only non-trivial relabelling preserving all five directed edges and this unoriented edge is the transposition $(1\ 4)$. Hence $|\text{Aut}(G)| = 2$, and the orbit-stabilizer theorem gives $|S_4|/|\text{Aut}(G)| = 12$ labelled realizations in this isomorphism class.

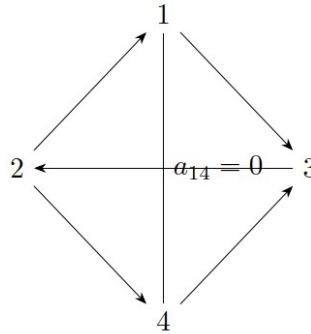


Figure 1. The partially oriented interaction graph.

For the skew-symmetric matrix A , $\det A = (ad - ce)^2$. Indeed, the Pfaffian of A is

$$\text{pf}(A) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = ad - (-c)(-e) + 0 = ad - ce,$$

and $\det A = \text{pf}(A)^2$. Thus $\det A = 0$ if and only if $ce = ad$. Put $u = h_1(x)$ and $v = h_2(x)$. Under $ce = ad$, we have $d = ce/a$. Hence

$$h_4 = \frac{e}{a}h_1, \quad h_3 = -\frac{b}{a}h_1 - \frac{c}{a}h_2 = -\frac{bu + cv}{a}. \quad (4)$$

Consequently, the full signature (h_1, h_2, h_3, h_4) is determined by the pair (u, v) . Set $S = a + b + c$, $T = b + d + e$, and define $C_1 = (\frac{b}{S}, \frac{c}{S}, \frac{a}{S}, 0)$, $C_2 = (0, \frac{d}{T}, \frac{e}{T}, \frac{b}{T})$. These points are obtained by solving $Ax = 0$ on the simplex. By (4), $h_1 = h_2 = 0$ implies $h_3 = h_4 = 0$; hence it suffices to solve $ax_2 = cx_3$, $-ax_1 + bx_3 - ex_4 = 0$. Thus, $x_2 = \frac{c}{a}x_3$, $x_1 = \frac{b}{a}x_3 - \frac{e}{a}x_4$. The endpoint $x_4 = 0$ gives C_1 , and the endpoint $x_1 = 0$ gives C_2 .

Theorem 1. (Kernel segment) Under condition (2), we have $\ker A \cap S^3 = [C_1, C_2]$.

Proof. Recall that $\ker A \cap S^3 = \{x \in S^3 : Ax = 0\}$. First, we prove that $[C_1, C_2] \subseteq \ker A \cap S^3$. Let $x \in [C_1, C_2]$. Then $x = \lambda_1 C_1 + \lambda_2 C_2$, $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$. Since $\lambda_2 = 1 - \lambda_1$, we have

$$x = \left(\frac{\lambda_1 b}{S}, \frac{\lambda_1 c}{S} + (1 - \lambda_1) \frac{d}{T}, \frac{\lambda_1 a}{S} + (1 - \lambda_1) \frac{e}{T}, (1 - \lambda_1) \frac{b}{T} \right).$$

For the second coordinate,

$$\begin{aligned} \lambda_1 cT + (1 - \lambda_1)dS &= \lambda_1(cT - dS) + dS = \lambda_1(c(b + d + e) - d(a + b + c)) + dS \\ &= \lambda_1(bc + ce - ad - bd) + dS = \lambda_1b(c - d) + dS, \end{aligned}$$

because $ce = ad$. Similarly,

$$\begin{aligned} \lambda_1 aT + (1 - \lambda_1)eS &= \lambda_1(aT - eS) + eS = \lambda_1(a(b + d + e) - e(a + b + c)) + eS \\ &= \lambda_1(ab + ad - be - ce) + eS = \lambda_1b(a - e) + eS. \end{aligned}$$

Therefore,

$$x = \left(\frac{\lambda_1 b}{S}, \frac{\lambda_1 b(c-d) + dS}{ST}, \frac{\lambda_1 b(a-e) + eS}{ST}, \frac{(1-\lambda_1)b}{T} \right).$$

By (4), it is enough to verify that $h_1(x) = h_2(x) = 0$. Indeed,

$$\begin{aligned} h_1(x) &= a \frac{\lambda_1 b(c-d) + dS}{ST} - c \frac{\lambda_1 b(a-e) + eS}{ST} = \frac{a\lambda_1 bc - a\lambda_1 bd + adS - c\lambda_1 ab + c\lambda_1 be - ceS}{ST} \\ &= \frac{\lambda_1 b(ce - ad) + (ad - ce)S}{ST} = 0. \end{aligned}$$

Also,

$$h_2(x) = -a \frac{\lambda_1 b}{S} + b \frac{\lambda_1 b(a-e) + eS}{ST} - e \frac{(1-\lambda_1)b}{T} = \frac{\lambda_1 b(-aT + b(a-e) + eS)}{ST} = \frac{\lambda_1 b(ce - ad)}{ST} = 0.$$

Hence, by (4), $h_1(x) = h_2(x) = h_3(x) = h_4(x) = 0$, and therefore $Ax = \mathbf{0}$. Thus, $[C_1, C_2] \subseteq \ker A \cap S^3$. Conversely, let $x = (x_1, x_2, x_3, x_4) \in \ker A \cap S^3$. Then $Ax = \mathbf{0}$, and in particular

$$h_1(x) = ax_2 - cx_3 = 0, \quad h_2(x) = -ax_1 + bx_3 - ex_4 = 0.$$

Hence $x_2 = \frac{c}{a}x_3$, $x_1 = \frac{b}{a}x_3 - \frac{e}{a}x_4$. Together with $x_1 + x_2 + x_3 + x_4 = 1$, these equations determine an affine line. Its intersection with the simplex S^3 is a closed line segment. Since $a, c > 0$, the equality $ax_2 = cx_3$ shows that $x_2 = 0$ if and only if $x_3 = 0$. If $x_2 = x_3 = 0$, then $h_2(x) = 0$ gives $ax_1 + ex_4 = 0$, which is impossible for $x \in S^3$. Therefore the endpoints of the segment occur when $x_4 = 0$ and when $x_1 = 0$. If $x_4 = 0$, then $ax_2 = cx_3$, $ax_1 = bx_3$, $x_1 + x_2 + x_3 = 1$, and hence $x = (\frac{b}{S}, \frac{c}{S}, \frac{a}{S}, 0) = C_1$. If $x_1 = 0$, then $ax_2 = cx_3$, $bx_3 = ex_4$, $x_2 + x_3 + x_4 = 1$. Using $ce = ad$, we obtain $x = (0, \frac{d}{T}, \frac{e}{T}, \frac{b}{T}) = C_2$. Thus the intersection has precisely the endpoints C_1 and C_2 . Consequently, $\ker A \cap S^3 = [C_1, C_2]$. Equivalently, the fixed segment can be parametrized by

$$X(t) = (1-t)C_1 + tC_2 = \left(\frac{(1-t)b}{S}, \frac{(1-t)c}{S} + \frac{td}{T}, \frac{(1-t)a}{S} + \frac{te}{T}, \frac{tb}{T} \right), \quad 0 \leq t \leq 1. \quad (5)$$

The proof is complete.

Theorem 2. The fixed-point set is $\text{Fix}(V) = [e_1, e_4] \cup [C_1, C_2] \cup \{e_2, e_3\}$.

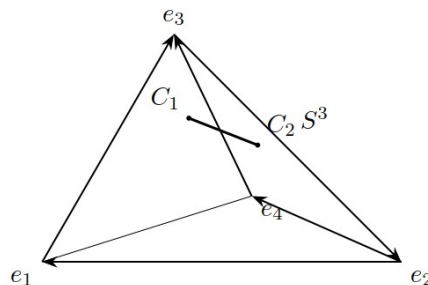


Figure 2. Schematic geometry of the two non-isolated fixed segments: the coordinate edge $[e_1, e_4]$ and the kernel segment $[C_1, C_2]$.

Proof. Every vertex is fixed. Since $a_{14} = 0$, every point of the edge $[e_1, e_4]$ is fixed. Theorem 1 shows that every point of $[C_1, C_2]$ is fixed. Conversely, let $x \in \text{Fix}(V)$. Then $x_i h_i(x) = 0$ for $i = 1, 2, 3, 4$. We distinguish the possibilities according to the support of x . If $|\text{supp } x| = 1$, then x is a vertex. If $|\text{supp } x| = 2$, an interior point of a coordinate edge can be fixed only when the interaction coefficient corresponding to that pair is zero. The only zero coefficient is a_{14} ; hence the only non-trivial fixed edge is $[e_1, e_4]$. Suppose $|\text{supp } x| = 3$. On relint Γ_{123} the fixed-point equations reduce to $ax_2 = cx_3$, $ax_1 = bx_3$, $x_1 + x_2 + x_3 = 1$, which have the unique solution C_1 . On relint Γ_{234} they reduce to $ax_2 = cx_3$, $bx_3 = ex_4$, $x_2 + x_3 + x_4 = 1$, and, using $ce = ad$, the unique solution is C_2 . The remaining two faces contain no fixed point in their relative interiors: on Γ_{124} one has $h_1 = ax_2 > 0$, while on Γ_{134} one has $h_1 = -cx_3 < 0$. Finally, if all four coordinates are positive, then $h_i(x) = 0$ for every i , so $x \in \ker A \cap S^3 = [C_1, C_2]$ by Theorem 1. This exhausts all possible supports and proves the assertion. The proof is completed.

Introduce the boundary points.

$$\begin{aligned} M_1 &= \left(0, \frac{c}{a+c}, \frac{a}{a+c}, 0 \right), & M_2 &= \left(\frac{b}{a+b}, 0, \frac{a}{a+b}, 0 \right), & M_3 &= \left(\frac{b}{b+c}, \frac{c}{b+c}, 0, 0 \right), \\ N_1 &= \left(0, 0, \frac{e}{b+e}, \frac{b}{b+e} \right), & N_2 &= \left(0, \frac{d}{b+d}, 0, \frac{b}{b+d} \right), & N_3 &= \left(0, \frac{d}{d+e}, \frac{e}{d+e}, 0 \right). \end{aligned} \quad (6)$$

Because $ce = ad$, one has $N_3 = M_1$. Using $u = h_1$ and $v = h_2$, we have $h_3 = -\frac{bu+cv}{a}$, $h_4 = \frac{e}{a}u$. Therefore the three lines

$$u = 0, \quad v = 0, \quad bu + cv = 0 \quad (7)$$

split the (u, v) -plane into exactly six open sectors.

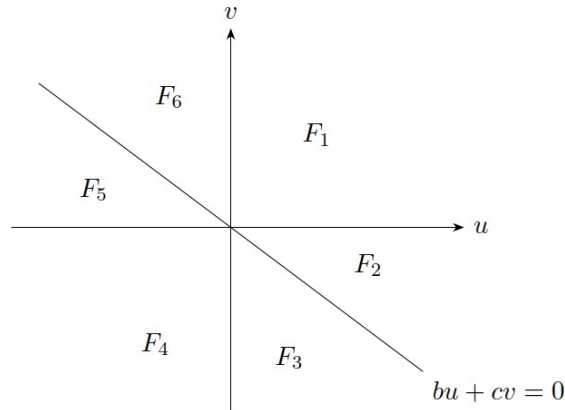


Figure 3. The six signature sectors in the (u, v) -plane.

Lemma 1. (Six strict signatures) Under $ce = ad$, there are exactly six non-empty strict signature domains. We denote their closures by $F_1, \dots, F_6$; the strict signatures on relint F_i are given in the following table.

Region	Conditions in the (u, v) -plane	Signature (h_1, h_2, h_3, h_4)
F_1	$u > 0, v > 0$	$(+, +, -, +)$
F_2	$u > 0, v < 0, bu + cv > 0$	$(+, -, -, +)$
F_3	$u > 0, v < 0, bu + cv < 0$	$(+, -, +, +)$
F_4	$u < 0, v < 0$	$(-, -, +, -)$
F_5	$u < 0, v > 0, bu + cv < 0$	$(-, +, +, -)$
F_6	$u < 0, v > 0, bu + cv > 0$	$(-, +, -, -)$

Proof. Since

$$h_4 = \frac{e}{a}h_1, \quad \frac{e}{a} > 0,$$

the signs of h_1 and h_4 always coincide. Thus, instead of the $2^4 = 16$ sign combinations that would be possible for four independent signs, there are at most $2^3 = 8$ possible sign combinations.

The two signatures $(+, +, +, +)$ and $(-, -, -, -)$ cannot occur. Indeed, in [9] given V maps S^3 into itself, $\sum_{i=1}^4 x'_i = \sum_{i=1}^4 x_i = 1$, and therefore $\sum_{i=1}^4 (x'_i - x_i) = 0$. Hence, in relint S^3 , the four differences $x'_i - x_i = x_i h_i(x)$, $i = 1, 2, 3, 4$, cannot all be strictly positive or all be strictly negative. Consequently, only six strict signatures are admissible. More explicitly, since $h_3 = -\frac{bu+cv}{a}$, the sign of h_3 is opposite to that of $bu + cv$. Hence the three lines in (7) $u = 0$, $v = 0$, $bu + cv = 0$ divide the (u, v) -plane into exactly six open sectors, which correspond precisely to the six signatures listed above.

For a strict sign vector $\sigma = (\sigma_1, \dots, \sigma_4) \in \{+, -\}^4$, its closed signature polytope is obtained by replacing each strict sign condition by the corresponding weak inequality. Thus, for example, $+$ means $h_i \geq 0$ and $-$ means $h_i \leq 0$.

Proposition 1. The six closed signature polytopes are

$$\begin{aligned} F_1 &= \text{co}\{e_2, M_1, C_1, C_2\}, & F_2 &= \text{co}\{e_2, N_2, M_3, C_1, C_2\}, \\ F_3 &= \text{co}\{e_1, e_4, N_2, M_3, C_1, C_2\}, & F_4 &= \text{co}\{e_1, e_4, N_1, M_2, C_1, C_2\}, \\ F_5 &= \text{co}\{e_3, N_1, M_2, C_1, C_2\}, & F_6 &= \text{co}\{e_3, M_1, C_1, C_2\}. \end{aligned}$$

Proof. Put $w = bu + cv$. Each F_i is the intersection of the three-dimensional simplex S^3 with three closed half-spaces bounded by the separating planes $u = 0, v = 0, w = 0$. Hence each F_i is a bounded convex polyhedron, and therefore a convex polytope.

We first determine all possible vertices of these intersections. We work in the affine hull

$x_1 + x_2 + x_3 + x_4 = 1$, which has dimension 3. At least three independent active boundary conditions must be active. These conditions are chosen from the coordinate planes $x_j = 0$ and the three separating planes above.

If three coordinate planes are active, we obtain one of the simplex vertices e_1, e_2, e_3, e_4 . If two coordinate planes and one separating plane are active, the point lies on a coordinate edge. The non-trivial interior edge intersections are exactly

edge	separating equation	intersection point
$[e_1, e_2]$	$w = 0$	M_3
$[e_1, e_3]$	$v = 0$	M_2
$[e_2, e_3]$	$u = 0$	M_1
$[e_2, e_4]$	$w = 0$	N_2
$[e_3, e_4]$	$v = 0$	N_1

These formulas follow immediately by solving the corresponding single linear equation on the indicated edge. The remaining edge $[e_1, e_4]$ is contained entirely in the plane $u = 0$, so this intersection produces no additional vertex. Also, the equation $h_4 = 0$ is equivalent to $u = 0$ by (4), and its intersection with $[e_2, e_3]$ gives $N_3 = M_1$ because $ce = ad$.

It remains to examine intersections involving two or more separating planes. Since $w = bu + cv$ with $b, c > 0$, any two equalities among $u = 0$, $v = 0$, $w = 0$ force $u = v = 0$. By Theorem 1, their intersection with S^3 is precisely the segment $[C_1, C_2]$. Intersecting this segment with a coordinate plane produces only its endpoints: $x_4 = 0$ gives C_1 and $x_1 = 0$ gives C_2 , while the planes $x_2 = 0$ and $x_3 = 0$ do not meet the kernel segment. If all three separating equations are imposed, one again obtains the whole segment $[C_1, C_2]$, not a new vertex.

This also shows that no extra vertices can arise from intersections of a separating plane with the relative interior of a two-dimensional face of the simplex. Such an intersection is a line segment; its endpoints occur on coordinate edges and have already been listed above. Thus the complete set of possible vertices is

$$\{e_1, e_2, e_3, e_4, M_1, M_2, M_3, N_1, N_2, C_1, C_2\}.$$

Finally, substituting these points into the weak sign conditions associated with the six sectors in Lemma 1 gives exactly the vertex sets displayed in the statement of the proposition. Since every bounded polyhedron is the convex hull of its vertices, the asserted convex-hull representations follow. The proof is complete.

Their union is S^3 , their strict interiors are pairwise disjoint, and different closed regions meet along common faces. In this precise sense they form a face-to-face polyhedral decomposition of the simplex. Given an initial point $x^{(0)}$, one computes $u_0 = ax_2^{(0)} - cx_3^{(0)}$, $v_0 = -ax_1^{(0)} + bx_3^{(0)} - ex_4^{(0)}$, and compares (u_0, v_0) with the three lines in (7) to determine its signature region. Each feasible point listed above is the unique solution of three independent active boundary equations; hence it is a vertex of the corresponding polytope.

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Proposition 2. All six closed polytopes contain the common edge $[C_1, C_2]$. Their two-dimensional common faces are

$$\begin{aligned} F_1 \cap F_2 &= \text{co}\{e_2, C_1, C_2\}, & F_2 \cap F_3 &= \text{co}\{M_3, N_2, C_1, C_2\}, \\ F_3 \cap F_4 &= \text{co}\{e_1, e_4, C_1, C_2\}, & F_4 \cap F_5 &= \text{co}\{M_2, N_1, C_1, C_2\}, \\ F_5 \cap F_6 &= \text{co}\{e_3, C_1, C_2\}, & F_6 \cap F_1 &= \text{co}\{M_1, C_1, C_2\}. \end{aligned} \quad (8)$$

Hence the adjacency graph is the six-cycle

$$F_1 - F_2 - F_3 - F_4 - F_5 - F_6 - F_1. \quad (9)$$

Proof. By Proposition 1, every F_i is a full-dimensional convex polytope in the affine hull of S^3 . Moreover, every point of $[C_1, C_2]$ satisfies $u = v = 0$, and hence also $bu + cv = 0$; consequently, the segment $[C_1, C_2]$ is contained in all six polytopes. We verify explicitly the dimension of one representative intersection. The regions F_1 and F_2 differ only in the sign of v . Hence $F_1 \cap F_2 = F_1 \cap \{v = 0\} = F_2 \cap \{v = 0\}$. By Proposition 1, the vertices of this common face are exactly e_2, C_1, C_2 , and therefore $F_1 \cap F_2 = \text{co}\{e_2, C_1, C_2\}$. The points C_1 and C_2 are distinct and span the kernel segment, whereas $e_2 \notin \ker A$ because $Ae_2 = (a, 0, -b, e)^T \neq 0$. Thus e_2, C_1, C_2 are affinely independent, so $\dim(F_1 \cap F_2) = 2$. Hence $F_1 \cap F_2$ is a common facet of F_1 and F_2 .

The remaining consecutive pairs are treated in exactly the same way. More precisely, the separating plane is $w = 0$ for $F_2 \cap F_3$, $u = 0$ for $F_3 \cap F_4$, $v = 0$ for $F_4 \cap F_5$, $w = 0$ for $F_5 \cap F_6$, and $u = 0$ for $F_6 \cap F_1$, where $w = bu + cv$. Each

set displayed in (8) lies in the corresponding separating plane, contains the line segment $[C_1, C_2]$, and contains at least one listed vertex outside that segment. Therefore each of these intersections has dimension exactly 2 and is a common facet.

If two regions are not consecutive in the cyclic order, their weak sign conditions force at least two of the separating equalities $u = 0$, $v = 0$, and $w = 0$. Any two of these imply $u = v = 0$, so the intersection is contained in $[C_1, C_2]$ and has dimension at most 1. Such regions are therefore not adjacent. Hence the adjacency graph is exactly the six-cycle (9).

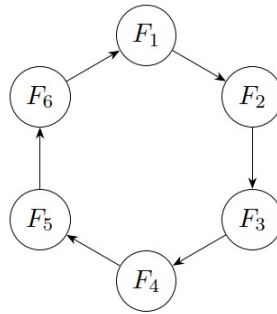


Figure 4. The underlying adjacency graph is a six-cycle. The arrows indicate the admissible first-crossing order for positive trajectories proved in Theorem 3.

Let $x^{(n+1)} = V(x^{(n)})$. From (3), we have $x_i^{(n+1)} - x_i^{(n)} = x_i^{(n)} h_i(x^{(n)})$, $i = 1, 2, 3, 4$. Therefore the signature determines which coordinates increase and which decrease at each iterate in a strict region.

Theorem 3. Suppose an orbit point lies in the relative interior of one of the six regions and its next iterate leaves that region by crossing a separating wall. Then the first possible neighbouring region follows the cyclic order (9).

Current region	Signs preserved after one step	Sign that may change	First possible next region
F_1	$h_1 > 0, h_3 < 0, h_4 > 0$	h_2	F_2
F_2	$h_1 > 0, h_2 < 0, h_4 > 0$	h_3	F_3
F_3	$h_2 < 0, h_3 > 0$	h_1, h_4	F_4
F_4	$h_1 < 0, h_3 > 0, h_4 < 0$	h_2	F_5
F_5	$h_1 < 0, h_2 > 0, h_4 < 0$	h_3	F_6
F_6	$h_2 > 0, h_3 < 0$	h_1, h_4	F_1

Proof. Put $y = V(x)$. From the definitions of the growth forms,

$$\begin{aligned} h_1(y) &= h_1(x) + ax_2h_2(x) - cx_3h_3(x), \\ h_2(y) &= h_2(x) - ax_1h_1(x) + bx_3h_3(x) - ex_4h_4(x), \\ h_3(y) &= h_3(x) + cx_1h_1(x) - bx_2h_2(x) + dx_4h_4(x), \end{aligned}$$

and, by (4), $h_4(y) = \frac{e}{a}h_1(y)$. We shall also use the skew-symmetry of A . Since $h(x) = Ax$ and $A^T = -A$,

$$x_1h_1(x) + x_2h_2(x) + x_3h_3(x) + x_4h_4(x) = x^T Ax = 0. \quad (10)$$

For brevity, in the following calculations we write $h_i = h_i(x)$.

Case F₁. Here $h_1 > 0$, $h_2 > 0$, $h_3 < 0$, $h_4 > 0$. Therefore $h_1(y) = h_1 + ax_2h_2 - cx_3h_3 > 0$. To control $h_3(y)$, use $0 < c, d \leq 1$. Since $x_1h_1 > 0$ and $x_4h_4 > 0$,

$$h_3(y) = h_3 + cx_1h_1 - bx_2h_2 + dx_4h_4 \leq h_3 + x_1h_1 - bx_2h_2 + x_4h_4.$$

By (10), $x_1h_1 + x_4h_4 = -x_2h_2 - x_3h_3$, and hence $h_3(y) \leq (1 - x_3)h_3 - (1 + b)x_2h_2 < 0$. Thus h_1, h_3, h_4 preserve their signs, while only h_2 may change sign. Consequently the first possible neighbouring region is F_2 .

Case F₂. Here $h_1 > 0$, $h_2 < 0$, $h_3 < 0$, $h_4 > 0$. First, $h_2(y) = h_2 - ax_1h_1 + bx_3h_3 - ex_4h_4 < 0$. Using (10) to eliminate x_2h_2 , we obtain

$$h_1(y) = h_1 + ax_2h_2 - cx_3h_3 = h_1(1 - ax_1) - (a + c)x_3h_3 - ax_4h_4 = h_1(1 - ax_1 - ex_4) - (a + c)x_3h_3,$$

where the last equality follows from $h_4 = (e/a)h_1$. Since $a, e \leq 1$,

$$1 - ax_1 - ex_4 \geq 1 - x_1 - x_4 = x_2 + x_3 > 0.$$

Thus $h_1(y) > 0$, and therefore $h_4(y) > 0$. Only h_3 may change sign, so the first possible neighbouring region is F_3 .

Case F₃. Here $h_1 > 0$, $h_2 < 0$, $h_3 > 0$, $h_4 > 0$. We immediately have $h_3(y) = h_3 + cx_1h_1 - bx_2h_2 + dx_4h_4 > 0$. Using (10) to eliminate x_3h_3 , we get

$$h_2(y) = h_2 - ax_1h_1 + bx_3h_3 - ex_4h_4 = h_2(1 - bx_2) - (a + b)x_1h_1 - (b + e)x_4h_4.$$

Since $b \leq 1$, we have $1 - bx_2 \geq 1 - x_2 = x_1 + x_3 + x_4 > 0$, so every term in the last expression is negative. Hence $h_2(y) < 0$. Therefore h_2 and h_3 preserve their signs, whereas h_1 and h_4 , which always have the same sign, may change simultaneously. The first possible neighbouring region is F_4 .

Case F₄. Here $h_1 < 0$, $h_2 < 0$, $h_3 > 0$, $h_4 < 0$. Clearly, $h_1(y) = h_1 + ax_2h_2 - cx_3h_3 < 0$. Since $x_1h_1 < 0$ and $x_4h_4 < 0$, the inequalities $c, d \leq 1$ now give

$$h_3(y) = h_3 + cx_1h_1 - bx_2h_2 + dx_4h_4 \geq h_3 + x_1h_1 - bx_2h_2 + x_4h_4.$$

Using (10) once more, $h_3(y) \geq (1 - x_3)h_3 - (1 + b)x_2h_2 > 0$. Hence h_1, h_3, h_4 preserve their signs, and only h_2 may change. The first possible neighbouring region is F_5 .

Case F₅. Here $h_1 < 0$, $h_2 > 0$, $h_3 > 0$, $h_4 < 0$. We have $h_2(y) = h_2 - ax_1h_1 + bx_3h_3 - ex_4h_4 > 0$. The same identity used in the case F_2 yields $h_1(y) = h_1(1 - ax_1 - ex_4) - (a + c)x_3h_3$. Since $1 - ax_1 - ex_4 > 0$, while $h_1 < 0$ and $h_3 > 0$, we obtain $h_1(y) < 0$, and hence $h_4(y) < 0$. Thus only h_3 may change sign, and the first possible neighbouring region is F_6 .

Case F₆. Here $h_1 < 0$, $h_2 > 0$, $h_3 < 0$, $h_4 < 0$. First, $h_3(y) = h_3 + cx_1h_1 - bx_2h_2 + dx_4h_4 < 0$. Using the identity obtained in the case F_3 , $h_2(y) = h_2(1 - bx_2) - (a + b)x_1h_1 - (b + e)x_4h_4$. Here $1 - bx_2 > 0$, $h_2 > 0$, and $h_1, h_4 < 0$,

so every term on the right-hand side is positive. Thus $h_2(y) > 0$. Hence h_2, h_3 preserve their signs, while h_1 and h_4 may change simultaneously. The first possible neighbouring region is F_1 .

The six cases prove that whenever a positive orbit leaves one strict signature region through a separating wall, the first neighbouring region it can enter is the next one in the cyclic order

$$F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow F_4 \longrightarrow F_5 \longrightarrow F_6 \longrightarrow F_1.$$

This proves the theorem.

Remark 1. A positive trajectory may remain in the same region for several iterates, hit a boundary, or converge toward an invariant subset without traversing the entire six-cycle. The theorem only restricts the first neighbouring region that can be entered when a crossing occurs.

For a negative trajectory, if $V(y) = x$, then

$$y_j - x_j = -y_j h_j(y). \quad (10)$$

Thus every backward coordinate displacement has the opposite sign from the corresponding forward displacement from y to x .

Proposition 3. If a predecessor y of x lies in one of the six relative interiors, the admissible backward crossing order is $F_1 \leftarrow F_2 \leftarrow F_3 \leftarrow F_4 \leftarrow F_5 \leftarrow F_6 \leftarrow F_1$.

Proof. Equation (10) reverses each forward coordinate direction, and the reverse crossing order follows from Theorem 3.

Theorem 4. Let $x^{(0)} \in \text{relint } S^3$. If $x^{(0)} \notin [C_1, C_2]$, then the positive trajectory $\{V^n(x^{(0)})\}_{n \geq 0}$ does not converge in S^3 . Equivalently, a convergent positive trajectory starting in $\text{relint } S^3$ is necessarily stationary, and its initial point belongs to $\text{relint}[C_1, C_2]$.

Proof. First observe that $\text{relint } S^3$ is forward invariant. Indeed, for $x \in \text{relint } S^3$ and $0 < a, b, c, d, e \leq 1$,

$$\begin{aligned} 1 + h_1(x) &= 1 + ax_2 - cx_3 \geq 1 - cx_3 > 0, \\ 1 + h_2(x) &= 1 - ax_1 + bx_3 - ex_4 \geq 1 - (x_1 + x_4) = x_2 + x_3 > 0, \\ 1 + h_3(x) &= 1 + cx_1 - bx_2 + dx_4 \geq 1 - bx_2 > 0, \\ 1 + h_4(x) &= 1 + ex_2 - dx_3 \geq 1 - dx_3 > 0. \end{aligned}$$

Hence all coordinates of $V(x)$ remain positive. Assume that $x^{(n)} = V^n(x^{(0)})$ converges to a point $p \in S^3$. Since V is continuous, $p = \lim_{n \rightarrow \infty} x^{(n+1)} = \lim_{n \rightarrow \infty} V(x^{(n)}) = V(p)$, so $p \in \text{Fix}(V)$. By Theorem 2, $p \in [e_1, e_4] \cup [C_1, C_2] \cup \{e_2, e_3\}$. We exclude these possibilities one by one. Suppose first that $p \in [C_1, C_2]$. Put $I(p) = \{i : p_i > 0\}$, $\Phi_p(x) = \prod_{i \in I(p)} \left(\frac{x_i}{p_i}\right)^{p_i}$.

Since $p \in \ker A \cap S^3$ and A is skew-symmetric, $p^T A = 0$. Therefore, for every $x \in \text{relint } S^3$, the weighted arithmetic-geometric mean inequality yields

$$\frac{\Phi_p(V(x))}{\Phi_p(x)} = \prod_{i \in I(p)} (1 + h_i(x))^{p_i} \leq \sum_{i \in I(p)} p_i (1 + h_i(x)) = 1 + p^T A x = 1.$$

Thus Φ_p is non-increasing along every positive trajectory in the relative interior. On the other hand,

$$\Phi_p(x) \leq \sum_{i \in I(p)} p_i \frac{x_i}{p_i} = \sum_{i \in I(p)} x_i \leq 1.$$

If $p \in \text{relint}[C_1, C_2]$, equality in AM-GM holds only for $x = p$; if $p = C_1$ or $p = C_2$, then one coordinate of p is zero while all coordinates of x are positive, so the last inequality is strict. Thus $x \neq p$ implies $\Phi_p(x) < 1$. Consequently, if $x^{(0)} \neq p$, then $\Phi_p(x^{(n)}) \leq \Phi_p(x^{(0)}) < 1$ ($n \geq 0$), whereas $x^{(n)} \rightarrow p$ would imply $\Phi_p(x^{(n)}) \rightarrow \Phi_p(p) = 1$, a contradiction. Hence a non-stationary interior trajectory cannot converge to any point of $[C_1, C_2]$.

Next, if $p = e_2$, then $h_1(e_2) = a > 0$. By continuity, there exist $\eta > 0$ and N such that $h_1(x^{(n)}) \geq \eta$ for all $n \geq N$. Hence $x_1^{(n+1)} \geq (1 + \eta)x_1^{(n)}$, $n \geq N$, which is impossible because $x_1^{(n)} > 0$ and convergence to e_2 requires $x_1^{(n)} \rightarrow 0$. Similarly, if $p = e_3$, then $h_2(e_3) = b > 0$, so the same argument applied to the second coordinate gives a contradiction. Finally, let $p \in [e_1, e_4]$. Then $p = (s, 0, 0, 1 - s)$ for some $0 \leq s \leq 1$, and $h_3(p) = cs + d(1 - s) > 0$. Hence $h_3(x^{(n)}) \geq \eta > 0$ for all sufficiently large n , so $x_3^{(n)}$ must eventually increase at least geometrically. This contradicts $x_3^{(n)} \rightarrow 0$. Therefore no fixed point can be the limit of a non-stationary positive trajectory starting in $\text{relint } S^3$. The theorem follows.

Theorem 5. Let $O^-(x^{(0)}) = \{x^{(-n)} : n \geq 0\}$ be a global negative trajectory contained in $\text{relint } S^3$. Then there exists a unique point $q = q(x^{(0)}) \in \text{relint}[C_1, C_2]$ such that $x^{(-n)} \rightarrow q$ as $n \rightarrow \infty$. Consequently, $\alpha(x^{(0)}) = \{q\}$, $\text{dist}(x^{(-n)}, [C_1, C_2]) \rightarrow 0$.

Proof. Put $z_n = x^{(-n)}$, so that $V(z_{n+1}) = z_n$. Fix $p \in \text{relint}[C_1, C_2]$ and define

$\Phi_p(x) = \prod_{i=1}^4 \left(\frac{x_i}{p_i}\right)^{p_i}$, with the convention $0^{p_i} = 0$. By Theorem 1, $Ap = 0$, and since $A^T = -A$, we also have $p^T A = 0$. Hence, by the weighted AM–GM inequality,

$$\frac{\Phi_p(V(x))}{\Phi_p(x)} = \prod_{i=1}^4 (1 + h_i(x))^{p_i} \leq \sum_{i=1}^4 p_i (1 + h_i(x)) = 1 + p^T Ax = 1.$$

Therefore $\Phi_p(z_n) \leq \Phi_p(z_{n+1}) \leq 1$, so $\Phi_p(z_n) \rightarrow L_p$ for some $L_p > 0$.

Set $R_p(x) = \prod_{i=1}^4 (1 + h_i(x))^{p_i}$. Since $\Phi_p(z_n) = \Phi_p(z_{n+1})R_p(z_{n+1})$, we have $R_p(z_{n+1}) \rightarrow 1$. By compactness of S^3 , let $z_{n_k} \rightarrow q$ for some accumulation point q . Then $R_p(q) = 1$. Equality in the weighted AM–GM inequality implies

$$1 + h_1(q) = 1 + h_2(q) = 1 + h_3(q) = 1 + h_4(q) = 1,$$

and hence $Aq = 0$. By Theorem 1, $q \in [C_1, C_2]$. Moreover, $\Phi_p(q) = L_p > 0$, whereas $\Phi_p(C_1) = \Phi_p(C_2) = 0$. Thus $q \in \text{relint}[C_1, C_2]$. It remains to prove uniqueness. Define $\Phi_q(x) = \prod_{i=1}^4 \left(\frac{x_i}{q_i}\right)^{q_i}$. As above, $\Phi_q(z_n)$ is non-decreasing and bounded above by 1. Since $z_{n_k} \rightarrow q$,

$$\Phi_q(z_{n_k}) \rightarrow \Phi_q(q) = 1,$$

and therefore $\Phi_q(z_n) \rightarrow 1$. On the other hand, weighted AM–GM gives $\Phi_q(x) \leq 1$, with equality if and only if $x = q$. Hence every accumulation point of $\{z_n\}$ must be q . By compactness, this implies $z_n \rightarrow q$. Thus $x^{(-n)} \rightarrow q$, and consequently $\alpha(x^{(0)}) = \{q\}$. Finally, since $q \in [C_1, C_2]$,

$$\text{dist}(x^{(-n)}, [C_1, C_2]) \leq d(x^{(-n)}, q) \rightarrow 0.$$

The proof is complete.

Remark 2. Theorems 4 and 5 show a clear asymmetry between the two time directions in the relative interior of the simplex: a non-stationary positive trajectory cannot converge, whereas every global negative trajectory contained in $\text{relint } S^3$ converges to a unique point of $\text{relint}[C_1, C_2]$. This is a negative-time statement and does not mean that $[C_1, C_2]$ is attracting for the forward dynamics in the sense of Definition 7.

For $V_i(x) = x_i(1 + h_i(x))$, the Jacobian $J(x) = DV(x)$ has entries $J_{ij}(x) = \delta_{ij}(1 + h_i(x)) + x_i a_{ij}$. Therefore,

$$J(x) = \begin{pmatrix} 1 + ax_2 - cx_3 & ax_1 & -cx_1 & 0 \\ -ax_2 & 1 - ax_1 + bx_3 - ex_4 & bx_2 & -ex_2 \\ cx_3 & -bx_3 & 1 + cx_1 - bx_2 + dx_4 & dx_3 \\ 0 & ex_4 & -dx_4 & 1 + ex_2 - dx_3 \end{pmatrix}.$$

On $[C_1, C_2]$, $Ax = 0$, so $J(x) = I + \text{diag}(x)A$. Using (5),

$$J(X(t)) = \begin{pmatrix} 1 & aX_1(t) & -cX_1(t) & 0 \\ -aX_2(t) & 1 & bX_2(t) & -eX_2(t) \\ cX_3(t) & -bX_3(t) & 1 & dX_3(t) \\ 0 & eX_4(t) & -dX_4(t) & 1 \end{pmatrix}.$$

Theorem 6. For $x \in [C_1, C_2]$, the eigenvalues of $J(x)$ are 1, 1, $1 \pm i\omega_*(x)$, where

$$\omega_*(x)^2 = a^2 x_1 x_2 + c^2 x_1 x_3 + b^2 x_2 x_3 + e^2 x_2 x_4 + d^2 x_3 x_4. \quad (11)$$

Here $\omega_*(x)$ denotes a spectral frequency and should not be confused with the ω -limit set notation of Definition 3.

Proof. Put $D = \text{diag}(x)$ and $B = DA$, so that $J(x) = I + B$. Let $P = D^{1/2}$ and $Q = D^{1/2}A$. Then

$$PQ = DA = B, \quad QP = D^{1/2}AD^{1/2} = K.$$

The matrices PQ and QP have the same characteristic polynomial; hence B and K have the same eigenvalues, including at the endpoints where some coordinates vanish. The matrix K is real skew-symmetric. Moreover, $\text{rank } K \leq \text{rank } A = 2$, because $ce = ad$, while $K \neq 0$ on the whole fixed segment. Therefore the eigenvalues of K are $0, 0, \pm i\omega_*(x)$ for some $\omega_*(x) > 0$.

For a real skew-symmetric matrix, $\text{tr}(K^2) = -2 \sum_{i < j} K_{ij}^2$. On the other hand, the above spectrum gives $\text{tr}(K^2) = -2\omega_*(x)^2$. Comparing these two identities yields exactly formula (11). Finally, adding the identity shifts the spectrum of B by 1, which yields the asserted eigenvalues of $J(x)$.

The eigenvalues of $J(C_1)$ are 1, 1, $1 \pm i\sqrt{\frac{abc}{a+b+c}}$. For the eigenvalue 1, two independent eigenvectors are C_1 and C_2 , viewed as vectors in $\mathbb{R}^4$. If $\mu_{\pm} = \pm i\sqrt{\frac{abc}{S}}$, $\lambda_{\pm} = 1 + \mu_{\pm}$, then corresponding eigenvectors are $w_{\pm}^{(1)} = (S\mu_{\pm} - ab, -ac - S\mu_{\pm}, a(b+c), 0)^T$.

The eigenvalues of $J(C_2)$ are 1, 1, $1 \pm i\sqrt{\frac{bde}{b+d+e}}$. For the eigenvalue 1, two independent eigenvectors are again C_1 and C_2 . If $\nu_{\pm} = \pm i\sqrt{\frac{bde}{T}}$, $\lambda_{\pm} = 1 + \nu_{\pm}$, then corresponding eigenvectors are

$$w_{\pm}^{(2)} = (0, T\nu_{\pm} - bd, -be - T\nu_{\pm}, b(d+e))^T.$$

Proposition 4. No point of the fixed segment $[C_1, C_2]$ is an attracting fixed point in the sense of Definition 7.

Proof. Let $p \in [C_1, C_2]$. Every neighbourhood of p in the relative topology of S^3 contains another point $q \in [C_1, C_2]$, $q \neq p$. By Theorem 1, q is fixed, so $V^n(q) = q$ for every $n \geq 0$. Hence

$$d(V^n(q), p) = d(q, p) > 0,$$

and therefore $V^n(q)$ does not converge to p . Thus p is not attracting. Since p was arbitrary, no point of $[C_1, C_2]$ is attracting.

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THE DERIVATION ALGEBRA OF THE EXTENDED CONFORMAL GALILEI ALGEBRA

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ABSTRACT. We investigate the derivation algebra of the mass-extended conformal Galilei algebra over the field of complex numbers. The algebra is realized as a one-dimensional central extension of the conformal Galilei algebra. We determine the derivations and identify the distinguished outer derivation which acts by one half on the translation module and by the identity on the central element. We then establish completeness of the derivation algebra and describe the associated derivation tower.

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Key words: Lie algebra, derivation, inner derivation.

Introduction

Conformal Galilei algebras arise as nonrelativistic extensions of the Galilei algebra incorporating an $\mathfrak{sl}_2$ -symmetry in the time variable. They provide a family of finite-dimensional Lie algebras describing nonrelativistic systems with conformal-type symmetries and have been studied extensively in mathematical physics, representation theory, and the theory of invariant differential equations. For a fixed spatial dimension d , these algebras form a family parametrized by a non-negative half-integer ℓ , which may be interpreted as the spin of the corresponding $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{so}(d, \mathbb{C})$ -module. The case $\ell = \frac{1}{2}$ is particularly important: it gives the Schrödinger algebra, while higher values of ℓ lead to higher-order conformal extensions of the Galilei algebra. Finite-dimensional nonrelativistic conformal Lie algebras can also be understood in terms of polynomial vector fields and are related to nonrelativistic conformal transformations and Newton–Hooke geometries.

Derivations of a Lie algebra encode infinitesimal symmetries of its algebraic structure. For a Lie algebra $\mathfrak{g}$, we denote by $\text{Der}(\mathfrak{g})$ the Lie algebra of all derivations of $\mathfrak{g}$, while $\text{ad}(\mathfrak{g})$ denotes the ideal of inner derivations. The quotient

$$\text{Out}(\mathfrak{g}) = \text{Der}(\mathfrak{g}) / \text{ad}(\mathfrak{g})$$

is the outer derivation algebra. Determining these spaces for non-semisimple Lie algebras is generally more subtle than in the semisimple case, where all derivations are inner by the classical Whitehead theorem.

In the present paper, we study the derivation algebra of the mass-extended conformal Galilei algebra $\widehat{\text{cga}}_\ell(d, \mathbb{C})$. The presence of the central element M introduces additional compatibility conditions for a derivation, since the bracket $[P_{n,i}, P_{m,j}]$ is no longer zero but is proportional to M . Consequently, a derivation must simultaneously preserve the action of the Levi factor, the Heisenberg-type structure of the radical, and the central extension. Our aim is to determine $\text{Der}(\widehat{\text{cga}}_\ell(d, \mathbb{C}))$ and to identify explicitly its inner and outer parts. The resulting description also clarifies how the mass central extension affects the derivation structure compared with the unextended conformal Galilei algebra.

Preliminaries

Let $\text{cga}_\ell(d, \mathbb{C})$ denote the conformal Galilei algebra in spatial dimension d . It has a Levi subalgebra

$$\mathfrak{s} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{so}(d, \mathbb{C})$$

and an abelian ideal

$$V = \text{span}\{P_{n,i} \mid 0 \leq n \leq 2\ell, 1 \leq i \leq d\}.$$

Thus, over a field of characteristic zero, one may write

$$\widehat{\text{cga}}_\ell(d, \mathbb{C}) = (\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{so}(d, \mathbb{C})) \ltimes V.$$

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As a module for $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{so}(d, \mathbb{C})$, the ideal V is naturally identified with

$$V \cong L(2\ell) \otimes \mathbb{C}^d,$$

where $L(2\ell)$ is the irreducible $\mathfrak{sl}_2(\mathbb{C})$ -module of highest weight 2ℓ and $\mathbb{C}^d$ is the natural $\mathfrak{so}(d, \mathbb{C})$ -module. In particular, the structure of $\mathfrak{cga}_\ell(d, \mathbb{C})$ combines a semisimple Levi factor with an abelian module, making these algebras non-semisimple and providing a natural setting for questions concerning their cohomology, deformations, central extensions, and derivations. This semidirect-product description is also useful from the representation-theoretic point of view.

Now, we presents the nontrivial brackets of $\mathfrak{cga}_\ell(d, \mathbb{C})$. Fix a basis $\{H, D, C, M_{ij}, P_{n,i}\}$ of $\mathfrak{cga}_\ell(d, \mathbb{C})$ where $1 \leq i, j \leq d$, $0 \leq n \leq 2\ell$. The generators H, D, C span a copy of $\mathfrak{sl}_2(\mathbb{C})$, while the elements M_{ij} span $\mathfrak{so}(d, \mathbb{C})$. The elements $P_{n,i}$ span the abelian ideal V . The we get the following nonzero brackets

$$\begin{aligned} [M_{ij}, M_{kr}] &= -\delta_{ik}M_{jr} - \delta_{jr}M_{ik} + \delta_{ir}M_{jk} + \delta_{jk}M_{ir}, \\ [D, H] &= 2H, \quad [C, H] = D, \quad [D, C] = -2C, \\ [M_{ij}, P_{n,k}] &= -\delta_{ik}P_{n,j} + \delta_{jk}P_{n,i}, \\ [H, P_{n,i}] &= -nP_{n-1,i}, \\ [D, P_{n,i}] &= 2(\ell - n)P_{n,i}, \\ [C, P_{n,i}] &= (2\ell - n)P_{n+1,i}. \end{aligned}$$

Generators outside the permitted range of the index n are understood to be zero.

In addition to their intrinsic Lie algebra structure, conformal Galilei algebras admit nontrivial one-dimensional central extensions for particular values of the parameters. Of particular importance is the so-called *mass central extension*. For every spatial dimension d and every half-integer ℓ , the abelian ideal V admits an invariant alternating bilinear form. Consequently, one may introduce a central element M and replace the trivial bracket on V by

$$[P_{n,i}, P_{m,j}] = \delta_{ij} \delta_{n+m, 2\ell} (-1)^{n+\ell+\frac{1}{2}} n!(2\ell - n)! M.$$

All other brackets remain unchanged, and M is central $[M, \widehat{\mathfrak{cga}}_\ell(d, \mathbb{C})] = 0$. The resulting Lie algebra will be denoted by $\widehat{\mathfrak{cga}}_\ell(d, \mathbb{C})$.

From the algebraic viewpoint, passing from $\mathfrak{cga}_\ell(d, \mathbb{C})$ to its mass extension changes an important structural feature: the ideal generated by the $P_{n,i}$ is no longer abelian. Instead, it becomes a Heisenberg-type algebra with one-dimensional center spanned by M . In particular, one can view the mass-extended algebra as

$$\widehat{\mathfrak{cga}}_\ell(d, \mathbb{C}) \cong (\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{so}(d, \mathbb{C})) \ltimes \mathfrak{h},$$

where $\mathfrak{h} = V \oplus \mathbb{C}M$ and the bracket on V is determined by the invariant symplectic form defining the central extension. This description is especially useful when studying automorphisms and derivations of the extended algebra.

The space of derivations of $\widehat{\mathfrak{cga}}_\ell(d, \mathbb{C})$

From now on, we use the following notations:

$$\mathfrak{g} = \mathfrak{cga}_\ell(d, \mathbb{C}), \quad \widehat{\mathfrak{g}} = \widehat{\mathfrak{cga}}_\ell(d, \mathbb{C}), \quad \mathfrak{s} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{so}(d, \mathbb{C}), \quad V = L(2\ell) \otimes \mathbb{C}^d, \quad W = V \oplus \mathbb{C}M.$$

Recall that a linear map $D : \mathfrak{a} \rightarrow \mathfrak{a}$ is a derivation if

$$D([x, y]) = [D(x), y] + [x, D(y)]$$

for all $x, y \in \mathfrak{a}$. The set of all derivations of $\mathfrak{a}$ forms a vector space and is denoted by $\text{Der}(\mathfrak{a})$. Moreover, $\text{Der}(\mathfrak{a})$ is a Lie algebra with respect to the commutator of linear operators. The operators ad_x defined as $\text{ad}_x(y) = [x, y]$ are derivations. Such derivations are called inner derivations. The set of inner derivations is an ideal of $\text{Der}(\mathfrak{a})$ and is denoted by $\text{ad}(\mathfrak{a})$. We now determine the derivations of $\widehat{\mathfrak{g}}$.

Lemma 2.1. Let $D \in \text{Der}(\widehat{\mathfrak{g}})$. Then $D(M) \in \mathbb{C}M$.

Proof. Let D be a derivation. Since M is central, we have

$$0 = D([M, x]) = [D(M), x]$$

for all $x \in \widehat{\mathfrak{cga}}_\ell(d, \mathbb{C})$. Therefore we obtain $D(M) \in Z(\widehat{\mathfrak{g}}) = \mathbb{C}M$. □

The next step is to remove the inner component of the restriction of D to $\mathfrak{s}$. Since $\mathfrak{s}$ is semisimple, the first Whitehead lemma gives

$$H^1(\mathfrak{s}, \mathfrak{s}) = 0, \quad H^1(\mathfrak{s}, V) = 0, \quad H^1(\mathfrak{s}, \mathbb{C}) = 0.$$

Consequently, after subtracting a suitable inner derivation, a derivation can be assumed to vanish on $\mathfrak{s}$.

Lemma 2.2. For every $D \in \text{Der}(\widehat{\mathfrak{g}})$ there exists $x \in \widehat{\mathfrak{g}}$ such that $D_0 = D - \text{ad}_x$ satisfies $D_0|_{\mathfrak{s}} = 0$.

Proof. Let us restrict D to $\mathfrak{s}$. Relative to $\widehat{\mathfrak{g}} = \mathfrak{s} \oplus V \oplus \mathbb{C}M$, the restriction has three components. The derivation identity for pairs of elements in $\mathfrak{s}$ shows that these components define 1-cocycles with coefficients in $\mathfrak{s}$, V , and the trivial module $\mathbb{C}$, respectively. By the Whitehead lemma all such 1-cocycles are coboundaries. Therefore the restriction $D|_{\mathfrak{s}}$ differs from an inner derivation by zero. Subtracting the corresponding inner derivation gives a derivation D_0 satisfying $D_0|_{\mathfrak{s}} = 0$. $\square$

Note that the radical of a Lie algebra is a characteristic ideal, i.e. any derivation maps the radical into itself (see [5]). Therefore any derivation D of $\widehat{\mathfrak{g}}$ induces a map $\widetilde{D} : W \rightarrow W$, where $W = V \oplus \mathbb{C}$.

Proposition 2.3. Suppose $D \in \text{Der}(\widehat{\mathfrak{g}})$ satisfies $D|_{\mathfrak{s}} = 0$. Then the induced map $\widetilde{D} : W \rightarrow W$ commutes with the $\mathfrak{s}$ -action.

Proof. For $x \in \mathfrak{s}$ and $v \in V$, we have

$$D([x, v]) = [D(x), v] + [x, D(v)] = [x, D(v)].$$

Since $[x, v] \in V$, this proves that $D(v)$ must transform under $\mathfrak{s}$ in the same way as v . The central component is allowed because M is an $\mathfrak{s}$ -trivial module. The noncentral component therefore satisfies $\widetilde{D}([x, v]) = [x, \widetilde{D}(v)] = x \cdot \widetilde{D}(v)$ for all $x \in \mathfrak{s}$ and $v \in V$. $\square$

Proposition 2.4. Let $D \in \text{Der}(\widehat{\mathfrak{g}})$, such that $D|_{\mathfrak{s}} = 0$. Then there exists $\lambda \in \mathbb{C}$ such that $D(v) = \lambda v$, ($v \in V$) and $D(M) = 2\lambda M$.

Proof. By the preceding proposition and Schur's lemma, we have

$$D(v) = av + \phi(v)M,$$

where $\phi : V \rightarrow \mathbb{C}$ is an $\mathfrak{s}$ -module homomorphism. Since V is a nontrivial irreducible $\mathfrak{s}$ -module, we get $\text{Hom}_{\mathfrak{s}}(V, \mathbb{C}) = 0$. Hence, we obtain $\phi = 0$. Therefore $D(v) = \lambda v$ for all $v \in V$. Now choose $v, w \in V$ for which $[v, w] \neq 0$. Since $[v, w] = \alpha M$, $\alpha \in \mathbb{C}$, the derivation identity gives

$$\begin{aligned} D([v, w]) &= [D(v), w] + [v, D(w)] \\ &= a[v, w] + a[v, w] = 2a[v, w]. \end{aligned}$$

On the other hand, $D([v, w]) = \alpha D(M)$. Because $\alpha \neq 0$, we obtain $D(M) = 2\lambda M$. $\square$

Now, we classify the derivations of $\widehat{\mathfrak{g}}$. Define a linear map $\tau : \widehat{\mathfrak{g}} \rightarrow \widehat{\mathfrak{g}}$ by

$$\tau|_{\mathfrak{s}} = 0, \quad \tau|_V = \frac{1}{2} \text{Id}_V, \quad \tau(M) = M. \quad (33)$$

In terms of the standard generators, τ acts on basis elements as follows

$$\tau(H) = 0, \quad \tau(D) = 0, \quad \tau(C) = 0, \quad \tau(M_{ij}) = 0, \quad \tau(P_{n,i}) = \frac{1}{2} P_{n,i}, \quad \tau(M) = M.$$

Proposition 2.5. The map τ is an outer derivation of $\widehat{\mathfrak{g}}$.

Proof. The derivation identity is immediate on $\mathfrak{s}$, since τ vanishes there. For $x \in \mathfrak{s}$, $v \in V$, we have

$$\tau([x, v]) = \frac{1}{2} [x, v]$$

while

$$[\tau(x), v] + [x, \tau(v)] = 0 + \frac{1}{2} [x, v] = \tau([x, v]).$$

Finally, for $v, w \in V$, we have

$$\tau([v, w]) = \tau(\beta M) = \beta M.$$

On the other hand,

$$[\tau(v), w] + [v, \tau(w)] = \frac{1}{2}[v, w] + \frac{1}{2}[v, w] = [v, w].$$

Thus the derivation identity holds.

Every inner derivation annihilates the center, i.e. $\text{ad}_x(M) = [x, M] = 0$ for all $x \in \mathfrak{g}$. However, we have $\tau(M) = M \neq 0$. Therefore $\tau \notin \text{ad}(\widehat{\mathfrak{g}})$. Hence τ is an outer derivation. $\square$

We can now state one of the main results of the present paper.

Theorem 2.6. Let $\ell \in \frac{1}{2} + \mathbb{N}_0$. Then for the mass-extended conformal Galilei algebra $\widehat{\mathfrak{g}}$, one has

$$\text{Der}(\widehat{\mathfrak{g}}) = \text{ad}(\widehat{\mathfrak{g}}) \oplus \mathbb{C}\tau. \quad (34)$$

Consequently, $\text{Out}(\widehat{\mathfrak{g}}) \cong \mathbb{C}$.

Proof. Let $D \in \text{Der}(\widehat{\mathfrak{g}})$. By the Lemma 2.2, there exists $x \in \widehat{\mathfrak{g}}$ such that $D_0 = D - \text{ad}_x$ vanishes on $\mathfrak{s}$. By the proposition above, we get $D_0(v) = \lambda v$ ($v \in V$) and $D_0(M) = 2\lambda M$. Thus we obtain $D_0 = 2\lambda\tau$. Hence $D = \text{ad}_x + 2\lambda\tau$. This proves $\text{Der}(\widehat{\mathfrak{g}}) = \text{ad}(\widehat{\mathfrak{g}}) + \mathbb{C}\tau$.

We claim that the sum is direct. Indeed, if $\text{ad}_x = c\tau$, then evaluating at M gives

$$0 = \text{ad}_x(M) = c\tau(M) = cM,$$

which implies $c = 0$. Since the center is $\mathbb{C}M$, from $\text{ad}_x = 0$ we obtain $x \in \mathbb{C}M$. Thus the intersection is zero. $\square$

Let $\mathcal{D} = \text{Der}(\widehat{\mathfrak{g}})$. be the Lie algebra of derivations of $\widehat{\mathfrak{g}}$. By the main theorem, we have $\mathcal{D} = \text{ad}(\widehat{\mathfrak{g}}) \oplus \mathbb{C}\tau$. Since $\ker(\text{ad}) = Z(\widehat{\mathfrak{g}}) = \mathbb{C}M$, we obtain $\text{ad}(\widehat{\mathfrak{g}}) \cong \widehat{\mathfrak{g}}/\mathbb{C}M$. Hence $\mathcal{D} \cong (\widehat{\mathfrak{g}}/\mathbb{C}M) \rtimes \mathbb{C}\tau$. On the other hand, we have $\mathfrak{g}_0 = \widehat{\mathfrak{g}}/\mathbb{C}M \cong \mathfrak{s} \ltimes V$, and the action of τ on $\mathfrak{g}_0$ is $[\tau, x] = 0$ ($x \in \mathfrak{s}$) and $[\tau, v] = \frac{1}{2}v$, $v \in V$. Thus

$$\mathcal{D} \cong (\mathfrak{s} \ltimes V) \rtimes \mathbb{C}\tau. \quad (35)$$

A Lie algebra $\mathfrak{a}$ is called *complete* if $Z(\mathfrak{a}) = 0$ and $\text{Der}(\mathfrak{a}) = \text{ad}(\mathfrak{a})$. A theorem of Su and Zhu states that the derivation algebra of a centerless perfect Lie algebra is complete.

Theorem 2.7.[1] Let $\mathfrak{a}$ be a centerless perfect Lie algebra over a field of arbitrary characteristic. Then $\text{Der}(\mathfrak{a})$ is complete.

We prove that Lie algebra of derivations of is complete. This result shows that Theorem 2.7 is true for perfect Lie algebras with trivial centers.

Theorem 2.8. The derivation algebra $\mathcal{D} = \text{Der}(\widehat{\mathfrak{g}})$ is complete.

Proof. The quotient $\mathfrak{g}_0 = \widehat{\mathfrak{g}}/Z(\widehat{\mathfrak{g}})$ is centerless and perfect. Moreover, the classification of derivations gives $\text{Der}(\widehat{\mathfrak{g}}) \cong \text{Der}(\mathfrak{g}_0)$. Under this identification, $\mathcal{D} \cong \text{Der}(\mathfrak{g}_0)$. By Theorem 2.7, $\text{Der}(\mathfrak{g}_0)$ is complete. Hence $\mathcal{D}$ is complete. $\square$

This theorem shows that the completeness of the Lie algebra of derivations holds for also perfect Lie algebras with nontrivial center.

Remark 2.9. The identification $\text{Der}(\widehat{\mathfrak{g}}) \cong \text{Der}(\mathfrak{g}_0)$ should not be interpreted as a formal consequence of quotienting by the center alone. The point is that, in the present situation, the derivation classification explicitly determines the lifting and shows that the derivation algebra is precisely the derivation algebra of the centerless quotient. This is the structural step needed before applying the Su–Zhu theorem.

Conclusion

We have determined the derivation algebra of the finite-dimensional mass-extended conformal Galilei algebra in the nonplanar cases. We proved that every derivation is the sum of an inner derivation and a scalar multiple of the distinguished outer derivation

$$\tau|_{\mathfrak{s}} = 0, \quad \tau|_V = \frac{1}{2}I_V, \quad \tau(M) = M.$$

Consequently, we obtained that $\text{Der}(\widehat{\mathfrak{g}}) = \text{ad}(\widehat{\mathfrak{g}}) \oplus \mathbb{C}\tau$, and $\text{Out}(\widehat{\mathfrak{g}}) \cong \mathbb{C}$. After factoring out the center, one obtains the centerless perfect Lie algebra $\mathfrak{g}_0 = \widehat{\mathfrak{g}}/Z(\widehat{\mathfrak{g}}) \cong \mathfrak{s} \ltimes V$. The completeness theorem of Su and Zhu then implies completeness of the derivation algebra. As a consequence, the derivation tower stabilizes at the first derivation algebra:

$$\widehat{\mathfrak{g}} \longrightarrow \text{Der}(\widehat{\mathfrak{g}}) \cong \text{Der}(\text{Der}(\widehat{\mathfrak{g}})) \cong \dots$$

The planar case $d = 2$ is structurally exceptional because $\mathfrak{so}_2(\mathbb{C})$ is abelian and the relevant module decomposes over $\mathbb{C}$. Therefore its derivation and derivation-tower structure requires a separate calculation rather than a direct application of the generic perfectness argument.

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HIGH-DIMENSIONAL DIAGONAL CLASSIFIERS

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ABSTRACT. The advancement in technology has necessitated the collection of massive high-dimensional data in bioinformatics, medical sciences, business, climatology, and magnetic resonance images. These collections have given rise to high-dimensional (p) and small sample(n) (HDSS) problems in group classifications. The classification coefficient of the HDSS data cannot be constructed due to the ill-conditioning or singularity of the covariance matrix. The curse of dimensionality suffered by the classical methods has rendered the formulation of the coefficient impossible as such different dimensionality reduction methods have been proposed to avert the curse of dimensionality and ill conditioning. Dimensionality reduction methods such as the principal component analysis have been applied as a prelude to reducing the dimension of the HDSS data to enhance constructing the coefficients of any classification methods. Some of these dimension reduction methods with infused constants as a benchmark selection, may increase or decrease computational time and improve classification accuracy at the cost of information loss. In this paper, we proposed two methods that can solve the ill-conditioning and singularity, or dimensionality problems associated to HDSS data directly. These methods are diagonal FLCM (DFLCM) and point extraction method (PEM). We investigated the robustness of these methods based on the optimal probability benchmark (OPB). The computational time (CPU time) of all the methods is also investigated. We applied three real data sets to determine the comparative performance, and the classification results revealed that the proposed DFLCM is very robust more than the Naïve Bayes classifier in terms of the probability of correct classification and CPU time. Therefore, this study affirmed that the OPB is suitable to determine the robustness of HDSS classification problems.

MSC (2020): 62H30, 62H12, 62R07.

Key words: High-dimensional data, Classification, Probability, CPU time, Robustness.

Introduction

Technological advancement has generated massive high-dimensional data set originating from health sciences, genetics, business, astronomy, magnetic resonance images, epidemiology, climatology, seismological study, chromatography, ecology [1, 2, 3 4], herbal medicine [5], bioinformatics [6] finance [7] tumour classification [8] and functional magnetic resonance images [9]. The availability and applications of these categories of data have rendered the classical classification methods unsuitable for computational and analytical purposes [10, 11]. As such, many of these classical methods have been modified and new classification methods have emerged to cope with the increasing data dimension (p) and small sample (n) otherwise referred to as HDSS ($p > n$). One of such method that could not cope with $p > n$ is the Fisher linear classification (FLCM) [3, 12]. Thus, problems are often encountered when the data dimension is greater than the sample size as such these classification techniques break down [13, 14]. For the HDSS problem, the numerical values of some of the variables do not contribute meaningfully to improve classification results, such variables decrease the proportion of correct classification of any methods. To improve the classification accuracy for the HDSS problems, researchers tend to find data reduction techniques to extract useful information from the data set thereby reducing the dimension of the data.

The principal component analysis [15, 16, 17, 18] is an established data dimension reduction technique that has been applied to extract useful information from HDSS problems. Variants of principal component analysis (PCA) for dimension reduction have been discussed [19, 20]. Several other regularized discriminant analyses methods have been proposed to solve HDSS for two groups classification problems [21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. The nearest shrunken centroids [31, 32] sparse partial least square discriminant analysis [33] are dimension reducing techniques to solve HDSS problems.

Other alternative methods such as the independent classification rule (ICR), diagonal linear classification (DLC), and diagonal quadratic classification (DQC) methods were proposed such that the diagonal of the covariance matrix is applied to construct the coefficient of the classification methods [3, 34]. These methods avert the curse of singularity or

dimensionality [6, 35] and the ill conditioning of the covariance matrix. Feature annealed independence rule (FAIR) was proposed as an alternative version of ICR [13, 14]. Besides the FAIR, the sure independence screening (SIS) was also proposed for dimensionality reduction as a step to applying other well-known lower-dimensional classification techniques to determine class separation [8].

Often the case, the above dimension reduction methods may result in information loss as such the classification accuracy of these methods may lack the exact information the data set would naturally portray [36]. The application of the HDSS data on the classical classification methods to perform classification task is impossible due to the difficulty in constructing the classification coefficient resulting from the ill conditioning of the covariance matrix. In this paper, we propose DFLCM and point extraction method (PEM), the DFLCM is a diagonal modification of the FLCM while the PEM extract vital data point from each sample variables based on group probabilities. The objective of the proposed methods is to avert the curse of singularity and ill conditioning of the HDSS data in other to construct the classification coefficient. The PEM shows the proportion of sparsity and density of the data matrix, as such, we can determine the overall proportion of the data points associated with the variables selected from each sample to construct the model. The structure of PEM allows us to determine the group probabilities and the extraction probabilities within each group. The PEM can be described as a method that extracts vital information based on within-group probability from each sample based on the numerical strength of the corresponding variables. This method adopts and utilizes vital data points from the entire data set to construct the classification coefficient via probability benchmark comparison. The PEM does not rely on the information from the covariance matrix to construct the classification coefficient for the HDSS problem. The PEM modulus operandis is described as follows; different samples with corresponding variables have different numerical strengths in the data matrix, as such deleting the entire data points associated with a variable corresponding to the samples may generate numerical differential in the data matrix. Therefore, PEM considers each variable data point as relevant until the benchmark isolates the irrelevant data points from the data matrix. Thus, the relevant data points are applied to construct the model. This can be expressed as; data points in group 1 may provide vital information(relevant) in variables 2,6,8,11,21, and group 2 may contain vital information in variables 1,4, 5,16,27,32,45 and so on for $p > n$ problem, this is illustrated in Table 1. We also considered the modification of the independent classification rule. The modified ICR is called the adjusted ICR (AICR). To give credence to the different methods, we applied real data set to investigate the comparative performance of the proposed methods, the ICR and the AICR, respectively. To determine the robustness of these methods, we applied the optimal probability benchmark to analyze their comparative performance.

This paper is structured as follows. In Section 2, we discussed the diagonal classifier (DFLCM), independence classification rule (ICR), adjusted independence classification (AICR), and finally in this section, we described the proposed point extraction method (PEM). Real data application, results, and discussion are presented in Section 3 followed by a concluding remark in Section 4.

Method

For the HDSS problem, developing the coefficient of the classical methods such as the FLCM becomes impractical to perform classification tasks due to the curse of singularity, dimensionality, and ill conditioning of the covariance matrix[37, 38]. In this section, we will navigate different procedures to perform classification for $p > n$ problems.

Diagonal FLCM (DFLCM)

Let X_{ij} and Y_{ij} such that $j > i$ are independent random samples from two different groups where n and p denote sample size and dimension. We assumed that the two groups have equal sample sizes, that is $n_k, k = 1, 2, i = 1, 2, \dots, n_k, n = \sum_{k=1}^2 n_k$. Let $p_k = \frac{n_k}{N}$, such that $\sum_{k=1}^2 p_k = 1$ be the priori probability from the respective groups.

$$\text{Let } \bar{X}_k = \frac{1}{n_j} \sum_{j=1}^p X_j, j = 1, \dots, p$$

$$S_k^2 = \frac{1}{n_j - 1} \sum_{i=1}^{n_k} (X_{ij} - \bar{X}_k)^2$$

$$S_p = \sum_{c=1}^{n_k} \frac{1}{(n_c - k)} (n_k - 1) S_k^2, c = k = 1, 2 \quad (1)$$

be the group mean vectors (X_k), covariance matrices (S_k^2) and pooled sampled covariance matrix (S_p), respectively. Computing the inverse of Equation (1) is difficult since $p > n$. To construct the model coefficient, we must use the diagonal of Equation (1), that is,

$$\hat{S} = \text{diag}(S_p) \quad (2)$$

Equation (2) allows us to solve the problem of dimensionality. Hence the model coefficient can be constructed as

$$\Delta_k = (\bar{X}_1 - \bar{X}_2) \hat{S}^{-1} X_{k,ij} \quad (3)$$

Equation (3) mimics the independence classification rule discussed in [13, 14]. To perform the classification task, we adopt a similar classification procedure discussed in [39]. Therefore, the decision rule is defined as; assign an object

to group one (G_1) if $\Delta_1 \geq \Delta_2$ otherwise, assign the object to group two $\Delta_1 < \Delta_2$. This process can be summarized as follow

$$\gamma = \begin{cases} X_{1j} \in G_1, \text{ if } \Delta_1 \geq \Delta_2 \\ \text{Otherwise } X_{1j} \in G_1, \text{ if } \Delta_1, \Delta_2 \end{cases}$$

The DFLCM is a blended concept of the FLCM, ICR, and predictive classifiers models.

Independence classification rule (ICR)

Since the coefficient of the FLCM could not be constructed for $p > n$, a unique distance classification measure called the independence classification rule (ICR) was proposed [14] whereby the diagonal of the covariance matrix was used to construct the coefficient of ICR. Due to the diagonal use, it is expected that the ICR would avert the curse of singularity. This method relies on the group mean vectors and the diagonal of the covariance matrix to construct the coefficient of the model. This is described as follows

$$\begin{aligned} \bar{X}_c &= \frac{1}{n_c} \sum_{i=1}^{n_c} X_{ci}, & \bar{\bar{X}} &= \frac{1}{k} \sum_{c=1}^k \bar{X}_c \\ S_c^2 &= \frac{1}{n_c-1} \sum_{i=1}^{n_c} (X_i - \bar{X}_c)^2 \\ S_p &= \frac{1}{\sum_{c=1}^k (n_c - k)} (n_c - 1) S_c^2, c = 1, \dots, k, i = 1, \dots, n_c \\ \hat{S} &= \text{diag}(S_p) \end{aligned} \quad (4)$$

Therefore, the ICR [3, 14] for discrimination and classification is given as

$$W_L = (\bar{X}_1 - \bar{X}_2) \hat{S}^{-1} (X_j - \bar{\bar{X}}) = \gamma d \quad (5)$$

where $\gamma = \bar{X}_2 \hat{S}^{-1}$, and $d = (X_j - \bar{\bar{X}})$.

By this rule, an object is assigned to G_1 if $W_L \geq 0$ otherwise assign to G_2 if $W_L < 0$. The ICR is very flexible, and its performance depends on the group data deviation (d) from the ground mean vector $\bar{\bar{X}}$. The independent rule method has a similar model coefficient (γ) with the Fisher linear classification method. Taking the absolute value of d may lead to a higher misclassification rate. Therefore, the strength of the independence classification rule depends on the signs of d . The ICR is also called the Naïve Baye classifier.

Adjusted ICR (AICR)

For the AICR, we require the definition of the group probabilities for the classification benchmark. Let $\pi = \sum_{c=1}^k \pi_c = 1$, be the overall probability and π_c be the group probabilities. The adjusted ICR is a direct derivation of the ICR with slight modification to the classification benchmark. The AICR is replicated from Equation (5) as

$$\hat{W}_L = (\bar{X}_1 - \bar{X}_2) \hat{S}^{-1} (X_j - \bar{\bar{X}}) = \gamma d \quad (6)$$

Therefore, the classification rule for the adjusted ICR is defined as assign X_1 to G_1 if $\hat{W}_L \geq \log(\pi)!$ otherwise, assign X_1 to G_2 if $\hat{W}_L < \log(\pi)!$ where $!$ denotes the factorial symbol.

Point extraction method (PEM)

This method applies within-group probability to extract relevant data points with a strong numerical value from each variable in each group to construct the classification coefficients. Based on the extraction, we can determine the sparsity and density of the data matrix. This allows us to determine the relevant data points used to build the model. The PEM can be described in detail as follow.

Let X_{ij} and Y_{ij} be $p > n$ be as defined above, $M = p$, let

$$P(X_i) = \frac{X_{ij}}{\sum_{i=1}^n X_{ij}} \quad (7)$$

$$P(Y_i) = \frac{Y_{ij}}{\sum_{i=1}^n Y_{ij}} \quad (8)$$

denote the first phase probability transformation for the two groups. Then the second phase is given as

$$C_X = \frac{P(X_i)}{M} \quad (9)$$

$$C_Y = \frac{P(Y_i)}{M} \quad (10)$$

Based on Equations (7-10), we can isolate the weight of the irrelevant data points from the data matrix for the two groups as follows.

$$W_X = \begin{cases} 1, C_X > P(X_i) \\ 0, C_X < P(X_i) \end{cases} \quad (11)$$

$$W_Y = \begin{cases} 1, C_Y > P(Y_i) \\ 0, C_Y < P(Y_i) \end{cases} \quad (12)$$

Therefore, Equation (11-12) contains the weight of the relevant data points. Equations (13-14) consist of the weighted relevant data points associated with each group and variables,

$$e_X = W_X P(X_i) \quad (13)$$

$$e_Y = W_Y P(Y_i) \quad (14)$$

Table 1: Point extraction matrix (Equation (13-14)) ($p > n$)

	X_1	X_2	X_3	X_4	X_5	X_6	X_7		X_p
1	w	w	0	w	0	0	w		w
2	0	0	w	0	w	w	0		0
3	w	0	w	0	w	0	w		w
:	:	:	:	:	:	:	:		:
n	0	w	0	w	w	w	w		w

The PEM retains the original dimension of the data matrix due to the zigzag relevant numerical strength associated with each sample and variables. It preserves all the variables based on the sample size information. The PEM cannot be described as a data reduction method rather a data point extraction method. Table 1 described Equations (13-14), respectively. Equations (13-14) is transformed to $n \times p$, from the above we have $n \times n$ transformed matrix form, that is

$$E_{n \times n} = e_X e_Y$$

$$\delta = \sum E_{n \times n}$$

$$\alpha = \frac{E_{n \times n}}{\delta}$$

$$\Delta = \frac{(E_{n \times n} - \alpha)(E_{n \times n} - \alpha)^T}{\delta}$$

$$\nabla_W = \frac{e_X}{p} - \frac{e_Y}{p}$$

$$W = \left(\frac{e_X}{p} - \frac{e_Y}{p} \right) \frac{(E_{n \times n} - \alpha)(E_{n \times n} - \alpha)^T}{\delta} \quad (15)$$

Equation (15) is the coefficient of the PEM, where W is $1 \times p$. The benchmark is given as

$$\beta = \frac{\phi}{k'} \quad (16)$$

$$\text{where } \phi = \frac{e_X}{p} + \frac{e_Y}{p}, \gamma_a = W e_X, \gamma_b = W e_Y.$$

Therefore, the decision rule is to assign an object to group one (G_1) if $\gamma_a \geq \beta$ otherwise assign an object to group two (G_2) if $\gamma_a < \beta$. Therefore, the classification summary is

$$\theta = \begin{cases} \gamma_a \geq \beta, & \text{if } \gamma_a \in G_1 \\ \gamma_b < \beta, & \text{if } \gamma_b \in G_2 \end{cases}$$

The optimal performance benchmark for high dimensional data can be determined by using the following formula. The probability of misclassification ($\mathfrak{M}$) is defined as

$$\mathfrak{M} = \left[\frac{(1-\tau)}{2 \times \tau} \right] \times \tau$$

where

$$\tau = \left(\frac{\sum_{i=1}^{n_1} \gamma_{ai} \in \Delta_1 + \sum_{i=1}^{n_2} \gamma_{bi} \in \Delta_2}{n_1 + n_2} \right) \quad (17)$$

Equation (17) explain the probability of the group membership being correctly classified and the proportion of membership prediction accuracy (ϕ) is defined as

$$\phi = \cap - \mathfrak{M} \quad (18)$$

where $\cap = 1$ denotes the optimal performance benchmark. The robustness of the high dimensional data can be evaluated using Equation (18).

Results and discussion

For this section, all data sets are uniformly reshuffled and divided into training 60% and testing 40% and seeded 123456. The results reported are based on the average classification accuracy over 1000 runs. Three real data set are considered viz: Musk version 2 data set [40,41 42, 43] sonar rock signal [44] and ozone level data set [45]. Equation (18) is applied to determine the robustness of any of these methods.

Applications

Musk version 2 data set.

The musk and non-musk main data set consist of $n=6598, p=166$, [43]. For $p > n$ problem we took $n=102$, and $p=166$. Previous studies [40, 41, 42] used this data set for $n > p$ problems. The neural network method achieved the highest classification accuracy of 91% [40] and the lowest classification accuracy was reported in [40] with 58.8% based on C4.5 prune. The objective is to determine whether the new molecules will be classified as musk and non-musk. The proportion of membership prediction accuracy (ϕ), and CPU time is reported in Table 2. The result reported in Table 2 showed that the DFLCM, ICR, and AICR performed comparably. Fig. 1 shows the CPU time with the AICR having the smallest CPU time followed by DFLCM, respectively.

Table 2: Performance analysis of the different methods

Methods	ϕ	CPU time
DFLCM	1.000	6.93
ICR	0.999	7.20
AICR	0.999	6.89
PEM	0.941	8.59

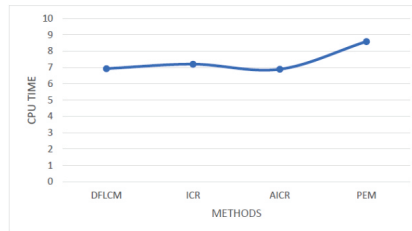


Fig 1: Comparative analysis of CPU time based on methods

Mines vs rock sonar signal

The objective is to classify sonar signal as bounced off a metal cylinder (M) and bounced off a roughly cylindrical rock (R) [44]. In Gorman and Sejnowski (1988), they used neural network techniques and nearest neighbor classifier and it was applied for $n > p$ problems with classification accuracy 90.4% and 82.7% based on the training set ($n=104$). For the $p > n$ problem, we used $p = 60, n = 50$. Table 3 contain the proportion of membership prediction accuracy of the different methods over 1000 runs. The DFLCM appeared to be the best classifier which is much better than the 90.4% [44] although with different study conditions. The ICR and AICR are more robust than 82.7%. For this data set, the DFLCM has better performance for both accuracy and time. Fig.2 depicts the comparative CPU time between DFLCM and AICR for the different methods.

Table 3: Performance analysis of the different methods

Methods	ϕ	CPU time
DFLCM	0.956	0.68
ICR	0.907	0.72
AICR	0.911	0.67
PEM	0.735	0.81

Ozone level detection

This data was previously used to forecast ozone days. It was reported that the data sets contain many irrelevant variables and sample bias[45]. This led us to apply this data set for $p > n$ problem. The original data set [45] contains $n = 2536, p = 72$ with group one ozone day and group two normal day. For $p > n$ problem, we elect to retain the dimension (p) and selected $n = 66$. Based on this, we applied these methods to classify ozone days and non-ozone days. Table 4 contains the classification results of the different methods with DFLCM as the best method followed by the ICR and AICR method. Fig.3 contains the CPU time of the different methods, the analysis showed that DFLCM appeared to have the fastest computational time followed by the ICR method.

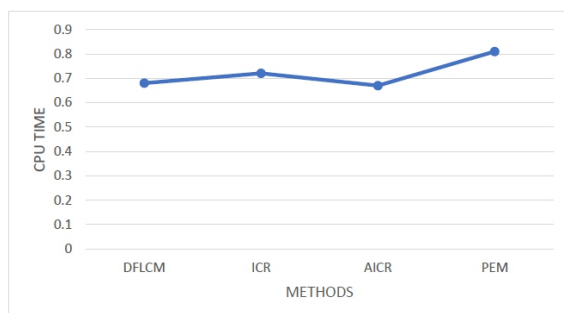


Fig 2: Comparative analysis of CPU time based on methods

Table 4: Performance analysis of the different methods

Methods	ϕ	CPU time
DFLCM	0.993	0.98
ICR	0.973	1.04
AICR	0.973	1.07
PEM	0.921	1.25

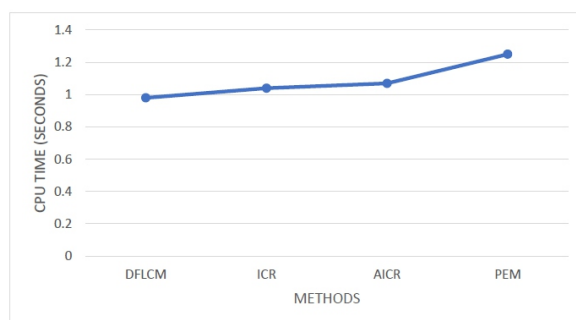


Fig 3: Comparative analysis of CPU time based on methods

It is vital to state that the performance of any classification method strictly relies on the nature of the dataset [6]. This implies that a particular model may result in minimum misclassification error due to certain dataset, also this model may result in high misclassification error for another data set. Based on this conviction, it is important for a researcher to consider a suitable model and data set to avoid the generalization of the under performance of a particular classification model. In this regard, model under performance should be investigated vigorously with different data set before generalization, or the results should be accompanied by detailed discussions. In general, no classification model performs excellently with any available data set hence model performance is data dependent. It is observed that the problems associated with $p > n$ are high computational cost and high misclassification error [8, 46, 47] due to irrelevant variables. In comparative terms, it implies that a low dimensional data set may result in a minimum computational cost and a better classification rate otherwise. In this study, we have observed that the PEM method is comparably suitable to do comparative performance analysis with the diagonal classifiers. Table 5 show the number of relevant (bold) and irrelevant data extracted for all the data set. The number in italics indicates the total data points in the data matrix. In future work, we will compare the PEM methods with dimension reduction techniques and variable selection techniques to determine further performance analysis.

Table 5: Number of selected data points (bold) by PEM

Data 1	$2 \times (p \times n)$	Relevant point by PEM	irrelevant point by PEM	Δ_1 relevant	Δ_2 relevant	Δ_1 irrelevant	Δ_2 irrelevant
Training (60%)	20,318.4	10,204	10,114.4	5,297	4,907	4,862.2	5,252.2
Testing (40%)	13,545.6	8,056	5,489.6	4,189	3,867	2,583.8	2,905.8
Total	33,864	18,260	15,604	9,486	8,774	7,446	8,158
Data 2	$2 \times (p \times n)$	Relevant point by PEM	irrelevant point by PEM	Δ_1 relevant	Δ_2 relevant	Δ_1 irrelevant	Δ_2 irrelevant
Training (60%)	3,600	2,010	1,590	1,012	998	788	802
Testing (40%)	2,400	1,611	789	820	791	380	409
Total	6,000	3,621	2,379	1,832	1,789	1,168	1,211
Data 3	$2 \times (p \times n)$	Relevant point by PEM	irrelevant point by PEM	Δ_1 relevant	Δ_2 relevant	Δ_1 irrelevant	Δ_2 irrelevant
Training (60%)	5,328	5,032	296	2,516	2,516	148	148
Testing (40%)	4,176	3,944	232	1,972	1,972	116	116
Total	9,504	8,976	528	4,488	4,488	264	264

$\Delta_i, i = 1, 2$ denotes groups.

Conclusions

Three diagonal classifiers for HDSS problems were considered for this paper. The point extraction method was also discussed. Three real data applications taken from the UCI machine learning repository were used with sample size modifications to suit HDSS problems. The results have shown that the proposed DFLCM methods appeared to be the best in terms of proportion of membership prediction accuracy and computational time (CPU time). The modified AICR method showed comparative performance followed by the conventional ICR and PEM in that order for these data sets. The analysis has demonstrated that the DFLCM is very capable to perform HDSS classification tasks based on these data set. It also indicated fast CPU computational time compared to the other method. We remark that the DFLCM and the other methods can be applied to investigate any HDSS problems with high-speed computers. This study concludes that these methods are suitable and comparable to carry out classification analysis based on the application of HDSS data.

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CRITICAL CUSP CANCELLATION AND ANALYTIC RESPONSE FOR A BERNOULLI RANDOM INTERVAL MAP

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ABSTRACT. We show that the critical cusp exponent $\beta = -1/2$ admits $W^{2,1}$ invariant densities in an explicit family of expanding full-branch interval maps. The usual estimates exclude this exponent because two zero-order coefficients in the second derivative of the transfer operator are individually non-integrable. Their signed combination, however, contains the factor $\beta(2\beta + 1)$ for a pure power cusp. We exploit its cancellation at $\beta = -1/2$ to prove a $W^{2,1}$ Lasota–Yorke inequality and a spectral gap for the explicit family. We also identify a separate obstruction caused by an interior cusp value. As an application, a small Bernoulli probability perturbation between two fixed maps has a stationary density that depends real-analytically on the perturbation parameter in $W^{2,1}$, with explicit formulas for all response derivatives.

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Key words: cusp maps, transfer operators, invariant density, Lasota–Yorke inequality, analytic response, random dynamical systems.

Introduction

Invariant and stationary densities describe the statistical behaviour of a dynamical system. For a density h_ε and an observable φ , the stationary expectation is $\int \varphi h_\varepsilon dx$. How does it change under a small perturbation? Statistical stability asks for continuity, whereas linear response asks for a first-order expansion. The latter allows perturbed averages to be approximated from the unperturbed density and its response, and quantifies the sensitivity of long-term statistics.

Uniform expansion alone does not ensure linear response for interval maps. Perturbations of tent-like maps changing the turning-point value can destroy differentiability when the one-sided slopes are finite [9]. Bahsoun and Galatolo [1] showed that an unbounded cusp derivative can instead restore linear response, because the vanishing inverse-derivative weights improve the regularity of the transfer operator. Oltiboev [4] obtained an L^p version for $p > 1$. Oltiboev, Rakhimov and Ruziboev [5] considered random compositions with a common cusp point and value. For $-1 < \beta < -1/2$, they obtained $W^{2,1}$ stationary densities and response in $W^{1,1}$ under perturbations of the probability law, following the viewpoint of [7].

The present note is motivated by the strict upper endpoint in [5]. Its second-order estimate bounds two singular zero-order terms separately by $t^{-2\beta-2}$, where t is the distance to the cusp. This weight is integrable for $\beta < -1/2$, but becomes t^{-1} at the endpoint. Does second-order regularity actually fail there, or do these estimates discard an essential cancellation? The divergence of a sufficient bound alone does not answer this question.

We answer it for an explicit full-branch family. The signed pure-power coefficient contains $\beta(2\beta + 1)$, which vanishes at $\beta = -1/2$. Adding a constant to the critical power in the branch derivative leaves an integrable remainder and yields a $W^{2,1}$ Lasota–Yorke inequality and a spectral gap. The family is a critical specialization of the unperturbed model of [5], also related to the model of [4]. The contribution is the endpoint analysis, rather than an unrelated construction.

An independent boundary issue remains. In [5], the first derivative of the transferred function has zero trace at the common cusp value. At the endpoint that trace can be nonzero, even if the cusp value is fixed. Its zero extension across an interior cusp value may then fail to belong to $W^{2,1}$. We avoid this obstruction by placing the cusp value at 1.

Finally, we vary the Bernoulli probabilities of two fixed maps. The operator is affine in the parameter and bounded on the same $W^{2,1}$ space, so the spectral gap and standard perturbation theory [3] give analytic response. The stronger response topology reflects this restricted perturbation: we do not extend the full deterministic or general random-law theorems of [4, 5] to the critical exponent.

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The critical coefficient

Let $I = [0, 1]$. Suppose that $f : I \rightarrow I$ has finitely many monotone branches, each C^1 with nonzero derivative in its interior. Its transfer operator is given, almost everywhere, by

$$(L_f \phi)(x) = \sum_{y \in f^{-1}(x)} \frac{\phi(y)}{|f'(y)|}.$$

It is positive, preserves Lebesgue integrals, and is an $L^1(I)$ contraction. An invariant probability density is a nonnegative fixed point with integral one. The Sobolev spaces used below are taken on $(0, 1)$, with no boundary conditions, and have norms $\|\phi\|_{W^{k,1}} = \sum_{j=0}^k \|\phi^{(j)}\|_1$, for $k = 1, 2$.

When the branches are C^3 in their interiors, the zero-order coefficient in the second derivative of $L_f \phi$, away from endpoints of branch images, is

$$D_f := 3 \frac{(f'')^2}{(f')^4} - \frac{f'''}{(f')^3}. \quad (1)$$

In the second-order estimate of [5], its absolute value is controlled by $3|f''|^2/|f'|^4 + |f'''|/|f'|^3 = O(t^{-2\beta-2})$. At the endpoint we must retain the signed combination in (1) instead.

Lemma 1. *Let c be the turning point and $t = |x - c|$. Suppose that the derivative magnitude on one branch is $p(t) = Kt^\beta$, where $K > 0$ and $\beta < 0$. Then, on either branch,*

$$D_f = 3 \frac{(p')^2}{p^4} - \frac{p''}{p^3} = \frac{\beta(2\beta+1)}{K^2} t^{-2\beta-2}. \quad (2)$$

In particular, this coefficient vanishes identically when $\beta = -1/2$.

Proof. On the increasing branch, $t = c - x$ and $(f', f'', f''') = (p, -p', p'')$. On the decreasing branch, $t = x - c$ and $(f', f'', f''') = (-p, -p', -p'')$. Both substitutions give the first equality in (2). Since $p' = K\beta t^{\beta-1}$ and $p'' = K\beta(\beta-1)t^{\beta-2}$, the second equality follows from $3\beta^2 - \beta(\beta-1) = \beta(2\beta+1)$.

An explicit critical cusp family

The unperturbed family in [5, Section 4] can be written, with $c = 1/2$, as

$$F_{\beta,\vartheta}(c \pm t) = 1 - \vartheta t - (1 - \vartheta/2)(2t)^{\beta+1}, \quad 0 \leq t \leq \frac{1}{2}.$$

Its specialization at $\beta = -1/2$ gives the square-root family below. For $\vartheta = 3/2$, it also agrees with the unperturbed formula in [4] at root parameter $k = 2$. These algebraic identities motivate the example; neither earlier theorem covers this critical case.

The derivative magnitude is a constant plus $Kt^{-1/2}$. Integrating from a cusp of height 1 and imposing zero endpoint values fixes K . Specifically, choose $1 < \vartheta < 2$ and set

$$K_\vartheta := \frac{1 - \vartheta/2}{\sqrt{2}} > 0, \quad f_\vartheta(c \pm t) := 1 - \vartheta t - 2K_\vartheta \sqrt{t}, \quad 0 \leq t \leq \frac{1}{2}. \quad (3)$$

Then $f_\vartheta(0) = f_\vartheta(1) = 0$ and $f_\vartheta(c) = 1$. For $t = |x - c| > 0$,

$$|f'_\vartheta(x)| = p_\vartheta(t) := \vartheta + K_\vartheta t^{-1/2}, \quad \inf_{x \neq c} |f'_\vartheta(x)| = 1 + \frac{\vartheta}{2} =: \lambda_\vartheta > 1. \quad (4)$$

Thus f_ϑ has two full branches, is uniformly expanding, and has cusp exponent $\beta = -1/2$. Write $\lambda_\vartheta := \lambda_\vartheta^{-1} < 1$.

The inverse branches are also explicit. For $x \in I$, define

$$u_\vartheta(x) := \frac{\sqrt{K_\vartheta^2 + \vartheta(1-x)} - K_\vartheta}{\vartheta}. \quad (5)$$

Then $g_{0,\vartheta}(x) = c - u_\vartheta(x)^2$ and $g_{1,\vartheta}(x) = c + u_\vartheta(x)^2$. Consequently,

$$(L_\vartheta \phi)(x) = \frac{u_\vartheta(x)}{K_\vartheta + \vartheta u_\vartheta(x)} [\phi(c - u_\vartheta(x)^2) + \phi(c + u_\vartheta(x)^2)]. \quad (6)$$

Here and below $L_\vartheta = L_{f_\vartheta}$.

Lemma 2. For the family (3), on either branch with $t = |x - c| > 0$,

$$D_{\vartheta}(t) := 3 \frac{(f_{\vartheta}'')^2}{(f_{\vartheta}')^4} - \frac{f_{\vartheta}'''}{(f_{\vartheta}')^3} = -\frac{3K_{\vartheta}\vartheta}{4\sqrt{t}(K_{\vartheta} + \vartheta\sqrt{t})^4}, \quad (7)$$

$$\frac{|f_{\vartheta}''|}{|f_{\vartheta}'|^2} = \frac{K_{\vartheta}}{2\sqrt{t}(K_{\vartheta} + \vartheta\sqrt{t})^2}, \quad \left| \frac{f_{\vartheta}'''}{(f_{\vartheta}')^3} \right| = \frac{K_{\vartheta}}{2(K_{\vartheta} + \vartheta\sqrt{t})^3}. \quad (8)$$

The coefficients in (7) and the first part of (8) belong to $L^1(0, 1/2)$, and the last coefficient is bounded.

Proof. From (4), $p'_{\vartheta} = -(K_{\vartheta}/2)t^{-3/2}$ and $p''_{\vartheta} = (3K_{\vartheta}/4)t^{-5/2}$. Substituting these expressions and $p_{\vartheta} = t^{-1/2}(K_{\vartheta} + \vartheta\sqrt{t})$ into $3(p'_{\vartheta})^2/p_{\vartheta}^4 - p''_{\vartheta}/p_{\vartheta}^3$ gives (7). The same substitution gives (8). Since $K_{\vartheta} > 0$, the first two coefficients are $O(t^{-1/2})$ and the last one is $O(1)$.

$W^{2,1}$ regularity at the endpoint

The estimate below follows the Lasota–Yorke strategy [10] in a Sobolev setting. The cancellation supplies the missing second-order integrability. There is also a first-order change from [5]: at the critical exponent the coefficient $|f_{\vartheta}''|/|f_{\vartheta}'|^2$ is in L^1 but not in L^2 , so the $(W^{1,1}, L^2)$ estimate used there is not imported. We instead use $W^{1,1}(I) \hookrightarrow L^{\infty}(I)$ to bound the lower-order terms, and obtain quasi-compactness directly from the second-order estimate.

Proposition 1. For each $\vartheta \in (1, 2)$, the operator L_{ϑ} is bounded on $W^{1,1}(I)$ and $W^{2,1}(I)$. There exists $C_{\vartheta} > 0$ such that

$$\|L_{\vartheta}\phi\|_{W^{2,1}} \leq \lambda_{\vartheta}^2 \|\phi\|_{W^{2,1}} + C_{\vartheta} \|\phi\|_{W^{1,1}}, \quad \phi \in W^{2,1}(I). \quad (9)$$

Moreover, the essential spectral radius of L_{ϑ} on $W^{2,1}(I)$ is at most λ_{ϑ}^2 .

Proof. For smooth ϕ , differentiation of (6), or of the inverse-branch formula, gives

$$(L_{\vartheta}\phi)' = L_{\vartheta} \left(\frac{\phi'}{f_{\vartheta}'} \right) - L_{\vartheta} \left(\frac{f_{\vartheta}''}{(f_{\vartheta}')^2} \phi \right) \quad (10)$$

and

$$\begin{aligned} (L_{\vartheta}\phi)'' &= L_{\vartheta} \left(\frac{\phi''}{(f_{\vartheta}')^2} \right) - 3L_{\vartheta} \left(\frac{f_{\vartheta}''}{(f_{\vartheta}')^3} \phi' \right) \\ &\quad + L_{\vartheta} \left(\left[3 \frac{(f_{\vartheta}'')^2}{(f_{\vartheta}')^4} - \frac{f_{\vartheta}'''}{(f_{\vartheta}')^3} \right] \phi \right). \end{aligned} \quad (11)$$

Both branch images are I , so no characteristic function with an interior endpoint occurs. The right-hand sides are integrable by Lemma 2; hence these classical formulas on $(0, 1)$ also give the weak derivatives there. In particular, the two terms in the final coefficient of (11) are kept together.

Using (4), Lemma 2 and the L^1 contraction property, we obtain

$$\begin{aligned} \|(L_{\vartheta}\phi)'\|_1 &\leq \lambda_{\vartheta} \|\phi'\|_1 + C \|\phi\|_{\infty} \leq C \|\phi\|_{W^{1,1}}, \\ \|(L_{\vartheta}\phi)''\|_1 &\leq \lambda_{\vartheta}^2 \|\phi''\|_1 + C \|\phi'\|_1 + 2\|D_{\vartheta}\|_{L^1(0,1/2)} \|\phi\|_{\infty} \leq \lambda_{\vartheta}^2 \|\phi''\|_1 + C \|\phi\|_{W^{1,1}}. \end{aligned}$$

Together with $\|L_{\vartheta}\phi\|_1 \leq \|\phi\|_1$, these estimates prove (9). Smooth approximation establishes the stated boundedness and inequality on the Sobolev spaces.

For the essential spectral radius, use the interpolation inequality $\|\phi\|_{W^{1,1}} \leq \eta \|\phi\|_{W^{2,1}} + C_{\eta} \|\phi\|_1$, valid for every $\eta > 0$. For each $b \in (\lambda_{\vartheta}^2, 1)$, (9) therefore gives $\|L_{\vartheta}\phi\|_{W^{2,1}} \leq b \|\phi\|_{W^{2,1}} + C_b \|\phi\|_1$. Iteration and L^1 contraction yield

$$\|L_{\vartheta}^n \phi\|_{W^{2,1}} \leq b^n \|\phi\|_{W^{2,1}} + \frac{C_b}{1-b} \|\phi\|_1. \quad (12)$$

The compact embedding $W^{2,1}(I) \Subset L^1(I)$ and Hennion's criterion [2] give $r_{\text{ess}} \leq b$. Letting $b \downarrow \lambda_{\vartheta}^2$ proves the claim.

Proposition 2. For every $\vartheta \in (1, 2)$, the map f_{ϑ} has a unique absolutely continuous invariant probability measure. Its density h_{ϑ} belongs to $W^{2,1}(I)$. On this space, 1 is an algebraically simple eigenvalue of L_{ϑ} , and the rest of the spectrum is contained in a disk of radius strictly smaller than one.

Proof. The weight $1/|f_{\vartheta}'|$, set equal to zero at the cusp, has bounded variation: its absolute derivative on each branch is the first coefficient in (8), which is integrable. Together with uniform expansion, this gives the hypotheses of Rychlik's theory [6]. An invariant probability density exists, and the transfer operator admits a finite cyclic Perron–Frobenius decomposition on $BV(I)$; see also [8, Chapter 3].

Each inverse branch has Lipschitz constant at most λ_{ϑ} . Thus a rank- n cylinder has diameter at most λ_{ϑ}^n , and its closure is mapped onto I by f_{ϑ}^n . Every nonempty open interval contains such a closed cylinder for sufficiently large n . Consequently f_{ϑ} is topologically exact.

To see why the cyclic decomposition is trivial, take a nonzero nonnegative $v \in \text{BV}(I)$ with $L_\vartheta^q v = v$ for some $q \geq 1$. There are an open interval U and $d > 0$ such that $v \geq d$ almost everywhere on U . Choose a cylinder $J \subset U$ of rank n divisible by q , and let $g_J : (0, 1) \rightarrow \text{int } J$ be its inverse branch. Its derivative is nonzero on $(0, 1)$, so $v = L_\vartheta^n v \geq d|g_J'| > 0$ almost everywhere there. Thus two nonnegative densities fixed by the same power cannot have disjoint supports. The finite cyclic decomposition reduces to a single mixing component: the invariant probability density is unique in BV and 1 is the only peripheral eigenvalue. The associated convergence extends from BV to L^1 by density and L^1 contraction, so there are no additional invariant probability densities in L^1 .

Spectra are understood on the complexified spaces. On $W^{2,1}(I)$ let $\ell(\phi) = \int_I \phi dx$. Since $L_\vartheta^* \ell = \ell$, the number 1 belongs to the spectrum. Proposition 1 puts it outside the essential spectral disk, so it is an eigenvalue. Every corresponding eigenvector lies in $\text{BV}(I)$ and is a nonzero multiple of h_ϑ . This proves $h_\vartheta \in W^{2,1}(I)$ and geometric simplicity. The same inclusion excludes all other eigenvalues of modulus one on $W^{2,1}(I)$.

A nontrivial Jordan block at 1 would give $(L_\vartheta - I)v = h_\vartheta$ after rescaling. Then $L_\vartheta^n v = v + nh_\vartheta$, contradicting $\|L_\vartheta^n v\|_1 \leq \|v\|_1$. The L^1 contraction also excludes eigenvalues of modulus greater than one. These observations, together with $r_{\text{ess}} < 1$, prove the asserted spectral gap.

The obstruction caused by an interior cusp value

In [5], the strict inequality $\beta < -1/2$ also gives $|f''|/|f'|^3 = O(t^{-2\beta-1}) \rightarrow 0$, which makes the first derivative of the transferred function vanish at the common cusp value. At the critical exponent this mechanism changes. The cancellation in (1) controls the interior differential expression, but does not by itself remove a nonzero derivative trace at an interior branch-image endpoint.

Proposition 3. *Suppose that a nonsingular map $f : I \rightarrow I$ has two monotone branches onto $[0, a]$, where $0 < a < 1$, and is symmetric near its cusp c . Assume that, for sufficiently small $t > 0$,*

$$f(c \pm t) = a - 2K\sqrt{t} + R(t), \quad K > 0, \quad (13)$$

where $R \in C^2((0, \delta))$ satisfies

$$R(t) = O(t), \quad R'(t) = O(1), \quad R''(t) = O(t^{-1}).$$

For $\phi \in C^1(I)$, let $L_f \phi$ be extended by zero to $(a, 1]$. Then

$$(L_f \phi)(x) = \frac{\phi(c)}{K^2}(a - x) + o(a - x), \quad \lim_{x \uparrow a} (L_f \phi)'(x) = -\frac{\phi(c)}{K^2}. \quad (14)$$

If $\phi(c) \neq 0$, this zero extension does not belong to $W^{2,1}(I)$.

Proof. Write $p(t) = Kt^{-1/2} - R'(t)$ and $q(t) = \phi(c - t) + \phi(c + t)$. If $t = t(x)$ is the inverse distance to the cusp, then

$$a - x = 2K\sqrt{t} + O(t), \quad (L_f \phi)(x) = \frac{q(t)}{p(t)}.$$

As $q(t) \rightarrow 2\phi(c)$ and $\sqrt{t} = (a - x)/(2K) + o(a - x)$, the first assertion in (14) follows. Moreover, $dt/dx = -1/p(t)$, so

$$(L_f \phi)'(x) = -\frac{q'(t)}{p(t)^2} + \frac{q(t)p'(t)}{p(t)^3}.$$

Here q' is bounded, $p'(t) = -(K/2)t^{-3/2} + O(t^{-1})$, and $p(t)^3 \sim K^3 t^{-3/2}$. This proves the derivative limit in (14). The derivative of the zero extension is zero on $(a, 1)$. If $\phi(c) \neq 0$, its left and right traces at a are different, whereas every $W^{2,1}(I)$ function has a continuous first derivative. This is a contradiction.

For (3), the cusp value is 1, so the nonzero derivative trace causes no interior obstruction. The full-branch choice and the algebraic cancellation therefore address distinct issues. Proposition 3 does not rule out exceptional inputs with $\phi(c) = 0$.

Bernoulli random perturbations and analytic response

We now apply the probability-law viewpoint of [5,7] to an affine perturbation of the critical family. Fix distinct parameters $\vartheta_0, \vartheta_1 \in (1, 2)$, and write $L_0 = L_{\vartheta_0}$, $L_1 = L_{\vartheta_1}$ and $D = L_1 - L_0$. Consider

$$L_\varepsilon = (1 - \varepsilon)L_0 + \varepsilon L_1 = L_0 + \varepsilon D. \quad (15)$$

For $0 \leq \varepsilon \leq 1$, this is the annealed transfer operator of the system which independently chooses f_{ϑ_0} with probability $1 - \varepsilon$ and f_{ϑ_1} with probability ε at each step. Negative values of ε are used only for analytic continuation; they do not define Bernoulli probabilities. Put

$$W_0^{2,1}(I) := \left\{ \phi \in W^{2,1}(I) : \int_I \phi dx = 0 \right\}.$$

Theorem 1. *There exists $\varepsilon_0 \in (0, 1)$ such that, for every real $|\varepsilon| < \varepsilon_0$, the operator L_ε has an isolated algebraically simple eigenvalue 1 on $W^{2,1}(I)$ and a unique normalized fixed vector $h_\varepsilon \in W^{2,1}(I)$, with $\int_I h_\varepsilon dx = 1$. The map $\varepsilon \mapsto h_\varepsilon \in W^{2,1}(I)$ is real analytic. For $0 \leq \varepsilon < \varepsilon_0$, this vector is nonnegative and hence a stationary density of the Bernoulli system.*

Let $h_0 = h_{\vartheta_0}$ and

$$R_0 = (I - L_0)^{-1} : W_0^{2,1}(I) \longrightarrow W_0^{2,1}(I).$$

Then

$$h'_0 = R_0 D h_0 = (I - L_0)^{-1} (L_1 - L_0) h_0, \quad (16)$$

and, for every $n \geq 1$,

$$\left. \frac{d^n}{d\varepsilon^n} h_\varepsilon \right|_{\varepsilon=0} = n! (R_0 D)^n h_0. \quad (17)$$

Proof. By Proposition 2, 1 is an isolated algebraically simple eigenvalue of L_0 on $W^{2,1}(I)$, with the remaining spectrum strictly inside the unit disk. Since D is bounded on this space, (15) is analytic in operator norm. Analytic perturbation theory [3] gives an analytic rank-one spectral projection and an eigenvector v_ε near h_0 . Its integral is nonzero for small ε . Integrating its eigenvalue equation and using integral preservation shows that the associated eigenvalue is exactly 1. Normalization yields the analytic family h_ε .

After decreasing ε_0 , spectral separation keeps the remaining spectrum inside a disk of radius less than one. The spectral projection is $P_\varepsilon \phi = h_\varepsilon \int_I \phi dx$, and therefore

$$L_\varepsilon^n \mathbf{1} \longrightarrow h_\varepsilon \quad \text{in } W^{2,1}(I), \quad n \rightarrow \infty. \quad (18)$$

For $0 \leq \varepsilon < \varepsilon_0$, each function on the left is nonnegative and has integral one. Thus the limit is a stationary probability density.

The spectral projection of L_0 is $P_0 \phi = h_0 \int_I \phi dx$, so the restriction of L_0 to $W_0^{2,1}(I) = \ker P_0$ has spectral radius less than one. Hence $R_0 = \sum_{k=0}^{\infty} L_0^k$ converges in operator norm on $W_0^{2,1}(I)$. Also, D maps $W^{2,1}(I)$ into this zero-integral subspace, because both L_0 and L_1 preserve integrals. Thus $R_0 D$ is well defined on $W^{2,1}(I)$.

Differentiating $L_\varepsilon h_\varepsilon = h_\varepsilon$ and $\int_I h_\varepsilon dx = 1$ at zero gives $(I - L_0)h'_0 = D h_0$ and $\int_I h'_0 dx = 0$, which proves (16). Since (15) is affine, repeated differentiation yields

$$(I - L_0)h_0^{(n)} = n D h_0^{(n-1)}, \quad \int_I h_0^{(n)} dx = 0, \quad n \geq 1.$$

Induction gives (17). Equivalently, for sufficiently small $|\varepsilon|$,

$$h_\varepsilon = h_0 + \sum_{n=1}^{\infty} \varepsilon^n (R_0 D)^n h_0 \quad \text{in } W^{2,1}(I). \quad (19)$$

Corollary 1. *For every bounded linear functional Φ on $W^{2,1}(I)$, $\varepsilon \mapsto \Phi(h_\varepsilon)$ is analytic near zero, and*

$$\left. \frac{d^n}{d\varepsilon^n} \Phi(h_\varepsilon) \right|_{\varepsilon=0} = n! \Phi((R_0 D)^n h_0), \quad n \geq 1.$$

In particular, for every $\varphi \in L^\infty(I)$,

$$\left. \frac{d}{d\varepsilon} \int_I \varphi h_\varepsilon dx \right|_{\varepsilon=0} = \int_I \varphi R_0 (L_1 - L_0) h_0 dx. \quad (20)$$

Concluding remarks

The example shows that the strict restriction $\beta < -1/2$ in the separate Sobolev coefficient estimates is not an intrinsic barrier for every cusp map. At the endpoint, the factor $2\beta + 1$ cancels the nominally non-integrable leading term, while the full-branch geometry avoids an interior jump in the first derivative. Together these properties give $W^{2,1}$ invariant densities and analytic response to a Bernoulli probability perturbation.

A natural continuation is to replace the explicit derivative $\vartheta + K\vartheta t^{-1/2}$ by an expansion $Kt^{-1/2} + A + O(t^{1/2})$ with compatible estimates on its first two derivatives. The resulting general endpoint theory, as well as deterministic perturbations of the maps themselves, is not pursued here.

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MONTE CARLO METHOD FOR THE SPATIALLY HOMOGENEOUS BOLTZMANN EQUATION

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ABSTRACT. This work presents a Monte Carlo computational approach for solving the spatially homogeneous nonlinear Boltzmann equation through a modified “particle-in-cells” model. Building upon the Belotserkovski-Yanitski and Kac models, the paper introduces a stochastic simulation scheme that synthesizes the principle of splitting operators with branching Markov processes. The method constructs unbiased estimators with small variance to compute molecular velocities and particle distributions, providing higher numerical accuracy and physical realism compared to traditional linear models. The developed algorithm enables efficient simulation of rarefied gas dynamics and demonstrates stable results across varying particle densities. Computational experiments, including shock wave structure modeling, show that this approach achieves improved convergence and fidelity in approximating spatially heterogeneous Boltzmann behavior.

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Key words: Spatially Homogeneous Boltzmann equation, Particle-in-Cell Model, Branching Markov process, Monte Carlo Method, Unbiased Estimator, Conjugated Computational Schemes

1. Introduction

This paper studies and proposes statistical algorithms for solving the nonlinear integral Boltzmann equation in measures, which describes the physical processes of particle collisions and coalescence. The equations under consideration are one of the integral forms of the kinetic equations of rarefied gas dynamics. Markov processes with simple interactions are used to construct unbiased estimates for the solution of the kinetic equation of rarefied gas dynamics. The well-known relationship between nonlinear integral equations and branching Markov processes is exploited [1].

Recently, many studies have explored the application of the Monte Carlo method to solving such problems. The proposed computational scheme is analogous to the adjoint scheme in the linear case. The proposed scheme is examined in detail for the case of "rigid" molecules. To describe the scheme, it is convenient to switch from an equation in measures to an equation in densities. Various computational schemes are considered and their effectiveness compared. The resulting computational schemes are tested in solving a problem of shock wave structure. A statistical algorithm is proposed using the nonlinear spatially homogeneous Boltzmann equation for the numerical implementation of the "particle-in-cell" statistical model, and methods for constructing unbiased, minimal-variance estimates for calculating the particle velocity in each cell are investigated.

2. Description of the Problem

Let's consider a nonlinear integral equation

$$F_t(dr, du) = F_0(dr - ut, du) \exp\{-H(r, u)t\} + \int_0^t [\exp\{-H(r, u)(t-s)\}] \times \\ \times \iint_D T_{t-s}(dr, du; r_1, u_1, r_2, u_2) H(r_1, u_1) F_s(dr_1, du_1) F_s(dr_2, du_2). \quad (1)$$

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where $(r, u) \in R \times R^3$, $(r_i, u_i) \in D$, $i = 1, 2$, $t \in R^1$, F_t and F_0 are measures in D , $H(r, u)$ is a non-negative function, $T_{t-s}(B, r_1, u_1, r_2, u_2)$ is a measurable function of the arguments (r_1, u_1, r_2, u_2) , at fixed B , and at fixed (r_1, u_1, r_2, u_2) is a measure in D . The measures F_t, F_0, T_{t-s} are considered non-negative.

Many works are dedicated to studying this kind of equation (for example [2], [3]), where probabilistic features of solutions of this type of equation were studied. Equations of the type (1) represent one of the integral forms of kinetic equations for rarefied gas dynamics. The questions of existence and uniqueness of the solution of equation (1) were studied in [3,4]. It's known [3], that if F_0 and T_{t-s} are probabilistic measures and $H(r, u) < c = \text{const}$, then equation (1) has a unique nonnegative solution, which will be a probability measure. Existence and sub stochasticity of the minimal nonnegative solution of the equation (1) were also proven in [3]. In [2] authors constructively built a random process whose one-dimensional distribution satisfied equation (1). The advantages of this (unlike to problems mentioned above) allowed using a Monte Carlo method. His method of constructing unbiased estimators is a method of "direct modeling". The advantage of the constructed computational scheme is that it permits the estimating of several functionals of the solution to equation (1) at one time. The disadvantage of this method is the difficulty constructing the random process that gives the minimal variance. Currently, this problem is not solved for that case. In the theory of Monte-Carlo methods, there are methods (like "importance sampling") that allow one to avoid these difficulties when solving concrete problems.

Below, for equations (1) we present the construction methods for unbiased estimators for functional of the type (h, f) , where $h(r, u, t)$ is a given function and $f(r, u, t)$ is a density of the measure F_t . The computational scheme that we derive here is an analogue of the conjugated scheme in the linear case. Furthermore, we will call this scheme a "conjugated scheme". It is shown that the application of the results of Ermakov and others [2] allow the construction of a computational scheme with small variance, in particular, when a prior information on solution of equation (1) is known. When applying the above scheme to real physical problems, we must also show the statistics conservation of mass, impulse and energy.

3. Construction of unbiased estimators

As is known, [5] among the statistical methods, which use Monte Carlo directly for modeling the flow of rarefied gas, the most efficient one is the statistical method of direct modeling the nonstationary flow. In [6], they described a statistical "particle-in cell" model, which was constructed base on the synthesis of ideas of splitting the physical processes in each Δt . This method was used by Harlow [7], and in statistical interpretations by Berd [8] and Kac [9]. The specific of the "particle-in-cell" method are the following. The state of every particle is characterized by vectors of coordinates r and speeds u at time, $k\Delta t$, where $k = 0, 1$. A discrete random process

$$\{r(t), u(t)\} = \{r_1(t), u_1(t), \dots, r_N(t), u_N(t)\}$$

should be constructed in the heterogeneous coordinate space at each time $k|\Delta t$. Here N is the number of molecules of monatomic gas in some volume, D . The principle of splitting operator is used for constructing the random process $r(t), u(t)$.

The sampling divides into two steps. Monte Carlo method is used both for the numerical simulation of collisions of the particles in cells (the first step), as well as for the collision-free motion (free-molecular) of particles (the second step). For the numerical realization of the second step in [10] an algorithm is given, which does not require saving the corresponding spatial coordinates of the particles. For the numerical realization of the first step, spatially homogenous model of Kac has been used. Thus, the synthesis of the idea of splitting, and model of Kac has occurred. That gives rise to the Markov model, which approximates the Boltzmann equation with accuracy that depends statistically (condition of molecular chaos) on the number of particles. In [11], Yanitski explained the value of the principle of splitting. The computational approach of Belotserkovski-Yanitski [6] is as efficient as the method of Berd, but has higher accuracy in approximating the Boltzmann equation. It's important to emphasize that "particle-in-cell" the statistical model of Belotserkovski-Yanitski is physical closer to the real problem. As new propose yet another computational scheme, which directly uses the nonlinear spatially homogenous Boltzmann equation for the numerical realization of the first step in the "particles-in-cell" model of Belotserkovski-Yanitski. The method of constructing unbiased estimators with small variance is next given for calculating of a velocity of a single molecule $u(k\Delta t)$ for $k = 0, 1, 2, \dots$

To find the velocity of the particle in each cell, we will use equation (1) in the spatially homogenous case. In this case, equation (1) has the form

$$\begin{aligned} F_{\Delta t}(du) = & F_0(du) \exp\{-H(u)\Delta t\} + \int_0^{\Delta t} ds \exp\{-H(u)(\Delta t - s)\} \times \\ & \times \iint_D T_{\Delta t-s}(du; u_1, u_2) H(u_1) F_s(du_1) F_s(du_2). \end{aligned} \quad (2)$$

where $F_0(du), F_t(du)$ are distribution particle velocity at time 0 and t respectively. Let $H(u) = c = \text{const.}$ As is known [4], this equation has a unique positive solution that is a probabilistic measure, and the iterative method starting with $F_0(du) \exp\{-c\Delta\}$ converges to the solution of equation (2). This is why we can apply the approach that was developed in [12] to equation (2). In [4], a direct computational scheme for solving equation (2) with certain side conditions was given. The results of [4] cannot be applied in our case, since the direct scheme, as given in [4], cannot estimate the value at the point (cell). In [4] a functional is estimated, not a point value.

Let's suppose that when the initial distribution, $F_0(du)$, has a density, the distribution $F_{\Delta t}(du)$ also has a density, $f(u, \Delta t)$. Making the transformation, which is analogous to the transformations [13], we obtain the following equation

$$\begin{aligned} f(u, \Delta t) = & f(u, 0) \exp\{-ct\} + \int_0^{\Delta t} ds \int d\bar{\omega} \int du_c \times \\ & \times (1 - \exp\{-c\Delta t\}) f_1(s) \frac{|u - u_c|c}{c} f(u_c - |u - u_c|\bar{\omega}, s) f(u_c + |u - u_c|\bar{\omega}, s) \times \\ & \times 0^{\Delta t} ds \int_{\mathbb{R}^3} (1 - \exp\{-c\Delta t\}) f_1(s) \left(1 - \frac{|u - u_c|c}{c}\right) f(u, s) f(u_2, s) du_2. \end{aligned} \quad (3)$$

For equation (3), we construct a branching Markov process, and thus we can derive various "conjugated" computational schemes for calculating a unbiased estimator of the functional (h, f) . Since $u = (u_x, u_y, u_z)$, we suppose $h(u, t) = u_x$, i.e. at time, Δt , the velocity is computed by the following formula

$$U(\Delta t) = \int u_x f(u, \Delta t) du$$

Now we describe the computational scheme. At time, Δt , one particle is born with initial density $p_0(u)$ at the point $(u, \Delta t) = (u_x, u_y, u_z, \Delta t)$. Here, the estimator has the form $\xi_{\Delta t}(u) = \frac{u_x}{p_0(u)}$. Further, with probability $\alpha_0(\Delta t) = \exp\{-c\Delta t\}$, the particle is absorbed at the point $(u, \Delta t)$, here estimator is computed by the formula $\xi_{\Delta t}(u) \leftarrow \varepsilon_{\Delta t}(u) f(u, 0)$.

If particle hasn't been absorbed, then the time $s_1 \in [0, \Delta t]$ is simulated with the probability density

$$f_1(s) = \frac{c \exp\{-c(\Delta t - s)\}}{1 - \exp\{-c\Delta t\}}.$$

Then with probability $\alpha_1(\Delta t) = \beta(1 - \exp\{c\Delta t\})$, where $\beta \in [0, 1]$, two particles will appear at the points $(u_c - |u - u_c|\varpi, s_1)$ and $(u_c + |u - u_c|\varpi, s_1)$. Here ϖ is an isotropic vector. The transition from u to u_c is done with the transition density $\psi_1(u_c)$. The density distribution at the point (u_c, s_1) , under the condition that collision has taken place (velocity changes), we'll write in the form of $P_1(u, \Delta t \rightarrow u_c, s_1) = f_1(s)\psi_1(u_c)$. Here the estimator is computed with the formula

$$\xi_{\Delta t}(u) \rightarrow \frac{\xi_{\Delta t}(u)(2|u - u_c|)\psi_1(u_c)}{\beta c}.$$

Further, with probability $\alpha_2(\Delta t) = (1 - \exp\{-c\Delta t\})(1 - \beta)$ two particles are created at the points (u, s_1) and (u_2, s_1) . Transition from u to u_2 is done with transition density $\psi_1(u_2)$. The density distribution of the point (u_2, s) with velocity remaining constant, can be written in the form $P_2(u, \Delta t \rightarrow u_2, s_1) = f_1(s)\psi_1(u_2)$. In this case the estimator is computed by the formula

$$\xi_{\Delta t}(u) \leftarrow \xi_{\Delta t}(u) \left(1 - \frac{|u - u_2|}{c}\right) (1 - \beta)^{-1} (\psi_1(u_2))^{-1}$$

Furthermore, new particles behave the same as the initial particle. Moreover, $s_0 = 0$, $s_k \in [0, t - \sum_{i=0}^{k-1} s_i]$, $t \leftarrow t - \sum_{i=0}^{k-1} s_i$, the moments of time s_k are chosen with probability density $f_1(s)$. At each step, the times $s_1, s_2, \dots$ are chosen and the process finishes, when all particles are absorbed, or the condition $t - \sum s_i \leq \varepsilon$, where ε is a small number which was given beforehand, is satisfied.

Applying the results of [13] to equation (3), we derive the following assertion. If a process is given with an initial density

$$P_0(u, \Delta t) = \frac{|u_x f(u, \Delta t)|}{(|u|_x, |f(u, \Delta t)|)},$$

the transition density

$$P_1(u, \Delta t \rightarrow u_c, s_1) = f_1(s) \frac{(2|u - u_c|c)}{c} f(u_c - |u - u_c|, s) \frac{f(u_c + |u - u_c|, s)}{f(u, \Delta t)}$$

$$P_2(u, \Delta t \rightarrow u_2, s_1) = f_1(s) \left(1 - \frac{|u - u_c|c}{c}\right) \frac{f(u, s)f(u_2, s)}{f(u, \Delta t)}$$

and absorption probability

$$g(u, \Delta t) = \frac{f(u, 0)}{f(u, t)} \exp\{-c\Delta t\}$$

then the variance of estimator $\xi_{\Delta t}(u)$ will be the solution to equation (3).

Now, let's study the construction of unbiased estimators with small variance, using [14] and known data on the solution at $t \rightarrow \infty$. It's known [15] that as $t \rightarrow \infty$ the gas approaches the equilibrium state. Then features of the gas do not depend on spatial coordinates or time. In this case, the solution of equation (3) has the form (Maxwellian)

$$f(u) = \frac{1}{(2\pi)^{3/2}} \exp\left(-\frac{u^2}{2}\right) \quad (4)$$

For large t , the solution of our problem is close to (4). If we put (4) into expressions for P_1 and P_2 this determines a Markov process. And gives us a use the small variance estimator $\xi_{\Delta t}(u)$. The density distribution of the point (u_c, s_1) , when collision has taken place. When particle velocities change, has the form

$$\tilde{P}_1(u, \Delta t \rightarrow u_c, s) = f_1(s) \frac{1}{(2\pi)^{3/2}} \exp\left\{-(u_c^2 + |u - u_c|^2) + \frac{u^2}{2}\right\}$$

If we make a change of variables in the following way: $\tilde{v} = 2v = 2(u_c - u)$, then we get for

$$\tilde{P}_1\left(u, \Delta t \rightarrow \frac{\tilde{v}}{2} + u, s_1\right) = f_1(s) \frac{1}{(2\pi)^{3/2}} \exp\left\{-\frac{(\tilde{v} - u)^2}{2}\right\} \quad (5)$$

and for the point (u_2, s_1) , the collision has taken place (case of constant velocity) and the expression for the density distribution, $\tilde{P}_2$, can be written as

$$\tilde{P}_2(u, \Delta t \rightarrow u_2, s_1) = f_1(s) \frac{1}{(2\pi)^{3/2}} \exp\left\{-\frac{u_2^2}{2}\right\} \quad (6)$$

here

$$g(u, \Delta t) = \exp\{-c\Delta t\} \quad (7)$$

Thus, velocities $\bar{v}$ and u_2 are distributed according to a normal law with densities

$$\psi_1(\bar{v}) = \frac{1}{(2\pi)^{3/2}} \exp\left\{-\frac{(\bar{v} - u)^2}{2}\right\}, \quad \psi_2(u_2) = \frac{1}{(2\pi)^{3/2}} \exp\left\{-\frac{u_2^2}{2}\right\}$$

and the optimal transition densities have the form (5), (6), (7).

For the construction of this algorithm, which is connected with densities (5), (6), (7), it is more convenient to apply equation (3), after the following change of variables

$$f(u, t) = \frac{1}{2} \psi(u) \varphi(u, \Delta t).$$

Here $\psi(u)$ is a known function, which is defined by (4). The computational scheme is simple. Because in computing the estimators there is no need to use weights that are larger than ours. Thus for (3) we derive

$$\begin{aligned} \varphi(u, \Delta t) &= \varphi(u, 0) \exp\{-c\Delta t\} + \int_0^{\Delta t} ds \int d\bar{\omega} \int_{R^3} du_c (1 - \exp\{-c\Delta t\}) \times \\ &\times f_1(s) \frac{(2|u - u_c|c)}{2c} \frac{\psi(u_c - |u - u_c|\bar{\omega})\psi(u_c + |u - u_c|\bar{\omega})}{\psi(u)} \times \\ &\times \varphi(u_c - |u - u_c|\bar{\omega}, s) \varphi(u_c + |u - u_c|\bar{\omega}, s) + \\ &+ \int_0^{\Delta t} ds \int_{R^3} du_2 (1 - \exp\{-c\Delta t\}) f_1(s) \left(1 - \frac{|u - u_c|c}{c}\right) \times \\ &\times \frac{1}{2} \frac{\psi(u)\psi(u_2)}{\psi(u)} \varphi(u, s) \varphi(u_2, s). \end{aligned} \quad (8)$$

where $u_c = u + \frac{\tilde{v}}{2}$. The algorithm for constructing the unbiased estimator $U(\Delta t) = \int u_x f(u, \Delta t) du$ described as follows. At time, Δt , one particle with initial density $p_0(u)$ at the point $(u, \Delta t) = (u_x, u_y, u_z, \Delta t)$ appears. The

particle at the point $(u, \Delta t)$ has a statistical "weight", which is equal to $\xi_{\Delta t}(u) = \frac{u_x}{p_0(u)}$. Furthermore, at the point $(u, \Delta t)$ the unbiased estimator, $\xi_{\Delta t}(u)$, is constructed for the considered equation in the following way. With probability $g(u, \Delta t) = \exp\{-c\Delta t\}$, the particle is absorbed at the point $(u, \Delta t)$. The estimator is computed by the formula $\xi_{\Delta t}(u) \leftarrow \varphi(u, 0)\xi_{\Delta t}(u)$. If the particle has not been absorbed, the time $s_1 \in [0, \Delta t]$ is simulated with probability density $f_1(s)$. Then, with probability $\frac{(1 - g(u, \Delta t))}{2}$ two particles appeared at points $\left(\left(\frac{\tilde{v}}{2} + u + \frac{\tilde{v}}{2}\tilde{\omega}\right), s_1\right)$ and $\left(\left(\frac{\tilde{v}}{2} + u - \frac{\tilde{v}}{2}\tilde{\omega}\right), s_1\right)$. Here the vector $\tilde{v} = u + \eta$, where η is sampled with a standard normal density. Here the estimator is computed by $\xi_{\Delta t}(u) \leftarrow \frac{\xi_{\Delta t}(u)|\tilde{v}|c}{8c}$. Furthermore, with probability $\frac{(1 - g(u, \Delta t))}{2}$ two particles appear at the points (u, s_1) and (u_2, s_1) . Here the vector u_2 is sampled from a standard normal distribution. In this case, the estimator is computed as $\xi_{\Delta t}(u) \leftarrow \xi_{\Delta t}(u) \left(1 - \frac{|u - u_c|c}{c}\right)$.

New particles behavior is the same as the original particles. The process finishes, when all particles are absorbed, or at $\Delta t - \sum s_i \leq \varepsilon$. When process completes, the estimator $\xi_{\Delta t}(u)$ is multiplied by a value of function $\frac{1}{2}\psi(u)$ at the point $(u, \Delta t)$, i.e. $\xi_{\Delta t}(u) \leftarrow \frac{\xi_{\Delta t}(u)\psi(u)}{2}$. Thus, in each cell we compute the velocity at each time, $k\Delta t$, where $k = 0, 1, 2, \dots$

4. Computational experiment

As is known, the nonlinear Boltzmann equation describes the behavior of rarefied gas much better than the linear Kac model. This is why we can expect that application of the nonlinear spatially homogenous Boltzmann equation in the "particles-in-cell" model allows us to do a computation, which gives a more exact approximation to the spatially heterogeneous Boltzmann equation.

We underline that boundary conditions at infinity can be taken into consideration by the method [6]. There computations show that the results of modeling are similar with known [5] solutions of the Boltzmann equation. As an example, computations for the wave shock structure have been done.

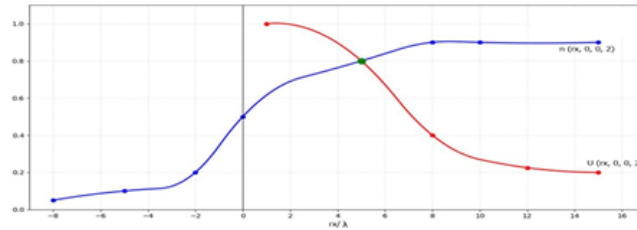


Figure 6: The calculated profile of density is $n(r_x, 0, 0, 2)$ and velocity is $u(r_x, 0, 0, 2)$. r_x is x axis.

The figure shows two graphs of density and velocity. The vertical line is the y-axis. Mach number is $M = 2$. The confidence level is 97%. The calculation was made with the following parameters. The number of particles in cells is $N1 = 8$. The ratio $\Delta t/\Delta r_x$, where $\Delta r = (\Delta r_x, \Delta r_y, \Delta r_z)$, is chosen for numerical stability. The parameters for the problem are the following:

$$t = 2, \Delta r_x = \Delta r_y = \Delta r_z = 0.2, \lambda = 1.$$

We note that in interval, where the Boltzmann equations "work" (intermediate interval), the "particle-in-cell" statistical model approximates the spatially heterogeneous Boltzmann equation better. At the same time, the approximation is independent at the number of particles, in D (even if a volume will only have one particle).

5. Conclusion

First, we note that for equation (3) we could apply a simple iterative process. For the computation of the integrals, which are obtained in the first and the second iteration, we can directly apply the Monte Carlo method. When comparing the density profiles at the first and the second iterations we noticed very little bias. This can be explained by the fact the iteration process to (3) converges slowly.

The next Direct Simulation Monte Carlo is a well-established method for the computer simulation of a gas flow at the molecular level. The Bird's Direct Simulation Monte Carlo (DSMC) method is a powerful computational technique for modeling rarefied gas flows. It was introduced fifty years ago and, since that time, most DSMC procedures have been superseded or significantly modified. By method Monte Carlo particle collisions, DSMC can accurately capture

non-equilibrium effects that are often present in rarefied flows. Further investigations and developments in this area were explored in the following papers [16,17,18].

The last is, gas dynamics problems can serve as examples of problems whose solution requires the use of modern multiprocessor computers. In particular, the multidimensional Boltzmann equation requires such technology for its solution. The complex nature of the kernel of this equation and the dependence of the solution on many variables necessitated the use of the Monte Carlo method. This method, as is well known, is particularly convenient for multiprocessor computers and allows for natural parallelization. Above, we describe an algorithm for solving the initial-boundary value problem for the Boltzmann equation, which could be effectively implemented on parallel computing systems.

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SOME GENERALIZED METRIC PROPERTIES OF THE SPACE OF G -PERMUTATION DEGREE

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ABSTRACT. In this paper we investigate some generalized metric properties of the space of G -permutation degree $SP_G^n X$. We prove that an infinite T_1 -space X has a σ -locally finite network if and only if the space $SP_G^n X$ has a σ -locally finite network. We also establish that X has a σ -locally finite base if and only if $SP_G^n X$ has a σ -locally finite base. As an application, we obtain characterizations of metrizability and σ -spaces for the space $SP_G^n X$. Finally, we prove that an infinite regular space X is a $\mathfrak{P}_0$ -space if and only if $SP_G^n X$ is a $\mathfrak{P}_0$ -space.

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Key words: locally finite family, σ -locally finite network, σ -locally finite base, σ -space, $\mathfrak{P}_0$ -space.

Introduction

Generalized metric spaces form one of the central topics of modern general topology. Various classes of spaces defined by network properties, such as spaces with σ -locally finite networks, σ -locally finite bases, σ -spaces and $\mathfrak{P}_0$ -spaces, naturally generalize metrizable spaces and play an important role in the study of some topological structures. These classes provide useful tools for describing spaces that are not metrizable while still retaining many properties characteristic of metrizable spaces. In particular, the existence of a σ -locally finite base is closely related to metrizability through the classical Nagata- Smirnov Metrization Theorem.

Banach introduced $\mathfrak{P}_0$ -spaces as regular spaces possessing a countable Pytkiev network and established several fundamental properties of this class, including its relationships with $\aleph_0$ -spaces and other generalized metric spaces (see [6]). Gabrielyan and Kąkol continued this line of research by introducing $\mathfrak{P}$ -spaces and studying their connections with $\mathfrak{P}_0$ -spaces, $\aleph$ -spaces and $\aleph_0$ -spaces. They proved that every $\mathfrak{P}$ -space has the strong Pytkiev property and investigated numerous properties and characterizations of these spaces (see [5]).

An important problem in topology is to determine which topological properties are preserved under various topological constructions. One such construction is the space of G -permutation degree $SP_G^n X$. Many topological and cardinal properties of spaces of G -permutation degree have been investigated in several papers. In particular, separation axioms and some other topological properties of the space $SP_G^n X$ were studied by R.B. Beshimov and R.M. Juraev (see [4] and [7]). Furthermore, some generalized metric properties and network-type properties of the space $SP_G^n X$ were investigated in [8], [9] and [10].

In [10], several generalized metric properties of the space $SP_G^n X$ were investigated. In particular, it was proved that the functor SP_G^n preserves developable spaces, Moore spaces, M_1 -spaces, M_2 -spaces, stratifiable spaces, Lašnev spaces and Nagata spaces. These results show that the functor SP_G^n preserves a wide range of important generalized metric properties of topological spaces.

In addition, in [10] It was proved that the functor SP_G^n preserves the point- G_δ property, r -spaces, cosmic spaces, $C(\kappa)$ -cosmic spaces, $\aleph_0$ -spaces, α -spaces and strongly first countable spaces. Consequently, it was shown that the functor SP_G^n preserves a broad class of generalized metric properties of topological spaces.

The purpose of this paper is to study other generalized metric properties of the spaces $SP_G^n X$.

Preliminaries

Let G be a subgroup of the finite permutation group S_n and let X^n be the n -th power of a topological space X . Define a relation $\sim_G$ on X^n as follows:

For elements $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ in X^n ,

$$\mathbf{x} \sim_G \mathbf{y} \iff \text{there exists } \sigma \in G \text{ such that } y_i = x_{\sigma(i)}, \quad 1 \leq i \leq n.$$

This relation is an equivalence relation on the set X^n and is called the G -symmetric equivalence relation, and the G -equivalence class (or orbit) of an element $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ is denoted by $[\mathbf{x}]_G$ or $[(x_1, \dots, x_n)]_G$.

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Denote by $\text{SP}_G^n X$ the quotient set $X^n / \sim_G$ equipped with the quotient topology (of the product topology on X^n). The quotient mapping is denoted by $\pi_{n,G}^s : X^n \rightarrow \text{SP}_G^n X$ and defined as

$$\pi_{n,G}^s([\mathbf{x}]_G) = \pi_{n,G}^s((x_1, \dots, x_n)) = [(x_1, \dots, x_n)]_G, \quad \mathbf{x} = (x_1, \dots, x_n) \in X^n.$$

This space is called *the space of G -permutation degree of X* . Observe that the points of the space $\text{SP}_G^n X$ are finite subsets of the product X^n . If $G = S_n$, then one writes $\text{SP}^n X$ instead of $\text{SP}_G^n X$ and this space is simply called *the space of permutation degree of X* . When $G = \{e\}$, where e is an identity element of the permutation group S_n , then $\text{SP}_G^n X$ is the space X^n (see [1]).

Let X be a T_1 -space and n be a positive integer. Note that the quotient mapping $\pi_{n,G}^s : X^n \rightarrow \text{SP}_G^n X$ is an open, closed and surjective map (see [2]).

Main Results

Definition 1. [3] A family $\mathcal{P}$ of subsets of a topological space X is said to be *locally finite* if every point $x \in X$ has a neighborhood U such that a collection $\{P \in \mathcal{P} : P \cap U \neq \emptyset\}$ is finite.

Definition 2. [3] A family $\mathcal{P}$ of subsets of a topological space X is called *σ -locally finite*, if there exists a sequence $\{\mathcal{P}_m\}_{m \in \mathbb{N}}$ of locally finite families such that $\mathcal{P} = \bigcup_{m=1}^{\infty} \mathcal{P}_m$.

A base $\mathcal{B}$ of a topological space X is called a *σ -locally finite base* if there exists a sequence $\{\mathcal{B}_m\}_{m \in \mathbb{N}}$ of locally finite families of open subsets of X such that $\mathcal{B} = \bigcup_{m \in \mathbb{N}} \mathcal{B}_m$. In other words, $\mathcal{B}$ is the union of countably many locally finite families.

Theorem 1. Let X be a T_1 -space, G be a subgroup of permutation group S_n and $n \in \mathbb{N}$. The topological space X has a σ -locally finite network if and only if the space $\text{SP}_G^n X$ has a σ -locally finite network.

Proof. ($\Rightarrow$) Suppose that topological space X has a σ -locally finite network $\mathcal{N} = \bigcup_{m=1}^{\infty} \mathcal{N}_m$, where each $\mathcal{N}_m$ is a locally finite family for each $m \in \mathbb{N}$. For every $m_1, \dots, m_n \in \mathbb{N}$ we can define a family $\mathcal{G}_{m_1, \dots, m_n} = \{[N_1 \times \dots \times N_n]_G : N_i \in \mathcal{N}_{m_i}, i = 1, \dots, n\}$. Put

$$\mathcal{G} = \bigcup_{m_1, \dots, m_n \in \mathbb{N}} \mathcal{G}_{m_1, \dots, m_n}.$$

We first prove that $\mathcal{G}$ is a network of $\text{SP}_G^n X$. Let $[\mathbf{x}]_G = [(x_1, \dots, x_n)]_G$ be an arbitrary point and let $\mathbf{U}$ be an arbitrary neighborhood of $[\mathbf{x}]_G$ in the space $\text{SP}_G^n X$.

Since $\pi_{n,G}^s : X^n \rightarrow \text{SP}_G^n X$ is continuous, $(\pi_{n,G}^s)^{-1}(\mathbf{U})$ is an open neighborhood of $(x_1, \dots, x_n)$ in X^n . Hence there exist open neighborhoods $V_1, \dots, V_n$ of $x_1, \dots, x_n$, respectively, such that $(x_1, \dots, x_n) \in V_1 \times \dots \times V_n \subset (\pi_{n,G}^s)^{-1}(\mathbf{U})$. Since $\mathcal{N}$ is a network of X , for every $i \in \{1, \dots, n\}$ there exists $N_i \in \mathcal{N}_{m_i}$ for some $m_i \in \mathbb{N}$ such that $x_i \in N_i \subset V_i$. Therefore, $[\mathbf{x}]_G \in [N_1 \times \dots \times N_n]_G \subset [V_1 \times \dots \times V_n]_G \subset \mathbf{U}$. Thus, $\mathcal{G}$ is a network of $\text{SP}_G^n X$.

It remains to prove that each $\mathcal{G}_{m_1, \dots, m_n}$ is locally finite. Fix $m_1, \dots, m_n \in \mathbb{N}$ and let $[\mathbf{x}]_G = [(x_1, \dots, x_n)]_G$ be an arbitrary point of $\text{SP}_G^n X$. For every $j, r \in \{1, \dots, n\}$, since $\mathcal{N}_{m_r}$ is locally finite, there exists an open neighborhood $W_{j,r}$ of x_j such that $\{N \in \mathcal{N}_{m_r} : N \cap W_{j,r} \neq \emptyset\}$ is finite. Put $W_j = \bigcap_{r=1}^n W_{j,r}$, $1 \leq j \leq n$. Then each W_j is a neighborhood of x_j , and for every $r \in \{1, \dots, n\}$ the family $\{N \in \mathcal{N}_{m_r} : N \cap W_j \neq \emptyset\}$ is finite.

Put $\mathbf{W} = [W_1 \times \dots \times W_n]_G$. Since $\pi_{n,G}^s$ is an open mapping, $\mathbf{W}$ is a neighborhood of $[\mathbf{x}]_G$ in $\text{SP}_G^n X$. Suppose that $[N_1 \times \dots \times N_n]_G \cap \mathbf{W} \neq \emptyset$ for some $N_i \in \mathcal{N}_{m_i}$. Then there exists a permutation $\tau \in G$ such that $N_{\tau(i)} \cap W_i \neq \emptyset$, $i = 1, \dots, n$. For each i , the set of members of $\mathcal{N}_{m_{\tau(i)}}$ which intersect W_i is finite. Since $i = 1, \dots, n$ ranges over only finitely many indices, there are only finitely many possible choices for $(N_1, \dots, N_n)$. Hence only finitely many members of $\mathcal{G}_{m_1, \dots, m_n}$ intersect $\mathbf{W}$.

Thus, $\mathcal{G}_{m_1, \dots, m_n}$ is locally finite. Since $\mathbb{N}^n$ is countable, $\mathcal{G} = \bigcup_{(m_1, \dots, m_n) \in \mathbb{N}^n} \mathcal{G}_{m_1, \dots, m_n}$ is a σ -locally finite network of $\text{SP}_G^n X$.

($\Leftarrow$) Conversely, suppose that $\mathcal{U} = \bigcup_{m=1}^{\infty} \mathcal{U}_m$ is a σ -locally finite network of $\text{SP}_G^n X$, where every $\mathcal{U}_m$ is locally finite. Consider a subset $\Delta = \{[(x, \dots, x)]_G : x \in X\}$ of $\text{SP}_G^n X$. Clearly, $(\pi_{n,G}^s)^{-1}([(x, \dots, x)]_G)$ consists of exactly one point. Hence the mapping $i : X \rightarrow \Delta$, $i(x) = [(x, \dots, x)]_G$ is a homeomorphism.

For every $m \in \mathbb{N}$, define $\mathcal{V}_m = \{U \cap \Delta : U \in \mathcal{U}_m\}$. Since the restriction of a locally finite family to a subspace is locally finite, each $\mathcal{V}_m$ is locally finite in Δ . Moreover, $\mathcal{V} = \bigcup_{m=1}^{\infty} \mathcal{V}_m$ is a network of the subspace Δ .

Now put $\mathcal{N}_m = \{i^{-1}(V) : V \in \mathcal{V}_m\}$. Since homeomorphisms preserve local finiteness, each $\mathcal{N}_m$ is locally finite in X . Let $\mathcal{N} = \bigcup_{m=1}^{\infty} \mathcal{N}_m$. We prove that $\mathcal{N}$ is a network of X . Let $x \in X$ and let O be an arbitrary neighborhood of x . Then $i(O)$ is open in Δ and contains $i(x)$. Since $\mathcal{V}$ is a network of Δ , there exists $V \in \mathcal{V}$ such that $i(x) \in V \subset i(O)$. Applying i^{-1} , we obtain $x \in i^{-1}(V) \subset O$. Hence $\mathcal{N}$ is a network of X . Therefore $\mathcal{N} = \bigcup_{m=1}^{\infty} \mathcal{N}_m$ is a σ -locally finite network of X . This completes the proof of Theorem 1.

Proposition 1. Let X be a T_1 -space, G be a subgroup of permutation group S_n and $n \in \mathbb{N}$. The topological space X has a σ -locally finite base if and only if the space $\text{SP}_G^n X$ has a σ -locally finite base.

Proof. ($\Rightarrow$) Suppose X has a σ -locally finite base. Let $\mathcal{B} = \bigcup_{m \in \mathbb{N}} \mathcal{B}_m$ be a σ -locally finite base of X . For fixed indices $m_1, \dots, m_n \in \mathbb{N}$, define the family:

$$\mathcal{C}_{m_1, \dots, m_n} = \{[B_1 \times \dots \times B_n]_G : B_i \in \mathcal{B}_{m_i}, i = 1, \dots, n\}.$$

Then, define the countable union $\mathcal{C} = \bigcup_{m_1, \dots, m_n \in \mathbb{N}} \mathcal{C}_{m_1, \dots, m_n}$.

Every member of $\mathcal{C}$ is open in $\text{SP}_G^n X$ because $B_1 \times \dots \times B_n$ is open in X^n and the quotient map $\pi_{n,G}^s : X^n \rightarrow \text{SP}_G^n X$ is open. To show the base property, let $[\mathbf{x}]_G = [(x_1, \dots, x_n)]_G$ be an arbitrary point in $\text{SP}_G^n X$ and let U be a neighborhood of $[\mathbf{x}]_G$. Choose a neighborhood $V_1 \times \dots \times V_n$ of the representative $\mathbf{x} = (x_1, \dots, x_n)$ contained in $(\pi_{n,G}^s)^{\leftarrow}(U)$. Since $\mathcal{B}$ is a base for X , for each $i \in \{1, \dots, n\}$, we can choose $B_i \in \mathcal{B}$ such that $x_i \in B_i \subset V_i$. It follows that $[\mathbf{x}]_G \in [B_1 \times \dots \times B_n]_G \subset U$. Thus, $\mathcal{C}$ is a base for $\text{SP}_G^n X$.

To show that each $\mathcal{C}_{m_1, \dots, m_n}$ is locally finite, let $[(x_1, \dots, x_n)]_G$ be an arbitrary point of $\text{SP}_G^n X$. Its inverse image under the mapping $\pi_{n,G}^s$ consists of finitely many points, $(\pi_{n,G}^s)^{\leftarrow}([(x_1, \dots, x_n)]_G) = \{\mathbf{y}^1, \mathbf{y}^2, \dots, \mathbf{y}^k\}$, where each $\mathbf{y}^j = (y_1^j, \dots, y_n^j)$. By using the corrected indexing argument for local finiteness, for each coordinate $j \in \{1, \dots, n\}$ and each $r \in \{1, \dots, n\}$, choose a neighborhood $W_{j,r}$ of x_j meeting only finitely many members of $\mathcal{B}_{m_r}$. Put $W_j = \bigcap_{r=1}^n W_{j,r}$. Let $\mathbf{W} = [W_1 \times \dots \times W_n]_G$, which is a neighborhood of $[(x_1, \dots, x_n)]_G$. By construction, every W_j meets only finitely many members of each family $\mathcal{B}_{m_1}, \dots, \mathcal{B}_{m_n}$. Consequently, by the permutation argument, only finitely many members of $\mathcal{C}_{m_1, \dots, m_n}$ intersect $\mathbf{W}$. Therefore, each $\mathcal{C}_{m_1, \dots, m_n}$ is locally finite, making $\mathcal{C}$ a σ -locally finite base.

($\Leftarrow$) Suppose $\mathcal{U} = \bigcup_{m=1}^{\infty} \mathcal{U}_m$ is a σ -locally finite base of $\text{SP}_G^n X$, where each $\mathcal{U}_m$ is locally finite. Consider the diagonal subset $\Delta = \{[(x, \dots, x)]_G : x \in X\} \subset \text{SP}_G^n X$. Since X is a T_1 -space, the mapping $i : X \rightarrow \Delta$ defined by $i(x) = [(x, \dots, x)]_G$ is a homeomorphism.

For every $m \in \mathbb{N}$, define $\mathcal{V}_m = \{U \cap \Delta : U \in \mathcal{U}_m\}$. Since the restriction of a locally finite family to a subspace remains locally finite, each $\mathcal{V}_m$ is locally finite in Δ , and $\mathcal{V} = \bigcup_{m=1}^{\infty} \mathcal{V}_m$ forms a base for the subspace Δ . Putting $\mathcal{B}_m = \{i^{\leftarrow}(V) : V \in \mathcal{V}_m\}$, and since homeomorphisms preserve local finiteness, each $\mathcal{B}_m$ is locally finite in X . Thus, $\mathcal{B} = \bigcup_{m=1}^{\infty} \mathcal{B}_m$ forms a σ -locally finite base for X . This completes the proof of Proposition 1.

Theorem 2. [4] Let X be a topological space, n be a natural number, and G be an arbitrary subgroup of the permutation group S_n . The topological space X is a T_i -space if and only if the space $\text{SP}_G^n X$ is a T_i -space, where $i = 1, 2, 3, 3\frac{1}{2}$.

Theorem 2 implies that a topological space X is regular if and only if the space $\text{SP}_G^n X$ is regular.

Theorem 3(Nagata-Smirnov Metrization Theorem). [3] A topological space X is metrizable if and only if X is regular and has a σ -locally finite base.

By applying Theorem 1, Theorem 2 and Theorem 3 we obtain following result.

Corollary 1. Let X be a regular space, G be a subgroup of permutation group S_n and $n \in \mathbb{N}$. The topological space X is metrizable if and only if $\text{SP}_G^n X$ is metrizable.

Definition 3. [5] A regular topological space X is called σ -space, if X has a σ -locally finite network.

By Definition 3 and Theorem 1 we obtain the following result.

Proposition 2. Let X be a regular space, G be a subgroup of permutation group S_n and $n \in \mathbb{N}$. The topological space X is σ -space if and only if $\text{SP}_G^n X$ is σ -space.

A family $\mathcal{F}$ of subsets of a topological space X is said to be *closure preserving* if for every subfamily $\mathcal{G} \subseteq \mathcal{F}$, $\bigcup \mathcal{G} = \bigcup \{\overline{G} : G \in \mathcal{G}\}$. A family $\mathcal{F}$ of subsets of a topological space X is called σ -closure preserving if $\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$, where each family $\mathcal{F}_n$ is closure preserving.

A family $\mathcal{F}$ of subsets of a topological space X is said to be *discrete* if every point $x \in X$ has a neighborhood U that intersects at most one member of $\mathcal{F}$. A family $\mathcal{F}$ of subsets of a topological space X is called σ -discrete if $\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$, where each family $\mathcal{F}_n$ is discrete.

Theorem 4. [11] For a regular space X the following are equivalent:

- i) X has a σ -closure preserving network;
- ii) X is a σ -space;
- iii) X has a σ -discrete network.

Proposition 2 and Theorem 4 yield the following corollaries.

Corollary 2. Let X be a regular space, G be a subgroup of permutation group S_n and $n \in \mathbb{N}$. The topological space X has σ -closure preserving network if and only if $SP_G^n X$ has σ -closure preserving network.

Corollary 3. Let X be a regular space, G be a subgroup of permutation group S_n and $n \in \mathbb{N}$. The topological space X has σ -discrete network if and only if $SP_G^n X$ has σ -discrete network.

We say that a subset A of a topological space X *accumulates at a point* $x \in X$ if each neighborhood $O_x \subset X$ of x contains infinitely many points of the set A . Each accumulation point belongs to the closure $\bar{A}$ of the set A in X . In a T_1 -space X a point $x \in X$ is an accumulation point of a set $A \subset X$ if and only if $x \in \bar{A} \setminus \{x\}$.

Definition 4. [6] Let X be a topological space and $x \in X$ be a point. A family $\mathcal{N}$ of subsets of X is called a *Pytkeev network at x* if $\mathcal{N}$ is a network at x and for every neighborhood $O_x \subset X$ of x and every subset $A \subset X$ accumulating at x there is a set $N \in \mathcal{N}$ such that $N \subset O_x$, $N \cap A$ is an infinite set. A family $\mathcal{N}$ is a Pytkeev network of X if it is a Pytkeev network at each point of X .

A regular topological space X is called $\mathfrak{P}_0$ -space [6] if X has a countable Pytkeev network.

Theorem 5. Let X be a regular space, G be a subgroup of permutation group S_n and $n \in \mathbb{N}$. The topological space X is a $\mathfrak{P}_0$ -space if and only if $SP_G^n X$ is a $\mathfrak{P}_0$ -space.

Proof. ($\Rightarrow$) Let X be a regular space with a countable Pytkeev network $\mathcal{P}$. By Theorem 3.5 of [6] we have that finite product of $\mathfrak{P}_0$ -spaces is also $\mathfrak{P}_0$ -space and hence a family $\mathcal{P}^n = \{P_1 \times \dots \times P_n : P_i \in \mathcal{P}, 1 \leq i \leq n\}$ is a countable Pytkeev network of X^n .

Now we prove that a family $SP_G^n \mathcal{P} = \{[P_1 \times \dots \times P_n]_G : P_i \in \mathcal{P}, 1 \leq i \leq n\}$ is a countable Pytkeev network of the space $SP_G^n X$, where $[P_1 \times \dots \times P_n]_G = \pi_{n,G}^s(P_1 \times \dots \times P_n)$. Let $\mathbf{A} \subset SP_G^n X$ be any set accumulating at a point $[\mathbf{x}]_G = [(x_1, \dots, x_n)]_G \in SP_G^n X$. Take any neighborhood $\mathbf{U}$ of $[\mathbf{x}]_G$.

Since X is regular, it is a T_1 -space. By Theorem 2, $SP_G^n X$ is also regular and hence T_1 . Choose, for every $\mathbf{y} \in \mathbf{A} \setminus \{[\mathbf{x}]_G\}$, a point $z_{\mathbf{y}} \in (\pi_{n,G}^s)^{\leftarrow}(\mathbf{y})$, and put $B = \{z_{\mathbf{y}} : \mathbf{y} \in \mathbf{A} \setminus \{[\mathbf{x}]_G\}\}$. Then $\pi_{n,G}^s(B) = \mathbf{A} \setminus \{[\mathbf{x}]_G\}$. Since $\mathbf{A}$ accumulates at $[\mathbf{x}]_G$ and $SP_G^n X$ is T_1 , we have $[\mathbf{x}]_G \in \overline{\mathbf{A} \setminus \{[\mathbf{x}]_G\}}$. Therefore, $[\mathbf{x}]_G \in \overline{\pi_{n,G}^s(B)}$. The mapping $\pi_{n,G}^s : X^n \rightarrow SP_G^n X$ is closed. Hence $\pi_{n,G}^s(\bar{B})$ is a closed subset of $SP_G^n X$ containing $\pi_{n,G}^s(B)$. Consequently, $[\mathbf{x}]_G \in \pi_{n,G}^s(\bar{B})$. Thus, there exists $z \in (\pi_{n,G}^s)^{\leftarrow}([\mathbf{x}]_G) \cap \bar{B}$. On the other hand, $B \subset (\pi_{n,G}^s)^{\leftarrow}(\mathbf{A} \setminus \{[\mathbf{x}]_G\})$, and therefore $B \cap (\pi_{n,G}^s)^{\leftarrow}([\mathbf{x}]_G) = \emptyset$. Hence $z \notin B$. Thus, $z \in \bar{B} \setminus B$.

Since X is T_1 , the product X^n is also a T_1 -space. If some neighborhood of z contained only finitely many points of B , then, using the T_1 property, we could remove these finitely many points and obtain a neighborhood of z disjoint from B , contradicting $z \in \bar{B}$. Hence every neighborhood of z contains infinitely many points of B . Thus, z is an accumulation point of B .

Since $\pi_{n,G}^s$ is continuous, $(\pi_{n,G}^s)^{\leftarrow}(\mathbf{U})$ is a neighborhood of z in X^n . Since $\mathcal{Q}$ is a Pytkeev network for X^n , there exists $Q \in \mathcal{Q}$ such that $z \in Q \subset (\pi_{n,G}^s)^{\leftarrow}(\mathbf{U})$ and $|Q \cap B|$ is infinite. It follows immediately that $\pi_{n,G}^s(Q) \subset \mathbf{U}$. Moreover, the mapping $\pi_{n,G}^s$ is finite-to-one, since every fiber has cardinality at most $|G|$. Therefore, the image of an infinite subset of X^n under $\pi_{n,G}^s$ is an infinite. Hence $|\pi_{n,G}^s(Q \cap B)|$ is an infinite. Since $\pi_{n,G}^s(Q \cap B) \subset \pi_{n,G}^s(Q) \cap \mathbf{A}$, we obtain $|\pi_{n,G}^s(Q) \cap \mathbf{A}|$ is an infinite. Therefore, for every neighborhood $\mathbf{U}$ of $[\mathbf{x}]_G$ and every subset $\mathbf{A}$ accumulating at $[\mathbf{x}]_G$, there exists $\pi_{n,G}^s(Q) \in SP_G^n \mathcal{Q}$ such that $[\mathbf{x}]_G \in \pi_{n,G}^s(Q) \subset \mathbf{U}$ and $|\pi_{n,G}^s(Q) \cap \mathbf{A}|$ is an infinite. Thus, $SP_G^n \mathcal{Q}$ is a countable Pytkeev network of $SP_G^n X$. Hence $SP_G^n X$ is a $\mathfrak{P}_0$ -space.

($\Leftarrow$) Let $\mathcal{N}$ be a countable Pytkeev network for the space $SP_G^n X$. For T_1 -space X , X is homeomorphic to the closed subspace $\Delta = \{[(x, \dots, x)]_G : x \in X\}$ of $SP_G^n X$. Therefore, it is enough to show that the subspace Δ has a countable Pytkeev network. By Proposition 3.1 of [6] Pytkeev network is preserved under every subspace. Hence, Δ has a countable Pytkeev network. It completes the proof of Theorem 5.

The relationships among the above classes of spaces can be summarized as follows.

Separable metrizable spaces $\longrightarrow \mathfrak{P}_0$ -spaces $\longrightarrow \sigma$ -spaces.

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WEIGHTED $\mathcal{P}_\alpha$ -CAPACITY

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ABSTRACT. In this paper, we study the weighted $\mathcal{P}_\alpha$ -capacity, which generalizes the classical $\mathcal{P}_\alpha$ -capacity. We establish its main properties, including continuity with respect to sets, convergence for compact sets, and subadditivity.

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Key words: α -subharmonic function, α -subharmonic measure, α -polar set, $\mathcal{P}_\alpha$ -capacity, weighted $\mathcal{P}_\alpha$ -capacity.

Introduction

Pluripotential theory constitutes an important branch of mathematics in which the study of classes of functions such as subharmonic (*sh*) and plurisubharmonic (*psh*) functions [12], as well as their generalizations, plays a fundamental role. These classes of functions are widely used in the investigation of differential forms, currents, capacity theory, and boundary value problems, and they are studied in close connection with one another.

An important generalization of the classical class of subharmonic functions is given by α -subharmonic functions (see [1]). Since subharmonic functions arise naturally in many branches of analysis and mathematical physics, their generalizations make it possible to extend classical methods and results to broader settings. For this reason, the study of α -subharmonic measures and the corresponding capacities is of particular interest in modern potential theory.

Recall that the class of α -subharmonic functions is defined by the positivity of the current $dd^c u \wedge \alpha$, where α is a closed, strictly positive differential form of degree $(n-1, n-1)$ in a bounded domain $D \subset \mathbb{C}^n$.

Definition 1. (see [1]) A function $u \in L^1_{loc}(D)$ is called α -subharmonic in D , if it is strongly upper semicontinuous in D , i.e., $u(z)$ is upper semicontinuous in D , in addition, for every $A \subset D$ which has full Lebesgue measure in a neighbourhood of $z^0 \in D$, the following relation

$$\limsup_{\substack{z \rightarrow z^0 \\ z \in A}} u(z) = u(z^0)$$

holds and the current $dd^c u \wedge \alpha$ is positive, i.e., for any positive test function ω in D , we have

$$\int u \alpha \wedge dd^c \omega \geq 0,$$

where $d = \partial + \bar{\partial}$, $d^c = \frac{\bar{\partial} - \partial}{4i}$.

The class of α -subharmonic functions in D is denoted by α -*sh*(D), and for convenience, the function $u(z) \equiv -\infty$ is also considered to be in this class.

Since the weighted (α, ψ) -subharmonic measure was introduced only recently [5], its associated capacity theory has not yet been developed. In this paper, we introduce the corresponding weighted $\mathcal{P}_\alpha$ -capacity and establish its fundamental properties, including continuity with respect to sets, convergence for compact sets, subadditivity, and its relation to α -polar sets. We now state the main results of the paper (see Section 2 and Section 3 for the definition of weighted $\mathcal{P}_\alpha$ -capacity and other relevant definitions).

Theorem 1. The following statements hold for weighted $\mathcal{P}_\alpha$ -capacity.

1. Assume that $\omega_\alpha = \omega_\alpha^*$ a.e on D . Let $\{K_j\}$ be a sequence of compact subsets of D such that $K_{j+1} \subset K_j$ and let ψ be bounded lower semicontinuous function in K_1 and $\sup_{K_1} \psi < 0$. Then, the relation

$$\mathcal{P}_\alpha(K, D, \psi) = \lim_{j \rightarrow \infty} \mathcal{P}_\alpha(K_j, D, \psi)$$

holds, where $K = \bigcap_{j=1}^{\infty} K_j$;

2. Let $U \subset D$ be an open set and let $U = \bigcup_{j=1}^{\infty} K_j$, where $K_j \subset K_{j+1}^{\circ}$ are compact sets. Assume that ψ is bounded upper semicontinuous function in U and $\sup_U \psi < 0$. Then

$$\lim_{j \rightarrow \infty} \mathcal{P}_{\alpha}(K_j, D, \psi) = \mathcal{P}_{\alpha}(U, D, \psi);$$

3. Let $E \subset D$ and ψ be bounded function on E with $\sup_E \psi < 0$. Then, the weighted $\mathcal{P}_{\alpha}$ -capacity of E is zero if and only if E is α -polar in D , that is,

$$\mathcal{P}_{\alpha}(E, D, \psi) \equiv 0 \iff E - \alpha - \text{polar}.$$

The α -subharmonic measure, defined in the class of α -subharmonic functions, has been studied in [10], where a number of its properties have been established. The $\mathcal{P}_{\alpha}$ -capacity in the unweighted case has also been investigated in [10] and the properties of α -subharmonic measure has been used to establish the characteristics of the $\mathcal{P}_{\alpha}$ -capacity. In the weighted setting, the weighted $\mathcal{P}_{\alpha}$ -capacity exhibits properties analogous to those of the $\mathcal{P}_{\alpha}$ -capacity and the corresponding proofs rely on the properties of the weighted α -subharmonic measure established in [5].

This paper is organized as follows: in Section 2, we present the properties of the weighted α -subharmonic measure; in Section 3, we define the capacity in the weighted and unweighted settings and prove Theorem 1. Furthermore, we present the relationship between the weighted and unweighted $\mathcal{P}_{\alpha}$ -capacities at the end.

2. The weighted α - subharmonic measure

Let $D \subset \mathbb{C}^n$ be a bounded α -regular domain, let $E \subset D$ be a fixed subset, and let ψ be a bounded function on E such that

$$\sup_{z \in E} \psi(z) < 0.$$

We denote by $\mathcal{U}(E, D, \psi)$ the class of all functions $u \in \alpha - sh(D)$ satisfying

$$u|_E \leq \psi, \quad u < 0 \quad \text{on } D.$$

We define the extremal function

$$\omega_{\alpha}(z, E, D, \psi) = \sup\{u(z) : u \in \mathcal{U}(E, D, \psi)\}.$$

Definition 2. The upper regularization

$$\omega_{\alpha}^{*}(z, E, D, \psi) = \overline{\lim_{w \rightarrow z}} \omega_{\alpha}(w, E, D, \psi)$$

is called (α, ψ) -subharmonic measure of the set E with respect to D .

By definition and according to the property of α -subharmonic functions (see [1]), the function $\omega_{\alpha}^{*}(z, E, D, \psi)$ is α -subharmonic in D . Note that, when $\psi \equiv -1$, (α, ψ) - subharmonic measure coincides with α -subharmonic measure, that is,

$$\omega_{\alpha}^{*}(z, E, D, -1) \equiv \omega_{\alpha}^{*}(z, E, D),$$

where

$$\omega_{\alpha}^{*}(z, E, D) = (\sup\{u(z) : u \in \alpha - sh(D), u|_E \leq -1, u|_D < 0\})^{*}$$

The (α, ψ) - measure has the following properties.

Proposition 1. [5] The following monotonicity properties hold:

1. if $E_1 \subset E_2 \subset D_1 \subset D_2$ and ψ is defined on E_2 , with $\sup_{E_2} \psi < 0$, then

$$\omega_{\alpha}^{*}(z, E_2, D_2, \psi) \leq \omega_{\alpha}^{*}(z, E_1, D_2, \psi) \leq \omega_{\alpha}^{*}(z, E_1, D_1, \psi), \quad \forall z \in D_1;$$

2. if $\psi_1|_E \leq \psi_2|_E$, then $\omega_{\alpha}^{*}(z, E, D, \psi_1) \leq \omega_{\alpha}^{*}(z, E, D, \psi_2)$, $\forall z \in D$;

3. Let $A \subset D$ be an α -polar set. Then, for any set $E \subset D$,

$$\omega_{\alpha}^{*}(z, E, D, \psi) = \omega_{\alpha}^{*}(z, E \cup A, D, \psi)$$

for all $z \in D$, where ψ is defined on $E \cup A$.

4. If $E \subset\subset D$, then

$$\lim_{z \rightarrow \partial D, z \in D} \omega_{\alpha}^{*}(z, E, D, \psi) = 0.$$

Apart from these properties presented above, the weighted α -subharmonic measure also possesses the properties stated in the following theorem.

Theorem 2. [5] The following properties hold for the weighted α -subharmonic measure.

1. Let $\{K_j\}$ be a sequence of compact subsets of D such that $K_{j+1} \subset K_j$ and let ψ be lower semicontinuous in K_1 . Then

$$\lim_{j \rightarrow \infty} \omega_\alpha(z, K_j, D, \psi) = \omega_\alpha(z, \bigcap_{j=1}^{\infty} K_j, D, \psi)$$

for all $z \in D$.

2. Let $E \subset \subset D_1$ and let $\{D_j\}$ be an increasing sequence of domains such that $D_j \subset D_{j+1}$, $D = \bigcup_{j=1}^{\infty} D_j$. Then

$$\lim_{j \rightarrow \infty} \omega_\alpha^*(z, E, D_j, \psi) = \omega_\alpha^*(z, E, D, \psi).$$

3. Let $E \subset D$ be an arbitrary set and let ψ be lower semicontinuous on E . Then, there exists a decreasing sequence of open sets $U_j \supset E$ such that

$$\left(\lim_{j \rightarrow \infty} \omega_\alpha^*(z, U_j, D, \tilde{\psi}) \right)^* = \omega_\alpha^*(z, E, D, \psi),$$

where $\tilde{\psi}$ is lower semicontinuous in some open neighborhood V of E and satisfies

$$\tilde{\psi}|_E = \psi|_E, \quad \sup_{z \in V} \tilde{\psi}(z) < 0.$$

4. Let $U \subset D$ be an open set and let $U = \bigcup_{j=1}^{\infty} K_j$, where $K_j \subset K_{j+1}^\circ$ are compact sets. Assume that ψ is upper semicontinuous in U . Then

$$\omega_\alpha^*(z, K_j, D, \psi) \downarrow \omega_\alpha^*(z, U, D, \psi).$$

The following lemma, which establishes the connection between weighted and unweighted α -subharmonic measures, is crucial for the study of $\mathcal{P}_\alpha$ -capacities.

Lemma 1. [5] For any set $E \subset D$, the following inequalities hold for all $z \in D$

$$-\inf_{w \in E} \psi(w) \omega_\alpha^*(z, E, D) \leq \omega_\alpha^*(z, E, D, \psi) \leq -\sup_{w \in E} \psi(w) \omega_\alpha^*(z, E, D).$$

3. The weighted $\mathcal{P}_\alpha$ -capacity and its properties

We first recall the definition of the $\mathcal{P}_\alpha$ -capacity. Let us be given any subset $E \subset D$ and its α -subharmonic measure $\omega_\alpha^*(z, E, D)$.

Definition 3. (see [10]). The quality

$$\mathcal{P}_\alpha(E, D) = - \int_D \omega_\alpha^*(z, E, D) dV$$

is called $\mathcal{P}_\alpha$ -capacity of E with respect to D , where dV is volume form in $\mathbb{C}^n$.

By the definition, the following estimate holds for the $\mathcal{P}_\alpha$ -capacity

$$0 \leq \mathcal{P}_\alpha(E, D) \leq m(D),$$

where m is Lebesgue measure in $\mathbb{C}^n$.

The $\mathcal{P}_\alpha$ -capacity was studied in [10] and the following properties were established.

1. the $\mathcal{P}_\alpha$ -capacity is monotone increasing and subadditive with respect to sets, i.e.,

$$E_1 \subset E_2 \implies \mathcal{P}_\alpha(E_1, D) \leq \mathcal{P}_\alpha(E_2, D)$$

and

$$\mathcal{P}_\alpha \left(\bigcup_{j=1}^{\infty} E_j, D \right) \leq \sum_{j=1}^{\infty} \mathcal{P}_\alpha(E_j, D);$$

2. for any subset $E \subset D$ and for any $\varepsilon > 0$, there exists an open set $U \supset E$ such that

$$\mathcal{P}_\alpha(U, D) - \mathcal{P}_\alpha(E, D) < \varepsilon;$$

3. for any decreasing sequence of compact sets $K_1 \supset K_2 \supset \dots$, the following relation holds:

$$\mathcal{P}_\alpha\left(\bigcap_{j=1}^{\infty} K_j, D\right) = \lim_{j \rightarrow \infty} \mathcal{P}_\alpha(K_j, D);$$

4. for any increasing sequence of open sets $G_1 \subset G_2 \subset \dots$, the following relation holds:

$$\mathcal{P}_\alpha\left(\bigcup_{j=1}^{\infty} G_j, D\right) = \lim_{j \rightarrow \infty} \mathcal{P}_\alpha(G_j, D);$$

Nevertheless, there remain numerous properties of the $\mathcal{P}_\alpha$ -capacity in the weighted setting that have not yet been fully explored, and the study of these properties continues to be of considerable interest in modern analysis. In the present work, we aim to contribute to this line of research by establishing and discussing several of its properties in the weighted setting.

We now begin the study of capacity in the weighted case.

Definition 4. The quality

$$\mathcal{P}_\alpha(E, D, \psi) = - \int_D \omega_\alpha^*(z, E, D, \psi) dV$$

is called the weighted $\mathcal{P}_\alpha$ -capacity of E with respect to D .

Note that, when $\psi \equiv -1$, the weighted and unweighted capacity coincide, that is,

$$\mathcal{P}_\alpha(E, D, -1) = \mathcal{P}_\alpha(E, D).$$

Since the weighted α -subharmonic measure $\omega_\alpha^*(z, E, D, \psi)$ satisfies the relations

$$\inf_{z \in E} \psi(z) \leq \omega_\alpha^*(z, E, D, \psi) \leq 0,$$

by definition of the weighted $\mathcal{P}_\alpha$ -capacity, we have

$$0 \leq \mathcal{P}_\alpha(E, D, \psi) \leq - \inf_{z \in E} \psi(z) \cdot m(D).$$

In addition, the following proposition characterizing the monotonicity properties of $\mathcal{P}_{(\alpha, \psi)}$ -capacity also holds.

Proposition 2. The following statements are true:

1. if $E_1 \subset E_2 \subset D$ and ψ is defined on E_2 , with $\sup_{E_2} \psi < 0$, then $\mathcal{P}_\alpha(E_1, D, \psi) \leq \mathcal{P}_\alpha(E_2, D, \psi)$;
2. if $\psi_1 \leq \psi_2$ and $\sup_E \psi_2 < 0$, then $\mathcal{P}_\alpha(E, D, \psi_2) \leq \mathcal{P}_\alpha(E, D, \psi_1)$;
3. if $E \subset D_1 \subset D_2$, then $\mathcal{P}_\alpha(E, D_1, \psi) \leq \mathcal{P}_\alpha(E, D_2, \psi)$ holds.

One can easily get the proof of the Proposition 2 by the definition and using the properties of the weighted (α, ψ) -subharmonic measure. This proposition shows the monotonicity properties of the weighted $\mathcal{P}_\alpha$ -capacity with respect to sets, domains and the weight functions, playing an important role in studying convergence properties.

Next, we give the proof of the Theorem 1.

Proof of Theorem 1. Now, we show the proof of the first assertion. By the theorem's assumption, Lebesgue measure of the set $\{\omega_\alpha < \omega_\alpha^*\}$ is 0, thus we find

$$\mathcal{P}_\alpha(K_j, D, \psi) = - \int_D \omega_\alpha^*(z, K_j, D, \psi) dV = - \int_D \omega_\alpha(z, K_j, D, \psi) dV$$

and

$$\mathcal{P}_\alpha(K, D, \psi) = - \int_D \omega_\alpha^*(z, K, D, \psi) dV = - \int_D \omega_\alpha(z, K, D, \psi) dV.$$

By the property 1 in Theorem 2,

$$\omega_\alpha(z, K_j, D, \psi) \uparrow \omega_\alpha(z, K, D, \psi).$$

Setting

$$f_j = \omega_\alpha(z, K_j, D, \psi) - \omega_\alpha(z, K_1, D, \psi) \geq 0,$$

and applying Levi's Theorem, we get

$$\lim_{j \rightarrow \infty} \int_D \omega_\alpha(z, K_j, D, \psi) dV = \int_D \omega_\alpha(z, K, D, \psi) dV.$$

Hence,

$$\lim_{j \rightarrow \infty} \mathcal{P}_\alpha(K_j, D, \psi) = \mathcal{P}_\alpha(K, D, \psi).$$

Next, we prove the second assertion. According to the property 4 in Theorem 2, we get the following relation

$$\omega_\alpha^*(z, K_j, D, \psi) \downarrow \omega_\alpha^*(z, U, D, \psi)$$

for all $z \in D$. Therefore, the nonnegative sequence $\{-\omega_\alpha^*(z, K_j, D, \psi)\}_{j=1}^\infty$ is increasing with respect to j . Thus, Levi's Theorem (see [11]) gives

$$\begin{aligned} \lim_{j \rightarrow \infty} \mathcal{P}_\alpha(K_j, D, \psi) &= - \lim_{j \rightarrow \infty} \int_D \omega_\alpha^*(z, K_j, D, \psi) dV \\ &= - \int_D \omega_\alpha^*(z, U, D, \psi) dV = \mathcal{P}_\alpha(U, D, \psi). \end{aligned}$$

Now, we turn to prove the property 3. First, let's assume that $\mathcal{P}_\alpha(E, D, \psi) = 0$. Since

$$\mathcal{P}_\alpha(E, D, \psi) = 0$$

and

$$\mathcal{P}_\alpha(E, D, \psi) = - \int_D \omega_\alpha^*(z, E, D, \psi) dV,$$

while

$$\omega_\alpha^*(z, E, D, \psi) \leq 0 \quad \text{in } D,$$

we obtain

$$\omega_\alpha^*(z, E, D, \psi) = 0 \quad \text{for almost every } z \in D.$$

We claim that

$$\omega_\alpha^*(z, E, D, \psi) \equiv 0 \quad \text{in } D.$$

Indeed, suppose, to the contrary, that there exists $z_0 \in D$ such that

$$\omega_\alpha^*(z_0, E, D, \psi) < 0.$$

Since $\omega_\alpha^*(z, E, D, \psi)$ is upper semicontinuous in D , there exist $\varepsilon > 0$ and a neighborhood $B(z_0, r) \Subset D$ such that

$$\omega_\alpha^*(z, E, D, \psi) < -\varepsilon \quad \text{for all } z \in B(z_0, r).$$

Consequently,

$$\int_D \omega_\alpha^*(z, E, D, \psi) dV \leq -\varepsilon m(B(z_0, r)) < 0,$$

which contradicts

$$\int_D \omega_\alpha^*(z, E, D, \psi) dV = 0.$$

Hence,

$$\omega_\alpha^*(z, E, D, \psi) \equiv 0 \quad \text{in } D.$$

By Lemma 1, we find that $\omega_\alpha^*(z, E, D) \equiv 0$ which implies that E is α -polar in D .

Now, we prove the converse. Suppose that E is α -polar in D . Then, by the property of the unweighted α -subharmonic measure, $\omega_\alpha^*(z, E, D) \equiv 0$ on D [10]. Again, using Lemma 1, we find that

$$\omega_\alpha^*(z, E, D, \psi) \equiv 0 \quad \text{on } D.$$

Therefore, by the definition of the weighted $\mathcal{P}_\alpha$ -capacity, we have

$$\mathcal{P}_\alpha(E, D, \psi) = - \int_D \omega_\alpha^*(z, E, D, \psi) dV = 0.$$

The proven property indicates that sets of zero $\mathcal{P}_\alpha$ -capacity are necessarily very small in a measure-theoretic sense; in particular, such sets have Lebesgue measure zero.

The following lemma on the countable subadditivity of weighted capacity and the relation between weighted and unweighted $\mathcal{P}_\alpha$ -capacities is crucial for establishing their properties and relationships.

Lemma 2. The following statements hold for weighted $\mathcal{P}_\alpha$ -capacity.

1. Let $E_j \subset D$ for $j = 1, 2, \dots$, $E = \bigcup_{j=1}^{\infty} E_j$ and ψ is defined on E , with $\sup_E \psi < 0$. Assume that the sets

$$\{z \in D : \omega_\alpha(z, E_j, D, \psi) < \omega_\alpha^*(z, E_j, D, \psi)\}$$

and

$$\{z \in D : \omega_\alpha(z, E, D, \psi) < \omega_\alpha^*(z, E, D, \psi)\}$$

have Lebesgue measure zero. Then

$$\mathcal{P}_\alpha\left(\bigcup_{j=1}^{\infty} E_j, D, \psi\right) \leq \sum_{j=1}^{\infty} \mathcal{P}_\alpha(E_j, D, \psi);$$

2. Let E be any subset of D . Then, the following relations hold

$$-\sup_{w \in E} \psi(w) \mathcal{P}_\alpha(E, D) \leq \mathcal{P}_\alpha(E, D, \psi) \leq -\inf_{w \in E} \psi(w) \mathcal{P}_\alpha(E, D).$$

Proof. To prove the first assertion, we take an arbitrary function $u_j(z)$ from each class $\mathcal{U}(E_j, D, \psi)$ and consider the function $u(z) = \sum_{j=1}^{\infty} u_j(z)$. Since for every $j \in \mathbb{N}$, $u_j|_D < 0$ is valid, so the sequence $S_k(z) = \sum_{j=1}^k u_j(z)$ is decreasing with respect to k and $S_k \in \alpha - sh(D)$. Therefore, as the limit function of decreasing sequence of α -subharmonic functions, $u(z)$ is α -subharmonic in D [1]. If we take a point $z \in E$, there exists a number $j_0 \in \mathbb{N}$ such that $z \in E_{j_0}$. Since $u_j|_D < 0$ for all j , the relation

$$u(z) = \sum_{j=1}^{\infty} u_j(z) \leq u_{j_0}(z) \leq \psi(z)$$

holds, which implies that

$$u \in \mathcal{U}\left(\bigcup_{j=1}^{\infty} E_j, D, \psi\right).$$

Therefore, for all $z \in D$, we have the following inequality

$$u(z) = \sum_{j=1}^{\infty} u_j(z) \leq \omega_\alpha\left(z, \bigcup_{j=1}^{\infty} E_j, D, \psi\right). \quad (1)$$

Fix $\varepsilon > 0$ and $z \in D$. By the definition of ω_α , for each $j \in \mathbb{N}$, we may choose u_j such that

$$u_j(z) > \omega_\alpha(z, E_j, D, \psi) - \frac{\varepsilon}{2^j}.$$

It follows that

$$\sum_{j=1}^{\infty} u_j(z) > \sum_{j=1}^{\infty} \omega_\alpha(z, E_j, D, \psi) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and the inequality (1) holds, we find that

$$\omega_\alpha\left(z, \bigcup_{j=1}^{\infty} E_j, D, \psi\right) \geq \sum_{j=1}^{\infty} \omega_\alpha(z, E_j, D, \psi). \quad (2)$$

By the theorem's condition, the sets

$$\{z \in D : \omega_\alpha(z, E_j, D, \psi) < \omega_\alpha^*(z, E_j, D, \psi)\}$$

and

$$\{z \in D : \omega_\alpha(z, \bigcup_{j=1}^{\infty} E_j, D, \psi) < \omega_\alpha^*(z, \bigcup_{j=1}^{\infty} E_j, D, \psi)\}$$

have Lebesgue measure zero. Then by the inequality (2) and according to the Levi's Theorem (see, for example [11]), we get

$$\begin{aligned} \mathcal{P}_\alpha \left(\bigcup_{j=1}^{\infty} E_j, D, \psi \right) &= - \int_D \omega_\alpha^* \left(z, \bigcup_{j=1}^{\infty} E_j, D, \psi \right) dV = - \int_D \omega_\alpha \left(z, \bigcup_{j=1}^{\infty} E_j, D, \psi \right) dV \\ &\leq - \int_D \left(\sum_{j=1}^{\infty} \omega_\alpha(z, E_j, D, \psi) \right) dV = - \sum_{j=1}^{\infty} \int_D \omega_\alpha(z, E_j, D, \psi) dV \\ &= - \sum_{j=1}^{\infty} \int_D \omega_\alpha^*(z, E_j, D, \psi) dV = \sum_{j=1}^{\infty} \mathcal{P}_\alpha(E_j, D, \psi). \end{aligned}$$

which proves the countably subadditiveness of the weighted $\mathcal{P}_\alpha$ -capacity.

Now, we prove the second assertion. According to Lemma 1, we have the following relation between the weighted and unweighted α -subharmonic measures

$$- \inf_{w \in E} \psi(w) \omega_\alpha^*(z, E, D) \leq \omega_\alpha^*(z, E, D, \psi) \leq - \sup_{w \in E} \psi(w) \omega_\alpha^*(z, E, D).$$

Multiplying the last inequality to (-1) and integrating over D , we get that

$$\sup_{w \in E} \psi(w) \int_D \omega_\alpha^*(z, E, D) dV \leq - \int_D \omega_\alpha^*(z, E, D, \psi) dV \leq \inf_{w \in E} \psi(w) \int_D \omega_\alpha^*(z, E, D) dV,$$

which is equivalent to

$$- \sup_{w \in E} \psi(w) \mathcal{P}_\alpha(E, D) \leq \mathcal{P}_\alpha(E, D, \psi) \leq - \inf_{w \in E} \psi(w) \mathcal{P}_\alpha(E, D).$$

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ERGODICITY OF ISOMETRIES MAPPING A SPHERE INTO ITSELF IN THE FIELD OF p -ADIC NUMBERS

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ABSTRACT. In this paper we study measure-preserving dynamical systems generated by isometries acting on spheres in the field of p -adic numbers. We focus on the boundary case $\rho(r) = |f(x) - x|_p = r$, which is not covered by the proper invariant-ball argument. Anashin's finite-level criterion is formulated for 1-Lipschitz self-maps of $\mathbb{Z}_p$ and therefore applies, after affine normalization, to dynamical systems on invariant balls. For $p \geq 3$, an invariant sphere is not a ball: it is a disjoint union of $p - 1$ balls of the next smaller radius. Hence the criterion on $\mathbb{Z}_p$ is not directly applicable to a map preserving the whole sphere. We establish an intrinsic finite-factor criterion for isometries of arbitrary p -adic spheres: the system is ergodic if and only if the permutations induced on all canonical partitions of the sphere consist of single cycles. We further prove a metric cyclic-lifting criterion: once the first finite factor consists of a single cycle, cyclicity at every subsequent level is equivalent to the return equalities $|f^{jN_n}(x_*) - x_*|_p = r/p^n$, $j = 1, \dots, p - 1$, at one point and at every level, where $N_n = (p - 1)p^{n-1}$. We apply these criteria to the borderline invariant spheres of $(2, 2)$ -rational functions with two fixed points and of $(2, 1)$ -rational functions with a unique fixed point. In both families we give explicit ergodic examples over $\mathbb{Q}_3$. The resulting ergodic systems are also minimal and uniquely ergodic.

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Key words: p -adic numbers, sphere, isometry, Haar measure, ergodicity, finite dynamical system.

Introduction

p -Adic ergodic theory is a part of non-Archimedean dynamics which studies measure-preserving transformations on ultrametric spaces. One of the important features of p -adic spaces is their hierarchical structure: every ball is a finite disjoint union of balls of smaller radius, and two balls either do not intersect or one of them contains the other. Therefore, the dynamics of a continuous or isometric map on a compact p -adic set can often be described by finite dynamical systems induced by partitions into balls.

For general background on ergodic transformations and non-Archimedean dynamical systems, see [1–4, 19]. Basic facts on p -adic analysis used in this paper can be found in [5, 6]. Rational p -adic dynamical systems and their ergodicity on invariant spheres were studied in [7–9, 13, 17]. Closely related p -adic mappings arising from statistical-mechanical models on Cayley trees, including Potts–Bethe and Ising–Potts type mappings, were investigated in [8, 10–12].

Most results on ergodicity of p -adic dynamical systems were first obtained for maps acting on the ring of p -adic integers $\mathbb{Z}_p$. This is natural, because $\mathbb{Z}_p$ is a compact topological group and carries a normalized Haar measure. At the same time, in rational p -adic dynamics the invariant sets of a map are often not equal to $\mathbb{Z}_p$; they can be balls and spheres with arbitrary centers and radii. For this reason, the study of Haar measure and ergodicity on arbitrary invariant spheres is a natural continuation of the classical problem on $\mathbb{Z}_p$.

In [18] the group structure of arbitrary balls and spheres was constructed, the Haar measure on such sets was described, and the dynamical system generated by an isometry on an invariant sphere was studied. In particular, if the quantity

$$\rho(r) = |f(x) - x|_p$$

does not depend on $x \in S_r(a)$ and satisfies $\rho(r) < r$, then the ergodicity problem was solved there. Namely, for $p \geq 3$ the system is not ergodic, and for $p = 2$ the system may be ergodic only in the case $r = 2\rho(r)$. Thus, the case $\rho(r) < r$ is not new; the novelty of the present work lies in the boundary case $\rho(r) = r$.

The aim of this paper is to study the remaining case

$$\rho(r) = r. \quad (1)$$

This case is more delicate, because the ball of radius $\rho(r)$ coincides with the whole sphere scale and does not automatically give a proper invariant ball of positive measure. To overcome this difficulty, we consider finite partitions of a sphere

into balls of smaller radii and reduce the ergodicity problem to the cyclicity of the induced permutations on these finite factors.

The distinction between dynamics on a ball and dynamics on a sphere is essential here. Anashin proved that a 1-Lipschitz self-map $F : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ is ergodic with respect to normalized Haar measure if and only if every reduction of F modulo p^n is a single-cycle permutation [1–3]. Since every p -adic ball is an affine copy of $\mathbb{Z}_p$, this criterion can be transferred, by normalization, to a map carrying a ball into itself. It also explains why the case $p = 2$ is exceptional: for every $k \in \mathbb{Z}$,

$$S_{2^{-k}}(a) = V_{2^{-(k+1)}}(a + 2^k),$$

so a 2-adic sphere is itself a ball; compare [4]. In contrast, for $p \geq 3$,

$$S_{p^{-k}}(a) = \bigsqcup_{i=1}^{p-1} V_{p^{-(k+1)}}(a + ip^k),$$

and the natural normalization $x = a + p^k u$ identifies the sphere with the unit group $\mathbb{Z}_p^*$ rather than with $\mathbb{Z}_p$. A map preserving the whole sphere may permute its $p - 1$ maximal ball components. Consequently, Anashin's criterion on $\mathbb{Z}_p$ is not directly applicable to the ergodicity problem for arbitrary sphere-preserving isometries when $p \geq 3$. Anashin also obtained ergodicity results for certain polynomial and locally analytic maps on spheres [1,2], but those results do not provide an intrinsic criterion for arbitrary isometries of a sphere.

The first main contribution of the present paper is therefore Theorem 5, an intrinsic finite-factor criterion formulated directly on $S_{p^{-k}}(a)$. Its canonical factor $X_{n,k}$ has

$$|X_{n,k}| = (p - 1)p^{n-1},$$

which reflects the decomposition of the sphere into the nonzero residue classes and their successive refinements. The second main contribution is the metric cyclic-lifting criterion of Theorem 6, which characterizes the passage from level n to level $n + 1$ by return-distance equalities at a single point. We then apply these criteria to the boundary case $\rho(r) = r$ for two classes of rational p -adic maps.

The paper is organized as follows. In Section 2 we recall basic notation. In Section 3 we formulate previous results on the group structure, Haar measure and the case $\rho(r) < r$. In Section 4 we consider finite factors of a sphere. In Section 5 we prove the ergodicity and cyclic-lifting criteria for the case $\rho(r) = r$. In Section 6 we give two model examples. In Section 7 we apply the criteria to two classes of rational functions.

Preliminaries

Let p be a prime number and let $\mathbb{Q}_p$ be the field of p -adic numbers with the p -adic absolute value $|\cdot|_p$. We write

$$\nu_p(x) = -\log_p |x|_p \quad (x \neq 0), \quad \nu_p(0) = +\infty,$$

and denote the group of p -adic units by

$$\mathbb{Z}_p^* = \{u \in \mathbb{Z}_p : |u|_p = 1\}.$$

For $a \in \mathbb{Q}_p$ and $r > 0$ we use the notation

$$U_r(a) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}, \quad V_r(a) = \{x \in \mathbb{Q}_p : |x - a|_p \leq r\},$$

$$S_r(a) = \{x \in \mathbb{Q}_p : |x - a|_p = r\}.$$

In what follows, every radius is assumed to belong to the value group of $\mathbb{Q}_p$; equivalently, $r = p^m$ for some $m \in \mathbb{Z}$.

Let $(X, \mathcal{B}, \mu)$ be a probability space. A measurable map $T : X \rightarrow X$ is measure-preserving if

$$\mu(T^{-1}(E)) = \mu(E), \quad E \in \mathcal{B}.$$

It is called ergodic if every measurable set E satisfying

$$\mu(T^{-1}(E) \triangle E) = 0$$

has measure either 0 or 1. Thus invariant sets are understood modulo sets of measure zero.

In this paper, an isometry of $S_r(a)$ means a bijection $f : S_r(a) \rightarrow S_r(a)$ satisfying

$$|f(x) - f(y)|_p = |x - y|_p, \quad x, y \in S_r(a). \quad (2)$$

Previous results

We recall several results which were proved in [18]. They are used in the present paper without repeating their proofs.

Let $a \in \mathbb{Q}_p$ and $r \in \{p^m : m \in \mathbb{Z}\}$. On the ball $V_r(a)$ and the sphere $S_r(a)$ define the operations

$$\begin{aligned} x \oplus y &= x + y - a, & x, y \in V_r(a), \\ x \odot y &= r(x - a)(y - a) + a, & x, y \in S_r(a). \end{aligned} \quad (3)$$

Here the number r in (3) is considered as an element of $\mathbb{Q}_p$.

Theorem 1. (see [18]). The following statements hold:

- (i) $(V_r(a), \oplus)$ is a compact topological additive Abelian group;
- (ii) $(S_r(a), \odot)$ is a compact topological multiplicative Abelian group.

Theorem 2. (see [18]). Any two balls with positive radii are isomorphic as groups. Any two spheres with positive radii are also isomorphic as groups.

Let $\mathcal{B}$ be the Borel σ -algebra generated by clopen subsets of $V_r(a)$ or $S_r(a)$. The measure $\bar{\mu}$ defined by

$$\bar{\mu}(V_\rho(s)) = \rho$$

is the Haar measure on the corresponding compact group. On the sphere $S_r(a)$ we use the normalized Haar measure

$$\mu(V_\rho(s)) = \frac{\bar{\mu}(V_\rho(s))}{\bar{\mu}(S_r(a))} = \frac{p\rho}{(p-1)r}.$$

Theorem 3. (see [18]). Let $f : S_r(a) \rightarrow S_r(a)$ be an isometry and let μ be the normalized Haar measure on $S_r(a)$. Then f preserves μ , i.e.

$$\mu(f(V_\rho(s))) = \mu(V_\rho(s))$$

for every ball $V_\rho(s) \subset S_r(a)$. Moreover, the orbit of any non-fixed initial point is not convergent.

Assume that the value $|f(x) - x|_p$ does not depend on the choice of $x \in S_r(a)$. We denote

$$\rho(r) = |f(x) - x|_p, \quad x \in S_r(a). \quad (4)$$

Theorem 4. (see [18]). Let $f : S_r(a) \rightarrow S_r(a)$ be an isometry and let (4) hold. If $\rho(r) < r$, then

- (i) the dynamical system $(S_r(a), f, \mu)$ is not ergodic for all $p \geq 3$;
- (ii) for $p = 2$ the dynamical system $(S_r(a), f, \mu)$ may be ergodic only if $r = 2\rho(r)$.

The proof of this theorem is based on the fact that a ball of radius $\rho(r)$ is a minimal invariant ball. Therefore, if $\rho(r) < r$, a proper invariant ball appears automatically. This is the reason why in the present paper we consider only the case (1).

Remark 1. Some rational p -adic dynamical systems are isometries on their invariant spheres. For instance, such phenomena were studied for rational functions in [7–9, 13, 17]. In those works the ergodicity of concrete rational maps was analyzed on invariant spheres. In the present paper we formulate the result for arbitrary isometries satisfying (1).

Finite partitions of a sphere

Let $k \in \mathbb{Z}$ and consider the sphere $S_{p^{-k}}(a)$. For every $n \geq 1$ this sphere can be represented as a disjoint union of balls of radius $p^{-(k+n)}$ in the following form:

$$S_{p^{-k}}(a) = \bigcup_{i_1=1}^{p-1} \bigcup_{i_2=0}^{p-1} \cdots \bigcup_{i_n=0}^{p-1} V_{p^{-(k+n)}} \left(a + p^k \sum_{l=1}^n i_l p^{l-1} \right). \quad (5)$$

Denote by $X_{n,k}$ the set of all balls appearing in the decomposition (5). Then

$$|X_{n,k}| = p^{n-1}(p-1).$$

To see the decomposition explicitly, write every $x \in S_{p^{-k}}(a)$ in its unique form

$$x - a = p^k (i_1 + i_2 p + \cdots + i_n p^{n-1} + p^n z), \quad z \in \mathbb{Z}_p,$$

where $i_1 \in \{1, \dots, p-1\}$ and $i_2, \dots, i_n \in \{0, \dots, p-1\}$. Fixing the first n digits gives one ball of radius $p^{-(k+n)}$, while two different digit strings give disjoint balls. This also explains directly why the number of atoms is $(p-1)p^{n-1}$.

Let $\nu_{n,k}$ be the normalized counting measure on $X_{n,k}$, i.e.

$$\nu_{n,k}(A) = \frac{|A|}{|X_{n,k}|}, \quad A \subset X_{n,k}.$$

Remark 2. No positivity assumption on k is needed. Indeed, the affine isometry $u \mapsto a + p^k u$ maps $\mathbb{Z}_p^* = S_1(0)$ onto $S_{p^{-k}}(a)$, and the decomposition (5) is valid for every $k \in \mathbb{Z}$. Consequently, all results in this and the next section hold for arbitrary $k \in \mathbb{Z}$. In examples based on integral congruence expansions we may retain $k \geq 1$ for convenience.

Lemma 1. Let $k \in \mathbb{Z}$, $n \geq 1$, and let $f : S_{p^{-k}}(a) \rightarrow S_{p^{-k}}(a)$ be an isometry. Then f induces a permutation on $X_{n,k}$.

Proof. Put

$$\varepsilon = p^{-(k+n)}$$

and let $B = V_\varepsilon(s)$ be an atom of $X_{n,k}$. If $x \in B$, then

$$|f(x) - f(s)|_p = |x - s|_p \leq \varepsilon.$$

Hence

$$f(B) \subset V_\varepsilon(f(s)). \quad (5a)$$

Because $\varepsilon < p^{-k} = |f(s) - a|_p$, the ultrametric inequality shows that every point of $V_\varepsilon(f(s))$ still belongs to $S_{p^{-k}}(a)$. Thus this ball is one of the atoms of $X_{n,k}$.

We now prove the reverse inclusion in (5a). Let $z \in V_\varepsilon(f(s))$. Since f is bijective, there is a point $x \in S_{p^{-k}}(a)$ such that $z = f(x)$. Using the isometry property once more, we get

$$|x - s|_p = |f(x) - f(s)|_p = |z - f(s)|_p \leq \varepsilon.$$

Therefore $x \in B$, and consequently $z = f(x) \in f(B)$. Hence

$$f(B) = V_\varepsilon(f(s)).$$

Thus f maps each atom of $X_{n,k}$ onto an atom of the same partition. If two atoms had the same image, their inverse images under the bijection f would coincide. Therefore the induced map on the finite set $X_{n,k}$ is injective, and hence it is a permutation.

We denote the induced permutation on $X_{n,k}$ by the same symbol f .

Remark 3. Let T be a permutation of a finite set X and let ν be the uniform measure. Every T -invariant subset of X is a union of cycles of T . Therefore (X, T, ν) is ergodic if and only if T has exactly one cycle. Applied to $(X_{n,k}, f, \nu_{n,k})$, this means that the induced permutation must act transitively on all balls of the partition (5).

Ergodicity in the case $\rho(r) = r$

In this section let $k \in \mathbb{Z}$ and let $f : S_{p^{-k}}(a) \rightarrow S_{p^{-k}}(a)$ be an isometry. The finite-factor criteria below are valid without any displacement assumption. For applications to the boundary case we shall additionally impose

$$\rho(r) = |f(x) - x|_p = r, \quad r = p^{-k}, \quad x \in S_r(a). \quad (6)$$

The following criterion is formulated intrinsically on the sphere. Under the normalization

$$x = a + p^k u,$$

the sphere $S_{p^{-k}}(a)$ is identified with $\mathbb{Z}_p^*$, and the partition $X_{n,k}$ is identified with the unit residue classes $(\mathbb{Z}/p^n\mathbb{Z})^*$. Thus

$$|X_{n,k}| = |(\mathbb{Z}/p^n\mathbb{Z})^*| = (p-1)p^{n-1}.$$

For $p \geq 3$ these factors are not the residue rings $\mathbb{Z}/p^n\mathbb{Z}$ occurring in Anashin's criterion on $\mathbb{Z}_p$. Hence the next theorem is not obtained by reducing the system to a self-map of one invariant ball; it is proved directly from the canonical partitions of the entire sphere.

For later use, put

$$N_n = |X_{n,k}| = (p-1)p^{n-1}, \quad n \geq 1. \quad (7)$$

Theorem 5. Let $k \in \mathbb{Z}$, let $f : S_{p^{-k}}(a) \rightarrow S_{p^{-k}}(a)$ be an isometry and let μ_k be the normalized Haar measure on $S_{p^{-k}}(a)$. Then the dynamical system

$$(S_{p^{-k}}(a), f, \mu_k)$$

is ergodic if and only if the finite dynamical system

$$(X_{n,k}, f, \nu_{n,k})$$

is ergodic for every $n \geq 1$.

Proof. We prove the two implications separately.

Assume first that $(S_{p-k}(a), f, \mu_k)$ is ergodic. Suppose, to the contrary, that for some $n \geq 1$ the induced permutation on $X_{n,k}$ has at least two cycles. Let Y be one of these cycles and define

$$A = \bigcup_{B \in Y} B.$$

The set A is clopen because it is a finite union of balls. It is also f -invariant, since the atoms in Y are permuted among themselves. Moreover, Y is nonempty and is not the whole of $X_{n,k}$, so

$$0 < \mu_k(A) < 1.$$

This contradicts ergodicity. Hence the induced permutation on every $X_{n,k}$ consists of a single cycle.

Conversely, assume that the induced permutation on $X_{n,k}$ consists of a single cycle for every $n \geq 1$. Let A be a measurable set invariant modulo measure zero, that is,

$$\mu_k(f^{-1}(A) \triangle A) = 0.$$

We shall prove that $\mu_k(A)$ is either 0 or 1.

First note that, for every integer $j \geq 1$,

$$\mu_k(f^{-j}(A) \triangle A) = 0. \quad (7a)$$

Indeed, the assertion is given for $j = 1$. If it holds for j , then

$$f^{-(j+1)}(A) \triangle A \subset f^{-1}(f^{-j}(A) \triangle A) \cup (f^{-1}(A) \triangle A).$$

Both sets on the right have measure zero because f preserves μ_k . Thus (7a) follows by induction.

Fix $n \geq 1$ and take any two atoms $B, C \in X_{n,k}$. Since the induced permutation is a single cycle, there exists an integer $j \geq 0$ such that $f^j(B) = C$. The map f is invertible and measure-preserving, and the invariance of A implies $f^{-j}(A) = A$ modulo a null set. Therefore

$$\begin{aligned} \mu_k(A \cap C) &= \mu_k(f^{-j}(A \cap C)) \\ &= \mu_k(f^{-j}(A) \cap B) \\ &= \mu_k(A \cap B). \end{aligned} \quad (7b)$$

Thus $\mu_k(A \cap B)$ has the same value for all atoms $B \in X_{n,k}$. There are $N_n = (p-1)p^{n-1}$ such atoms, each of measure $1/N_n$. Summing (7b) over all atoms gives

$$\mu_k(A \cap B) = \frac{\mu_k(A)}{N_n} \quad (B \in X_{n,k}),$$

and hence

$$\frac{\mu_k(A \cap B)}{\mu_k(B)} = \mu_k(A). \quad (7c)$$

Suppose, for contradiction, that

$$0 < a := \mu_k(A) < 1.$$

Finite unions of atoms of the partitions $X_{n,k}$ form an algebra that generates the Borel σ -algebra. Since Haar measure on the compact sphere is regular and the atoms form a clopen base, for every $\varepsilon > 0$ there exist a level n and a union U of atoms of $X_{n,k}$ such that

$$\mu_k(A \triangle U) < \varepsilon. \quad (7d)$$

Here are the details. By regularity, choose a compact set $K \subset A$ and an open set $O \supset A$ such that

$$\mu_k(O \setminus K) < \varepsilon.$$

For every $x \in K$, the partition balls form a neighbourhood base, so there is an atom $B_{n_x}(x)$ satisfying

$$x \in B_{n_x}(x) \subset O.$$

Compactness of K gives a finite subcover. Let n be at least as large as all levels occurring in this subcover and refine each selected ball into level- n atoms. If U is the union of the resulting atoms, then

$$K \subset U \subset O.$$

Consequently,

$$A \triangle U \subset O \setminus K,$$

which proves (7d).

Let $t = \mu_k(U)$. Since U is a union of level- n atoms, (7c) gives

$$\mu_k(A \cap U) = at.$$

Therefore

$$\begin{aligned} \mu_k(A \triangle U) &= \mu_k(A) + \mu_k(U) - 2\mu_k(A \cap U) \\ &= a + t - 2at. \end{aligned} \tag{7e}$$

If $0 < a \leq 1/2$, then $a + t - 2at \geq a$. If $1/2 \leq a < 1$, then $a + t - 2at \geq 1 - a$. Thus in all cases

$$\mu_k(A \triangle U) \geq \min\{a, 1 - a\} > 0,$$

which contradicts (7d) for $\varepsilon < \min\{a, 1 - a\}$. Hence $\mu_k(A)$ cannot lie strictly between 0 and 1. The system is therefore ergodic.

Corollary 1. For an isometry $f : S_{p^{-k}}(a) \rightarrow S_{p^{-k}}(a)$, the following statements are equivalent:

- (i) $(S_{p^{-k}}(a), f, \mu_k)$ is ergodic;
- (ii) the induced permutation on $X_{n,k}$ consists of a single cycle for every $n \geq 1$;
- (iii) every orbit is dense in $S_{p^{-k}}(a)$;
- (iv) at least one orbit is dense in $S_{p^{-k}}(a)$.

If these conditions hold, μ_k is the unique invariant Borel probability measure.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is precisely Theorem 5.

Assume (ii) and fix $x \in S_{p^{-k}}(a)$. Let B be any nonempty open subset of the sphere. Since the partition balls form a base of the p -adic topology, there exist $n \geq 1$ and an atom $C \in X_{n,k}$ such that $C \subset B$. The induced permutation on $X_{n,k}$ is a single cycle, so the orbit of the atom $B_n(x)$ eventually reaches C . Hence $f^j(x) \in C \subset B$ for some $j \geq 0$. Thus the orbit of every point is dense, and (ii) $\Rightarrow$ (iii).

The implication (iii) $\Rightarrow$ (iv) is immediate. Now assume (iv) and let the orbit of x_* be dense. For a fixed n , this orbit meets every atom of $X_{n,k}$. The sequence of atoms containing $f^j(x_*)$ is exactly the orbit of $B_n(x_*)$ under the induced finite permutation. Since it visits all atoms, that permutation consists of a single cycle. This holds for every n , so (iv) $\Rightarrow$ (ii).

It remains to prove unique ergodicity. Let ν be any invariant Borel probability measure on the sphere. Fix n . If $B, C \in X_{n,k}$, then $C = f^j(B)$ for some j . Since f is bijective, invariance under inverse images also gives invariance under forward images: for every Borel set E ,

$$\nu(f(E)) = \nu(f^{-1}(f(E))) = \nu(E).$$

Applying this identity repeatedly, we obtain

$$\nu(C) = \nu(f^j(B)) = \nu(B).$$

Thus all N_n atoms have the same ν -measure, necessarily $1/N_n$. The normalized Haar measure μ_k has the same value on every atom of every partition.

Let $\mathcal{P}$ consist of the empty set, the whole sphere, and all atoms of all partitions $X_{n,k}$. Any two nonempty members of $\mathcal{P}$ are either disjoint or one contains the other; hence $\mathcal{P}$ is a π -system. It is also a base of clopen sets and therefore generates the Borel σ -algebra. Since ν and μ_k agree on $\mathcal{P}$, the uniqueness theorem for probability measures on a generating π -system gives $\nu = \mu_k$.

We now give a metric form of the criterion. Recall the number N_n defined in (7). For $x \in S_{p^{-k}}(a)$, denote by $B_n(x)$ the unique element of $X_{n,k}$ containing x . If the induced permutation on $X_{n,k}$ consists of a single cycle, then after N_n iterates the ball $B_n(x)$ returns to itself. The return map therefore permutes the p balls of level $n + 1$ contained in $B_n(x)$.

Lemma 2. Assume that the induced permutation on $X_{n,k}$ consists of a single cycle. Then the following statements are equivalent:

- (i) the induced permutation on $X_{n+1,k}$ consists of a single cycle;

- (ii) the return map f^{N_n} is a p -cycle on the p balls of level $n+1$ contained in one element of $X_{n,k}$;
 (iii) for one $x \in S_{p-k}(a)$ one has

$$|f^{jN_n}(x) - x|_p = p^{-(k+n)}, \quad j = 1, \dots, p-1;$$

- (iv) the same equalities hold for every $x \in S_{p-k}(a)$.

Proof. Fix a parent ball $B \in X_{n,k}$ and denote by $\mathcal{C}(B)$ the set of its p children in $X_{n+1,k}$. Since the level- n permutation is a single cycle of length N_n , we have

$$f^{N_n}(B) = B.$$

Therefore f^{N_n} induces a permutation

$$R_B : \mathcal{C}(B) \longrightarrow \mathcal{C}(B), \quad R_B(C) = f^{N_n}(C),$$

which will be called the return permutation over B .

The return permutations over different parent balls have the same cycle structure. Indeed, if $B' = f^q(B)$, then the map

$$\Phi_q : \mathcal{C}(B) \rightarrow \mathcal{C}(B'), \quad \Phi_q(C) = f^q(C),$$

is a bijection, and the commutativity of the iterates of f gives

$$R_{B'} = \Phi_q \circ R_B \circ \Phi_q^{-1}. \quad (8a)$$

Thus the return permutations are conjugate.

We first prove (i) $\Leftrightarrow$ (ii). Let $C \in \mathcal{C}(B)$ and let ℓ be the length of the cycle of C under R_B . We claim that the cycle of C under the full level- $(n+1)$ permutation has length exactly ℓN_n . Indeed, if $f^q(C) = C$, then projection to the parent level gives $f^q(B) = B$. Since the parent-level cycle has length N_n , this forces $q = tN_n$ for some integer $t \geq 1$. The equality $f^{tN_n}(C) = C$ is exactly $R_B^t(C) = C$, whose least positive solution is $t = \ell$. Therefore the least return time of C is ℓN_n .

There are exactly pN_n atoms at level $n+1$. Hence the full level- $(n+1)$ permutation consists of a single cycle if and only if $\ell N_n = pN_n$, or equivalently $\ell = p$. The latter condition says that R_B is a p -cycle. By (8a), it is enough to check this over one parent ball. This proves (i) $\Leftrightarrow$ (ii).

We next relate the return permutation to the metric equalities. Fix $x \in S_{p-k}(a)$, put $B = B_n(x)$, and let $C = B_{n+1}(x) \in \mathcal{C}(B)$. For every $j \geq 1$, both x and $f^{jN_n}(x)$ belong to the parent ball B , while $f^{jN_n}(x)$ belongs to the child $R_B^j(C)$. Two points in the same level- n ball belong to the same level- $(n+1)$ child exactly when their distance is at most $p^{-(k+n+1)}$. If they belong to different children, their distance is larger than $p^{-(k+n+1)}$ but no larger than $p^{-(k+n)}$. Since the nonzero values of the p -adic norm are integral powers of p , the only possibility is that their distance equals $p^{-(k+n)}$. Hence

$$|f^{jN_n}(x) - x|_p = p^{-(k+n)} \iff R_B^j(C) \neq C. \quad (8b)$$

The equalities in (iii) hold for $j = 1, \dots, p-1$ if and only if the child C does not return to itself during the first $p-1$ iterates of R_B . Since R_B acts on a set of p elements, this is equivalent to saying that the orbit of C has length p . In that case R_B is a p -cycle. Thus (iii) $\Leftrightarrow$ (ii).

Finally, if (iii) holds for one point, then R_B is a p -cycle. By (8a), every return permutation is a p -cycle, and (8b) then holds for every child and every point. Hence (iii) $\Rightarrow$ (iv). The reverse implication is immediate. All four statements are therefore equivalent.

Theorem 6. Fix $x_* \in S_{p-k}(a)$. All finite dynamical systems $(X_{n,k}, f, \nu_{n,k})$ are ergodic if and only if the following conditions hold:

- (i) $(X_{1,k}, f, \nu_{1,k})$ is ergodic;
 (ii) for every $n \geq 1$ and every $j = 1, \dots, p-1$,

$$|f^{jN_n}(x_*) - x_*|_p = p^{-(k+n)}. \quad (8)$$

Proof. Assume first that every finite factor is ergodic. Then, by Remark 3, the induced permutations on both $X_{n,k}$ and $X_{n+1,k}$ are single cycles for every $n \geq 1$. Condition (i) is immediate. Applying Lemma 2 to the fixed point x_* gives

$$|f^{jN_n}(x_*) - x_*|_p = p^{-(k+n)}, \quad j = 1, \dots, p-1,$$

for every n , so condition (ii) also holds.

Conversely, assume (i) and (ii). By (i), the induced permutation on $X_{1,k}$ is a single cycle. Suppose inductively that the permutation on $X_{n,k}$ is a single cycle. Condition (ii) supplies exactly the metric equalities in part (iii) of

Lemma 2 at the point x_* . Lemma 2 therefore implies that the permutation on $X_{n+1,k}$ is also a single cycle. Starting from $n = 1$ and repeating this argument proves by induction that every finite factor is ergodic.

Remark 4. For $p \geq 5$, checking only the equality in (8) corresponding to $j = 1$ at a single point is not sufficient: it merely shows that the child containing that point is not fixed. Even if the $j = 1$ equality is verified at every point, it only excludes fixed children, whereas a fixed-point-free permutation of p elements may decompose into several cycles. The full family $j = 1, \dots, p-1$ in (8) is therefore a simple uniform criterion, although it is not claimed to be a minimal set of tests. For $p = 2$, the single value $j = 1$ is sufficient. For $p = 3$, the equality corresponding to $j = 2$ at one point already rules out return cycles of lengths 1 and 2 and is therefore sufficient; alternatively, one may check the $j = 1$ equality at every point.

Combining Theorems 5 and 6, we obtain the following criterion for the case $\rho(r) = r$.

Corollary 2. Let $f : S_{p-k}(a) \rightarrow S_{p-k}(a)$ be an isometry and assume that

$$|f(x) - x|_p = p^{-k}, \quad x \in S_{p-k}(a).$$

Then the system $(S_{p-k}(a), f, \mu_k)$ is ergodic if and only if conditions (i) and (ii) of Theorem 6 hold.

Proof. By Theorem 5, ergodicity of the system on the sphere is equivalent to ergodicity of all finite factors. By Theorem 6, the latter is equivalent to conditions (i) and (ii) stated there. The displacement assumption identifies this result with the boundary case $\rho(r) = r$.

For $p = 3$ the criterion has the following convenient form.

Corollary 3. Let $p = 3$ and assume that $|f(x) - x|_3 = 3^{-k}$ on $S_{3-k}(a)$. Then the first finite factor is the transposition of its two balls. The system is ergodic if and only if, for one point $x_* \in S_{3-k}(a)$ and every $n \geq 1$,

$$|f^{2 \cdot 3^{n-1}}(x_*) - x_*|_3 = |f^{4 \cdot 3^{n-1}}(x_*) - x_*|_3 = 3^{-(k+n)}. \quad (9)$$

Proof. The first partition of a 3-adic sphere has $p-1 = 2$ atoms, each of radius $3^{-(k+1)}$. If x and $f(x)$ belonged to the same atom, then

$$|f(x) - x|_3 \leq 3^{-(k+1)},$$

contrary to the assumption $|f(x) - x|_3 = 3^{-k}$. Thus every point is sent to the other atom, and the induced permutation on $X_{1,k}$ is the transposition. In particular, condition (i) of Theorem 6 is automatic. For $p = 3$ we have

$$N_n = (p-1)p^{n-1} = 2 \cdot 3^{n-1}$$

and $j = 1, 2$. Substituting these values into condition (ii) of Theorem 6 gives exactly the two equalities in (9). The assertion now follows from Corollary 2.

The following proposition will be used in the 3-adic examples.

Proposition 1. Let $m \geq 2$ and suppose that T is analytic on a ball containing $V_{3-m}(x_*)$. Assume that on this ball T has a Taylor expansion

$$T(x_* + h) = T(x_*) + Ah + \sum_{q=2}^{\infty} c_q h^q, \quad A, c_q \in \mathbb{Z}_3,$$

and that

$$\nu_3(T(x_*) - x_*) = m, \quad A = T'(x_*) \equiv 1 \pmod{3}. \quad (10)$$

Then

$$\nu_3(T^j(x_*) - x_*) = m, \quad j = 1, 2, \quad (11)$$

and

$$\nu_3(T^3(x_*) - x_*) = m + 1, \quad (T^3)'(x_*) \equiv 1 \pmod{3}. \quad (12)$$

Proof. Put

$$d = T(x_*) - x_*, \quad A = T'(x_*).$$

By assumption, $\nu_3(d) = m$. Since all higher Taylor coefficients are 3-adic integers, the expansion may be written in the form

$$T(x_* + h) = T(x_*) + Ah + h^2 E(h), \quad E(h) \in \mathbb{Z}_3 \quad (13)$$

whenever $\nu_3(h) \geq m$.

Taking $h = d$ in (13), we obtain

$$\begin{aligned} T^2(x_*) - x_* &= d + Ad + d^2 E(d) \\ &= d(1 + A + dE(d)). \end{aligned}$$

Now $A \equiv 1 \pmod{3}$ and $d \equiv 0 \pmod{3}$, so

$$1 + A + dE(d) \equiv 2 \pmod{3}.$$

The factor in parentheses is therefore a 3-adic unit. Hence

$$\nu_3(T^2(x_*) - x_*) = \nu_3(d) = m.$$

Together with the hypothesis for $j = 1$, this proves (11).

Let $h_2 = T^2(x_*) - x_*$. The preceding calculation gives

$$h_2 = d(1 + A) + O(d^2),$$

where $O(d^2)$ denotes a term of valuation at least $2m$. Applying (13) to $h = h_2$ yields

$$\begin{aligned} T^3(x_*) - x_* &= d + Ah_2 + O(h_2^2) \\ &= d(1 + A + A^2) + O(d^2). \end{aligned}$$

Write $A = 1 + 3c$ with $c \in \mathbb{Z}_3$. Then

$$1 + A + A^2 = 3(1 + 3c + 3c^2),$$

so

$$\nu_3(1 + A + A^2) = 1.$$

Consequently, the main term $d(1 + A + A^2)$ has valuation $m + 1$. The remainder has valuation at least $2m$, and

$$2m \geq m + 2 > m + 1$$

because $m \geq 2$. The remainder therefore cannot cancel the main term, and

$$\nu_3(T^3(x_*) - x_*) = m + 1.$$

It remains to calculate the derivative. Differentiating the Taylor series shows that, for $\nu_3(h) \geq 1$,

$$T'(x_* + h) = A + hE_1(h), \quad E_1(h) \in \mathbb{Z}_3.$$

The points $T(x_*)$ and $T^2(x_*)$ differ from x_* by elements of valuation $m \geq 2$. Hence

$$T'(T(x_*)) \equiv T'(T^2(x_*)) \equiv A \equiv 1 \pmod{3}.$$

By the chain rule,

$$(T^3)'(x_*) = T'(T^2(x_*)) T'(T(x_*)) T'(x_*) \equiv 1 \pmod{3}.$$

This proves (12).

Example 1.

Let

$$p = 3, \quad k \in \mathbb{Z},$$

and consider the sphere $S_{3^{-k}}(a)$, where $a \in \mathbb{Q}_3$. Define

$$f(x) = a + 2(x - a). \tag{14}$$

For $x \in S_{3^{-k}}(a)$,

$$|f(x) - a|_3 = |2|_3 |x - a|_3 = 3^{-k},$$

so the sphere is invariant. For any x, y on the sphere,

$$|f(x) - f(y)|_3 = |2(x - y)|_3 = |x - y|_3,$$

and hence f is an isometry. Moreover,

$$|f(x) - x|_3 = |x - a|_3 = 3^{-k},$$

so this is exactly the case $\rho(r) = r$.

Under the coordinate change $x = a + 3^k u$, the sphere is identified with $\mathbb{Z}_3^*$ and the map becomes $u \mapsto 2u$. On the first finite factor the two possible nonzero residues are 1 and 2, and

$$1 \mapsto 2, \quad 2 \mapsto 1.$$

Thus the first finite factor consists of a single cycle.

For $N_n = 2 \cdot 3^{n-1}$, direct iteration of (14) gives

$$f^{jN_n}(x) - x = (2^{jN_n} - 1)(x - a), \quad j = 1, 2. \quad (15)$$

Since

$$|x - a|_3 = 3^{-k}, \quad (16)$$

it remains to calculate the valuation of $2^{jN_n} - 1$. We have

$$2^{jN_n} - 1 = 4^{j3^{n-1}} - 1.$$

The lifting-the-exponent formula for the odd prime 3 gives

$$\begin{aligned} \nu_3(4^{j3^{n-1}} - 1) &= \nu_3(4^j - 1) + \nu_3(3^{n-1}) \\ &= 1 + (n - 1) = n, \end{aligned}$$

for $j = 1, 2$, because $4 - 1 = 3$ and $4^2 - 1 = 15$ both have 3-adic valuation one. Therefore

$$\nu_3(2^{jN_n} - 1) = n, \quad j = 1, 2. \quad (17)$$

Combining (15)–(17), we obtain

$$|f^{jN_n}(x) - x|_3 = |2^{jN_n} - 1|_3 |x - a|_3 = 3^{-n} 3^{-k} = 3^{-(k+n)}, \quad j = 1, 2.$$

All conditions of Corollary 3 are satisfied. Hence

$$(S_{3-k}(a), f, \mu_k)$$

is ergodic. By Corollary 1, it is also minimal and uniquely ergodic.

Example 2.

Let

$$p = 3, \quad k \geq 1, \quad \theta \in \mathbb{Q}_3, \quad |\theta|_3 > 1,$$

and consider

$$f(x) = \frac{\theta^2 x^2 + 2x + 1}{x^2 + 2x + \theta^2}, \quad x \in S_{3-k}(1). \quad (18)$$

Put

$$u = x - 1, \quad |u|_3 = 3^{-k}, \quad \lambda = \theta^{-2}.$$

Since $|\theta|_3 > 1$, we have $\nu_3(\lambda) \geq 2$, and hence $\lambda \in 9\mathbb{Z}_3$. A direct substitution gives

$$f(1 + u) - 1 = u \frac{(1 - \lambda)(2 + u)}{1 + \lambda(3 + 4u + u^2)}. \quad (19)$$

We first verify invariance. Since $u \in 3\mathbb{Z}_3$, the numbers $1 - \lambda$ and $2 + u$ are 3-adic units. Also,

$$|\lambda(3 + 4u + u^2)|_3 < 1,$$

so the denominator in (19) is a unit. Therefore

$$|f(x) - 1|_3 = |u|_3 = 3^{-k}.$$

Thus $S_{3-k}(1)$ is invariant.

We next prove that f is an isometry on this sphere. For $x, y \in S_{3-k}(1)$, a direct calculation gives

$$f(x) - f(y) = \frac{(\theta - 1)(\theta + 1)(x - y)(\theta^2 x + \theta^2 y + 2xy + x + y + 2)}{(\theta^2 + x^2 + 2x)(\theta^2 + y^2 + 2y)}. \quad (20)$$

Because $x \equiv y \equiv 1 \pmod{3}$, we have

$$|x|_3 = |y|_3 = |x + y|_3 = 1.$$

Moreover, $|\theta \pm 1|_3 = |\theta|_3$ and

$$|\theta^2 + x^2 + 2x|_3 = |\theta^2 + y^2 + 2y|_3 = |\theta|_3^2.$$

In the remaining factor of the numerator, the term $\theta^2(x+y)$ has norm $|\theta|_3^2$, whereas $2xy + x + y + 2$ has norm at most 1. Hence

$$|\theta^2x + \theta^2y + 2xy + x + y + 2|_3 = |\theta|_3^2.$$

Taking norms in (20), all powers of $|\theta|_3$ cancel and we obtain

$$|f(x) - f(y)|_3 = |x - y|_3.$$

Thus f is an isometry.

Formula (19) also gives the displacement. The quotient multiplying u is congruent to 2 modulo 3, so that quotient minus 1 is a unit. Hence

$$|f(x) - x|_3 = |f(1+u) - 1 - u|_3 = |u|_3 = 3^{-k}.$$

Reducing (19) modulo 3^{k+1} gives

$$f(1+u) - 1 \equiv 2u \pmod{3^{k+1}}. \quad (21)$$

Indeed, $u^2 \in 3^{2k}\mathbb{Z}_3 \subset 3^{k+1}\mathbb{Z}_3$, while every term containing λu belongs to $3^{k+2}\mathbb{Z}_3$. After writing $u = 3^k v$ with $v \in \mathbb{Z}_3^*$, congruence (21) becomes $v \mapsto 2v$ modulo 3. Therefore the first finite factor is the transposition of its two atoms.

For the higher-level lifting, set

$$G(u) = f(1+u) - 1 = u \frac{(1-\lambda)(2+u)}{1+\lambda(3+4u+u^2)}. \quad (22)$$

Since the denominator is a unit and $\lambda \in 9\mathbb{Z}_3$, there exists an integral analytic function $E(u)$ such that

$$\frac{1-\lambda}{1+\lambda(3+4u+u^2)} = 1 + \lambda E(u).$$

Consequently,

$$G(u) \equiv 2u + u^2 \pmod{3^{k+2}}. \quad (23)$$

Put $P(u) = 2u + u^2 = u(2+u)$. Since $2+u$ is a unit, $P(u)$ belongs to the same sphere $|u|_3 = 3^{-k}$. By (23), the points $G(u)$ and $P(u)$ are congruent modulo 3^{k+2} . Since G is an isometry on this sphere, $G(G(u))$ and $G(P(u))$ are congruent modulo the same power. Applying (23) once more, now at the point $P(u)$, gives

$$G(P(u)) \equiv P(P(u)) \pmod{3^{k+2}}.$$

Combining the last two congruences, we obtain

$$G^2(u) \equiv P(P(u)) \pmod{3^{k+2}}.$$

A direct calculation gives

$$\begin{aligned} P(P(u)) - u &= 3u + 6u^2 + 4u^3 + u^4 \\ &= 3u(1+2u) + u^3(4+u). \end{aligned}$$

Since $k \geq 1$, we have $\nu_3(u^3) \geq 3k \geq k+2$. Thus, for $H = G^2$,

$$H(u) - u \equiv 3u(1+2u) \pmod{3^{k+2}}. \quad (24)$$

The factor $1+2u$ is a unit, so

$$\nu_3(H(u) - u) = k+1.$$

A direct differentiation of (22), followed by reduction modulo 3, gives

$$G'(u) \equiv 2 \pmod{3}.$$

Since $G(u)$ also has valuation k , the chain rule yields

$$H'(u) = G'(G(u))G'(u) \equiv 2^2 \equiv 1 \pmod{3}. \quad (25)$$

The denominator in (22) is a unit on $3\mathbb{Z}_3$. To justify the integrality assertion used below, fix $u_0 \in 3\mathbb{Z}_3$ and write the denominator at $u_0 + h$ as $D(u_0) + hD_1(h)$, where $D(u_0) \in \mathbb{Z}_3^*$ and $D_1(h) \in \mathbb{Z}_3[[h]]$. Then

$$\frac{1}{D(u_0 + h)} = \frac{1}{D(u_0)} \sum_{q=0}^{\infty} \left(-\frac{hD_1(h)}{D(u_0)} \right)^q,$$

so the local Taylor coefficients of G belong to $\mathbb{Z}_3$. The same property is preserved under composition; hence it holds for every iterate of G . Define

$$R_1 = H = G^2, \quad R_{n+1} = R_n^3 = G^{2 \cdot 3^n}.$$

For R_1 , formulae (24)–(25) give

$$\nu_3(R_1(u) - u) = k + 1, \quad R_1'(u) \equiv 1 \pmod{3}.$$

Applying Proposition 1 to R_1 gives the same valuation for $R_1^2(u) - u$ and increases the valuation by one for $R_2(u) - u = R_1^3(u) - u$; it also gives $R_2'(u) \equiv 1 \pmod{3}$. Repeating the same argument inductively yields

$$R_n = G^{2 \cdot 3^{n-1}}$$

and, for every $n \geq 1$ and $j = 1, 2$,

$$|R_n^j(u) - u|_3 = 3^{-(k+n)}. \quad (26)$$

Corollary 3 now implies that

$$(S_{3^{-k}}(1), f, \mu_k)$$

is ergodic. By Corollary 1, it is also minimal and uniquely ergodic.

The same rational function has another invariant sphere on which its behaviour is different. Let $|\theta|_3 = 3^m$, $m \geq 1$, and write $\theta = 3^{-m}\tau$ with $\tau \in \mathbb{Z}_3^*$. Every point of $S_{|\theta|_3^2}(1)$ has the form

$$x = 1 + 3^{-2m}y, \quad y \in \mathbb{Z}_3^*.$$

A direct calculation gives

$$3^{2m}(f(x) - 1) = \frac{y(y + 2 \cdot 3^{2m})(\tau^2 - 3^{2m})}{y^2 + 3^{2m}(\tau^2 + 4y) + 3^{4m+1}}. \quad (27)$$

The denominator is congruent to y^2 modulo 3, while the numerator is congruent to $y^2\tau^2$. Since every unit square is congruent to 1 modulo 3, we obtain

$$3^{2m}(f(x) - 1) \equiv 1 \pmod{3}. \quad (28)$$

In the normalized coordinate y , the first partition has two atoms, corresponding to $y \equiv 1$ and $y \equiv 2 \pmod{3}$. Formula (28) shows that the image of the entire sphere is contained in the first of these atoms. That atom has normalized Haar measure $1/2$. If it is denoted by B , then $f^{-1}(B)$ is the whole sphere, and hence

$$\mu(f^{-1}(B)) = 1 \neq \frac{1}{2} = \mu(B).$$

Thus f is not measure-preserving on $S_{|\theta|_3^2}(1)$ and therefore cannot be ergodic there.

Remark 5. For any odd prime p , the same reduction of (19) gives multiplication by 2 on the first factor $\mathbb{F}_p^*$. Therefore a necessary condition for ergodicity is that the residue class of 2 generate the cyclic group $\mathbb{F}_p^*$.

Remark 6. Consider the affine isometry

$$f(x) = a + g(x - a), \quad g \in \mathbb{Z}_p^*.$$

After the normalization $x = a + p^k u$, the map becomes $u \mapsto gu$ on $\mathbb{Z}_p^*$. The atoms of the level- n partition correspond exactly to the unit residue classes modulo p^n , and the induced permutation is multiplication by g on $(\mathbb{Z}/p^n\mathbb{Z})^*$. It is a single cycle precisely when the class of g has order equal to the number of units, namely

$$\text{ord}_{p^n}(g) = (p - 1)p^{n-1}.$$

Consequently, the affine system is ergodic if and only if this equality holds for every $n \geq 1$.

Applications to rational functions

We now apply Theorem 6 to two classes of rational functions. In both classes the displacement on the considered invariant sphere is equal to the radius.

First consider the nondegenerate $(2, 2)$ -rational function

$$f(x) = \frac{ax^2 + bx}{x^2 + dx + b}, \quad ab \neq 0, \quad a \neq d, \quad a^2 - ad + b \neq 0. \quad (29)$$

Assume that the denominator splits over $\mathbb{Q}_p$, and let $\xi_1, \xi_2 \in \mathbb{Q}_p$ be its poles. Put

$$\alpha = \min\{|\xi_1|_p, |\xi_2|_p\}, \quad \beta = \max\{|\xi_1|_p, |\xi_2|_p\}, \quad \delta = |a|_p, \quad s = |a - d|_p.$$

The fixed points are 0 and $a - d$. If the denominator is irreducible over $\mathbb{Q}_p$, then its two roots are conjugate over $\mathbb{Q}_p$. The p -adic absolute value has a unique extension to every finite algebraic extension, so conjugation preserves that absolute value. The two roots therefore have the same norm. Hence the strict inequality $\alpha < \beta$ considered below can occur only in the split case.

Theorem 7. Let $p \geq 3$, let $\xi_1, \xi_2 \in \mathbb{Q}_p$, and assume that

$$\delta = \beta, \quad \alpha < r < \beta, \quad s = \beta. \quad (30)$$

Choose $\lambda \in \mathbb{Q}_p^*$ with $|\lambda|_p = r$ and define

$$G_\lambda(u) = \lambda^{-1} f(\lambda u) = \frac{a\lambda u^2 + bu}{\lambda^2 u^2 + d\lambda u + b}. \quad (31)$$

Fix $u_* \in \mathbb{Z}_p^*$ and put $N_n = (p-1)p^{n-1}$. Then $(S_r(0), f, \mu_{0,r})$ is ergodic if and only if

- (i) the residue class $\overline{a/d}$ is a generator of $\mathbb{F}_p^*$;
- (ii) for every $n \geq 1$ and every $j = 1, \dots, p-1$,

$$|G_\lambda^{jN_n}(u_*) - u_*|_p = p^{-n}. \quad (32)$$

If these conditions hold, the system is minimal and uniquely ergodic.

Proof. The invariant spheres of this family were studied in [17]. For completeness, we verify directly all properties needed here.

Relabel the poles, if necessary, so that

$$|\xi_1|_p = \alpha, \quad |\xi_2|_p = \beta.$$

Since

$$x^2 + dx + b = (x - \xi_1)(x - \xi_2),$$

Vieta's formulae and the strict inequality $\alpha < \beta$ give

$$|d|_p = |\xi_1 + \xi_2|_p = \beta, \quad |b|_p = |\xi_1 \xi_2|_p = \alpha\beta. \quad (32a)$$

Let $x \in S_r(0)$. From $\alpha < r < \beta$ we obtain

$$|x - \xi_1|_p = r, \quad |x - \xi_2|_p = \beta. \quad (32b)$$

Also,

$$|ax|_p = \beta r > |b|_p = \alpha\beta,$$

so $|ax + b|_p = \beta r$. Consequently,

$$|f(x)|_p = \frac{|x|_p |ax + b|_p}{|x - \xi_1|_p |x - \xi_2|_p} = \frac{r \cdot \beta r}{r\beta} = r.$$

Thus $S_r(0)$ is invariant.

We now prove the isometry property. A direct subtraction gives

$$f(x) - f(y) = \frac{(x - y)Q(x, y)}{(x^2 + dx + b)(y^2 + dy + b)}, \quad (32c)$$

where

$$Q(x, y) = ab(x + y) + adxy + b^2 - bxy.$$

For $x, y \in S_r(0)$, the term $adxy$ has norm

$$|adxy|_p = \beta^2 r^2.$$

All other terms have strictly smaller norm:

$$\begin{aligned} |ab(x + y)|_p &\leq \alpha\beta^2 r < \beta^2 r^2, \\ |b^2|_p &= \alpha^2 \beta^2 < \beta^2 r^2, \\ |bxy|_p &= \alpha\beta r^2 < \beta^2 r^2. \end{aligned}$$

Therefore $|Q(x, y)|_p = \beta^2 r^2$. By (32b), each denominator in (32c) has norm $r\beta$, and hence

$$|f(x) - f(y)|_p = |x - y|_p.$$

Thus the restriction of f to $S_r(0)$ is an isometry.

The displacement is obtained from the identity

$$f(x) - x = \frac{x^2(a - d - x)}{(x - \xi_1)(x - \xi_2)}.$$

Since $|a - d|_p = \beta > r$, we have $|a - d - x|_p = \beta$. Together with (32b), this gives

$$|f(x) - x|_p = \frac{r^2\beta}{r\beta} = r. \quad (32d)$$

Now consider the affine isometry

$$h_\lambda : \mathbb{Z}_p^* \rightarrow S_r(0), \quad h_\lambda(u) = \lambda u.$$

It maps the canonical partitions of $\mathbb{Z}_p^*$ onto the partitions of $S_r(0)$ and transports normalized Haar measure to $\mu_{0,r}$. The conjugate map is

$$h_\lambda^{-1} \circ f \circ h_\lambda = G_\lambda.$$

Using (32a), we rewrite (31) as

$$G_\lambda(u) = \frac{a}{d}u \frac{1 + b/(a\lambda u)}{1 + \lambda u/d + b/(d\lambda u)}.$$

For $u \in \mathbb{Z}_p^*$, the correction terms satisfy

$$\left| \frac{b}{a\lambda u} \right|_p = \left| \frac{b}{d\lambda u} \right|_p = \frac{\alpha}{r} < 1, \quad \left| \frac{\lambda u}{d} \right|_p = \frac{r}{\beta} < 1.$$

Thus both correction factors are congruent to 1 modulo p , and

$$G_\lambda(u) \equiv \frac{a}{d}u \pmod{p}. \quad (32e)$$

Since $|a/d|_p = 1$, multiplication by $\overline{a/d}$ permutes $\mathbb{F}_p^*$. This permutation is a single cycle if and only if $\overline{a/d}$ generates the cyclic group $\mathbb{F}_p^*$. Hence condition (i) is exactly the first-level condition of Theorem 6.

Under the conjugacy h_λ , the metric equality

$$|G_\lambda^{jN_n}(u_*) - u_*|_p = p^{-n}$$

is equivalent to

$$|f^{jN_n}(h_\lambda(u_*)) - h_\lambda(u_*)|_p = rp^{-n}.$$

Therefore condition (ii) is precisely the cyclic-lifting condition of Theorem 6 for the original sphere. Theorem 5 and Theorem 6 now prove the equivalence stated in the theorem. Finally, Corollary 1 gives minimality and unique ergodicity.

Example 3.

Consider

$$f(x) = \frac{2x^2 + 9x}{x^2 + x + 9} \quad \text{on } S_{1/3}(0) \subset \mathbb{Q}_3. \quad (33)$$

We verify the conditions of Theorem 7 with the fixed normalized point $u_* = 1$.

The discriminant of the denominator is

$$\Delta = 1 - 36 = -35.$$

Since $-35 \equiv 1 \pmod{3}$ and the derivative of $z^2 + 35$ at $z = 1$ is $2 \not\equiv 0 \pmod{3}$, Hensel's lemma gives a square root of -35 in $\mathbb{Q}_3$. Hence the denominator splits over $\mathbb{Q}_3$. Its two roots have sum -1 and product 9 . Suppose that their norms were equal to some number t . From the product relation we would have $t^2 = |9|_3 = 1/9$, hence $t = 1/3$. The ultrametric inequality would then give

$$|\xi_1 + \xi_2|_3 \leq \frac{1}{3},$$

contrary to $|\xi_1 + \xi_2|_3 = |-1|_3 = 1$. Thus the two norms are unequal. For two summands of unequal norm, the norm of their sum equals the larger norm; consequently the larger root norm is 1 . The product relation then gives the smaller root norm $1/9$. Therefore

$$\alpha = \frac{1}{9}, \quad \beta = 1.$$

Furthermore,

$$\delta = |2|_3 = 1, \quad s = |2 - 1|_3 = 1.$$

Thus (30) holds with $r = 1/3$.

Putting $x = 3u$ gives the normalized map

$$G(u) = \frac{u(2u + 3)}{3u^2 + u + 3}. \quad (34)$$

For $u \in \mathbb{Z}_3^*$, the denominator is congruent to u modulo 3 and is therefore a unit. Reducing (34) modulo 3 yields

$$G(u) \equiv 2u \pmod{3}.$$

Thus the first factor is the transposition of 1 and 2.

At the next modulus, direct substitution gives the single cycle

$$1 \mapsto 2 \mapsto 4 \mapsto 8 \mapsto 7 \mapsto 5 \mapsto 1 \pmod{9}. \quad (35)$$

In particular,

$$G^2(1) \equiv 4 \pmod{9}, \quad G^4(1) \equiv 7 \pmod{9},$$

so

$$\nu_3(G^2(1) - 1) = \nu_3(G^4(1) - 1) = 1. \quad (36)$$

These are exactly the two return equalities required at level $n = 1$.

A direct calculation modulo 27 gives

$$G^6(1) \equiv 10 \pmod{27}.$$

Hence

$$\nu_3(G^6(1) - 1) = \nu_3(9) = 2. \quad (36a)$$

Moreover,

$$G'(u) = -\frac{7u^2 - 12u - 9}{(3u^2 + u + 3)^2} \equiv 2 \pmod{3}. \quad (37)$$

Every iterate of a unit remains a unit because $G(u) \equiv 2u \pmod{3}$. Therefore, by the chain rule,

$$(G^6)'(1) = \prod_{i=0}^5 G'(G^i(1)) \equiv 2^6 \equiv 1 \pmod{3}. \quad (37a)$$

The denominator of G is a unit on $\mathbb{Z}_3^*$. Expanding its reciprocal by the same geometric-series argument used in Example 2 shows that the Taylor coefficients of G on every unit residue ball belong to $\mathbb{Z}_3$. This property is preserved under composition, so it also holds for all iterates of G .

Set

$$R_2 = G^6.$$

Equations (36a) and (37a) allow us to apply Proposition 1 to R_2 at the point 1 with $m = 2$. We obtain

$$\nu_3(R_2^j(1) - 1) = 2, \quad j = 1, 2,$$

and $R_3 = R_2^3 = G^{18}$ satisfies

$$\nu_3(R_3(1) - 1) = 3, \quad R_3'(1) \equiv 1 \pmod{3}.$$

Repeating this argument inductively with

$$R_n = G^{2 \cdot 3^{n-1}}$$

gives

$$\nu_3(R_n^j(1) - 1) = n, \quad n \geq 2, \quad j = 1, 2. \quad (38)$$

Together with (36), these equalities verify condition (ii) of Theorem 7 for every $n \geq 1$. Consequently,

$$(S_{1/3}(0), f, \mu_{0,1/3})$$

is ergodic, minimal and uniquely ergodic.

Now consider the $(2, 1)$ -rational function

$$f(x) = \frac{x^2 + ax + b}{x + c}, \quad a \neq c, \quad c^2 - ac + b \neq 0. \quad (39)$$

Its unique fixed point is

$$x_0 = \frac{b}{c-a}.$$

Put

$$\eta = x_0 + a, \quad \zeta = x_0 + c, \quad \alpha = |\eta|_p, \quad \beta = |\zeta|_p.$$

Theorem 8. Let $p \geq 3$ and assume that

$$\alpha = \beta, \quad |a - c|_p = \beta, \quad 0 < r < \beta. \quad (40)$$

Choose $\lambda \in \mathbb{Q}_p^*$ with $|\lambda|_p = r$ and define

$$G_\lambda(t) = \lambda^{-1}(f(x_0 + \lambda t) - x_0) = t \frac{\lambda t + \eta}{\lambda t + \zeta}. \quad (41)$$

Put $q = \eta/\zeta$, fix $t_* \in \mathbb{Z}_p^*$, and let $N_n = (p-1)p^{n-1}$. Then $(S_r(x_0), f, \mu_{x_0, r})$ is ergodic if and only if

- (i) the residue class $\bar{q}$ is a generator of $\mathbb{F}_p^*$;
- (ii) for every $n \geq 1$ and every $j = 1, \dots, p-1$,

$$|G_\lambda^{jN_n}(t_*) - t_*|_p = p^{-n}. \quad (42)$$

If these conditions hold, the system is minimal and uniquely ergodic.

Proof. The invariant spheres of this family were studied in [14]. We give a direct verification of all properties used in the ergodicity criterion.

Write

$$u = x - x_0.$$

A direct substitution of $x = x_0 + u$ into (39), using $x_0 = b/(c-a)$, gives the centred identity

$$f(x) - x_0 = u \frac{u + \eta}{u + \zeta}. \quad (42a)$$

Let $x \in S_r(x_0)$, so $|u|_p = r$. Since $r < |\eta|_p = |\zeta|_p = \beta$, we have

$$|u + \eta|_p = |u + \zeta|_p = \beta.$$

Taking norms in (42a) gives

$$|f(x) - x_0|_p = r,$$

so the sphere is invariant.

We next prove that the restriction is an isometry. For $u = x - x_0$ and $v = y - x_0$, subtraction of the two expressions in (42a) gives

$$f(x) - f(y) = (u - v) \frac{\eta\zeta + uv + \zeta(u + v)}{(u + \zeta)(v + \zeta)}. \quad (42b)$$

The term $\eta\zeta$ has norm β^2 , while

$$|uv|_p = r^2 < \beta^2, \quad |\zeta(u + v)|_p \leq \beta r < \beta^2.$$

Thus the numerator factor in (42b) has norm β^2 , and the denominator also has norm β^2 . Therefore

$$|f(x) - f(y)|_p = |u - v|_p = |x - y|_p.$$

To compute the displacement from x , subtract $u = x - x_0$ from (42a):

$$\begin{aligned} f(x) - x &= u \left(\frac{u + \eta}{u + \zeta} - 1 \right) \\ &= u \frac{\eta - \zeta}{u + \zeta} \\ &= u \frac{a - c}{u + \zeta}. \end{aligned} \quad (42c)$$

Since $|a - c|_p = \beta$ and $|u + \zeta|_p = \beta$, it follows that

$$|f(x) - x|_p = r. \quad (42d)$$

Now define

$$h_\lambda : \mathbb{Z}_p^* \rightarrow S_r(x_0), \quad h_\lambda(t) = x_0 + \lambda t.$$

This is an isometry carrying the canonical partitions and normalized Haar measure of $\mathbb{Z}_p^*$ to those of the sphere. Its conjugate map is $G_\lambda = h_\lambda^{-1} \circ f \circ h_\lambda$, as in (41). Since $|\lambda|_p = r < \beta$, we can write

$$G_\lambda(t) = qt \frac{1 + \lambda t/\eta}{1 + \lambda t/\zeta}, \quad q = \frac{\eta}{\zeta},$$

and both correction terms have norm less than 1. Hence

$$G_\lambda(t) \equiv qt \pmod{p}. \quad (42e)$$

The element q is a unit because $|\eta|_p = |\zeta|_p$. Multiplication by $\bar{q}$ is a single-cycle permutation of $\mathbb{F}_p^*$ exactly when $\bar{q}$ is a generator. Thus condition (i) is the first-level condition of Theorem 6.

Finally,

$$|G_\lambda^{jN_n}(t_*) - t_*|_p = p^{-n}$$

is equivalent, after multiplying by $|\lambda|_p = r$, to the return equality at the point $h_\lambda(t_*)$ on $S_r(x_0)$. Hence condition (ii) is precisely the cyclic-lifting condition of Theorem 6. Theorems 5 and 6 prove the stated ergodicity criterion, and Corollary 1 gives minimality and unique ergodicity.

Example 4.

Consider

$$f(x) = \frac{x^2 + x}{x + 2} \quad \text{on } S_{1/3}(0) \subset \mathbb{Q}_3. \quad (43)$$

Here

$$x_0 = 0, \quad \eta = 1, \quad \zeta = 2,$$

and therefore

$$\alpha = |\eta|_3 = 1, \quad \beta = |\zeta|_3 = 1, \quad |a - c|_3 = |1 - 2|_3 = 1.$$

Thus (40) holds for $r = 1/3$. Taking $\lambda = 3$, we obtain

$$G(t) = \frac{t(3t + 1)}{3t + 2}. \quad (44)$$

The denominator is a unit for every $t \in \mathbb{Z}_3^*$. Modulo 3,

$$G(t) \equiv \frac{t}{2} \equiv 2t \pmod{3},$$

so the first finite factor is the transposition of the two unit residues.

Direct calculations give

$$G^2(1) \equiv 7 \pmod{9}, \quad G^4(1) \equiv 4 \pmod{9}, \quad G^6(1) \equiv 19 \pmod{27}. \quad (45)$$

Therefore

$$\nu_3(G^2(1) - 1) = \nu_3(6) = 1, \quad \nu_3(G^4(1) - 1) = \nu_3(3) = 1,$$

and

$$\nu_3(G^6(1) - 1) = \nu_3(18) = 2. \quad (45a)$$

Moreover,

$$G'(t) = \frac{9t^2 + 12t + 2}{(3t + 2)^2} \equiv 2 \pmod{3}. \quad (46)$$

Since all iterates of 1 are units, the chain rule gives

$$(G^6)'(1) = \prod_{i=0}^5 G'(G^i(1)) \equiv 2^6 \equiv 1 \pmod{3}. \quad (46a)$$

The denominator of G is a unit on $\mathbb{Z}_3^*$. A local geometric-series expansion of its reciprocal therefore gives integral Taylor coefficients on every unit residue ball, and composition preserves this property for all iterates of G .

Set $R_2 = G^6$. By (45a)–(46a), Proposition 1 applies to R_2 at 1 with $m = 2$. It gives

$$\nu_3(R_2^j(1) - 1) = 2, \quad j = 1, 2,$$

and shows that $R_3 = R_2^3$ has displacement valuation 3 and derivative congruent to 1 modulo 3. Repeating the argument for

$$R_n = G^{2 \cdot 3^{n-1}}$$

yields

$$\nu_3(R_n^j(1) - 1) = n, \quad n \geq 2, \quad j = 1, 2.$$

Together with the two level-1 valuations in (45), this verifies condition (42) for every $n \geq 1$. Therefore

$$(S_{1/3}(0), f, \mu_{0,1/3})$$

is ergodic, minimal and uniquely ergodic.

Remark 7. For the function $ax/(x^2 + a)$ and for the $(3, 1)$ -rational family with a unique fixed point, the displacement on every invariant sphere is strictly smaller than its radius; see [15,16]. Hence these families do not give additional cases with $\rho(r) = r$.

Conclusion

In this paper we studied isometric dynamical systems on p -adic spheres in the case $\rho(r) = r$. We established an intrinsic finite-factor criterion for isometries preserving an arbitrary p -adic sphere. This criterion is essential for odd primes: when $p \geq 3$, a sphere is a disjoint union of $p - 1$ maximal balls and its natural normalized model is $\mathbb{Z}_p^*$, not $\mathbb{Z}_p$. Therefore the classical criterion for self-maps of $\mathbb{Z}_p$, or equivalently of a single invariant ball, cannot be applied directly to the whole sphere. We also proved the metric cyclic-lifting criterion: after the first finite factor consists of a single cycle, the return map must act as a p -cycle on the children of every partition ball. A convenient one-point criterion valid uniformly for all primes is obtained by checking the return equalities for $j = 1, \dots, p - 1$.

We applied the criterion to affine isometries and to rational functions. In particular, we described the borderline invariant spheres of $(2, 2)$ -rational functions with two fixed points and the equal-radius boundary spheres of $(2, 1)$ -rational functions with a unique fixed point. The two explicit 3-adic rational examples show that ergodicity in the case $\rho(r) = r$ actually occurs. In all resulting ergodic cases the system is also minimal and uniquely ergodic.

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ON QUADRUPLER OF INFINITE-DIMENSIONAL MANIFOLDS IN THE SPACE OF PROBABILITY MEASURES

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ABSTRACT. By investigating the geometric and topological properties of the space $P(X)$ of all probability measures defined on an infinite compactum X , we identify several subspaces of $P(X)$ that are homeomorphic to triples and quadruples of infinite-dimensional manifolds. Furthermore, we distinguish pairs of subspaces in the space of probability measures that are homeomorphic to pairs of infinite-dimensional manifolds.

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Key words: Triples and quadruples of infinite-dimensional manifolds, probability measures, pair of manifolds, homotopy density, bitopological space, probability measures with finite supports, convex set.

Introduction

In this paper, we study topological properties such as manifolds, connectivity, infinite-dimensionality, homogeneity, and homeomorphy, as well as $A(N)$ -geometric properties such as convexity, local convexity, homotopy density, contractibility, and boundary sets of the space of all probability measures $P(X)$, defined on a given infinite compactum X . We also investigate its everywhere dense subsets $B \subset P(X)$ and the set $P_\omega(X)$ of all probability measures with finite supports.

We study how geometric properties of a topological space influence the properties of probability measures on this space. The construction of a general theory of infinite-dimensional manifolds significantly increased interest in infinite-dimensional objects occurring in nature. It is known that the space $P(X)$ of all probability measures on the considered compactum X is a convex subset of a locally convex infinite-dimensional space. The study of the above-listed properties of the space $P(X)$ of all probability measures and its dense subsets is conducted at the intersection of at least the following (main) mathematical disciplines: topology, functional analysis, geometry, and measure theory.

It turns out that probability measures are not only objects of study but also tools for research, through which A. Pelczynski laid the foundations of the theory of Dugundji spaces. [1-2]. We consider both geometric and topological properties of compact and σ -compact subsets of the space $P(X)$ related to their nesting in each other [3-4], geometric and topological properties of boundary, homotopically dense subsets of the space $P(X)$ [5], for subsets E, M, N , $P(X) \supset E \supset M \supset N$ of the space $P(X)$ of topological equivalence of the triple $(P(X), E, N)$, (E, M, N) to the triples (Q, S, Σ) , (Q, Σ, σ) , (S, Σ, σ) .

For an infinite ANR compactum X , the topological equivalence of pairs of manifolds $(P_\infty(X), P_\omega(X))$ to the pairs of manifolds (R^∞, σ) and (Q^∞, Σ) is studied.

Recall that a topological space Y is called an absolute (neighborhood) retract in the class K (written $Y \in A(N)R(K)$ [6], if $Y \in K$ and for every homeomorphism h mapping Y onto a closed subset $h(Y)$ of the space X from the class K , the set $h(Y)$ is a (neighborhood) retract of the space X .

Recall that a topological space X is called a manifold modeled on the space Y , or a Y -manifold [6], if every point of the space X has a neighborhood homeomorphic to an open subset of the space X .

Usually, a Q -manifold is called a separable metric space locally homeomorphic to the Hilbert cube Q , where $Q = \prod_{i=1}^\infty [-1, 1]_i$ is the Hilbert cube [7], $W_i^\pm = \{(g_j) \in Q | g_i = \pm 1\}$ is the i -th face of the Hilbert cube Q , $BdQ = \bigcup_{i=1}^\infty W_i^\pm$ is called the pseudo-boundary of the cube Q , and $S = Q \setminus BdQ$ is the pseudo-interior of the cube Q . It is known that $S \cong \prod_{i=1}^\infty (-1, 1)_i$ [8].

In the theory of infinite-dimensional manifolds, the following objects play an important role: the Hilbert cube Q , the separable Hilbert space ℓ_2 , Σ – the linear span of the standard brick $Q' = \prod_{n=1}^\infty [0, \frac{1}{2^n}]$ in the Hilbert space ℓ_2 . By ℓ_2^f or σ is denoted the linear subspace of the Hilbert space ℓ_2 , consisting of all points only a finite number of coordinates of

which are different from zero, and Q^f — the subspace of the Hilbert cube Q , consisting of all points only a finite number of coordinates of which are different from zero.

By the Anderson-Kadec theorem [8], the Hilbert space ℓ_2 is homeomorphic to S . From the results of Bessaga-Pelczynski, it follows that $\sum$ is homeomorphic to $\text{rint}Q$ and $\sigma \simeq \ell_2^f \simeq Q^f$ [9]. Here, by $\text{rint}Q$ is denoted the set $\{x = (x_n) \in Q \mid |x_n| < t < 1 \text{ for all } n \in N\}$. Further, it is obvious that $\text{rint}Q \approx BdQ$, hence it is true: $BdQ \approx \sum$.

It is known that the spaces Q , $\sum$ and ℓ_2 are strongly infinite-dimensional, while the spaces ℓ_2^f , σ and Q^f are weakly infinite-dimensional and all these spaces are homogeneous.

Definition. [8] A closed set A of a space X is called a Z -set in X , if the identity mapping id_X of the space X can be approximated arbitrarily closely by mappings $f : X \rightarrow X \setminus A$.

A countable union of Z -sets in X is called a σ - Z -set in X .

Following [7], a σ - Z -set, B of the Hilbert cube Q is called a boundary set in Q (denoted by $B(Q)$), if $Q \setminus B \approx \ell_2$. More generally, a boundary set in a Q -manifold is called a σ - Z -set, the complement of which is an ℓ_2 -manifold.

From the above, it follows that the pseudo-boundary BdQ of the Hilbert cube Q is its boundary set.

Let X be a topological space.

A set $A \subset X$ is called homotopically dense in X [5], if there exists a homotopy $h(x, t) : X \times [0, 1] \rightarrow X$ such that $h(x, 0) = id_X$ and $h(X \times (0, 1]) \subset A$.

A set $A \subset X$ is homotopically negligible in X , if $X \setminus A$ is homotopically dense in X .

An embedding $e : Y \rightarrow X$ is homotopically dense (correspondingly, homotopically negligible), if $e(Y)$ is a homotopically dense set (correspondingly homotopically negligible) in X [5].

For a compactum X , there is a simple topological classification of the spaces $P(X)$ of all probability measures. In the case of a finite n -point space $X = \{n\}$ the points μ of the space $P(n) = P_n(n)$ are convex linear combinations of Dirac measures:

$$\mu = m_0\delta(0) + m_1\delta(1) + \dots + m_{n-1}\delta(n-1)$$

Therefore, they are naturally identified with the points of the $(n-1)$ dimensional simplex σ^{n-1} . Here the Dirac $\delta(i)$ form the vertices of the simplex, and the masses m_i , placed at the points i , are the barycentric coordinates of the measure μ . Thus, the compactum $P(n)$ is affinely homeomorphic to the simplex σ^{n-1} .

In the case of an infinite compactum X , the space $P(X)$ is also a compactum [1]. Further, containing simplices of arbitrarily large number of dimensions, they are infinite-dimensional. By Cayley's theorem, a convex compactum $P(X) \subset R^{C(X)}$ affinely embeds into ℓ_2 . Consequently, by Keller's theorem, the compactum $P(X)$, as an infinite-dimensional convex compactum lying in ℓ_2 , is homeomorphic to the Hilbert cube $Q = I^{\aleph_0}$. On the other hand, the space $P(X)$ of all probability measures on a compactum X is called the set of all regular Borel probability measures on X , equipped with the weakest of the topologies for which each functional $f_u : C(X) \rightarrow R$ is continuous, mapping the measure μ to $\mu(U)$ (U is an open set in X).

It is also known that for any χ_1 -degree of a non-degenerate compactum K the space of all probability measures $P(K^{\chi_1})$ is homeomorphic to the Tychonoff cube I^{χ_1} , $P(K^{\chi_1}) = I^{\chi_1}$, I is the interval $[0, 1]$. Note, in particular, that all these spaces are topologically homogeneous. But for spaces $P(K^{\chi_1})$ at $\tau > \chi_1$ the situation is different [10].

For an arbitrary compactum X and measure $\mu \in P(X)$, its support $\text{supp}(\mu)$ is defined — it is the smallest of the closed sets $F \subset X$ for which $\mu(F) = \mu(X)$ i.e. $\text{supp}(\mu) = \bigcap \{A : A = \bar{A}, \mu \in P(A)\}$;

$P_n(X) = \{\mu \in P(X) : |\text{supp} \mu| \leq n\}$ — the set of all measures μ with no more than n supports, $P_\omega(X) = \bigcup_{n=1}^\infty P_n(X)$ — the set of all probability measures μ with finite supports. Recall that the space $P_f(X) \subset P(X)$ consists of all probability measures [11]

$$\mu = m_1\delta(x_1) + m_2\delta(x_2) + \dots + m_n\delta(x_n)$$

with finite supports, for each of which $m_i \geq \frac{n}{n+1}$ for some i ;

$$P_{f,n}(X) = \{\mu \in P_f(X) : |\text{supp} \mu| \leq n\}.$$

From here, the compact subspace $P_f(X)$ is also a union of compacta $P_{f,n}(X)$, i.e., $P_f(X) = \bigcup_{n=1}^\infty P_{f,n}(X)$. It is obvious that $P_{f,n}(X) \subseteq P_n(X)$ and $P_f(X) \subseteq P_\omega(X)$.

It is obvious that for a metric compactum X and any $n \in \mathbb{N}$, the sets $P_n(X)$, $P_f(X)$, $P_{f,n}(X)$ are closed in $P(X)$.

Consequently, the subspace $P_\omega(X) \subset P(X)$ is σ -compact and everywhere dense in $P(X)$.

Note that the subspaces $P(X) \setminus P_n(X)$, $P(X) \setminus P_{f,n}(X)$ and $P(X) \setminus P_f(X)$ are Q -manifolds for any infinite compactum X and any $n \in \mathbb{N}$. In the particular case, $P(Q) \setminus P_n(Q)$, $P(Q) \setminus P_{f,n}(Q)$, $P(Q) \setminus P_f(Q)$ are Q -manifolds. And the subspace $P(X) \setminus P_\omega(X)$ is a G_δ -set in $P(X)$.

For subsets $F \subset E$ and $G \subset F \subset E$, the pair (E, F) or triple (E, F, G) of spaces is called an (E, F) -pair of manifolds or an (E, F, G) -triple (X, M, N) of spaces, if there is an open cover U of the space X and an open embedding $\Phi_u : U \rightarrow E, u \in E$ such that $\Phi_u(M \cap U) = F \cap \Phi_u(U)$ and, accordingly, $\Phi_u(N \cap U) = G \cap \Phi_u(U)$ [12].

Let R^∞ and Q^∞ be the direct limit of spaces R^n and Q^n respectively, i.e., $R^\infty = \lim_{\rightarrow} R^n$ and $Q^\infty = \lim_{\rightarrow} Q^n$, where Q is the Hilbert cube and R is the real line; $R^n = \mathbb{R}^n \times \{0\} \subset R^{n+1}$, $R^\infty = \bigcup_{n=1}^\infty R^n$, $Q^\infty = \bigcup_{n=1}^\infty Q^n$, $Q^n = Q^n \times \{0\} \subset Q^{n+1}$ [13].

On the families $\{R^n : n \in \mathbb{N}\}$ and $\{Q^n : n \in \mathbb{N}\}$, the topology is weak with respect to the hereditary topology in R^ω and Q^ω [13], where $R^\omega = \prod_{n=1}^\infty R^n$ and $Q^\omega = \prod_{n=1}^\infty Q^n$.

Definition.[13] A bitopological space is a pair (X_1, X_2) defined on a single set X ($X_1 = X_2$) or a space (X, τ_1, τ_2) with two topologies τ_1 and τ_2 , denoted as $X_i = (X, \tau_i)$, $i = \overline{1, 2}$.

Let X be a metric compactum and $P_\infty(X) = \varprojlim P_n(X)$ be the direct upper limit of compacta (tower) $P_1(X) \subset P_2(X) \subset \dots$. In this case, there is a bitopological space $(P_\infty(X), P_\omega(X))$ of all probability measures with finite supports. Note that the space $P_\infty(X)$ is a free convex set over the compactum X possessing the property: every continuous mapping $f : X \rightarrow C$ has a unique affine extension $\tilde{f} : P_\infty(X) \rightarrow C$, where C is a convex set in a linear topological space. The subspace $P_\omega(X)$ is a locally convex free convex set over the compactum X possessing the following property: every continuous mapping $f : X \rightarrow C$ has a unique continuous extension $\tilde{f} : P_\omega(X) \rightarrow C$, where C is a convex set of a locally convex linear topological space.

A bitopological space $M = (M, \omega, m)$ is called an (R^∞, σ) -manifold if the topology m is metrizable, $m \subset \omega$, and for every point of the space M there is an m -open neighborhood U and a bitopological open embedding $h : (U, \omega, m) \rightarrow (R^\infty, \sigma)$ such that h is not open for all open embeddings U_m in σ , but is an open embedding for U_ω in R^∞ .

A triple $M = (M, \omega, m)$ is called bilocally equiconnected (denoted as *bi-LEC*) if the diagonal $\Delta_M \subset M \times M$ has an m -neighborhood U (i.e., a neighborhood in the product $M_m \times M_m$) such that there is a bilocally equiconnecting function $\lambda : U \times [0, 1] \rightarrow M$, where $\lambda(x, y, 0) = x$, $\lambda(x, y, 1) = y$, and $\lambda(x, x, t) = x$ for every $(x, y) \in U$ and $t \in [0, 1]$.

Theorem 5. [13] A bitopological space M is an (R^∞, σ) -manifold (or a (Q^∞, Σ) -manifold) if and only if M is *bi-LEC*, M_ω is an R^∞ -manifold (or a Q^∞ -manifold), and every compactum in M_m is a strong Z -set.

Main part

The results of works [15-17] show that for an infinite compactum X , the bitopological pair $(P_\infty(X), P_\omega(X))$ is a *bi-LEC* pair (i.e., satisfies the condition of Theorem 5 [13]) in the following cases:

- a) X is finite-dimensional;
- b) $A \subset X$, A is an F_σ -set or A is σ -compact and $\bar{A} = X$;
- v) $U \subset X$, U is open and $\bar{U} = X$;
- g) A is countable and $\bar{A} = X$;
- i.e., a) $(P_\infty(X), P_\omega(A)) \simeq (Q^\infty, \sigma)$;
- b) $(P_\infty(X), P_\omega(A)) \simeq (R^\infty, \sigma)$;
- v) $(P_\infty(X), P_\omega(U)) \simeq (R^\infty, \sigma)$;
- g) $(P_\infty(X), P_\omega(A)) \simeq (R^\infty, \sigma)$.

In the following cases, the pair $(P_\infty(X), P_\omega(A))$ is homeomorphic to the pair (Q^∞, Σ) if X is an *ANR* compactum:

- a) $P_\omega(X)$ contains a Hilbert cube;
 - b) X contains a Hilbert cube Q ;
- In particular, $(P_\infty(Q), P_\omega(Q)) \simeq (Q^\infty, \Sigma)$.

We have noticed that in many cases for subsets $A \subset X$ of an infinite compactum X , the subspace $P_\omega(A)$ is homeomorphic to σ . There exist different infinite compacta X and Y for which $P_\omega(X)$ is homeomorphic to $P_\omega(Y)$, but $P_\infty(X) \neq P_\infty(Y)$ [13]. That is, the following holds:

Proposition 2.8.[13] For different compacta X and Y , from $P_\infty(X) \simeq P_\infty(Y)$ follows $P_\omega(X) \simeq P_\omega(Y)$.

In the work [12], there is the following:

Theorem 3.1.[12] $\Sigma \setminus \sigma \simeq \Sigma \times S$.

We have noticed that for an infinite compactum X , various everywhere dense subspaces $Y \subset P(X)$ of the Hilbert cube space $P(X)$ are its boundary sets. Consequently, $P_\omega(X)$ is always a boundary set of the compactum $P(X)$. However, the subspace $P_\omega(X)$ can be homeomorphic to either Σ or σ . That is, for some infinite compacta X , if the subspace $P_\omega(X)$ preserves the Hilbert cube Q , then $P_\omega(X) \simeq \Sigma$. But there exists an everywhere dense subset $A \subset X$ different from X such that $P_\omega(A)$ is a boundary set. In this case, the subspace $P_\omega(X)$ is homeomorphic to σ . In these cases, the subspace $P_\omega(X) \setminus P_\omega(A)$ is a $\Sigma \times S$ manifold. That is, the following holds:

Theorem 1. Let X be such an infinite compactum that $P_\omega(X)$ contains the Hilbert cube Q , and $A \subset X$, $A \neq X$, $\bar{A} = X$ and $|A| = \aleph_0$. Then $P_\omega(X) \setminus P_\omega(A) \simeq \Sigma \times S$.

Corollary 1. Let X be an infinite compactum containing the Hilbert cube Q and $A \subset X$, $A \neq X$, $\bar{A} = X$ and $|A| = \aleph_0$. Then $P_\omega(X) \setminus P_\omega(A) \simeq \Sigma \times S$.

Corollary 2. If $A \subset Q$, A is countable and $\bar{A} = X$, then $P_\omega(Q) \setminus P_\omega(A) \simeq \Sigma \times S$.

In works [3,16], it was shown that for an everywhere dense subset $A \subset X$ of a compactum X , the subspaces $P(A)$ and $P_\omega(A)$ are boundary sets of the compactum $P(X)$, and $P(A) \cong \Sigma$ and $P_\omega(A) \simeq \sigma$. On the other hand, the following holds: $P_\omega(A) \subset P(A)$. Thus, the following is true:

Theorem 2. Let X be an infinite compactum and $A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq \dots$ be closed subsets such that $A = \bigcup_{i=1}^\infty A_i$ is everywhere dense in X . a) $P(A) \simeq \Sigma$, if $P(A)$ contains the Hilbert cube Q ; Then b) $P(A) \simeq \ell_2^f$, if $P(A)$ does not contain the Hilbert cube Q .

Theorem 3. Let X be an infinite compactum and $A_1 \subset A_2 \subset \dots \subset A_n \subset \dots$ be a sequence of closed sets such that $A \neq X$ and $A = \bigcup_{n=1}^\infty A_i$ is everywhere dense in X .

Then a) $P_\omega(A)$ is homeomorphic to ℓ_2^f , if $P_\omega(A)$ does not contain the Hilbert cube Q ;

b) $P_\omega(A) \simeq \Sigma$, if $P_\omega(A)$ contains the Hilbert cube Q .

Corollary 3. Under the conditions of Theorem 2, let N be a countable everywhere dense subset. Then $P(A) \setminus P(N) \simeq \Sigma \times S$, if $P(A)$ contains the Hilbert cube Q ;

Corollary 4. Under the conditions of Theorem 3, let N be a countable everywhere dense subset. Then $P_\omega(A) \setminus P_\omega(N) \simeq \Sigma \times S$, if $P_\omega(A)$ contains the Hilbert cube Q ;

Theorem 4. Let X be a finite-dimensional compactum, and $A_1 \subset A_2 \subset \dots \subset A_n \subset \dots$ be a sequence of its closed subsets such that $A \neq X, \bar{A} = X$. Then $P(A) \setminus P(N) \simeq \Sigma \times S$, if there exists an infinite A_{i_0} , where N is a countable subset in X .

Corollary 5. Let X be a finite-dimensional compactum, and $A_1 \subset A_2 \subset \dots \subset A_n \subset \dots$ be its closed subsets such that $A = \bigcup_{n=1}^\infty A_i$ is everywhere dense and A is different from X .

Then $P(A) \setminus P_\omega(A) \simeq \Sigma \times S$ and $P(A) \setminus P_\omega(X) \simeq \Sigma \times S$, if there exists an index i_0 such that A_{i_0} is infinite.

In work [1], there is the following:

Proposition 3.27.[1] If for the triple (X, Y, Z) the pair (X, Y) is homeomorphic to the pair (Q, S) , and the pair (X, Z) is homeomorphic to the pair $(Q, rintQ)$, then the triple (X, Y, Z) is homeomorphic to the triple $(Q, S, rintQ)$.

In work [12], there is the following:

Theorem 3.6.[12] Let X be a Q -manifold (or S -manifold) and $N \subset M \subset X$. Then it is a (Q, Σ, σ) -triple (or (S, Σ, σ) -triple) of manifolds if and only if the pair (M, N) is a (Σ, σ) -manifold and M or N is embedded densely in X .

This theorem can be stated in the following way:

(X, M, N) is a (Q, Σ, σ) - (or (S, Σ, σ))- triple of manifolds, if and only if the pair (X, M) is a (Q, Σ) - (or, (S, Σ))-pair (or, (X, N) is a (Q, σ) - (or (S, σ))- manifold pair and (M, N) is a (Σ, σ) pair.

Proposition 1. For any infinite compactum X , the spaces a) $CP_\omega(X)$, $S_P(A)$, $P(X) \setminus P(A)$ and $P(X \setminus A)$ are homotopically dense in $P(X)$, where $CP_\omega(X) = P(X) \setminus P_\omega(X)$, $A \subset X$, $A \neq \emptyset$, $A \neq X$;

b) For any $n \in \mathbb{N}$ the space $P_\omega(X) \setminus P_n(X)$ is homotopically dense in $P_\omega(X)$; v) For any $n \in \mathbb{N}$ the space $P_\omega(X \setminus A)$ is homotopically dense in $P_\omega(X)$ where $A \subset X$, $A \neq \emptyset$, $A \neq X$;

Proof. Let X be an infinite compact space; then $P(X) \simeq Q$. We shall show that the subspace $CP_\omega(X)$, consisting of the complement of the set of all probability measures with finite supports, is homotopically dense in $P(X)$.

Take a point $\mu_0 \in P(X) \setminus P_\omega(X)$. The support $supp\mu_0$ of this measure μ_0 is not finite, i.e., $|supp\mu_0| \geq \chi_0$. Construct a homotopy $h(\mu, t) : P(X) \times [0, 1] \rightarrow P(X)$ by setting $h(\mu, t) = t\mu_0 + (1 - t)\mu$.

At $t = 0$, $h(\mu, 0) = 0 \cdot \mu_0 + (1 - 0)\mu = \mu$, i.e., $h(\mu, 0) = id_{P(X)}$.

At $t = 1$, then $h(\mu, 1) = \mu_0$.

For $t > 0$, the measure $h(\mu, t) \notin P_\omega(X)$, since the support of the measure $h(\mu, t)$ consists of the set $supp\mu \cup supp\mu_0$. This set is infinite. Thus, the subspace $P_\omega(X)$ is homotopically dense in $P(X)$.

In the case of the homotopy density of the subsets $S_P(A)$, $P(X) \setminus P(A)$ and $P(X \setminus A)$, take a point μ_0 in the complement sets of these sets, i.e., $\mu_0 \in P(X) \setminus S_P(A)$, $\mu_0 \in P(A)$, $\mu_0 \in P(X) \setminus P(X \setminus A)$. The rest of the proof is identical. That is, the sets $S_P(A)$, $P(X) \setminus P(A)$ and $P(X \setminus A)$ are homotopically dense in $P(X)$.

b) It is known that $P_\omega(X)$ is a σ - Z -set in $P(X)$ and $P_\omega(X)$ is a boundary set of the space $P(X)$. On the other hand, $P_\omega(X)$ is an AR -space and $P_\omega(X)$ is convex. Consequently, the space $P_\omega(X)$ is strongly universal [4]. From this it follows that $P_\omega(X)$ possesses the compact Z -property, i.e., any compactum in $P_\omega(X)$ is a Z -set in $P_\omega(X)$, including $P_n(X)$ which is also a Z -set. From the result [6] it follows that $P_\omega(X) \setminus P_n(X)$ is homeomorphic to $P_\omega(X)$.

v) Now we show that $P_\omega(X) \setminus P_n(X)$ is homotopically dense in $P_\omega(X)$. Take a measure $\mu_0 \in P(X)$ such that $|supp\mu_0| = \{x_1, x_2, \dots, x_{n+3}\}$. Construct the required homotopy $h(\mu, t) : P_\omega(X) \times [0, 1] \rightarrow P_\omega(X)$ by setting

$$h(\mu, t) = t\mu_0 + (1 - t)\mu$$

At $t = 0$, $h(\mu, 0) = \mu$, i.e., $h(\mu, 0) = id_{P_\omega(X)}$.

For $t \in (0, 1]$ the measure $h(\mu, t)$ has more than $n+3$ distinct points, i.e., $h(\mu, (0, 1]) \notin P_n(X)$. Hence, $h(\mu, (0, 1]) \in P_\omega(X) \setminus P_n(X)$. Thus, the subspace $P_\omega(X) \setminus P_n(X)$ is homotopically dense in $P_\omega(X)$.

g) It is obvious that $P_\omega(X \setminus A)$ is convex and an AR -space. Take a measure $\mu_0 \in P_\omega(X) \setminus P_\omega(X \setminus A)$ such that $\text{supp } \mu_0$ consists of $n+3$ points. Let the measure μ_0 have finite support, i.e., $\text{supp } \mu_0 = \{x_1, x_2, \dots, x_{n+3}\}$ and $x_i \notin A$. The required homotopy is $h(\mu, t) : P_\omega(X) \times [0, 1] \rightarrow P_\omega(X)$.

Construct it using the formula $h(\mu, t) = t\mu_0 + (1-t)\mu$. A simple verification shows that the subspace $P_\omega(X \setminus A)$ is homotopically dense in $P_\omega(X)$. $\square$

Theorem 5. For any infinite compactum X and its closed non-empty infinite subset A different from the compactum X itself, there exist such dense subsets $\Sigma_P(A)$ and $\sigma_P(A)$ in the space $S_P(A)$ that $\Sigma_P(A) \cong \Sigma$ and $\sigma_P(A) \cong \sigma$. The triple $(S_P(A), \Sigma_P(A), \sigma_P(A)) \simeq (S, \Sigma, \sigma)$.

Proof. Let X be an infinite compactum, and $A \subset X$, $A = \bar{A}$ and $A \neq X$, A being infinite. Then $P(A) \simeq Q$ and $P(X) = Q$. On the other hand, the set $P(A) \subset S_P(A)$. It is known that $S_P(A) \subset P(X)$ and $S_P(A)$ is everywhere dense and homotopically dense in $P(X)$. From the homeomorphism of $P(A)$ to the Hilbert cube Q , we can consider $P(A)$ as a Keller cube. We may assume that $O \in P(A)$, where O is the zero point. Take a set $K = \{\mu_1, \mu_2, \dots, \mu_k\}$ in $S_P(A) \setminus P(A)$. Now for each $n \in \mathbb{N}$ set $S_P^n(A) = \text{conv}\{P(A), \mu_1, \mu_2, \dots, \mu_n\}$, where $S_P^n(A)$ is the convex hull of the set $\{P(A), \mu_1, \mu_2, \dots, \mu_n\}$. From Lemma 5.3.14 [5] it follows that $S_P^n(A)$ is a Keller cube and the zero point O belongs to $S_P^n(A)$.

Consequently, $S_P^n(A) \simeq Q$. Now denote $\bigcup_{n=1}^\infty S_P^n(A)$ by $\Sigma_P(A)$. It is obvious that $S_P^1(A), S_P^2(A), \dots, S_P^n(A), \dots$ is a skeletoid in $S_P(A)$ and $\Sigma_P(A)$ is a convex σ -compact AR space containing the Hilbert cube Q . In this case, from Theorem 5.3.13 [5] it follows that the subspace $\Sigma_P(A)$ is strongly μ_0 -universal. Consequently, from Corollary 5.3.15 [5] we have $\Sigma_P(A) \simeq \Sigma$.

Now let $\{\mu_1, \mu_2, \dots, \mu_k, \dots\}$ be a countable everywhere dense subset in $P(A)$. For each $n \in \mathbb{N}$ set $X_n = \text{conv}\{\mu_1, \mu_2, \dots, \mu_n\}$ – the convex hull of the points $\{\mu_1, \mu_2, \dots, \mu_n\}$. We have the sequence

$$X_1 \subset X_2 \subset \dots \subset X_n \subset \dots \quad (9)$$

The sequence (9) is a strongly universal space by Lemma 5.3.11 [5]. Denote $\bigcup_{i=1}^\infty X_i$ by $\sigma_P(A)$. In this case, for $n \in \mathbb{N}$, X_n is convex and X is σ -compact and A is a convex set. Note that it is an AR space. By Corollary 5.3.12 [5] it follows that $\sigma_P(A) \simeq \sigma$. Thus, the triple $(S_P(A), \Sigma_P(A), \sigma_P(A)) \simeq (Q, \Sigma, \sigma)$. $\square$

Theorem 6. Let X be an infinite compactum. In the space $P_{\nabla\omega}(X) = P(X) \setminus P_\omega(X)$ there exist such dense subsets $\Sigma(\omega)$ and $\ell_2^f(\omega)$ that $\Sigma(\omega) \simeq \Sigma$, $\ell_2(\omega) \simeq \ell_2$ and the triple $(P_{\nabla\omega}(X), \Sigma(\omega), \ell_2^f(\omega)) \simeq (S, \Sigma, \sigma)$.

Proposition 2. Let X be an infinite compactum. For any non-empty subset $A \subset X$ different from X , the subspace $P(X) \setminus P(A)$ is homotopically dense in $P(X)$.

Proof. Let X be an infinite compactum $A \subset X$ and $A \neq X$. In this case $P(A)$ is convex in $P(X)$ and $P(X) \simeq Q$. Take an arbitrary point $x_0 \in X \setminus A$. Construct a homotopy $h(\mu, t) : P(X) \times [0, 1] \rightarrow P(X)$ by setting $h(\mu, t) = (1-t)\mu + t\delta_{x_0}$.

At $t = 0$, $h(\mu, 0) = (1-0)\mu + 0 \cdot \delta_{x_0} = \mu$, i.e., $h(\mu, 0) = id_{P(X)}$.

At $t = 1$, $h(\mu, 1) = \delta_{x_0}$.

For $t \in (0, 1]$, i.e., $t > 0$, the measure $h(\mu, t) \notin P(A)$, since $\text{supp } h(\mu, t)$ consists of the set $\text{supp } \mu \cup \{x_0\}$. Thus, the subspace $P(X) \setminus P(A)$ is homotopically dense in $P(X)$. $\square$

Proposition 3. For any infinite compactum X and any $n \in \mathbb{N}$, the subspace $P(X) \setminus P_n(X)$ is homotopically dense in $P(X)$.

Proof. Let X be infinite and $n \in \mathbb{N}$. Take an arbitrary measure $\mu_0 \in P(X) \setminus P_n(X)$ such that $|\text{supp } \mu_0| \geq n+3$. Construct a homotopy $h(\mu, t) : P(X) \times [0, 1] \rightarrow P(X)$ by setting $h(\mu, t) = (1-t)\mu + t\mu_0$.

At $t = 0$ the measure $h(\mu, 0) = \mu$, i.e., $h(\mu, 0) = id_{P(X)}$.

At $t = 1$ the measure $h(\mu, 1) = \mu_0$.

For $t > 0$ the measure $h(\mu, t) \notin P_n(X)$ because the support $\text{supp } h(\mu, t)$ of the measure $h(\mu, t)$ consists of the set $\text{supp } \mu \cup \text{supp } \mu_0$. Thus, the subspace $P(X) \setminus P_n(X)$ is homotopically dense in $P(X)$. $\square$

Proposition 4. For any infinite compactum X , the subspace $P(X) \setminus P_f(X)$ is homotopically dense in $P(X)$.

Proof. Let X be an infinite compactum. It is known that $P_f(X) \subset P(X)$ and $P(X) \setminus P_f(X)$ is convex. Take an arbitrary point $\mu_0 \in P(X) \setminus P_f(X)$ where the support $\text{supp } \mu_0$ is infinite; or the measure μ_0 has the form $\mu_0 = \frac{1}{n}\delta_{x_1} + \frac{1}{n}\delta_{x_2} + \dots + \frac{1}{n}\delta_{x_n}$, where $n > 2$, i.e., $\text{supp } \mu_0 = \{x_1, x_2, \dots, x_n\}$. Construct a homotopy $h(\mu, t) : P(X) \times [0, 1] \rightarrow P(X)$ by setting $h(\mu, t) = (1-t)\mu + t\mu_0$.

At $t = 0$, the measure $h(\mu, 0) = \mu$, i.e., $h(\mu, 0) = id_{P(X)}$.

At $t = 1$ the measure $h(\mu, 1) = \mu_0$.

For $t > 0$ the measure $h(\mu, t) \notin P_f(X)$, since the support of the measure $h(\mu, t)$ contains the set $\text{supp } \mu \cup \text{supp } \mu_0$.

Thus, the subspace $P(X) \setminus P_f(X)$ is homotopically dense in $P(X)$. $\square$

We have noticed that for an infinite compactum X and its everywhere dense subset A different from X , the subspaces $P_\omega(X)$ and $P_\omega(A)$ are boundary sets of the compactum $P(X)$, i.e., $P(X) \setminus P_\omega(X)$ and $P(X) \setminus P_\omega(A)$ are homeomorphic to ℓ_2 . In this case, the following can be asserted:

- a) for any $n \in \mathbb{N}$ the space $P_\omega(X) \setminus P_n(X)$ is homeomorphic to $P_\omega(X)$;
- b) for any $n \in \mathbb{N}$ the space $P_\omega(A) \setminus P_n(A)$ is homeomorphic to $P_\omega(A)$;

It is known that $P_f(X)$ is compact and is a countable union of compacta $P_{f,n}(X)$ which are subsets of the compactum $P_n(X)$. Consequently, the compacta $P_{f,n}(X)$ are Z -sets in $P_\omega(X)$. In this case, it can be asserted:

- v) for any infinite compactum X , the space $P_\omega(X) \setminus P_f(X)$ is homeomorphic to the space $P_\omega(X)$.

In work [3], we proved that for any infinite compactum X and any of its closed subsets A different from X , the space $S_P(A)$ is homeomorphic to the Hilbert space ℓ_2 .

If B is a compactum in $S_P(A)$, then B is a Z -set in $S_P(A)$. Then, by the theorem in [6], the space $S_P(A) \setminus B$ is homeomorphic to ℓ_2 . In particular, $S_P(A) \setminus P(A)$ is homeomorphic to ℓ_2 .

Theorem 7. For any infinite metric compactum X and any everywhere dense, non-open G_δ -subset Y different from X , the subspace $P(X) \setminus P(Y)$ is homeomorphic to the space Σ .

Proof. Let X be an infinite compactum and $Y \subset X$, where Y is a G_δ -set, $Y \neq X$, and Y is not open. In this case, we have: $Y = \bigcap_{i=1}^{\infty} U_i$, where U_i is open in X and $\bar{U}_i = X$. Note that $U_1 \supset U_2 \supset \dots \supset U_n \supset \dots$. Hence $X \setminus U_1 \subset X \setminus U_2 \subset \dots \subset X \setminus U_n \subset \dots$. For each $i \in \mathbb{N}$, the set $X \setminus U_i$ is closed and $X \setminus U_i$ is infinite. Consequently, $X \setminus U_i$ is compact.

By the equality $P(X \setminus A) = P(X) \setminus S_P(A)$, where $A \subset X, A \neq X$, we have:

$$P(X) \setminus P(Y) = \{\mu \in P(X) : \text{supp } \mu \cap (X \setminus Y) \neq \emptyset\}.$$

Thus, $P(X) \setminus P(Y) = S_P(X \setminus Y) = S_P(X \setminus \bigcap_{i=1}^{\infty} U_i) = S_P(\bigcup_{i=1}^{\infty} (X \setminus U_i)) = \bigcup_{i=1}^{\infty} S_P(X \setminus U_i) = \bigcup_{i=1}^{\infty} P(X \setminus U_i)$.

Hence, the space $P(X) \setminus P(Y) = \bigcup_{i=1}^{\infty} P(X \setminus U_i)$ is σ -compact. Note that $P(X \setminus U_i) \subset P(X \setminus U_{i+1})$. By virtue of the infinity of $X \setminus U_i$, the space $P(X \setminus U_i) \simeq Q$. From the results of work [4], the subspace $P(X) \setminus P(Y)$ is a σ - Z -set in $P(X)$.

It follows from the definition that $P(X) \setminus P(Y)$ is locally homotopically negligible in $P(X)$. Thus, the subspace $P(X) \setminus P(Y)$ is an increasing sequence of its infinite-dimensional convex compacta such that:

- 1) $P(X \setminus U_i)$ is a Z -set in $P(X \setminus U_{i+1})$ [4];
- 2) $\bigcup_{i=1}^{\infty} P(X \setminus U_i)$ is everywhere dense in $P(X)$.

Then the pair $(P(X), P(Y))$ is homeomorphic to the pair $(Q, \text{rint } Q)$ by Lemma 2.10 [18]. By Theorem 2.2 [18], the pair $(P(X), P(Y))$ is homeomorphic to the pair (Q, S) .

Thus, the following holds:

Theorem 8. For any infinite metric compactum X and its proper non-open everywhere dense G_δ subset $Y \subset X$, the pair $(P(X), P(Y))$ is homeomorphic to the pair (Q, S) .

Corollary.[18] For space X , we have $P(X) \simeq S$ if and only if X is a complete separable non-compact metric space.

Using the functorial values of the probability measure functor $P : \text{Comp} \rightarrow \text{Comp}$ and the relative positions of compacta [16-17], Theorems 7-8, the Corollary in [18], and Corollary 16 in [18] is proved as follows:

Theorem 9. The quadruple $(P(X), P(Y), P(A), P_\omega(L))$ is homeomorphic to the quadruple (Q, s, Σ, σ) if and only if X is an infinite separable compactum, Y is a non-open dense G_δ -set with $Y \neq X$, A is open and dense, and L is a countable dense subspace in X .

Theorem 10. For any compactum X and any proper subset $A \subset X$, the subspace $S_{P_n}(A)$ ($n > 1$) is homotopically dense in $P_n(X)$.

Proof. Let X be an arbitrary compactum. It is known that $P_n : \text{Comp} \rightarrow \text{Comp}$. Consider the normal functor and the set $S_{P_n}(A) = \{\mu \in P_n(X) : \text{supp } \mu \cap A \neq \emptyset\}$, where A is a proper subset of the compactum X .

If X is a finite set, i.e., $|X| = n$, then $P_n(X) = \sigma^{n-1}$ – a simplex of dimension $n - 1$.

Note that the set $S_{P_n}(A)$ is everywhere dense in $P_n(X)$ for any proper subset $A \subset X$.

The required homotopy $h(\mu, t) : P_n(X) \times [0, 1] \rightarrow P_n(X)$ is constructed as $h(\mu, t) = (1 - t)\mu + t\mu_0$, where $\mu_0 \in P(A)$, i.e., $\text{supp } \mu_0 \subseteq A$.

If $\mu \in P_n(X)$, then $h(\mu, 0) = (1 - 0)\mu + 0 \cdot \mu_0 = \mu$, i.e., $h(\mu, 0) = \text{id}_{P_n(X)}$.

If $t > 0$, then the measure $h(\mu, t) = (1 - t)\mu + t\mu_0$ has a support $\text{supp } h(\mu, (0, 1])$ that intersects the set A , that is, $\text{supp } h(\mu, (0, 1]) \cap A \neq \emptyset$. Hence, $h(\mu, t) \in S_{P_n}(A)$ for $t > 0$. This means that $S_{P_n}(A)$ is homotopically dense in $P_n(X)$. $\square$

If X is a compactum and $A \subset X$ is its proper subset, then $S_{P_n}(A)$ is a convex subset in $P_n(X)$. Indeed, if $\mu_0 \in S_{P_n}(A)$ and $\mu_1 \in S_{P_n}(A)$, then the measure $(1-t)\mu + t \cdot \mu_1$ belongs to $S_{P_n}(A)$ because the support intersects with A .

On the other hand, the set $S_{P_n}(A)$ is locally convex. Thus, for a proper subset $A \subset X$ in the subspace $S_{P_n}(A)$ of the space $P_n(X)$, we have:

Theorem 11. For any compactum X and any of its proper subsets $A \subset X$, the subspace $S_{P_n}(A)$ is an absolute retract, i.e., $S_{P_n}(A) \in AR$.

It is known that for any compactum X , the set $P_{n,n-1}(X)$ is an open subspace in $P_n(X)$. If $\mu \in P_{n,n-1}(X)$, then its support consists of exactly n points, that is, $\text{supp } \mu = \{x_1, x_2, x_3, \dots, x_n\}$. The set of all measures μ whose supports consist of exactly n points forms an n -dimensional simplex σ^n , and its interior is $\text{int } \sigma^n$. It is known that $\text{int } \sigma^n$ is homeomorphic to the Euclidean space $\mathbb{R}^n$ of dimension n .

Thus, $\text{int } \sigma^n \subset P_{n,n-1}(X)$.

Take an arbitrary $\mu_0 \in P_n(X) \setminus P_{n-1}(X)$, then $\text{supp } \mu_0 = \{x_1, x_2, \dots, x_n\}$. Let $O(\mu_0)$ be an arbitrary neighborhood of the measure μ_0 in $P_n(X)$. Then $O(\mu_0) \cap (P_n(X) \setminus P_{n-1}(X)) = O'(\mu_0)$, where $O'(\mu_0)$ is open in $P_n(X) \setminus P_{n-1}(X)$. We may assume that $O'(\mu_0)$ is an open subset in $\text{int } \sigma^n \simeq \mathbb{R}^n$. Every open subset of $\text{int } \sigma^n$ is homeomorphic to $\mathbb{R}^n$.

The following holds:

Theorem 12. For any compactum X , the subspace $P_{n,n-1}(X)$ of the space $P_n(X)$ is an $\mathbb{R}^n$ -manifold.

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