



# O'ZMU XABARLARI

## ВЕСТНИК НУУЗ

### АСТА NUUZ

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Bosh muharrir:

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Bosh muharrir o'rinbosari:

**ERGASHOV Y. S.** — f.-m.f.d., professor

Tahrir hay'ati:

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**Xudoyberdiyev A.** – f.-m.f.d. prof.

**Xudoyberganov G.** – f.-m.f.d., prof.

**Matyakubov A. S.** – f.-m.f.d., prof.

Mas'ul kotib: f.-m.f.d. (PhD) **G'aybullayev R.Q.**

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**CONTENTS**  
**СОДЕРЖАНИЕ**

<b>Akhmedova D. D.</b> Dynamics of Legendre polynomials .....	3
<b>Bozorqulov A. A.</b> $k_0^{(m)}$ -translation-invariant gibbs measures for one the fertile three-state Hard-Core model .....	11
<b>Dekhkonov F. N.</b> On the control problem associated with a fourth order pseudo-parabolic equation .....	23
<b>Husenova J. T.</b> CHekli o‘lchamli Fredgolm operatori darajalarining spektri va rezolventasi .....	35
<b>Ismoilova D. E.</b> Uchinchi tartibli operatorli matritsa muhim spektrining chegaralari uchun baholashlar .....	44
<b>Ismoilov G. I.</b> Spectral properties of the one-particle Schrödinger operator perturbed with non-local Delta-potential .....	52
<b>Kuldoshev K. K., Narzillaev N. Kh.</b> $(m, \psi, \delta)$ – regularity of compacts in $\mathbb{C}^n$ .....	65
<b>Madraximova Z. S., Khayrullaev U. B., Temirova S. B.</b> Numerical methods for the differential equations fractional order .....	77
<b>Mutalliyev N. N.</b> Translation invariant gibbs measures for three state Hard-Core models in the case wand .....	84
<b>Okhunova M. O.</b> On a family of Volterra cubic stochastic operators .....	96
<b>Rahmatullaev M. M., Rasulova M. A., Hakimova M. A.</b> Periodic ground states for the CHui-Weeks model on the cayley tree of order two .....	104
<b>Sharipova M. Sh.</b> Investigation of the essential spectrum of a $3 \times 3$ operator matrix .....	111
<b>Umirqulova G. H.</b> Panjaradagi uch zarrachali sistemaga mos model hamiltonian muhim spektrining tavsifi .....	118
<b>Yilihamujiang Yusupu, Matyakubov A.</b> Application of Matrices in Plant Recognition and Artificial Intelligence: A PYNQ-Z2-Based Solution .....	128
<b>Икрамов А. М.</b> Компьютерное моделирование термоупругого деформирования тел с отверстием .....	139
<b>Маматова Н. Т.</b> Обратная задача о динамическом нагружении стержня с учетом отражения нелинейных волн .....	147
<b>Муминов У. Р.</b> Динамика вырожденных отображений Лотки-Вольтерра действующих на симплексе $S^4$ .....	157
<b>Мустапокулов Х. Я., Мамадалиев Н. А.</b> Построение п-стратегий в игре простого преследования-убегания с импульсным управлением .....	164
<b>Сагдуллаева М. М.</b> Нелокальная задача с интегральным условием для уравнения третьего порядка .....	175
<b>Файзиев А. К.</b> Начальная задача для импульсных систем интегродифференциальных уравнений .....	182
<b>Худайбергенов К. К.</b> Сходимость рекуррентных нейронных сетей множествами весовыми связями сети .....	191

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## DYNAMICS OF LEGENDRE POLYNOMIALS

Akhmedova D. D. \*

## RESUME

Legendre polynomials have been applied in image reconstruction since the early years. In image processing, Legendre Polynomials can be used for color image compression. In this paper, we studied the dynamics of Legendre polynomials, which are orthogonal in the interval  $[-1, 1]$ . We computed fixed points and orbits of some points using Wolfram Mathematica for the case when  $n=2$ . The Legendre polynomial is an even function, so it is sufficient to compute it in the interval  $[0, 1]$ .

**Key words:** Dynamical system, legendre polynomials, fixed points, orbits, periodic point.

## 1. Introduction

The late 19th century witnessed the development of the field of orthogonal polynomials, with numerous sets of orthogonal polynomials being derived from the examination of particular physical problems. One of them are Legendre polynomials. They are orthogonal on the interval  $[-1, 1]$ .

It's no secret that in recent decades, researchers have been using Legendre polynomials to process medical images, as well as their signals [3]-[5]. In [3]-[5] the Legendre polynomial method is used for 1D modeling, and the extended formulation in [3]-[12] is used for 2D modeling. In medical images, Legendre polynomials are used mainly in color image compression [3], as well as in image reconstruction based on its blur invariant [6]. For instance, they are used 2D Legendre polynomials in the classification MRI images for hybrid technique [12]. In image processing, Legendre Polynomials can be used for color image compression [3], construction of image from its blur invariant [3, 8]. [4, 5] presents a method of how to calculate Legendre moments for the grayscale image. In this paper, we learnt dynamical systems of Legendre polynomials. This dictates the relevance of the task of this article. Paper is organized as follows. In section 2 short description of Legendre polynomials and their dynamical systems. In section 3 we found their fixed points and orbits in case  $n = 2$ . Furthermore, they are shown in images. Finally the conclusion is presented in section 4.

## 2. Main part

**Definition 2.1.** Two function  $f_i(x)$  and  $f_j(x)$  are called orthogonal on an interval  $[a, b]$ , if satisfies this equation

$$\int_a^b f_i(x)f_j(x)dx = 0$$

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\*Akhmedova D. D – Andijan State University, dilafruz.ahmedova.0695@gmail.com

A set of orthogonal polynomials in the set are orthogonal to each other, that is

$$\int_a^b f_i(x)f_j(x)dx = 0, \quad \text{where } i \neq j$$

These polynomials arise as solutions to Legendre's differential equation (Adrien -Marie Legendre 1752-1883)

$$(1 - x^2) \frac{\partial^2 P(x)}{\partial x^2} - 2x \frac{\partial P(x)}{\partial x} + \lambda P(x) = 0 \quad (1)$$

where  $\lambda = n(n + 1)$

This equation is commonly encountered in physics, especially in cases where spherical polar coordinates are utilized and the problem exhibits cylindrical symmetry

Legendre polynomials are integral components of the quantum mechanical solution for rotational motion.

We have following polynomials in cases  $n = 1, 2, 3, \dots$

$$P_0(x) = 1;$$

$$P_1(x) = x;$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1);$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x);$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3);$$

... ..

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

Our main interest is learning dynamics of these polynomials.

First thing to do is that we need to find it's fixed points.

**Definition 2.2** The point  $x$  is called a fixed point for  $f$ , if  $f(x) = x$ .

**Definition 2.3** Suppose  $x_0$  is a fixed point for  $F$ . Then  $x_0$  is an *attracting fixed point* if  $|F'(x_0)| < 1$ . The point  $x_0$  is a *repelling fixed point* if  $|F'(x_0)| > 1$ . Finally, if  $|F'(x_0)| = 1$ , the fixed point is called *neutral* or *indifferent*.

Let  $n = 1$ , then we have following sequences

$$P_1(x) = P_1^2(x) = P_1^3(x) = \dots = P_1^k(x) = x$$

for any  $k \in N$

So any iterate is definitely equal to  $x$  value.

Next we consider that  $n = 2$ , then we have as we defined above

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \quad (2)$$

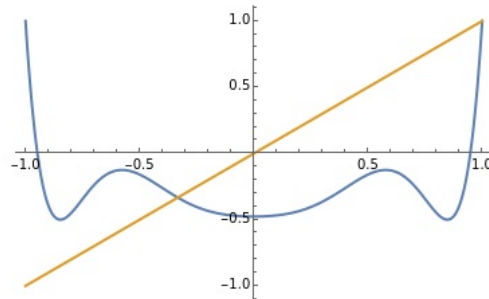


Fig 3. Fixed points of  $P_2^3(x)$

We solve this equation to find it's fixed points

$$P_2(x) = x$$

$$-\frac{1}{2} + \frac{3x^2}{2} - x = 0$$

and we have  $x_1 = -\frac{1}{3}$  and  $x_2 = 1$  solutions So this equation has two real roots.

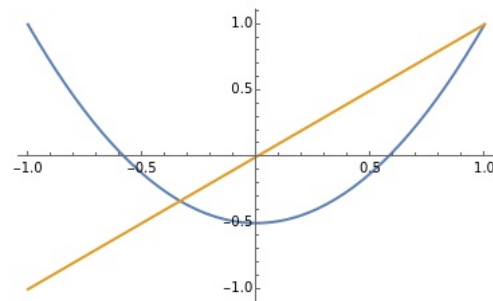


Fig 1. Fixed points of  $P_2(x)$

After then we find fixed points of second iterate of that:

$$P_2^2(x) = -\frac{1}{8} - \frac{9x^4}{4} + \frac{27x^4}{8} = x \quad (3)$$

in this case equation has four  $x = -\frac{1}{3}$  multiple and  $x = 1$  real roots

Similary,  $P_2^3(x) = x$  equation has two real and six complex roots.

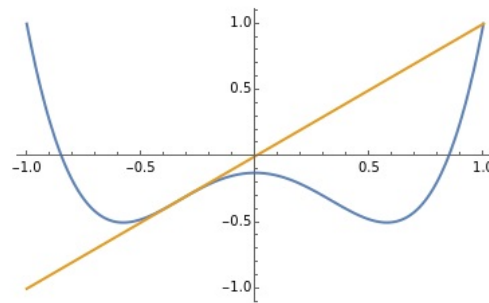


Fig 2. Fixed points of  $P_2^2(x)$

**Theorem 2.4.** *Let us*

$$F_2^n(x) = x \quad (4)$$

equation for  $\forall n \in N$ , then 1) It has two real roots, if  $n = 2k + 1$ , 2) It has four real roots, if  $n = 2k$ .

Next we check type of these fixed points. For this we compute the magnitude of the first derivative  $P'(x_0)$

$$P'(x_0) = \left(\frac{1}{2}3x^2 - 1\right)' = 3x$$

We have  $|P'(x_0)| = 1$  in case  $x_0 = -\frac{1}{3}$  and  $|P'(x_0)| > 1$  for  $x_0 = 1$ . Then  $x_0 = -\frac{1}{3}$  is indifferent point and  $x_0 = 1$  is repelling fixed point by definition, but we will see  $x_0 = -\frac{1}{3}$  is attracting fixed point by graphical analysis in next section.

Given  $x_0 \in R$ , we define the orbit of this point under F to be the sequence of points

$$x_0, x_1 = F(x_0), x_2 = F^2(x_0), \dots, x_n = F^n(x_0)$$

The point  $x_0$  is called the seed of the orbit.

There are many different kinds of orbits in a typical dynamical system. Obviously, the most important kind of orbit is the fixed point. The orbit of the fixed point is the constant sequence  $x_0, x_0, x_0, \dots$

So The orbits of the fixed points of  $-\frac{1}{3}$  and 1 we identified above are respectively

$$-\frac{1}{3}, -\frac{1}{3}, \dots, -\frac{1}{3}$$

and

$$1, 1, \dots, 1$$

Next using computer research we find orbits for  $x \in [0, 1]$ . We assume  $x_0 = 0.1$ , then it's orbit, which consist of 100 points of  $x_0, F(x_0), F^2(x_0), \dots, F^{100}(x_0)$  is

it shows that this orbit in  $[-0.5, 0.1]$  interval.

Similary

```
{0.1, -0.485, -0.147163, -0.467515, -0.172145, -0.455549, -0.188712,
-0.446581, -0.200847, -0.43949, -0.210272, -0.433678, -0.217885, -0.428789,
-0.224209, -0.424595, -0.229578, -0.420941, -0.234213, -0.417716, -0.23827,
-0.414841, -0.24186, -0.412255, -0.245068, -0.409912, -0.247958, -0.407775,
-0.250579, -0.405815, -0.252971, -0.404009, -0.255165, -0.402336,
-0.257189, -0.400781, -0.259062, -0.39933, -0.260803, -0.397973, -0.262427,
-0.396698, -0.263945, -0.395499, -0.265371, -0.394368, -0.266711,
-0.393298, -0.267975, -0.392284, -0.26917, -0.391321, -0.270302, -0.390406,
-0.271375, -0.389533, -0.272396, -0.388701, -0.273368, -0.387905,
-0.274294, -0.387144, -0.275179, -0.386415, -0.276026, -0.385715,
-0.276836, -0.385043, -0.277613, -0.384396, -0.278359, -0.383774, -0.279076,
-0.383175, -0.279766, -0.382597, -0.28043, -0.382039, -0.281069, -0.3815,
-0.281687, -0.380979, -0.282283, -0.380475, -0.282858, -0.379987, -0.283415,
-0.379514, -0.283954, -0.379055, -0.284476, -0.37861, -0.284981, -0.378179,
-0.285472, -0.377759, -0.285947, -0.377351, -0.286409, -0.376955, -0.286858}
```

```
{0.2, -0.44, -0.2096, -0.434102, -0.217333, -0.429149, -0.223746, -0.424906,
-0.229182, -0.421213, -0.233869, -0.417958, -0.237967, -0.415058, -0.24159,
-0.412451, -0.244826, -0.41009, -0.247739, -0.407938, -0.25038, -0.405965,
-0.252789, -0.404147, -0.254998, -0.402464, -0.257034, -0.4009, -0.258919,
-0.399442, -0.260669, -0.398077, -0.262302, -0.396797, -0.263829,
-0.395592, -0.265261, -0.394455, -0.266608, -0.39338, -0.267878, -0.392362,
-0.269078, -0.391396, -0.270214, -0.390476, -0.271292, -0.389601,
-0.272317, -0.388765, -0.273292, -0.387967, -0.274222, -0.387203,
-0.275111, -0.386471, -0.27596, -0.385769, -0.276773, -0.385095, -0.277553,
-0.384447, -0.278301, -0.383823, -0.27902, -0.383221, -0.279712, -0.382642,
-0.280378, -0.382082, -0.28102, -0.381542, -0.281639, -0.38102, -0.282236,
-0.380514, -0.282813, -0.380025, -0.283372, -0.379551, -0.283912,
-0.379091, -0.284435, -0.378645, -0.284942, -0.378212, -0.285433,
-0.377792, -0.28591, -0.377383, -0.286373, -0.376986, -0.286822, -0.376599,
-0.287259, -0.376223, -0.287684, -0.375857, -0.288098, -0.3755, -0.2885}
```

**100 points of orbits of  $x=0.2$**

Next we describe it using the dynamics of one-dimensional maps, which called graphical analysis

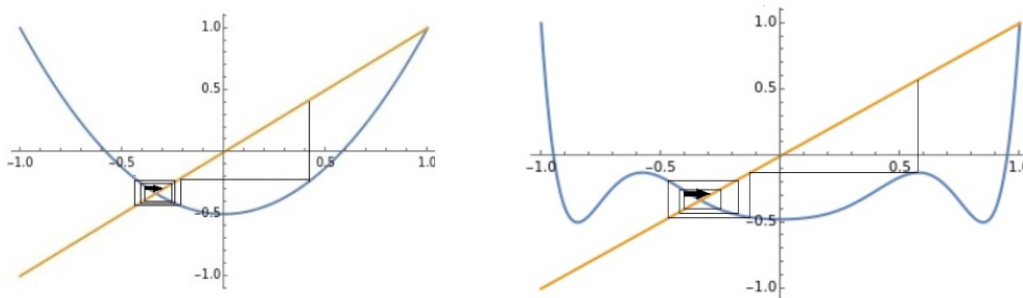
Assume we have the graph of a function  $F$  and wish to display the orbit of a given point  $x_0$ . We begin by superimposing the diagonal line  $y = x$  on the graph of  $F$ . As we know, the points of intersection of the diagonal with the graph give us the fixed points of  $F$ . To find the orbit of  $x_0$ , we begin at the point  $(x_0, x_0)$  on the diagonal directly above  $x_0$  on the  $x$ -axis. We first draw a vertical line to the graph of  $F$ . When this line meets the graph, we have reached the point  $(x_0, F(x_0))$ . We then draw a horizontal line from this point to the diagonal. We reach

```
{0.3, -0.365, -0.300163, -0.364854, -0.300323, -0.364709, -0.300481,
-0.364567, -0.300636, -0.364427, -0.30079, -0.364288, -0.300941, -0.364152,
-0.30109, -0.364017, -0.301237, -0.363884, -0.301383, -0.363753, -0.301526,
-0.363623, -0.301667, -0.363495, -0.301807, -0.363369, -0.301944,
-0.363244, -0.30208, -0.363121, -0.302214, -0.363, -0.302347, -0.36288,
-0.302478, -0.362761, -0.302607, -0.362644, -0.302734, -0.362528, -0.30286,
-0.362414, -0.302985, -0.3623, -0.303108, -0.362189, -0.303229, -0.362078,
-0.303349, -0.361969, -0.303468, -0.361861, -0.303585, -0.361754,
-0.303701, -0.361649, -0.303815, -0.361545, -0.303928, -0.361441, -0.30404,
-0.361339, -0.304151, -0.361238, -0.30426, -0.361139, -0.304368, -0.36104,
-0.304475, -0.360942, -0.304581, -0.360845, -0.304686, -0.36075, -0.304789,
-0.360655, -0.304892, -0.360562, -0.304993, -0.360469, -0.305093,
-0.360377, -0.305193, -0.360286, -0.305291, -0.360196, -0.305388,
-0.360107, -0.305484, -0.360019, -0.305579, -0.359932, -0.305673, -0.359846,
-0.305767, -0.35976, -0.305859, -0.359675, -0.30595, -0.359592, -0.306041}
```

**100 points of orbits of  $x=0.3$**

the diagonal at the point whose  $y$ -coordinate is  $F(x_0)$ , and so the  $x$ -coordinate is also  $F(x_0)$ . Thus we reach the diagonal directly over the point whose  $x$ -coordinate is  $F(x_0)$ , the next point on the orbit of  $x_0$ . Now we continue this procedure. Draw a vertical line from  $(F(x_0), F(x_0))$  on the diagonal to the graph: this yields the point  $(F(x_0), F^2(x_0))$ . Then a horizontal line to the diagonal reaches the diagonal at  $(F^2(x_0), F^2(x_0))$ , directly above the next point in the orbit. To display the orbit of  $x_0$  geometrically, we thus continue this procedure: we first draw a vertical line from the diagonal to the graph, then a horizontal line from the graph back to the diagonal. The resulting “staircase” or “cobweb” provides an illustrative picture of the orbit of  $x_0$ .

Figure 4 shows graphical analysis of  $P_2(x)$  function, which we learnt above. such as that



**Fig4. Graphical analysis of  $P_2(x)$  and  $P_2^2(x)$**

if we find points of orbits of in  $[0, 1]$ , then they are always in  $[-0.5, 1]$

**Conclusion**



In our everyday lives, we are constantly surrounded by a vast array of information, which is transmitted, processed, and analyzed through various media. Digital images, in particular, play a crucial role in communication as they represent a finite set of digital values known as pixels. These images contain valuable information and features and are widely utilized in diverse industrial sectors, including medicine, social media networks, and security environments. The interpretation of image information holds substantial significance, leading to a rapid increase in the demand for image processing and analysis methods. We can find solution to such problems by studying thr dynamic systems of Legendre polynomials. In this paper, we present dynamics of legendre polynomials in case  $n = 2$ . We know that the polynomials we have learned are orthogonal on the interval  $[-1, 1]$ . However, since the function is even when  $n = 2$ , we divided the cross section by two into intervals  $[-1, 0]$  and  $[0, 1]$ . All results in the interval  $[0, 1]$  correspond to tje interval  $[-1, 0]$ .

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### REZYUME

Lejandr ko'pxadlari tasvirni qayta tiklashda dastlabki yillardan beri qo'llanilgan. Tasvirni qayta ishlashda Lejandr ko'pxadlarini rangli tasvirni siqish uchun ishlatish mumkin. Ushbu maqolada biz  $[-1, 1]$  oralig'ida ortogonal bo'lgan Legendre ko'phadlarining dinamikasini o'rgandik. Biz  $n=2$  bo'lgan holat uchun Wolfram Mathematica dasturi yordamida ba'zi nuqtalarning qo'zg'almas nuqtalari va orbitalarini hisobladik. Bu holda lejandr ko'pxadi juft funktsiyadir, shuning uchun uni  $[0, 1]$  oralig'ida hisoblash kifoya.

**Kalit so'zlar:** Dinamik sistema, Lejandr ko'phadi, qo'zg'almas nuqta, trayektoriya, davriy nuqta, davriy trayektoriya.

### РЕЗЮМЕ

Полиномы Лежандра применялись при реконструкции изображений с первых лет. При обработке изображений полиномы Лежандра могут использоваться для сжатия цветных изображений. В данной статье мы исследовали динамику полиномов Лежандра, ортогональных в интервале  $[-1, 1]$ . Мы рассчитали неподвижные точки и орбиты некоторых точек с помощью Wolfram Mathematica для случая, когда  $n=2$ . Полином Лежандра — четная функция, поэтому его достаточно вычислить на интервале  $[0, 1]$ .

**Ключевые слова:** Динамическая система, полиномы Лежандра, неподвижные точки, орбиты, периодическая точка.

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$k_0^{(m)}$ -TRANSLATION-INVARIANT GIBBS MEASURES FOR ONE THE  
FERTILE THREE-STATE HARD-CORE MODEL

Bozorqulov A. A. \*

RESUME

We consider fertile three-state Hard-Core (HC) models with the activity parameter  $\lambda > 1$  on a Cayley tree. It is known that there exist four types of such models: wrench, wand, hinge, and pipe. We study  $k_0^{(m)}$ -translation-invariant Gibbs measures for the considered model in the case wand on the Cayley tree. Such measures are constructed by means of translation-invariant Gibbs measures. In the case wand for  $k_0 = 2$  and  $m = 1, 2, 3$  on the Cayley tree the existence of  $k_0^{(m)}$ -translation-invariant Gibbs measure proved and the exact number of such measures found.

**Key words:** Cayley tree, configuration, fertile graph, Hard-core model, Gibbs measure,  $k_0^{(m)}$ -translation-invariant Gibbs measure.

INTRODUCTION

Solutions to problems encountered in studying the thermodynamic properties of physical and biological systems primarily lead to challenges within the theory of Gibbs measures. These measures are instrumental in characterizing the macroscopic behavior of systems by providing insights into their equilibrium states, statistical dependencies, and phase transitions. It is known that each limit Gibbs measure corresponds to a particular phase of the physical system. This association enables the characterization of distinct thermodynamic phases, such as solid, liquid, or gas, each represented by a unique Gibbs measure. Therefore, in the theory of Gibbs measures, one of the important problems is the existence of a phase transition, i.e., when the physical system changes its state when the temperature changes. This occurs when the Gibbs measure is not unique (see [1]-[4]).

Moreover, it is known that for continuous Hamiltonians (see [5]), the set of Gibbs measures forms a non-empty, convex, and compact subset within the space of all probability measures, equipped with the weak topology (see, e.g., [1, chapter 7]). The set of the Gibbs measures on  $\mathbb{Z}^d$  is the convex hull of the set of all limit Gibbs measures (see [6]).

Hard constraints arise in diverse fields such as combinatorics, statistical mechanics, queuing theory, and telecommunications. In particular, the hard-core model appears in various applications, such as the study of random independent sets within a graph [7], [8], the modeling of gas molecules on a lattice [9], and the analysis of multicasting in telecommunication networks [10], [11]. A.E. Mazel and Yu.M. Suhov introduced and conducted an in-depth study of the hard-core (HC) model on the  $d$ -dimensional lattice  $\mathbb{Z}^d$  [12].

There are many works devoted to studying translation-invariant and periodic Gibbs measures for HC models with three states on the Cayley tree of order  $k \geq 1$  (see [13]-[19]). In particular, in the work [17] in the case wand is considered translation-invariant splitting Gibbs

\*Bozorqulov A. A. – Fergana State University, adhamjonbozorqulov@gmail.com

measures for the HC model with three states on a Cayley tree of arbitrary order and explicit forms of such measures are found on the Cayley tree of order  $k = 2$ .

In work [20], the authors constructed some Gibbs measures (hereinafter referred to as the Gibbs measures obtained by the ART-construction) for the Ising model on the Cayley tree. In papers [21], [22] for the Ising model and in work [23] for the Potts model by means of the translation-invariant Gibbs measure on the Cayley tree of order  $k_0$ , a new Gibbs measure on the Cayley tree of order  $k$ ,  $k_0 < k$ , was constructed and it was called a  $(k_0)$ -translation-invariant Gibbs measure.

In this paper, for the HC model with three states in the case wand, similar to the works of [21]-[23], using known solutions of functional equations corresponding to translation-invariant Gibbs measures on a Cayley tree of order  $k_0 = 2$  known from the work of [17], we construct new measures, which we call  $k_0^{(m)}$ -translation-invariant Gibbs measures, where  $m$  is the number of times of solutions corresponding known translation-invariant Gibbs measures on the set of direct successors of vertex  $x$ . More precisely, in the case for  $m = 1$ ,  $k \geq 3$ , for  $m = 2$ ,  $k = 5, \dots, 9$  and for  $m = 3$ ,  $k = 7, 8, 9$  conditions for the existence of  $k_0^{(m)}$ -translation-invariant Gibbs measure and its exact number found.

## PRELIMINARIES

The Cayley tree  $\mathfrak{S}^k$  of order  $k \geq 1$  is an infinite tree, i.e., a connected graph without cycles, such that exactly  $k + 1$  edges originate from each vertex. Let  $\mathfrak{S}^k = (V, L, i)$ , where  $V$  is the set of vertices  $\mathfrak{S}^k$ ,  $L$  is the set of edges and  $i$  is the incidence function setting each edge  $l \in L$  into correspondence with its endpoints  $x, y \in V$ . If  $i(l) = \{x, y\}$ , then the vertices  $x$  and  $y$  are called the *nearest neighbors*, denoted by  $l = \langle x, y \rangle$ .

For a fixed point  $x^0 \in V$ ,

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^n W_m,$$

where  $d(x, y)$  is the distance between vertices  $x$  and  $y$  on a Cayley tree, i.e., the number of edges of the shortest path connecting  $x$  and  $y$ .

Write  $x \prec y$ , if the path from  $x^0$  to  $y$  goes through  $x$ . Call vertex  $y$  a direct successor of  $x$  if  $y \succ x$  and  $x, y$  are nearest neighbors. Note that in  $\mathfrak{S}^k$  any vertex  $x \neq x^0$  has  $k$  direct successors and  $x^0$  has  $k + 1$  direct successors. Denote by  $S(x)$  the set of direct successors of  $x$ , i.e. if  $x \in W_n$ , than

$$S(x) = \{y_i \in W_{n+1} \mid d(x, y_i) = 1, i = 1, 2, \dots, k\}.$$

*HC model.* Let  $\Phi = \{0, 1, 2\}$  and  $\sigma \in \Omega = \Phi^V$  be a configuration on  $V$ . In this model, each vertex  $x$  is assigned one of the values  $\sigma(x) \in \Phi = \{0, 1, 2\}$ . The values  $\sigma(x) = 1, 2$  mean that the vertex  $x$  is ‘occupied’, and  $\sigma(x) = 0$  means that  $x$  is ‘vacant’. We let  $\Omega$  ( $\Omega_{V_n}$ ) denote the set of all configurations on  $V$  ( $V_n$ ).

We consider the set  $\Phi$  as the set of vertices of a graph  $G$ . We use the graph  $G$  to define a  $G$ -admissible configuration as follows. A configuration  $\sigma$  is called a *G-admissible configuration*

on the Cayley tree (in  $V_n$ ), if  $\{\sigma(x), \sigma(y)\}$  is the edge of the graph  $G$  for any pair of nearest neighbors  $x, y$  in  $V$  (in  $V_n$ ). We let  $\Omega^G$  ( $\Omega_{V_n}^G$ ) denote the set of  $G$ -admissible configurations.

The activity set [13] for a graph  $G$  is a function  $\lambda : G \rightarrow R_+$  from the vertices of  $G$ . The value  $\lambda_i$  of the function  $\lambda$  at the vertex  $i \in \{0, 1, 2\}$  is called the vertex activity.

For given  $G$  and  $\lambda$  we define the formal Hamiltonian of the  $G$ -HC model as

$$H_G^\lambda(\sigma) = \begin{cases} - \sum_{x \in V} \log \lambda_{\sigma(x)}, & \text{if } \sigma \in \Omega^G, \\ +\infty, & \text{if } \sigma \notin \Omega^G. \end{cases}$$

In this paper we consider the case  $\lambda_0 = 1$ ,  $\lambda_1 = \lambda_2 = \lambda$ . For  $\sigma_n \in \Omega_{V_n}^G$  we let

$$\#\sigma_n = \sum_{x \in V_n} \mathbf{1}(\sigma_n(x) \geq 1)$$

denote the number of occupied vertices in  $V_n$ .

Let  $z : x \mapsto z_x = (z_{0,x}, z_{1,x}, z_{2,x}) \in R_+^3$  be a vector-valued function on  $V$ . For  $n = 1, 2, \dots$  and  $\lambda > 0$ , we consider the probability measure  $\mu^{(n)}$  on  $\Omega_{V_n}^G$  defined as

$$\mu^{(n)}(\sigma_n) = \frac{1}{Z_n} \lambda^{\#\sigma_n} \prod_{x \in W_n} z_{\sigma(x), x}, \quad (1)$$

where  $Z_n$  is a normalization divisor,

$$Z_n = \sum_{\tilde{\sigma}_n \in \Omega_{V_n}^G} \lambda^{\#\tilde{\sigma}_n} \prod_{x \in W_n} z_{\tilde{\sigma}(x), x}.$$

The probability measure  $\mu^{(n)}$  is said to be consistent if for all  $n \geq 1$  and any  $\sigma_{n-1} \in \Omega_{V_{n-1}}^G$ :

$$\sum_{\omega_n \in \Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n) \mathbf{1}(\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}^G) = \mu^{(n-1)}(\sigma_{n-1}). \quad (2)$$

In this case, by the theorem Kolmogorov (see [5] or [24] there is a unique measure  $\mu$  on  $(\Omega^G, \mathbf{B})$  such that

$$\mu(\{\sigma : \sigma|_{V_n} = \sigma_n\}) = \mu^{(n)}(\sigma_n)$$

for all  $n$  and any  $\sigma_n \in \Omega_{V_n}^G$ .

**Definition 1.** A measure  $\mu$  defined by formula (1) with consistency condition (2) is called a splitting hard core Gibbs measure with activity  $\lambda > 0$ , corresponding to the function  $z : x \in V \setminus \{x^0\} \mapsto z_x$ . In this case, a HC Gibbs measure corresponding to a constant function  $z_x \equiv z$  is said to be translation-invariant.

**Definition 2.** [13] A graph is said to be fertile if there is a set of activities  $\lambda$  such that the corresponding Hamiltonian has at least two translation-invariant Gibbs measures.

Let  $L(G)$  be the set of edges of a graph  $G$ . We let  $A \equiv A^G = (a_{ij})_{i,j=0,1,2}$  denote the adjacency matrix of the graph  $G$ , i.e.,

$$a_{ij} \equiv a_{ij}^G = \begin{cases} 1, & \text{if } \{i, j\} \in L(G), \\ 0, & \text{if } \{i, j\} \notin L(G). \end{cases}$$

The following theorem presents a condition on  $z_x$  ensuring that the measure  $\mu^{(n)}$  is consistent.

**Theorem 1.** [17] *The probability measures  $\mu^{(n)}$ ,  $n = 1, 2, \dots$ , defined by formula (1) are consistent if and only if the following relations hold for any  $x \in V \setminus \{x^0\}$ :*

$$\begin{aligned} z'_{1,x} &= \lambda \prod_{y \in S(x)} \frac{a_{10} + a_{11}z'_{1,y} + a_{12}z'_{2,y}}{a_{00} + a_{01}z'_{1,y} + a_{02}z'_{2,y}}, \\ z'_{2,x} &= \lambda \prod_{y \in S(x)} \frac{a_{20} + a_{21}z'_{1,y} + a_{22}z'_{2,y}}{a_{00} + a_{01}z'_{1,y} + a_{02}z'_{2,y}}, \end{aligned} \quad (3)$$

where  $z'_{i,x} = \lambda z_{i,x} / z_{0,x}$ ,  $i = 1, 2$ .

In (3), we assume that  $z_{0,x} \equiv 1$  and  $z_{i,x} = z'_{i,x} > 0$  for  $i = 1, 2$ . Then by Theorem 1 there exists a unique  $G$ -HC Gibbs measure  $\mu$  if and only if for any functions  $z : x \in V \mapsto z_x = (z_{1,x}, z_{2,x})$  the equality holds:

$$z_{i,x} = \lambda \prod_{y \in S(x)} \frac{a_{i0} + a_{i1}z_{1,y} + a_{i2}z_{2,y}}{a_{00} + a_{01}z_{1,y} + a_{02}z_{2,y}}, \quad i = 1, 2. \quad (4)$$

### $k_0^{(m)}$ -TRANSLATION-INVARIANT SPLITTING GIBBS MEASURES IN THE CASE $G = \mathit{wand}$

In this paper we consider the case fertile graph  $G = \mathit{wand}$ :

$$\mathit{wand} : \{0, 1\}\{0, 2\}\{1, 1\}\{2, 2\}.$$

In [17] for any  $k$  the translation-invariant Gibbs measures for the HC model with three states in the case of a fertile graph  $G = \mathit{wand}$  studied and considered the following system of equations (see (3.12) of [17]):

$$\begin{cases} z_1 = \lambda \left( \frac{1+z_1}{z_1+z_2} \right)^k, \\ z_2 = \lambda \left( \frac{1+z_2}{z_1+z_2} \right)^k. \end{cases} \quad (5)$$

It is also known from the work of [17] that the system (5) for  $k = 2$  and  $\lambda > 1$  has three solutions, two of them  $h = (\tilde{z}, \tilde{t})$  and  $l = (\tilde{t}, \tilde{z})$  correspond to translation-invariant splitting Gibbs measure  $\mu_1, \mu_2$ , where

$$z_1 = \tilde{z} = \left( \frac{1 + \sqrt{1 - 4a_0^2}}{2a_0} \right)^2, \quad z_2 = \tilde{t} = \left( \frac{1 - \sqrt{1 - 4a_0^2}}{2a_0} \right)^2.$$

Here  $a_0 = 2/(\sqrt{\lambda + 8} + \sqrt{\lambda})$ . It is easy to verify that  $\tilde{z}\tilde{t} = 1$ ,  $\tilde{z} + \tilde{t} = 1/a_0^2 - 2$ .

Using these known solutions  $\tilde{z}, \tilde{t}$  we construct new measures on a Cayley tree that differ from the previously known ones.

We consider the half-tree. Namely the root  $x^0$  has  $k$  nearest neighbors. We construct below new solutions of the functional equation (4).

The boundary condition  $z = \{z_x, x \in G_k\}$  with fields taking values  $h, l$  defined by the following steps (for an example of such a function see Fig. 1):

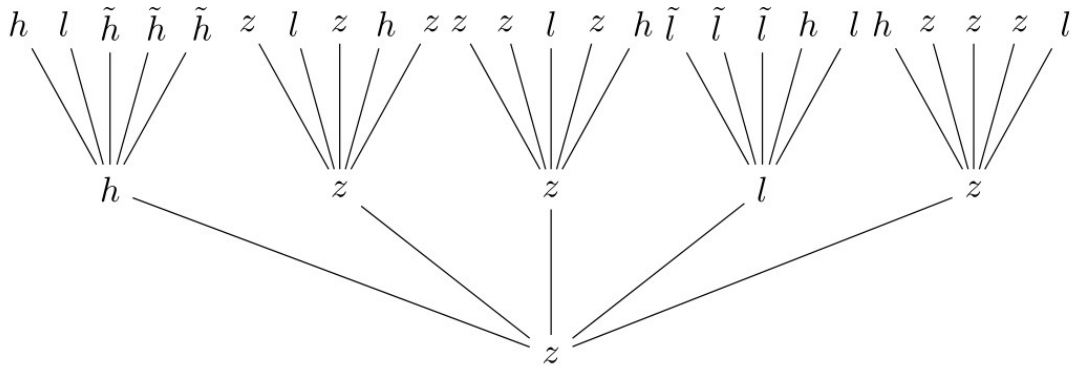


Figure 1. In this figure the values of function  $z_x$  on the vertices of the Cayley tree of order 5 are shown. This is the case when  $m = 1$ .

Step-1. If at vertex  $x$  we have  $z_x = z$ , then the function  $z_y$ , which gives real values to each vertex  $y \in S(x)$  by the following rule

$$\begin{cases} h \text{ on } m \text{ vertices of } S(x), \\ l \text{ on } m \text{ vertices of } S(x), \\ z \text{ on } k - 2m \text{ remaining vertices.} \end{cases}$$

Step-2. If at vertex  $x$  we have  $z_x = h$ , then the function  $z_y$ , which gives real values to each vertex  $y \in S(x)$  by the following rule

$$\begin{cases} h \text{ on } m \text{ vertices of } S(x), \\ l \text{ on } m \text{ vertices of } S(x), \\ \tilde{h} \text{ on } k - 2m \text{ remaining vertices.} \end{cases}$$

Step-3. If at vertex  $x$  we have  $z_x = l$ , then the function  $z_y$ , which gives real values to each vertex  $y \in S(x)$  by the following rule

$$\begin{cases} h \text{ on } m \text{ vertices of } S(x), \\ l \text{ on } m \text{ vertices of } S(x), \\ \tilde{l} \text{ on } k - 2m \text{ remaining vertices.} \end{cases}$$

Here  $\tilde{h} = (\tilde{h}_1, \tilde{h}_2)$ ,  $\tilde{l} = (\tilde{l}_1, \tilde{l}_2)$  and  $2m < k$ .

The measure corresponding to the set of values  $z = \{z_x, x \in G_k\}$  constructed in this way is called  $k_0^{(m)}$ -translation-invariant splitting Gibbs measure, where  $k_0$  is the order of the tree on which the translation-invariant measures  $\mu_1, \mu_2$  exist, and  $m$  is the number of times of solutions  $h, l$  on the vertices of  $S(x)$ .

In the second and third steps, from (4) for  $\tilde{h}$  and  $\tilde{l}$ , respectively, we obtain the following systems of equations

$$\begin{cases} \tilde{z} = \frac{1}{\lambda^{m-1}} \cdot \left( \frac{1+\tilde{h}_1}{\tilde{h}_1+\tilde{h}_2} \right)^{k-2m}, \\ \tilde{t} = \frac{1}{\lambda^{m-1}} \cdot \left( \frac{1+\tilde{h}_2}{\tilde{h}_1+\tilde{h}_2} \right)^{k-2m} \end{cases}, \tag{6}$$

and

$$\begin{cases} \tilde{t} = \frac{1}{\lambda^{m-1}} \cdot \left( \frac{1+\tilde{l}_1}{\tilde{l}_1+\tilde{l}_2} \right)^{k-2m}, \\ \tilde{z} = \frac{1}{\lambda^{m-1}} \cdot \left( \frac{1+\tilde{l}_2}{\tilde{l}_1+\tilde{l}_2} \right)^{k-2m}, \end{cases} \quad (7)$$

the solution is as follows:

$$\tilde{h}_1 = \frac{c^2 + cd - 1}{c^2 - cd + 1}, \quad \tilde{h}_2 = -\frac{c^2 - cd - 1}{c^2 - cd + 1}, \quad \tilde{l}_1 = -\frac{c^2 - cd - 1}{c^2 - cd + 1}, \quad \tilde{l}_2 = \frac{c^2 + cd - 1}{c^2 - cd + 1},$$

where

$$c = {}^{k-2m}\sqrt{\left( \frac{1 + \sqrt{1 - 4a_0^2}}{2a_0} \right)^2} = L^{\frac{2}{k-2m}}(\lambda), \quad d = {}^{k-2m}\sqrt{\frac{1}{\lambda^{m-1}}},$$

$$L(\lambda) = \frac{\sqrt{\lambda + 8} + \sqrt{\lambda} + \sqrt{(\sqrt{\lambda + 8} + \sqrt{\lambda})^2 - 16}}{4}, \quad \lambda > 1.$$

Clearly, the systems of equations (6) and (7) make sense if  $\tilde{h}_1, \tilde{h}_2, \tilde{l}_1, \tilde{l}_2$  are positive.

**Lemma 1.** *Let  $k > 2m$ ,  $k, m \in \mathbb{Z}$  and  $\lambda > 1$ . Then  $\tilde{h}_1 = \tilde{l}_2 > 0$ .*

*Proof.* Firstly, we show that  $c^2 + cd - 1 > 0$ .

$$c^2 + cd - 1 = L^{\frac{4}{k-2m}}(\lambda) + L^{\frac{2}{k-2m}}(\lambda) \cdot {}^{k-2m}\sqrt{\frac{1}{\lambda^{m-1}}} - 1.$$

It is easy to verify that  $L(\lambda) > 1$ . Therefore  $c^2 + cd - 1 > 0$ .

Similarly, we can show that  $c^2 - cd + 1 > 0$ .

It follows that our inequality is true for all  $\lambda > 1$ :

$$\tilde{h}_1 = \tilde{l}_2 = \frac{c^2 + cd - 1}{c^2 - cd + 1} > 0.$$

Let  $C(k, m) = \{\lambda : \lambda > 1, \tilde{h}_2 = \tilde{l}_1 > 0\}$ .

**The case  $m = 1$ .**

**Lemma 2.** *Let  $m = 1$  and  $k > 2m$ . Then there exists a single  $\lambda^*(k, m)$  such that  $C(k, m) = \{\lambda : 1 < \lambda < \lambda^*\} \neq \emptyset$ .*

*Proof.* Let  $m = 1$  and  $k > 2m$ . Let's find the values of  $\lambda$  satisfying the following inequality

$$\tilde{h}_2 = \tilde{l}_1 = \frac{-c^2 + cd + 1}{c^2 - cd + 1} > 0.$$

So, it is sufficient to solve the inequality  $-c^2 + cd + 1 > 0$ .

$$-c^2 + cd + 1 = f(\lambda) = -L^{\frac{4}{k-2}} + L^{\frac{2}{k-2}} + 1.$$

Now let us check the sign of the derivative of the function  $f(\lambda)$ ,

$$f'(\lambda) = -\frac{2}{k-2} L^{\frac{2}{k-2}-1} L' \left( 2L^{\frac{2}{k-2}} - 1 \right).$$



It is known  $L(\lambda) > 1$  and

$$L'(\lambda) = \frac{\sqrt{\lambda+8} + \sqrt{\lambda}}{8\sqrt{\lambda^2+8\lambda}} \left( 1 + \frac{\sqrt{\lambda+8} + \sqrt{\lambda}}{\sqrt{(\sqrt{\lambda+8} + \sqrt{\lambda})^2 - 16}} \right) > 0,$$

from them  $f'(\lambda) < 0$ .

Thus, the function  $f(\lambda)$  is monotonically decreasing, and  $f(1) = 1$ . Note that  $\lambda > 1$  for that reason  $\sup_{1 < \lambda < \infty} f(\lambda) = 1$ .

It is easily checked that

$$\lim_{\lambda \rightarrow +\infty} f(\lambda) = L^{\frac{2}{k-2}}(\lambda)[1 - L^{\frac{2}{k-2}}(\lambda)] + 1 = -\infty.$$

Then there exists  $\lambda^*(k, 1)$  intersecting the axis  $\lambda$  such that  $(1, \lambda^*(k, 1))$  has  $f(\lambda) > 0$ .

Now we determine  $\lambda^*(k, 1)$ ,

$$f(\lambda) = -L^{\frac{4}{k-2}}(\lambda) + L^{\frac{2}{k-2}}(\lambda) + 1 = 0.$$

If we consider this as a quadratic equation with respect to  $L^{\frac{2}{k-2}}(\lambda)$ :

$$L(\lambda) = \left( \frac{\sqrt{5} + 1}{2} \right)^{\frac{k-2}{2}}.$$

Let us introduce the following notation  $\sqrt{\lambda+8} + \sqrt{\lambda} = t$ . From this notation we find the following

$$L(\lambda) = \frac{t + \sqrt{t^2 - 16}}{4}, \quad \lambda = \left( \frac{t^2 - 8}{2t} \right)^2, \quad t = 2L(\lambda) + \frac{2}{L(\lambda)}. \quad (8)$$

From these

$$\lambda^*(k, 1) = \left( \frac{\left( \frac{\sqrt{5}+1}{2} \right)^{2(k-2)} + 1}{\left( \frac{\sqrt{5}+1}{2} \right)^{\frac{k-2}{2}} \left( \left( \frac{\sqrt{5}+1}{2} \right)^{k-2} + 1 \right)} \right)^2. \quad (9)$$

Now we check the sign of the derivative of  $\lambda$  with respect to the variable  $k = x$  ( $x \in \mathbb{N}$ ).

$$(\lambda^*(x))' = \frac{\left( \left( \frac{\sqrt{5}+1}{2} \right)^{2(x-2)} + 1 \right) \ln \frac{\sqrt{5}+1}{2} \left( \left( \frac{\sqrt{5}+1}{2} \right)^{x-2} - 1 \right) \left( \left( \frac{\sqrt{5}+1}{2} \right)^{2(x-2)} + 4 \left( \frac{\sqrt{5}+1}{2} \right)^{x-2} + 1 \right)}{\left( \frac{\sqrt{5}+1}{2} \right)^{x-2} \left( \left( \frac{\sqrt{5}+1}{2} \right)^{x-2} + 1 \right)^3},$$

from this, it is easy to see that  $(\lambda^*(x))' > 0$  since  $x \geq 3$ . Hence, as  $k$  increases, the value of  $\lambda^*(k, 1)$  also increases.

**Remark 1.** From the formula (9), the following values for  $\lambda^*(k, 1)$  can be obtained:

$$\lambda^*(3, 1) \approx 1.18034; \quad \lambda^*(4, 1) = 1.8; \quad \lambda^*(5, 1) \approx 3.09017; \quad \lambda^*(6, 1) \approx 5.44444;$$

$$\lambda^*(7, 1) \approx 9.48382; \lambda^*(8, 1) \approx 16.19999; \lambda^*(9, 1) \approx 27.19763.$$

It is clear that the set  $D(k, m) = C(k, m) \cap (1, \infty) = C(k, m)$ . For Gibbs measures constructed by the first step of the above construction, the system (4) has the form

$$\begin{cases} z_1 = \frac{1}{\lambda^{m-1}} \left( \frac{1+z_1}{z_1+z_2} \right)^{k-2m}, \\ z_2 = \frac{1}{\lambda^{m-1}} \left( \frac{1+z_2}{z_1+z_2} \right)^{k-2m}, \end{cases} \quad (10)$$

where  $z_1 > 0$ ,  $z_2 > 0$ ,  $\lambda \in C(k, m)$ .

Now we consider the system of equations (10) for  $m = 1$ . The following theorem is true.

**Theorem 2.** *For the Hard-Core model in the case  $G = \text{wand}$  when  $k_0 = 2$  and  $m = 1$ , the following statements are true:*

1. *If  $\lambda \in C(3, 1)$  or  $\lambda \in C(4, 1)$ , then there exists one  $k_0^{(m)}$ -translation-invariant Gibbs measure.*

2. *For  $k \geq 5$  and  $\lambda \in C(k, 1)$  there are exactly three  $k_0^{(m)}$ -translation-invariant Gibbs measures  $\mu_1^*, \mu_2^*, \mu_3^*$ .*

*Proof.* 1. If  $m = 1$ , the following system, independent of  $\lambda$ , is formed from the system (10)

$$\begin{cases} z_1 = \left( \frac{1+z_1}{z_1+z_2} \right)^{k-2}, \\ z_2 = \left( \frac{1+z_2}{z_1+z_2} \right)^{k-2}. \end{cases} \quad (11)$$

Subtracting the second equation from the first, we obtain

$$(z_1 - z_2) \left( 1 - \frac{(1+z_1)^{k-3} + \dots + (1+z_2)^{k-3}}{(z_1+z_2)^{k-2}} \right) = 0. \quad (12)$$

Therefore  $z_1 = z_2$  or

$$(z_1 + z_2)^{k-2} = (1+z_1)^{k-3} + \dots + (1+z_2)^{k-3}. \quad (13)$$

In the case  $z_1 = z_2 = z$ , from (11) we obtain the equation

$$z = f(z) = \left( \frac{1+z}{2z} \right)^{k-2},$$

which has a unique positive solution  $z^*$ , i.e. the system of equations (11) has a solution  $(z^*, z^*)$ .

By (12) for  $k = 3$  we obtain  $z_1 = z_2$  and  $z_1 + z_2 = 1$ . With  $z_1 + z_2 = 1$  the system (11) has no solution. Hence,  $k = 3$  has a single solution in the case  $z_1 = z_2 = z$ .

From (12) for  $k = 4$  we get the equalities  $z_1 = z_2$  and  $(z_1 + z_2)^2 = z_1 + z_2 + 2$ . From the second equality and by (11) we have  $z_1 = z_2 = 1$  since  $z_1 > 0$ ,  $z_2 > 0$ . Therefore, even when  $k = 4$ , there is a single solution in the case  $z_1 = z_2 = z$ .

2. To prove this, if we assume that  $\lambda = 1$  in system (5) in the work [25] and the value of system (11) in this paper  $\lambda(t) = 1$  and  $k_1 = k - 2$ , then the expression  $\lambda_{cr} = \frac{1}{k-3} \left( \frac{2}{k-2} \right)^{k-2}$  is for  $k \geq 5$ ,  $\lambda_{cr} < 1 = \lambda$ . Thus, as concluded in the paper [25], since the value  $\lambda = 1$  is greater

than  $\lambda_{cr}$ , than for  $\lambda \in C(k, 1)$  we have exactly three  $k_0^{(1)}$ -translation-invariant Gibbs measures  $\mu_1^*, \mu_2^*, \mu_3^*$  corresponding three solutions (11).

**The case  $m = 2$ .** In this case, the following lemma is valid.

**Lemma 3.** *The following are true:*

$C(5, 2) = \{\lambda : 1 < \lambda < \lambda^*(5, 2)\}$ , where  $\lambda^*(5, 2) \approx 1.13996$ .

$C(6, 2) = \{\lambda : 1 < \lambda < \lambda^*(6, 2)\}$ , where  $\lambda^*(6, 2) \approx 1.51753$ .

$C(7, 2) = \{\lambda : 1 < \lambda < \lambda^*(7, 2)\}$ , where  $\lambda^*(7, 2) \approx 2.16334$ .

$C(8, 2) = \{\lambda : 1 < \lambda < \lambda^*(8, 2)\}$ , where  $\lambda^*(8, 2) \approx 3.17116$ .

$C(9, 2) = \{\lambda : 1 < \lambda < \lambda^*(9, 2)\}$ , where  $\lambda^*(9, 2) \approx 4.68427$ .

*Proof.* Let's find  $\lambda^*(5, 2)$ . To do this, solve the inequality

$$\tilde{h}_2 = \tilde{l}_1 = \frac{-c^2 + cd + 1}{c^2 - cd + 1} > 0.$$

For this purpose it is sufficient to solve the inequality  $-c^2 + cd + 1 > 0$ .

$$-c^2 + cd + 1 = f(\lambda) = -L^4(\lambda) + \frac{1}{\lambda}L^2(\lambda) + 1.$$

$$f'(\lambda) = -4L^3L' + \frac{2}{\lambda}LL' - \frac{1}{\lambda^2}L^2 = -2LL' \left( 2L^2 - \frac{1}{\lambda} + \frac{L}{2\lambda^2L'} \right) < 0$$

because  $\lambda > 1$ ,  $L(\lambda) > 1$  and  $L'(\lambda) > 0$ . Thus, the function  $f(\lambda)$  is monotonically decreasing, and  $f(1) = 1$ . Note that  $\lambda > 1$  for that reason  $\sup_{1 < \lambda < \infty} f(\lambda) = 1$ .

$$\lim_{\lambda \rightarrow +\infty} f(\lambda) = L^2(\lambda) \left( \frac{1}{\lambda} - L^2(\lambda) \right) + 1 = -\infty.$$

Then there exists  $\lambda^*(5, 2)$  intersecting the axis  $\lambda$  such that  $f(\lambda) > 0$  in the interval  $(1, \lambda^*(5, 2))$ .

Now let's find the value of  $\lambda^*(5, 2)$ . To do this, we solve the equation  $f(\lambda) = 0$  using (8) we can get

$$f(\lambda) = -L^4(\lambda) + \frac{1}{\lambda}L^2(\lambda) + 1 = \frac{-L^4(L^4 + 1)^2 + L^4(L^2 + 1)^2 + (L^4 + 1)^2}{(L^4 + 1)^2} = 0.$$

Denoting  $L^2 \equiv L^2(\lambda) = s$  from last equation can get  $s^5 - s^4 + s^3 - 3s^2 + s - 1 = 0$ , which has the only positive root  $s \approx 1.53057$ . From this we obtain  $\lambda^*(5, 2) \approx 1.13996$ .

The values of  $\lambda^*(6, 2)$ ,  $\lambda^*(7, 2)$ ,  $\lambda^*(8, 2)$ ,  $\lambda^*(9, 2)$  are also found in the same way as above.

The following theorem is true.

**Theorem 3.** *For the Hard-Core model in the case  $G = \text{wand}$  when  $k_0 = 2$  and  $m = 2$ , the following statements are true:*

1. *If  $\lambda \in C(5, 2)$  or  $\lambda \in C(6, 2)$ , then there exists one  $k_0^{(m)}$ -translation-invariant Gibbs measure.*

2. *If  $\lambda \in C(7, 2)$  or  $\lambda \in C(8, 2)$  or  $\lambda \in C(9, 2)$ , then there are exactly three  $k_0^{(m)}$ -translation-invariant Gibbs measures  $\mu_1^*, \mu_2^*, \mu_3^*$ .*

*Proof.* 1. For  $m = 2$  the system (10) has the form

$$\begin{cases} z_1 = \frac{1}{\lambda} \left( \frac{1+z_1}{z_1+z_2} \right)^{k-4}, \\ z_2 = \frac{1}{\lambda} \left( \frac{1+z_2}{z_1+z_2} \right)^{k-4}. \end{cases} \quad (14)$$

For  $k = 5$ , subtracting the second equation from the first in system (14), the equalities  $z_1 - z_2 = 0$  or  $z_1 + z_2 = \frac{1}{\lambda}$  are formed, and for  $z_1 - z_2 = 0$  the system has only one solution  $(z^*, z^*)$ , in the case  $z_1 + z_2 = \frac{1}{\lambda}$  the system has no solution.

For  $k = 6$ , the following equalities  $z_1 - z_2 = 0$  and  $z_1 + z_2 = \frac{1+\sqrt{1+8\lambda}}{2\lambda}$  are obtained. In this case  $z_1 = z_2$  there is only one solution  $(z^*, z^*)$  and in the case  $z_1 + z_2 = \frac{1+\sqrt{1+8\lambda}}{2\lambda}$  the system no solution because the condition  $\lambda > 1$  is not satisfied. Hence, in the case  $\lambda \in C(5, 2)$  or  $\lambda \in C(6, 2)$  we have one  $k_0^{(2)}$ -translation-invariant Gibbs measure corresponding unique solution  $(z^*, z^*)$  of the system (14).

2. For  $k = 7$  to solve the system (14) we use Theorem 4 presented when solving system (13) in [14].

In [14] if we look  $\lambda$  as  $\frac{1}{\lambda}$  in system (13), we find  $\lambda_{cr} = 27/4$  for the system (14) above.

Also,  $\lambda_{cr} > \lambda^*(7, 2)$ . Hence, for  $k = 7$  the system (14) has three solutions for  $1 < \lambda < \lambda^*(7, 2)$ .

Using the above method we can find the existence of three  $k_0^{(m)}$ -translation-invariant Gibbs measures in the sets  $C(8, 2)$ ,  $C(9, 2)$ .

Thus, in the case  $\lambda \in C(7, 2)$  or  $\lambda \in C(8, 2)$  or  $\lambda \in C(9, 2)$  we have three  $k_0^{(2)}$ -translation-invariant Gibbs measure corresponding three solutions of the system (14).

**The case  $m = 3$ .** In this case, the following lemma is true.

**Lemma 4.** *The following are true:*

$C(7, 3) = \{\lambda : 1 < \lambda < \lambda^*(7, 3)\}$ , where  $\lambda^*(7, 3) \approx 1.11728$ .

$C(8, 3) = \{\lambda : 1 < \lambda < \lambda^*(8, 3)\}$ , where  $\lambda^*(8, 3) \approx 1.39816$ .

$C(9, 3) = \{\lambda : 1 < \lambda < \lambda^*(9, 3)\}$ , where  $\lambda^*(9, 3) \approx 1.83764$ .

*Proof.* The proof of this lemma is similar to the proof of Lemma 3.

The following theorem is true.

**Theorem 4.** *For the Hard-Core model in the case  $G = \text{wand}$  when  $k_0 = 2$  and  $m = 3$ , the following statements are true:*

1. *If  $\lambda \in C(7, 3)$  or  $\lambda \in C(8, 3)$ , then there exists one  $k_0^{(m)}$ -translation-invariant Gibbs measure.*

2. *If  $\lambda \in C(9, 3)$ , then there are exactly three  $k_0^{(m)}$ -translation-invariant Gibbs measures  $\mu_1^*$ ,  $\mu_2^*$ ,  $\mu_3^*$ .*

*Proof.* The proof is similar to the proof of Theorem 3.

*All Gibbs measures obtained in Theorems are new and different from previously known Gibbs measures.*

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### REZYUME

Ushbu ishda Keli daraxtida unumdor graf aktivlik parametri  $\lambda > 1$  bo'lganda, uch holatli Hard-Core (HC) modelini ko'rib chiqamiz. Ma'lumki, uch holatli HC modelida unumdor graflarning to'rt turi mavjud, ular: kalit, jezl, sirtmoq va hushtak. Bunda biz Keli daraxtida  $k_0^{(m)}$ -translyatsion-invariant Gibbs o'lchovlari jezl bo'lgan holatni o'rganamiz. Bunday o'lchovlar translyatsion -invariant Gibbs o'lchovlari yordamida quriladi. Natijada graf Keli daraxtida jezl bo'lgan hol uchun  $k_0 = 2$  va  $m = 1, 2, 3$  bo'lganda  $k_0^{(m)}$ -translation-invariant Gibbs o'lchovlarining aniq soni topildi va ularning mavjudligi isbotlandi.

**Kalit so'zlar:** Keli daraxti, konfiguratsiya, unumdor graf, Hard-core modeli, Gibbs o'lchovi,  $k_0^{(m)}$ -translation-invariant Gibbs o'lchovi.

### РЕЗЮМЕ

Мы рассматриваем плодородные Hard-Core (HC) модели с тремя состояниями и с параметром активности  $\lambda > 1$  на дереве Кэли. Известно, что существует четыре типа таких моделей: петля, свисток, жезл и ключ. Мы изучаем  $k_0^{(m)}$ -трансляционно-инвариантные меры Гиббса для рассматриваемой модели в случае графа жезл на дереве Кэли. Такие меры строятся с помощью трансляционно-инвариантных мер Гиббса. В случае жезл при  $k_0 = 2$  и  $m = 1, 2, 3$  на дереве Кэли доказано существование  $k_0^{(m)}$ -трансляционно-инвариантных мер Гиббса и найдено точное количество таких мер.

**Ключевые слова:** Дерево Кэли, конфигурация, плодородный граф, модель Hard-core, мера Гиббса,  $k_0^{(m)}$ -трансляционно-инвариантная мера Гиббса.

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ON THE CONTROL PROBLEM ASSOCIATED WITH A FOURTH ORDER  
PSEUDO-PARABOLIC EQUATION

Dekhkonov F. N. \*

## RESUME

In this paper, we consider boundary control problem for a fourth order pseudo-parabolic type equation in a bounded two-dimensional domain. A control function is given at the boundary of the considered domain. To achieve the average temperature in the domain, it is required to find a control function. Using the Fourier method, the control problem is reduced to the Volterra integral equation of the second type. The existence of the control function is proved using the method of successive approximation.

**Key words:** Fourth order pseudo-parabolic equation, Volterra integral equation, admissible control, initial-boundary problem, method of successive approximation.

## INTRODUCTION

It is known that in recent years, due to the increasing interest in physics and mathematics, the boundary problems related to pseudo-parabolic equations were widely studied. For this purpose, various boundary problems for parabolic and pseudo-parabolic equations have been widely studied by many researchers.

It is well known that fourth-order pseudo-parabolic equations describe a variety of important physical processes, such as heat conduction in materials, electric signals in a nonlinear telegraph line with nonlinear damping, viscous flow in materials with memory [1], vibration of a nonlinear elastic rod with viscosity [2], nonlinear bidirectional shallow water waves [3], the velocity evolution of ionacoustic waves in a collisionless plasma when an ion viscosity is invoked [4], and so on.

The boundary control of a linear pseudo-parabolic equation and compare the results to those of parabolic equations was studied in [5]. The stability, uniqueness, and existence of solutions of some classical problems for the considered equation are studied in [6]. In [7], the point control problems for linear pseudo-parabolic and parabolic type equations are considered.

The optimal control problem for the parabolic type equations was studied by Fattorini and Friedman [8, 9]. Control problems for the infinite-dimensional case were studied by Egorov [10], who generalized Pontryagin's maximum principle to a class of equations in Banach space, and the proof of a bang-bang principle was shown in the particular conditions.

The boundary control problem for a parabolic equation with a piecewise smooth boundary in an  $n$ -dimensional domain was studied in [11] and an estimate for the minimum time required to reach a given average temperature was found. In [12], the considered the heat conduction equation with the Robin boundary condition and developed a mathematical model of the process

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\*Dekhkonov F. N. – Namangan State University, Namangan, Uzbekistan,  
National University of Uzbekistan, Tashkent, Uzbekistan, f.n.dehqonov@mail.ru

of heating a cylindrical domain. Control problems for the heat transfer equation in the three-dimensional domain are studied in [13]. Basic information on optimal control problems is given in detail in monographs by Lions and Fursikov [14, 15]. Boundary control problems for linear pseudo-parabolic type equations were studied in works [16-18], and it was proved that there is a control function for heating the domain to the average temperature.

In [19], the null boundary control problem associated with a fourth-order parabolic equation in a one-dimensional bounded domain was considered by the method of reducing the control problem to well-posed problems proposed by Guo and Littman [20]. In [21], the null interior controllability for a fourth order parabolic equation was studied. The method they used is based on Lebeau-Rabbiano inequality. Optimal time problems for the fourth-order parabolic equation in the two-dimensional domain are studied in works [22, 23].

For the fourth order nonlinear pseudo-parabolic equation, there are also some results about initial boundary value problem and Cauchy problem, especially on the global existence, nonexistence and asymptotic behavior of the solutions [24-26].

In 1978, Bakiyevich and Shadrin [24] considered the following problem

$$\begin{cases} u_t - a_1 u_{xx} - a_2 u_{xxt} + a_3 u_{xxxx} = f(x, t), & x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) = \varphi(x), & x \in \mathbb{R}, \end{cases}$$

where  $a_1, a_2 > 0$ ,  $a_3 \geq 0$  are constants. They showed that the solutions of this problem are expressed through the sum of convolutions of functions  $\varphi(x)$  and  $f(x, t)$  with corresponding fundamental solutions of the problem. In 2009, Khudaverdiyev and Farhadova [25] discussed the following fourth order semilinear pseudo-parabolic equation

$$u_t - a_1 u_{xxt} + u_{xxxx} = f(x, t, u, u_x, u_{xx}, u_{xxx}), \quad 0 \leq x \leq 1, \quad 0 \leq t \leq T < +\infty,$$

where  $a_1 > 0$  is a fixed number. They proved the existence in large theorem for generalized solution by means of Schauder stronger fixed point principle. Zhao and Xuan [26] studied the following

$$u_t - a_1 u_{xx} - a_2 u_{xxt} + a_3 u_{xxxx} + f(u)_x = 0, \quad x \in \mathbb{R}, \quad t > 0.$$

They obtained the existence and convergence behavior of the global smooth solutions.

In this work, we considered the problem of boundary control for the fourth-order pseudo-parabolic equation in the quadratic domain. For this, the control problem was first reduced to the Volterra integral equation of the second kind, and the continuity of the integral in the given domain was shown. As a result, the existence of the control function was proved using the method of successive approximation. At the end of the article, the control function is found in the exact values of the parameters.

## STATEMENT OF PROBLEM

In this article, we consider the following fourth order pseudo-parabolic equation in the domain  $\Omega = (0, \pi) \times (0, \pi)$

$$u_t - \Delta u - \Delta u_t + \Delta^2 u = 0, \quad (x, y, t) \in \Omega_T := \Omega \times (0, T), \quad (1)$$



with boundary value conditions

$$u_x(0, y, t) = -\Phi(y)\nu(t), \quad u_x(\pi, y, t) = 0, \quad u_y(x, 0, t) = 0, \quad u_y(x, \pi, t) = 0, \quad (2)$$

and

$$u_{xxx}(0, y, t) = 0, \quad u_{xxx}(\pi, y, t) = 0, \quad u_{yyy}(x, 0, t) = 0, \quad u_{yyy}(x, \pi, t) = 0, \quad (3)$$

and initial value condition

$$u(x, y, 0) = 0, \quad 0 \leq x, y \leq \pi, \quad (4)$$

where  $\Delta u = u_{xx}(x, y, t) + u_{yy}(x, y, t)$ ,  $\Delta^2 u = u_{xxxx}(x, y, t) + 2u_{xxyy}(x, y, t) + u_{yyyy}(x, y, t)$ ,  $\Phi(y)$  is a given function and  $\nu(t)$  is the control function

Let  $M > 0$  be some given constant. It is called that the control function  $\nu(t) \in C^1[0, T]$  is *admissible*, if it fulfills the following conditions

$$\nu(0) = 0, \quad |\nu(t)| \leq M, \quad t \in [0, T]. \quad (5)$$

Assume, the given function  $\Phi \in C^5[0, \pi]$  satisfies the conditions

$$\Phi^{(1)}(0) = \Phi^{(1)}(\pi) = \Phi^{(3)}(0) = \Phi^{(3)}(\pi) = 0, \quad \frac{1}{\pi} \int_0^{\pi} \Phi(y) dy = 1. \quad (6)$$

**Control Problem.** Let  $a, b > 1$  and assume that function  $\phi(t)$  is a given. Then find the control function  $\nu(t)$  from the condition

$$\int_0^{\pi/a} \int_0^{\pi/b} u(x, y, t) dx dy = \phi(t), \quad 0 \leq t \leq T, \quad (7)$$

where  $u(x, y, t)$  is a solution of the problem (1)-(4) and it depends on the control function  $\nu(t)$ .

We now offer the primary theorem for demonstrating admissible control's existence.

**Theorem 1.** *There exists a constant  $\gamma > 0$  such that for any function  $\phi(t) \in C[0, T]$  satisfying the conditions*

$$\phi(0) = 0, \quad |\phi(t)| \leq \frac{M}{\gamma}, \quad t \in [0, T],$$

*the solution  $\nu(t)$  of the equation (7) exists, unique and satisfies the conditions (5).*

We will consider the proof of Theorem 1 step by step in the next sections.

## INTEGRAL EQUATION FOR CONTROL FUNCTION

In this section, we consider the reduction of the control problem to a Volterra integral equation of the second kind.

**Definition 1.** *By solution to initial-boundary problem (1)-(4), we understand the function  $u(x, y, t)$  represented in the form*

$$u(x, y, t) = \nu(t) \Phi(y) \frac{(\pi - x)^2}{2\pi} - w(x, y, t), \quad (8)$$

where the function  $w(x, y, t)$  with the regularity  $w(x, y, t) \in C_{x,y,t}^{4,4,1}(\Omega_T) \cap C(\bar{\Omega}_T)$  and  $w_{xxx}, w_{yyy} \in C(\bar{\Omega})$  is the solution to the mixed problem

$$w_t - \Delta w - \Delta w_t + \Delta^2 w = \frac{(\pi - x)^2}{2\pi} \Phi(y) \nu'(t) - \frac{(\pi - x)^2}{2\pi} \Phi^{(2)}(y) \nu'(t) - \frac{1}{\pi} \Phi(y) \nu'(t) - \\ - \frac{(\pi - x)^2}{2\pi} \Phi^{(2)}(y) \nu(t) + \frac{2}{\pi} \Phi^{(2)}(y) \nu(t) - \frac{1}{\pi} \Phi(y) \nu(t) + \frac{(\pi - x)^2}{2\pi} \Phi^{(4)}(y) \nu(t),$$

with boundary value conditions

$$w_x(0, y, t) = w_x(\pi, y, t) = 0, \quad w_y(x, 0, t) = w_y(x, \pi, t) = 0,$$

and

$$w_{xxx}(0, y, t) = w_{xxx}(\pi, y, t) = 0, \quad w_{yyy}(x, 0, t) = w_{yyy}(x, \pi, t) = 0,$$

and initial value condition

$$w(x, y, 0) = 0.$$

As a result, we get (see [27])

$$w(x, y, t) = \left( \frac{\pi}{6} - \frac{1}{\pi} \right) \nu(t) - \frac{1}{\pi} \int_0^t \nu(s) ds + \\ + \frac{2}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(1 + n^2) \Phi_n}{m^2 (1 + n^2 + m^2)} \left( \int_0^t e^{-\mu_{m,n}(t-s)} \nu'(s) ds \right) \cos mx \cos ny + \\ + \frac{2}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(n^2 + n^4) \Phi_n}{m^2 (1 + n^2 + m^2)} \left( \int_0^t e^{-\mu_{m,n}(t-s)} \nu(s) ds \right) \cos mx \cos ny,$$

where  $\mu_{m,n}$  as follows

$$\mu_{m,n} = \frac{(m^2 + n^2)^2 + m^2 + n^2}{1 + m^2 + n^2} = m^2 + n^2.$$

**Lemma 1.** Let the function  $\Phi(y) \in C^5[0, \pi]$  satisfy the conditions (6). Then the following estimate is holds

$$|\Phi_n| \leq \frac{C}{n^5}, \quad n = 1, 2, \dots,$$

where  $\Phi_n$  is the Fourier coefficient of the function  $\Phi(y)$  and  $C$  is a positive constant.

**Proof.** It is known that the Fourier coefficient of the function  $\Phi(y)$  in the interval  $(0, \pi)$  is defined as follows

$$\Phi_n = \frac{2}{\pi} \int_0^{\pi} \Phi(y) \cos ny dy, \quad n = 1, 2, \dots \quad (9)$$

By (9) and condition (6), we might write

$$\Phi_n = \frac{2}{\pi} \int_0^{\pi} \Phi(y) \cos ny dy = \frac{2}{\pi n} \left( \Phi(y) \sin ny \Big|_{x=0}^{x=\pi} - \int_0^{\pi} \Phi^{(1)}(y) \sin ny dy \right) =$$

$$\begin{aligned}
&= \frac{2}{\pi n^2} \left( \Phi^{(1)}(y) \cos ny \Big|_{x=0}^{x=\pi} - \int_0^\pi \Phi^{(2)}(y) \cos ny dy \right) = \\
&= -\frac{2}{\pi n^3} \left( \Phi^{(2)}(y) \sin ny \Big|_{x=0}^{x=\pi} - \int_0^\pi \Phi^{(3)}(y) \sin ny dy \right) = \\
&= -\frac{2}{\pi n^4} \left( \Phi^{(3)}(y) \cos ny \Big|_{x=0}^{x=\pi} - \int_0^\pi \Phi^{(4)}(y) \cos ny dy \right) = \\
&= \frac{2}{\pi n^5} \left( \Phi^{(4)}(y) \sin ny \Big|_{x=0}^{x=\pi} - \int_0^\pi \Phi^{(5)}(y) \sin ny dy \right) = \\
&= -\frac{2}{\pi n^5} \int_0^\pi \Phi^{(5)}(y) \sin ny dy.
\end{aligned}$$

Then we get

$$|\Phi_n| \leq \frac{C}{n^5}, \quad n = 1, 2, \dots$$

Lemma 1 is proved.

**Lemma 2.** Let  $\nu(t) \in C^1[0, T]$ . Then the solution to problem (1)-(4) is

$$\begin{aligned}
u(x, y, t) &= \frac{(\pi - x)^2}{2\pi} \Phi(y) \nu(t) - \left( \frac{\pi}{6} - \frac{1}{\pi} \right) \nu(t) + \frac{1}{\pi} \int_0^t \nu(s) ds - \\
&- \frac{2}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(1 + n^2) \Phi_n}{m^2 (1 + n^2 + m^2)} \left( \int_0^t e^{-\mu_{m,n}(t-s)} \nu'(s) ds \right) \cos mx \cos ny - \\
&- \frac{2}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(n^2 + n^4) \Phi_n}{m^2 (1 + n^2 + m^2)} \left( \int_0^t e^{-\mu_{m,n}(t-s)} \nu(s) ds \right) \cos mx \cos ny, \quad (10)
\end{aligned}$$

**Proof.** Let us prove that the Fourier series of the function  $w(x, y, t)$  belongs to the class  $C_{x,y,t}^{4,4,1}(\Omega_T) \cap C(\bar{\Omega}_T)$  and  $w_{xxx}, w_{yyy} \in C(\bar{\Omega})$ . It is sufficient to prove that the corresponding series converge uniformly. Due to conditions imposed on the expression of the function  $\nu(t)$  for  $w(x, y, t)$  converges uniformly, so  $w(x, y, t) \in C(\bar{\Omega}_T)$ . It is logically proved that  $w_{xxx}, w_{yyy} \in C(\bar{\Omega}_T)$ ,  $w_t, \Delta w, \Delta w_t, \Delta^2 w \in C(\Omega_T)$  and  $w(x, y, t)$  satisfy the equation.

Q.E.D.

Lemma 2 is proved.

From the solution (10) and the condition (7), we can write

$$\phi(t) = \int_0^{\pi/a} \int_0^{\pi/b} u(x, y, t) dx dy =$$

$$\begin{aligned}
&= \nu(t) \int_0^{\pi/a} \int_0^{\pi/b} \Phi(y) \frac{(\pi-x)^2}{2\pi} dx dy - \frac{\pi^2}{ab} \left( \frac{\pi}{6} - \frac{1}{\pi} \right) \nu(t) + \frac{\pi}{ab} \int_0^t \nu(s) ds - \\
&\quad - \frac{2}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(1+n^2) \Phi_n \sin \frac{m\pi}{a} \sin \frac{n\pi}{b}}{m^3 n (1+n^2+m^2)} \int_0^t e^{-\mu_{m,n}(t-s)} \nu'(s) ds - \\
&\quad - \frac{2}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(n^2+n^4) \Phi_n \sin \frac{m\pi}{a} \sin \frac{n\pi}{b}}{m^3 n (1+n^2+m^2)} \int_0^t e^{-\mu_{m,n}(t-s)} \nu(s) ds.
\end{aligned}$$

According to the condition  $\nu(0) = 0$ , we get

$$\begin{aligned}
\phi(t) &= \nu(t) \int_0^{\pi/a} \int_0^{\pi/b} \Phi(y) \frac{(\pi-x)^2}{2\pi} dx dy - \frac{\pi^2}{ab} \left( \frac{\pi}{6} - \frac{1}{\pi} \right) \nu(t) + \\
&\quad + \frac{\pi}{ab} \int_0^t \nu(s) ds - \nu(t) \frac{2}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(1+n^2) \Phi_n \sin \frac{m\pi}{a} \sin \frac{n\pi}{b}}{m^3 n (1+n^2+m^2)} + \\
&\quad + \frac{2}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(1+n^2) \Phi_n \sin \frac{m\pi}{a} \sin \frac{n\pi}{b}}{m n (1+n^2+m^2)} \int_0^t e^{-\mu_{m,n}(t-s)} \nu(s) ds. \tag{11}
\end{aligned}$$

According to Parseval equality, we get

$$\int_0^{\pi/a} \int_0^{\pi/b} \Phi(y) \frac{(\pi-x)^2}{2\pi} dx dy = \frac{\pi^3}{6ab} + \frac{2}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\Phi_n \sin \frac{m\pi}{a} \sin \frac{n\pi}{b}}{m^3 n}. \tag{12}$$

By (11) and (12), we have

$$\begin{aligned}
\phi(t) &= \nu(t) \frac{\pi}{ab} + \nu(t) \frac{2}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\Phi_n \sin \frac{m\pi}{a} \sin \frac{n\pi}{b}}{m n (1+n^2+m^2)} + \\
&\quad + \frac{\pi}{ab} \int_0^t \nu(s) ds + \frac{2}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(1+n^2) \Phi_n \sin \frac{m\pi}{a} \sin \frac{n\pi}{b}}{m n (1+n^2+m^2)} \int_0^t e^{-\mu_{m,n}(t-s)} \nu(s) ds.
\end{aligned}$$

We set

$$K(t) = \frac{\pi}{ab} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Psi_{m,n} e^{-\mu_{m,n}t}, \quad t > 0, \tag{13}$$

where

$$\Psi_{m,n} = \frac{2(1+n^2) \Phi_n \sin \frac{m\pi}{a} \sin \frac{n\pi}{b}}{\pi m n (1+n^2+m^2)}, \tag{14}$$

and

$$\beta = \frac{\pi}{ab} + \frac{2}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\Phi_n \sin \frac{m\pi}{a} \sin \frac{n\pi}{b}}{m n (1 + n^2 + m^2)}. \quad (15)$$

Then, we get the Volterra integral equation of the second kind

$$\nu(t) + \alpha \int_0^t K(t-s) \nu(s) ds = \alpha \phi(t), \quad t > 0, \quad (16)$$

where  $\alpha = \beta^{-1}$ .

### PROOF OF THEOREM 1

In this section, we consider the existence of a solution to the Volterra integral equation of the second kind. Then we prove the admissibility of the control function.

**Lemma 3.** *Let the conditions of Lemma 1 hold. Then, the function  $K(t)$  is continuous on the half line  $t \geq 0$ .*

**Proof.** Using Lemma 1 and equality (14), we can write

$$|\Psi_{m,n}| \leq \frac{2C}{\pi} \frac{1+n^2}{m n^5 (1+n^2+m^2)}$$

Then, we have the estimate

$$\begin{aligned} |K(t)| &\leq \frac{\pi}{ab} + \left| \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Psi_{m,n} e^{-\mu_{m,n} t} \right| \leq \\ &\leq \frac{\pi}{ab} + \frac{2C}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1+n^2}{m n^5 (1+n^2+m^2)} = A, \end{aligned}$$

where  $A$  is a positive number and it follows

$$A = \frac{\pi}{ab} + \frac{2C}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1+n^2}{m n^5 (1+n^2+m^2)}. \quad (17)$$

Therefore, the function  $K(t)$  is continuous on the half-line  $t \geq 0$ .

Lemma 3 is proved.

Since the kernel of integral equation (16) is continuous, according to the general theory, the solution of this integral equation is unique.

**Lemma 4.** *Let  $\phi(t)$  be continuous on the half-line  $t \geq 0$ . Then the equation (16) has the solution.*

**Proof.** The Lemma 3 states that the function  $K(t)$  is bounded in  $t \in [0, T]$ . Therefore, the integral equation (16) can be solved using the method of successive approximation.

Set

$$\nu_0(t) = \alpha \phi(t), \quad \nu_k(t) = \alpha \int_0^t K(t-s) \nu_{k-1}(s) ds, \quad k = 1, 2, \dots \quad (18)$$

Then, we have the solution

$$\nu(t) = \sum_{k=0}^{\infty} (-1)^k \nu_k(t). \quad (19)$$

Indeed, we can see that this function satisfies Eq. (16)

$$\begin{aligned} \alpha \int_0^t K(t-s) \nu(s) ds &= \alpha \sum_{k=0}^{\infty} (-1)^k \int_0^t K(t-s) \nu_k(s) ds = \\ &= \sum_{k=0}^{\infty} (-1)^k \nu_{k+1}(t) = - \sum_{k=1}^{\infty} (-1)^k \nu_k(t) = \\ &= - \sum_{k=0}^{\infty} (-1)^k \nu_k(t) + \nu_0(t) = -\nu(t) + \alpha \phi(t). \end{aligned}$$

Lemma 4 is proved.

It is known  $\alpha$  is finite. According to Lemma 1,  $\beta$  is a finite constant number. Indeed, we may write

$$\begin{aligned} |\beta| &\leq \frac{\pi}{ab} + \frac{2}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{|\Phi_n|}{m n (1 + n^2 + m^2)} \leq \\ &\leq \frac{\pi}{ab} + \frac{2C}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m n^5 (1 + n^2 + m^2)} < +\infty. \end{aligned}$$

Set

$$\|\phi\|_T = \max_{0 \leq t \leq T} |\phi(t)|.$$

**Lemma 5.** *Let  $\phi \in C[0, T]$ . Then, the following estimate is valid:*

$$|\nu_k(t)| \leq |\alpha|^{k+1} A^k \frac{t^k}{k!} \|\phi\|_T, \quad k = 0, 1, \dots,$$

where  $\alpha = \beta^{-1}$  and  $\beta, A$  are defined by (15), (17), respectively.

**Proof.** Now we prove the Lemma using the induction method.

**I.** It is not difficult to see that the inequality holds for  $k = 0$ .

**II.** We prove for  $k + 1$ :

$$\begin{aligned} |\nu_{k+1}(t)| &\leq |\alpha| \int_0^t |K(t-s)| |\nu_k(s)| ds \leq \\ &\leq |\alpha|^{k+2} A^k \|\phi\|_T \int_0^t |K(t-s)| \frac{s^k}{k!} ds \leq \end{aligned}$$

$$\leq |\alpha|^{k+2} A^{k+1} \|\phi\|_T \int_0^t \frac{s^k}{k!} ds = |\alpha|^{k+2} A^{k+1} \frac{t^{k+1}}{(k+1)!} \|\phi\|_T.$$

Then, we get the required estimate

$$|\nu_k(t)| \leq |\alpha|^{k+1} A^k \frac{t^k}{k!} \|\phi\|_T, \quad k = 0, 1, \dots,$$

Now we present the proof of the main theorem 1.

**Proof of Theorem 1.** Using the lemma 5 and (18), (19), we get

$$|\nu(t)| \leq \sum_{k=0}^{\infty} |\nu_k(t)| \leq \|\phi\|_T \sum_{k=0}^{\infty} |\alpha|^{k+1} A^k \frac{t^k}{k!} = |\alpha| e^{|\alpha| A t} \|\phi\|_T.$$

Then using the inequality  $|\phi(t)| \leq \frac{M}{\gamma}$ , we have

$$|\nu(t)| \leq |\alpha| e^{|\alpha| A T} \|\phi\|_T \leq M, \quad t \in [0, T],$$

where as  $\gamma$  we took

$$\gamma = |\alpha| e^{|\alpha| A T}, \quad 0 < T < +\infty.$$

From the integral equation (16) and condition  $\phi(0) = 0$ , we have  $\nu(0) = 0$ . Thus, we have proved the admissibility of the control function  $\nu(t)$ .

### EXAMPLE

Let  $\Phi(y) = 1 + \cos 2y$ ,  $y \in (0, \pi)$  in initial-boundary problem (1)-(4). If we take  $a = \pi$  and  $b = 2$  in the control problem, as a result, the expressions given by equalities (14) and (15) are determined as follows:

$$\beta = \frac{1}{2}, \quad \Psi_{m,n} = 0.$$

Thus, the function  $K(t)$  defined by (13), we can write

$$K(t) = \frac{1}{2}.$$

We consider the following function:

$$\phi(t) = t e^{-t}, \quad t \in [0, T].$$

The physical meaning of the function  $\phi(t)$  is the average temperature in the domain.

In this case, we can write the main integral equation (16) as follows:

$$\nu(t) + \int_0^t \nu(s) ds = 2 t e^{-t}, \quad t > 0.$$

Thus, we obtain the following solution:

$$\nu(t) = (t - t^2) e^{-t}, \quad t \geq 0.$$

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## REZYUME

Ushbu maqolada biz chegaralangan ikki o'lchamli sohada to'rtinchi tartibli psevdoparabolik tipdagi tenglama uchun chegaraviy boshqaruv masalasini ko'rib chiqamiz. Ko'rib chiqilayotgan soha chegarasida boshqaruv funksiyasi berilgan. Sohadagi o'rtacha temperaturaga erishish uchun boshqaruv funksiyasini topish kerak. Furye usuli yordamida boshqaruv masalasi ikkinchi turdagi Volterra integral tenglamasiga keltiriladi. Boshqarish funksiyasining mavjudligi ketma-ket yaqinlashish usuli yordamida isbotlangan.

**Kalit so'zlar:** To'rtinchi tartibli psevdoparabolik tenglama; Volterra integral tenglamasi; joiz boshqaruv; boshlang'ich-chegaraviy masala, ketma-ket yaqinlashish usuli.

### РЕЗЮМЕ

В данной работе рассматривается задача граничного управления для уравнения псевдопараболического типа четвертого порядка в ограниченной двумерной области. На границе рассматриваемой области задана функция управления. Для достижения средней температуры в области требуется найти функцию управления. С помощью метода Фурье задача управления сводится к интегральному уравнению Вольтерра второго рода. Существование функции управления доказывается с помощью метода последовательных приближений.

**Ключевые слова:** Псевдопараболическое уравнение четвертого порядка; интегральное уравнение Вольтерра; допустимое управление; начально-краевая задача, метод последовательных приближений.

UDC 517.918

CHEKLI O‘LCHAMLI FREDGOLM OPERATORI DARAJALARINING  
SPEKTRI VA REZOLVENTASI

Husenova J. T. \*

## REZYUME

Ushbu maqolada  $L^2[-\pi; \pi]$  Hilbert fazosida ta’sir qiluvchi chekli o‘lchamli  $T$  integral operatorning (Fredgolm operatorining) istalgan  $n$  natural soni uchun  $T^n$  darajalari topilgan.  $T^n$  operatorning xos qiymatlari va mos xos funksiyalari aniqlangan.  $T^n$  operatorga mos rezolventa operatori qurilgan.

**Kalit so‘zlar:** integral operator, Fredgolm operatori, chekli o‘lchamli operator, xos qiymat, karralik, xos funksiya, rezolventa operatori.

## 1. Kirish.

Elementlari funksiyalardan iborat fazoga funksional fazo deyiladi [1]. Masalan,  $C[-\pi; \pi] - [-\pi; \pi]$  kesmada aniqlangan barcha uzluksiz funksiyalar to‘plami,  $L^2[-\pi; \pi] - [-\pi; \pi]$  kesmada aniqlangan barcha kvadrati bilan integrallanuvchi (umuman olganda kompleks qiymatlarni qabul qiluvchi) funksiyalar to‘plami funksional to‘plamlarga misol bo‘la oladi. Funksional fazoda tenglama berilgan bo‘lib, noma’lum element funksiyadan iborat bo‘lsa, bunday tenglamaga funksional tenglama deyiladi.  $f(x) = f(-x)$  - juftlik tenglamasi;  $f(x + L) = f(x)$  - davriylik tenglamasi;  $f(x + y) = f(x) + f(y)$  - additivlik tenglamasi va boshqa tenglamalar funksional tenglamalarga eng sodda misollar bo‘la oladi.  $f(x + y) = f(x) + f(y)$  - Koshi tenglamasi;  $f(x + y) + f(x - y) = 2[f(x) + f(y)]$  - kvadratik tenglama yoki parallelogramm qoidasi;  $f((x + y)/2) = (f(x) + f(y))/2$  - Yensen tenglamasi;  $f(x + y) \cdot f(x - y) = f^2(x)$  - Lobachevskiy tenglamasi;  $f(x + y) + f(x - y) = 2f(x)f(y)$  - Dalamber tenglamasi mashhur funksional tenglamalarga misol bo‘la oladi.

Agar funksional tenglamada noma’lum funksiya integral belgisi ostida bo‘lsa, u holda bunday tenglamaga integral tenglama deyiladi. Masalan,

$$f(x) = \int_{-\pi}^{\pi} K(x, t)f(t)dt \quad (1)$$

tenglama  $f(x)$  funksiyaga nisbatan chiziqli integral tenglamadir, bu yerda  $K(\cdot, \cdot) - [-\pi; \pi]^2$  da aniqlangan berilgan funksiya.

(1) tenglamaning yadrosi deb ataluvchi  $K(\cdot, \cdot)$  funksiya o‘lchovli va

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |K(x, y)|^2 dx dy < \infty$$

shartni qanoatlantirsin, ya’ni  $K(\cdot, \cdot)$  kvadrati bilan integrallanuvchi funksiya bo‘lsin.

\*Husenova J. T. – Buxoro davlat universiteti

$L^2[-\pi; \pi]$  Hilbert fazosida aniqlangan

$$(Tf)(x) = \int_{-\pi}^{\pi} K(x, t)f(t)dt \quad (2)$$

operatorga  $K$  yadroli Fredgolm operatori deyiladi.

Ko'p hollarda  $T$  operator Fridriks modelida qo'zg'alish operatori [2-7] sifatida o'rganiladi.

Ushbu maqolada (2) ko'rinishdagi  $T$  Fredgolm operatori yadrosi

$$K(x, y) = \sum_{i=1}^m v_i(x)v_i(y)$$

kabi aniqlangan hol o'rganiladi, bu yerda  $v_i(\cdot)$ ,  $i = \overline{1, m} - [-\pi; \pi]$  kesmada aniqlangan haqiqiy qiymatli o'zaro chiziqli bog'lanmagan uzluksiz funksiyalar,  $m \in \mathbb{N}$ . Dastlab ixtiyoriy  $n$  natural soni uchun berilgan  $T$  operatorning  $T^n$  darajalari aniqlangan. Topilgan  $T^n$  operator uchun  $\lambda = 0$  soni cheksiz karrali xos qiymat, ko'pi bilan  $m$  ta (karraligi bilan qo'shib hisoblaganda) xos qiymatlarga ega ekanligi ko'rsatilgan va mos xos funksiyalari tavsiflangan.

## 2. Chekli o'lchamli Fredgolm operatorining xos qiymatlari va mos xos funksiyalari

$L^2[-\pi; \pi]$  Hilbert fazosida

$$(T_i f)(x) = v_i(x) \int_{-\pi}^{\pi} v_i(t)f(t)dt, \quad i = \overline{1, m} \quad (3)$$

tenglik yordamida ta'sir qiluvchi operatorni qaraymiz. Bunda  $v_i(\cdot)$ ,  $i = \overline{1, m} - [-\pi; \pi]$  kesmada aniqlangan haqiqiy qiymatli o'zaro chiziqli bog'lanmagan uzluksiz funksiyalar.

Bizga Funktsional analiz kursidan yaxshi ma'lumki, ixtiyoriy  $f, g \in L^2[-\pi; \pi]$  elementlar uchun ularning skalyar ko'paytmasi

$$(f, g) = \int_{-\pi}^{\pi} f(t)\overline{g(t)}dt$$

tenglik yordamida,  $f \in L^2[-\pi; \pi]$  elementning normasi esa

$$\|f\| = \left( \int_{-\pi}^{\pi} |f(t)|^2 dt \right)^{\frac{1}{2}}$$

tenglik yordamida aniqlanadi. Tegishli ta'riflardan foydalanib,  $L^2[-\pi; \pi]$  Hilbert fazosida

$$T := T_1 + \dots + T_m$$

kabi aniqlangan  $T$  operator chiziqli, chegaralangan va o'z-o'ziga qo'shma operator ekanligini ko'rsatish mumkin.

Ta'rifga ko'ra

$$(Tf)(x) = \sum_{i=1}^m v_i(x) \int_{-\pi}^{\pi} v_i(t)f(t)dt$$

tenglik o‘rinlidir.

Maqola matnida foydalanish uchun  $\sigma(\cdot)$ ,  $\sigma_{\text{ess}}(\cdot)$  va  $\sigma_{\text{pp}}(\cdot)$  simvollar orqali mos ravishda chegaralangan va o‘z-o‘ziga qo‘shma operatorning spektri, muhim spektri va nuqtali spektri belgilangan.

Ushbu maqolaning asosiy natijalaridan biri quyidagi teoremda o‘z aksini topgan.

**1-teorema.**  $\lambda = 0$  soni  $T$  operator uchun cheksiz karrali xos qiymat bo‘ladi.

**Isbot.** Berilgan  $T$  operator uchun xos qiymatga nisbatan  $Tf = 0$  tenglamani qaraymiz. Avvalo bu tenglamani

$$\sum_{i=1}^m v_i(x) \int_{-\pi}^{\pi} v_i(t) f(t) dt = 0 \quad (4)$$

ko‘rinishda yozib olamiz.  $v_i(\cdot)$ ,  $i = \overline{1, m}$  funksiyalar o‘zaro chiziqli bog‘lanmagan uzluksiz funksiyalar ekanligidan (4) tenglik faqat va faqat

$$\int_{-\pi}^{\pi} v_i(t) f(t) dt = 0, \quad i = \overline{1, m} \quad (5)$$

bo‘lgandagina bajarilishi kelib chiqadi.

Endi ixtiyoriy  $k$  natural soni uchun  $k$  ta chiziqli bog‘lanmagan, (5) shartni qanoatlantiruvchi funksiyalar mavjud ekanligini ko‘rsatamiz. Buning uchun matematik induksiya metodidan foydalanamiz.  $L^2[-\pi; \pi]$  cheksiz o‘lchamli fazo bo‘lgani uchun shunday  $w_1 \in L^2[-\pi; \pi]$  funksiya topilib,

$$\int_{-\pi}^{\pi} v_i(t) w_1(t) dt = 0, \quad i = \overline{1, m}$$

shart bajariladi. Xuddi shuningdek,  $L^2[-\pi; \pi]$  cheksiz o‘lchamli fazo bo‘lgani uchun shunday  $w_2 \in L^2[-\pi; \pi]$  funksiya topilib,  $w_1$  va  $w_2$  funksiyalar chiziqli bog‘lanmagan hamda

$$\int_{-\pi}^{\pi} v_i(t) w_2(t) dt = 0, \quad i = \overline{1, m}$$

shart bajariladi. Faraz qilaylik, ixtiyoriy  $k \geq 3$  soni uchun  $w_1, w_2, \dots, w_{k-1}$  funksiyalar qurilgan bo‘lib,  $\{w_1, w_2, \dots, w_{k-1}\}$  chiziqli bo‘lanmagan sistema va ixtiyoriy  $j \in \{1, \dots, k-1\}$  uchun

$$\int_{-\pi}^{\pi} v_i(t) w_j(t) dt = 0, \quad i = \overline{1, m}$$

shart bajarilsin. U holda shunday  $w_k \in L^2[-\pi; \pi]$  funksiya topilib,  $\{w_1, w_2, \dots, w_{k-1}, w_k\}$  chiziqli bo‘lanmagan sistema va

$$\int_{-\pi}^{\pi} v_i(t) w_k(t) dt = 0, \quad i = \overline{1, m}$$

shart bajariladi. Shunday qilib,  $\{w_1, w_2, \dots, w_k\}$  chiziqli bog‘lanmagan sistema mavjud bo‘lib, uning elementlari (5) shartni qanoatlantiradi. Bu esa  $\lambda = 0$  soni  $T$  operator uchun cheksiz karrali xos qiymat ekanligini bildiradi. 1-teorema to‘liq isbotlandi.

1-teorema isbotidan ko‘rinib turibdiki, (5) tenglikni qanoatlantiruvchi har qanday  $f \in L^2[-\pi; \pi]$  funksiya  $T$  operatorning  $\lambda = 0$  xos qiymatiga mos xos funksiya bo‘ladi.

$T$  operatorning nolmas xos qiymatlarini aniqlash maqsadida

$$\Delta(\lambda) := \det(\lambda\delta_{ij} - (v_i, v_j))$$

funksiyalarni kiritamiz, bu yerda

$$\delta_{ij} := \begin{cases} 1, & \text{agar } i = j \text{ bo'lsa,} \\ 0, & \text{agar } i \neq j \text{ bo'lsa.} \end{cases}$$

Odatda  $\Delta(\cdot)$  funksiyaga  $T$  operatorga mos Fredgolv determinanti deyiladi.

Quyidagi teorema  $T$  operatorning xos qiymatlari va  $\Delta(\cdot)$  funksiyaning nollari orasidagi munosabatni ifodalaydi.

**2-teorema.**  $\lambda \neq 0$  soni  $T$  operatorning xos qiymati bo'lishi uchun  $\Delta(\lambda) = 0$  bo'lishi zarur va yetarlidir.

**Isbot.** Faraz qilaylik,  $\lambda \neq 0$  soni  $T$  operatorning xos qiymati,  $f \in L^2[-\pi; \pi]$  esa unga mos xos funksiya bo'lsin. U holda  $f$  funksiya  $Tf = \lambda f$  xos qiymatga nisbatan tenglamani, ya'ni

$$\sum_{i=1}^m v_i(x) \int_{-\pi}^{\pi} v_i(t) f(t) dt = \lambda f(x) \quad (6)$$

tenglamani qanoatlantiradi.

Ixtiyoriy  $\lambda \neq 0$  uchun (6) tenglamadan  $f$  funksiyani

$$f(x) = \frac{1}{\lambda} \sum_{i=1}^m C_i v_i(x) \quad (7)$$

ko'rinishda topamiz, bu yerda

$$C_i := \int_{-\pi}^{\pi} v_i(t) f(t) dt, \quad i = \overline{1, m}. \quad (8)$$

$f$  funksiya uchun topilgan (7) ifodani (8) belgilashga qo'yib, (6) tenglama nolmas yechimga ega bo'lishi uchun quyidagi

$$\begin{cases} \sum_{j=1}^m (\lambda\delta_{1j} - (v_1, v_j)) = 0 \\ \sum_{j=1}^m (\lambda\delta_{2j} - (v_2, v_j)) = 0 \\ \dots\dots\dots \\ \sum_{j=1}^m (\lambda\delta_{mj} - (v_m, v_j)) = 0 \end{cases}$$

$m$  ta noma'lumli  $m$  ta chiziqli tenglamalar sistemasi yoki

$$(\lambda\delta_{ij} - (v_i, v_j))_{i,j=1}^m \begin{pmatrix} C_1 \\ \vdots \\ C_m \end{pmatrix} = 0$$

matritsaviy tenglama nolmas  $(C_1, \dots, C_m) \in \mathbb{C}^m$  yechimga ega bo‘lishi, ya’ni  $\Delta(\lambda) = 0$  bo‘lishi zarur va yetarli ekanligini hosil qilamiz, bu yerda  $\mathbb{C}^m$  orqali  $\mathbb{C}$  to‘planning  $m$ -darajasi belgilangan. 2-teorema to‘liq isbotlandi.

Isbotlangan 2-teoremadan  $T$  operatorning diskret spektri uchun

$$\sigma_{\text{disc}}(T) = \{\lambda \neq 0 : \Delta(\lambda) = 0\}$$

tenglikni hosil qilamiz.

Shunday qilib,  $T$  operatorning xos qiymatlarini topish masalasi  $\Delta(\cdot)$  funksiyaning nollarini topish masalasiga keltirildi. Aniqlanishiga ko‘ra,  $\Delta(\cdot)$  funksiya  $\lambda$  o‘zgaruvchiga nisbatan  $m$ -darajali algebraik ko‘phad bo‘ladi. Algebraning asosiy teoremasiga ko‘ra  $\Delta(\cdot)$  funksiya ko‘pi bilan  $m$  ta (karraligi bilan qo‘shib hisoblaganda) haqiqiy nolga ega bo‘ladi. Demak,  $T$  operator bitta  $\lambda = 0$  cheksiz karrali xos qiymatga, ko‘pi bilan  $m$  ta (karraligi bilan qo‘shib hisoblaganda) nolmas xos qiymatlarga ega bo‘ladi.

1-2-teoremalardan quyidagi tasdiqlarni hosil qilamiz:

$$\sigma_{\text{ess}}(T) = \{0\}, \quad \sigma_{\text{disc}}(T) = \{\lambda \neq 0 : \Delta(\lambda) = 0\},$$

$$\sigma(T) = \sigma_{\text{pp}}(T) = \{0\} \cup \{\lambda \neq 0 : \Delta(\lambda) = 0\}.$$

Shunday qilib,  $T$  sof nuqtali spektrga ega operator ekan.

Navbatdagi izlanishlarimizda qulaylik uchun

$$(v_i, v_j) = 0, \quad i \neq j, \quad i, j = \overline{1, m} \quad (9)$$

shart bajarilishini talab qilamiz. U holda  $\Delta(\cdot)$  funksiyani

$$\Delta(\lambda) = (\lambda - \|v_1\|^2)(\lambda - \|v_2\|^2) \cdots (\lambda - \|v_m\|^2)$$

ko‘rinishda yozish mumkin va 2-teoremaga ko‘ra

$$\sigma_{\text{disc}}(T) = \{\|v_1\|^2, \|v_2\|^2, \dots, \|v_m\|^2\}$$

tenglikni hosil qilamiz. Shunday qilib, agar (9) shart bajarilsa, u holda  $T$  operator sof nuqtali spektrga ega bo‘lib,

$$\sigma(T) = \sigma_{\text{pp}}(T) = \{0, \|v_1\|^2, \|v_2\|^2, \dots, \|v_m\|^2\}$$

munosabatlar o‘rinlidir.

1-2-teoremalarning isbotida qo‘llanilgan mulohazalar yordamida ixtiyoriy  $j \in \{1, 2, \dots, m\}$  uchun

$$\sigma(T_j) = \sigma_{\text{pp}}(T_j) = \{0, \|v_j\|^2\}$$

munosabatlar o‘rinlidir. Demak, agar (9) shart bajarilsa, u holda  $T$  va  $T_j$ ,  $j = \overline{1, m}$  operatorlarning spektrlari o‘rtasida

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_2) \cup \dots \cup \sigma(T_m)$$

bog‘lanish o‘rinlidir. Shunday qilib,  $T$  operatorning spektrini o‘rganish masalasi nisbatan sodda ko‘rinishga ega  $T_j$ ,  $j = \overline{1, m}$  operatorlarning spektrlarini o‘rganish masalasiga keltirildi.

### 3. $T$ operator darajalarining spektri va rezolventasi.

Endi (9)-shart asosida  $T$  operator darajalarining xos qiymatlarini aniqlash masalasini qaraymiz. Dastlab  $T^2$  operatorni aniqlaymiz:

$$(T^2 f)(x) = T(Tf)(x) = \sum_{i=1}^m \|v_i\|^2 v_i(x) \int_{-\pi}^{\pi} v_i(t) f(t) dt.$$

Oxirgi tenglikda  $\bar{v}_i(x) := \|v_i\| v_i(x)$  belgilash olib

$$(T^2 f)(x) = \sum_{i=1}^m \bar{v}_i(x) \int_{-\pi}^{\pi} \bar{v}_i(t) f(t) dt$$

ifodani hosil qilamiz. Shu sababli, agar (9) shart bajarilsa, u holda 1-2-teoremlarga ko'ra  $T^2$  operator sof nuqtali spektrga ega va

$$\sigma(T^2) = \sigma_{pp}(T^2) = \{0, \|\bar{v}_1\|^2, \|\bar{v}_2\|^2, \dots, \|\bar{v}_m\|^2\},$$

ya'ni

$$\sigma(T^2) = \sigma_{pp}(T^2) = \{0, \|v_1\|^4, \|v_2\|^4, \dots, \|v_m\|^4\}$$

munosabatlar o'rinlidir.

Endi  $T^3$  operatorni aniqlaymiz:

$$(T^3 f)(x) = T^2(Tf)(x) = \sum_{i=1}^m \|v_i\|^4 v_i(x) \int_{-\pi}^{\pi} v_i(t) f(t) dt.$$

Oxirgi tenglikda  $\tilde{v}_i(x) := \|v_i\|^2 v_i(x)$  belgilash olib,  $T^3$  operator uchun

$$(T^3 f)(x) = \sum_{i=1}^m \tilde{v}_i(x) \int_{-\pi}^{\pi} \tilde{v}_i(t) f(t) dt$$

ifodani hosil qilamiz. Shu sababli, agar (9) shart bajarilsa, u holda 1-2-teoremlarga ko'ra  $T^3$  operator sof nuqtali spektrga ega va

$$\sigma(T^3) = \sigma_{pp}(T^3) = \{0, \|\tilde{v}_1\|^2, \|\tilde{v}_2\|^2, \dots, \|\tilde{v}_m\|^2\},$$

ya'ni

$$\sigma(T^3) = \sigma_{pp}(T^3) = \{0, \|v_1\|^6, \|v_2\|^6, \dots, \|v_m\|^6\}$$

munosabatlar o'rinlidir.

Ta'kidlash joizki, agar (9) shart bajarilsa, u holda ixtiyoriy  $n \geq 2$  natural soni uchun  $T^n$  operator

$$(T^n f)(x) = \sum_{i=1}^m \|v_i\|^{2(n-1)} v_i(x) \int_{-\pi}^{\pi} v_i(t) f(t) dt \quad (10)$$

tenglik yordamida ta'sir qiladi.

Yuqoridagi mulohazalardan foydalanib, quyidagi teoremani hosil qilamiz.



**3-teorema.** Agar (9) shart bajarilsa, u holda ixtiyoriy  $n \geq 2$  natural soni uchun  $T^n$  operator sof nuqtali spektrga ega va

$$\sigma(T^n) = \sigma_{pp}(T^n) = \{0, \|v_1\|^{2n}, \|v_2\|^{2n}, \dots, \|v_m\|^{2n}\}$$

munosabatlar o‘rinli.

Quyidagi teorema  $T^n$  operatorga mos rezolventa operatorini tavsiflaydi.

**4-teorema.** Agar (9) shart bajarilsa, u holda  $L^2[-\pi; \pi]$  Hilbert fazosida (10) tenglik yordamida ta’sir qiluvchi  $T^n$  operatorga mos  $R_\lambda(T^n)$  rezolventa operatori

$$(R_\lambda(T^n)g)(x) = -\frac{1}{\lambda}g(x) - \frac{1}{\lambda} \sum_{i=1}^m \frac{\|v_i\|^{2n}}{\lambda - \|v_i\|^{2n}} v_i(x) \int_{-\pi}^{\pi} v_i(t)g(t)dt$$

formula yordamida aniqlanadi.

**Isbot.** Fiksirlangan  $\lambda \in \mathbb{C} \setminus \sigma(T^n)$  soni uchun  $f, g \in L^2[-\pi; \pi]$  funksiyalarga nisbatan

$$\sum_{i=1}^m \|v_i\|^{2(n-1)} v_i(x) \int_{-\pi}^{\pi} v_i(t)f(t)dt - \lambda f(x) = g(x) \quad (11)$$

tenglamani qaraymiz.

$\lambda \neq 0$  bo‘lgani bois (11) tenglamadan  $f(\cdot)$  funksiya uchun

$$f(x) = -\frac{1}{\lambda}g(x) + \frac{1}{\lambda} \sum_{i=1}^m C_i \|v_i\|^{2(n-1)} v_i(x) \quad (12)$$

ifodani topamiz, bu yerda  $C_i$  sonlari (8) tenglik yordamida aniqlanadi.

$f(x)$  funksiya uchun topilgan (12) ifodani (8) belgilashga qo‘yib, (9) shartga ko‘ra

$$\begin{aligned} (\lambda - \|v_1\|^{2n})C_1 &= - \int_{-\pi}^{\pi} v_1(t)g(t)dt; \\ (\lambda - \|v_2\|^{2n})C_2 &= - \int_{-\pi}^{\pi} v_2(t)g(t)dt; \\ \dots\dots\dots \\ (\lambda - \|v_m\|^{2n})C_m &= - \int_{-\pi}^{\pi} v_m(t)g(t)dt \end{aligned}$$

tenglamalar sistemasiga ega bo‘lamiz.

Bundan esa o‘z navbatida  $\lambda \notin \sigma(T^n)$  ekanligidan

$$\begin{aligned} C_1 &= -\frac{1}{\lambda - \|v_1\|^{2n}} \int_{-\pi}^{\pi} v_1(t)g(t)dt; \\ C_2 &= -\frac{1}{\lambda - \|v_2\|^{2n}} \int_{-\pi}^{\pi} v_2(t)g(t)dt; \\ \dots\dots\dots \\ C_m &= -\frac{1}{\lambda - \|v_m\|^{2n}} \int_{-\pi}^{\pi} v_m(t)g(t)dt \end{aligned}$$

tenglamalar sistemasi hosil bo'ladi.

$C_i$  uchun topilgan yechimlarni (12) tenglikka qo'yib,  $f(x)$  funksiya uchun

$$f(x) = -\frac{1}{\lambda}g(x) - \frac{1}{\lambda} \sum_{i=1}^m \frac{\|v_i\|^{2n}}{\lambda - \|v_i\|^{2n}} v_i(x) \int_{-\pi}^{\pi} v_i(t)g(t)dt$$

tenglikni hosil qilamiz.

Ta'rifga ko'ra  $T^n$  operatorga mos  $R_\lambda(T^n)$  rezolventa operatori  $L^2[-\pi; \pi]$  Hilbert fazosida

$$(R_\lambda(T^n)g)(x) = -\frac{1}{\lambda}g(x) - \frac{1}{\lambda} \sum_{i=1}^m \frac{\|v_i\|^{2n}}{\lambda - \|v_i\|^{2n}} v_i(x) \int_{-\pi}^{\pi} v_i(t)g(t)dt$$

formula bilan aniqlanishini hosil qilamiz. 4-teorema to'liq isbotlandi.

Endi asosiy natijalar bayonida foydalanilgan (9) shart bajariladigan  $v_1, \dots, v_m$  funksiyalarga misol keltiramiz.

Faraz qilaylik,  $m = 2$  bo'lsin. U holda  $v_1(x) = \sin x$  va  $v_2(x) = \cos x$  funksiyalar (9) shartni qanoatlantiradi, ya'ni

$$(v_1, v_2) = \int_{-\pi}^{\pi} \sin t \cos t dt = 0.$$

Faraz qilaylik,  $m = 3$  bo'lsin. U holda

$$\begin{aligned} v_1(x) &= \begin{cases} \sin(3x), & \text{agar } x \in [-\pi; -\pi/3] \text{ bo'lsa;} \\ 0, & \text{agar } x \in [-\pi/3; \pi] \text{ bo'lsa;} \end{cases} \\ v_2(x) &= \begin{cases} \sin(3x), & \text{agar } x \in [-\pi/3; \pi/3] \text{ bo'lsa;} \\ 0, & \text{agar } x \in [-\pi; -\pi/3] \cup [\pi/3; \pi] \text{ bo'lsa;} \end{cases} \\ v_3(x) &= \begin{cases} \sin(3x), & \text{agar } x \in [\pi/3; \pi] \text{ bo'lsa;} \\ 0, & \text{agar } x \in [-\pi; \pi/3] \text{ bo'lsa} \end{cases} \end{aligned}$$

funksiyalar (9) shartni qanoatlantiradi, ya'ni

$$(v_1, v_2) = (v_1, v_3) = (v_2, v_3) = 0$$

tengliklar bajariladi.

**Xulosa.** Ushbu maqolada  $L^2[-\pi; \pi]$  Hilbert fazosida ta'sir qiluvchi chekli o'lchamli ajralgan yadroli  $T$  integral operator (Fredgolm operator) uchun  $\lambda = 0$  soni cheksiz karrali xos qiymat bo'lishi isbotlangan. Nollari  $T$  Fredgolm operatorining xos qiymatlari bilan ustma-ust tushuvchi Fredgolm determinanti qurilgan. (9) shartdan foydalanib,  $n = 2$ ,  $n = 3$  va ixtiyoriy  $n > 3$  natural soni uchun  $T^n$  operatorning ta'sir formulasi topilgan. Ularning xos qiymatlari va mos xos funksiyalari aniqlangan. Iсталgan  $n$  natural soni uchun  $T^n$  operatorga mos rezolventa operatorining aniq ta'sir formulasi keltirib chiqarilgan. Maqola matni so'ngida  $n = 2$  va  $n = 3$  hollarda (9) shart bajariladigan parametr funksiyalarga misollar keltirilgan.

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### РЕЗЮМЕ

В этой статье для конечномерного интегрального оператора  $T$  (оператора Фредгольма), действующего в гильбертовом пространстве  $L^2[-\pi; \pi]$ , найден  $T^n$  для любого натурального числа  $n$ . Определены собственные значения и соответствующие им собственные функции оператора  $T^n$ . Построена резольвентная оператор, соответствующий оператору  $T^n$ .

**Ключевые слова:** интегральный оператор, оператор Фредгольма, конечномерный оператор, собственное значение, кратность, собственная функция, оператор резольвенты.

### RESUME

In this paper for the finite-dimensional integral operator  $T$  (Fredholm operator) acting in Hilbert space  $L^2[-\pi; \pi]$  the  $n$ th power  $T^n$  is found for any natural number  $n$ . The eigenvalues and corresponding eigenfunctions of the operator  $T^n$  are defined. The resolvent operator corresponding to the  $T^n$  operator is constructed.

**Key words:** integral operator, Fredholm operator, finite dimensional operator, eigenvalue, multiplicity, eigenfunction, resolvent operator.

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**UCHINCHI TARTIBLI OPERATORLI MATRITSA MUHIM SPEKTRINING  
CHEGARALARI UCHUN BAHOLASHLAR**

Ismoilova D. E. \*

**REZYUME**

Ushbu maqolada fermionli Fok fazoda chiziqli, chegaralangan va o'z-o'ziga qo'shma bo'lgan uchinchi tartibli operatorli matritsa qaralgan. Bu operatorli matritsaning muhim spektri tavsiflangan. Ta'sirlashish parametrining qiymatlarining qiymatlariga nisbatan muhim spektrning quyi va yuqori chegaralari uchun baholashlar olingan.

**Kalit so'zlar:** fermion, operatorli matritsa, Fredgolm determinanti, xos qiymat, muhim spektr, quyi chegara, yuqori chegara, spektral baho.

**1. Kirish.**

Elementlari Banax yoki Hilbert fazolarida ta'sir qiluvchi chiziqli operatorlardan iborat bo'lgan operatorli matritsalarining spektral xossalari ko'plab ilmiy va amaliy masalalarda o'rganilgan [1]. Qattiq jismlar fizikasi [2], kvant maydon nazariyasi [3], statistik fizika [4], kvant mexanikasi [5], magnitogidrodinamika [6] va boshqa ko'plab sohalarda operatorli matritsalar, xususan, panjaradagi soni saqlanmaydigan chekli sondagi zarrachalar sistemasiga mos Hamiltonianlarning muhim va diskret spektrlari bilan bog'liq dolzarb muammolar paydo bo'ladi. Shuning uchun bunday turdagi operatorli matritsalarining spektral xossalarini o'rganish muhim masalalardan biri hisoblanadi.

Mazkur maqolada panjaradagi soni saqlanmaydigan va uchtadan oshmaydigan zarrachalar sistemasi bilan bog'liq  $\mathcal{A}$  uchinchi tartibli operatorli matritsa qaralgan.  $\mathcal{A}$  operatorli matritsani u ta'sir qiladigan Hilbert fazo xususiyatidan kelib chiqib oltinchi tartibli operatorli matritsa ko'rinishida ham tasvirlash mumkin. Unitar akslantirish yordamida  $\mathcal{A}$  operatorli matritsani diagonal elementlari fermionli Fok fazoning qirqilgan uch zarrachali qism fazosida ta'sir qiluvchi uchinchi tartibli operatorli matritsalar bo'lgan ikkinchi tartibli diagonal operatorli matritsaga unitar ekvivalet ekanligini ko'rsatish mumkin [7]. Bu xossa  $\mathcal{A}$  operatorli matritsa muhim, diskret va nuqtali spektrlarini tadqiq qilishda qulay hisoblanadi. Dastlab  $\mathcal{A}$  operatorli matritsa muhim spektrining joylashuv o'rni aniqlangan hamda uning ikki va uch zarrachali tarmoqlari topilgan.  $\alpha$  ta'sirlashish parametrining qiymatlariga nisbatan muhim spektrning quyi va yuqori chegaralari uchun spektral baholashlar olingan.

**2. Uchinchi tartibli operatorli matritsa va uning muhim spektri.**

Dastlab ushbu maqolada ishlatiladigan ayrim belgilashlarni kiritamiz.  $\mathbb{T}^d := (-\pi; \pi]^d$  orqali  $d$  o'lchamli torni,  $\mathcal{H}_0 := \mathbb{C}$  orqali bir o'lchamli kompleks fazoni,  $\mathcal{H}_1 := L_2(\mathbb{T}^d)$  orqali  $\mathbb{T}^d$  to'plamda aniqlangan kvadrati bilan integrallanuvchi (umuman olganda kompleks

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\*Ismoilova D. E. – Buxoro davlat universiteti

qiymatlarni qabul qiluvchi) funksiyalarning Hilbert fazosini va  $\mathcal{H}_2 := L_2^{\text{as}}((\mathbb{T}^d)^2)$  orqali  $(\mathbb{T}^d)^2$  to'plamda aniqlangan kvadrati bilan integrallanuvchi antisimmetrik (umuman olganda kompleks qiymatlarni qabul qiluvchi) funksiyalar fazosini belgilaymiz.

Quyidagi Hilbert fazolarini qaraymiz:

$$\mathcal{F}_{\text{as}}^{(1)}(L_2(\mathbb{T}^d)) := \mathcal{H}_0 \oplus \mathcal{H}_1; \quad \mathcal{F}_{\text{as}}^{(2)}(L_2(\mathbb{T}^d)) := \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2.$$

Yuqoridagi tengliklar yordamida aniqlangan, ya'ni  $\mathcal{F}_{\text{as}}^{(n)}(L_2(\mathbb{T}^d))$ ,  $n = 1, 2$ , ko'rinishidagi fazoga  $\mathcal{F}_{\text{as}}(L_2(\mathbb{T}^d))$  fermionli Fok fazoning  $(n + 1)$  zarrachali qirqilgan qism fazosi deyiladi.  $\mathcal{F}_{\text{as}}^{(2)}(L_2(\mathbb{T}^d))$  fazoning elementlari  $f = (f_0, f_1, f_2)$  vektor-funksiya ko'rinishida aniqlangan bo'lib, bu yerda  $f_i \in \mathcal{H}_i$ ,  $i = 0, 1, 2$ .

$\mathbb{C}^2 \otimes \mathcal{F}_{\text{as}}^{(2)}(L_2(\mathbb{T}^d))$  fazoning elementlari  $f = \{f_0^{(s)}, f_1^{(s)}, f_2^{(s)}, s = \pm\}$  kabi aniqlangan bo'lib, ushbu fazoning ixtiyoriy ikkita

$$f = \{f_0^{(s)}, f_1^{(s)}, f_2^{(s)}, s = \pm\} \quad \text{va} \quad g = \{g_0^{(s)}, g_1^{(s)}, g_2^{(s)}, s = \pm\}$$

elementlari uchun skalyar ko'paytma amali quyidagicha kiritilgan

$$\langle f, g \rangle := \sum_{s=\pm} \left( f_0^{(s)} \overline{g_0^{(s)}} + \int_{\mathbb{T}^d} f_1^{(s)}(k_1) \overline{g_1^{(s)}(k_1)} dk_1 + \frac{1}{2} \int_{(\mathbb{T}^d)^2} f_2^{(s)}(k_1, k_2) \overline{g_2^{(s)}(k_1, k_2)} dk_1 dk_2 \right).$$

$\mathbb{C}^2 \otimes \mathcal{F}_{\text{as}}^{(2)}(L_2(\mathbb{T}^d))$  Hilbert fazoda

$$\mathcal{A} := \begin{pmatrix} A_{00} & A_{01} & 0 \\ A_{01}^* & A_{11} & A_{12} \\ 0 & A_{12}^* & A_{22} \end{pmatrix}$$

tenglik bilan ta'sir qiluvchi uch diagonalli uchinchi tartibli operatorli matritsani qaraymiz.  $\mathcal{A}$  operatorli matritsaning  $A_{ij}$  elementlari

$$\begin{aligned} A_{00} f_0^{(s)} &= s\varepsilon f_0^{(s)}, & A_{01} f_1^{(s)} &= \alpha \int_{\mathbb{T}^d} v(t) f_1^{(-s)}(t) dt, \\ (A_{11} f_1^{(s)})(k_1) &= (s\varepsilon + w(k_1)) f_1^{(s)}(k_1), & (A_{12} f_2^{(s)})(k_1) &= \alpha \int_{\mathbb{T}^d} v(t) f_2^{(-s)}(k_1, t) dt, \\ (A_{22} f_2^{(s)})(k_1, k_2) &= (s\varepsilon + w(k_1) + w(k_2)) f_2^{(s)}(k_1, k_2), & f &= \{f_0^{(s)}, f_1^{(s)}, f_2^{(s)}; s = \pm\} \end{aligned}$$

kabi aniqlangan. Bu yerda  $A_{ij}^*$  orqali  $A_{ij}$  operatorga qo'shma operator belgilangan,  $\varepsilon$  - fiksirlangan musbat haqiqiy son,  $w(\cdot)$  va  $v(\cdot)$  funksiyalar  $\mathbb{T}^d$  torda aniqlangan haqiqiy qiymatli uzluksiz funksiyalar,  $\alpha > 0$  esa ta'sirlashish parametri hisoblanadi.

Parameterlarga qo'yilgan bunday shartlarda  $\mathbb{C}^2 \otimes \mathcal{F}_{\text{as}}^{(2)}(L_2(\mathbb{T}^d))$  Hilbert fazosida aniqlangan  $\mathcal{A}$  operatorli matritsa chiziqli, chegaralangan va o'z-o'ziga qo'shma bo'ladi.

Eslatib o'tish joizki, simmetrik funksiyalar yordamida qurilgan bozonli Fok fazoda  $\mathcal{A}$  operatorli matritsaga o'xshash panjaradagi ko'pi bilan ikkita fotonga ega spin-bozon modeli [8-10] maqolalarda tadqiq qilingan.

Quyida  $\mathcal{A}$  operatorli matritsaning spektral xossalarini o'rganish uchun qulay bo'lgan  $\mathcal{F}_{\text{as}}^{(2)}(L_2(\mathbb{T}^d))$  Hilbert fazosida aniqlangan  $\mathcal{A}^{(s)}$ ,  $s = \pm$  chiziqli, chegaralangan, o'z-o'ziga qo'shma bo'lgan ikkita uchinchi tartibli operatorli matritsalarini qaraymiz:

$$\mathcal{A}^{(s)} := \begin{pmatrix} \widehat{A}_{00}^{(s)} & \widehat{A}_{01} & 0 \\ \widehat{A}_{01}^* & \widehat{A}_{11}^{(s)} & \widehat{A}_{12} \\ 0 & \widehat{A}_{12}^* & \widehat{A}_{22}^{(s)} \end{pmatrix}.$$

Uning elementlari

$$\begin{aligned} \widehat{A}_{00}^{(s)} f_0 &= s\varepsilon f_0, & \widehat{A}_{01} f_1 &= \alpha \int_{\mathbb{T}^d} v(t) f_1(t) dt, \\ (\widehat{A}_{11}^{(s)} f_1)(k_1) &= (-s\varepsilon + w(k_1)) f_1(k_1), & (\widehat{A}_{12} f_2)(k_1) &= \alpha \int_{\mathbb{T}^d} v(t) f_2(k_1, t) dt, \\ (\widehat{A}_{22}^{(s)} f_2)(k_1, k_2) &= (s\varepsilon + w(k_1) + w(k_2)) f_2(k_1, k_2), & (f_0, f_1, f_2) &\in \mathcal{F}_{\text{as}}^{(2)}(L_2(\mathbb{T}^d)) \end{aligned}$$

kabi aniqlangan. Bizga funksional analiz kursidan ma'lum bo'lgan ta'riflar yordamida

$$\begin{aligned} (\widehat{A}_{01}^* f_0)(k_1) &= \alpha v(k_1) f_0; \\ (\widehat{A}_{12}^* f_1)(k_1, k_2) &= \alpha (v(k_2) f_1(k_1) - v(k_1) f_1(k_2)), & (f_0, f_1) &\in \mathcal{F}_{\text{as}}^{(1)}(L_2(\mathbb{T}^d)). \end{aligned}$$

tengliklar o'rinli ekanligini osongina ko'rsatish mumkin.

Qulaylik uchun  $\sigma(\cdot)$ ,  $\sigma_{\text{ess}}(\cdot)$ ,  $\sigma_{\text{disc}}(\cdot)$  va  $\sigma_{\text{pp}}(\cdot)$  to'plamlar orqali chiziqli, chegaralangan va o'z-o'ziga qo'shma operatorning mos ravishda spektr, muhim spektr, diskret spektr va nuqtali spektrini belgilaymiz.

Endi  $\mathcal{A}$  va  $\mathcal{A}^{(s)}$ ,  $s = \pm$  operatorli matritsalarining spektrlari orasidagi bog'liqlikni o'rganamiz.

Bizga yaxshi ma'lumki,  $\mathcal{A}$  operatorli matritsaning spektri uchun  $\sigma(\mathcal{A}) = \sigma(\mathcal{A}^{(+)}) \cup \sigma(\mathcal{A}^{(-)})$  tenglik o'rinli. Bundan tashqari,  $\mathcal{A}$  operatorli matritsaning muhim, nuqtali va diskret spektrlari uchun

$$\begin{aligned} \sigma_{\text{ess}}(\mathcal{A}) &= \sigma_{\text{ess}}(\mathcal{A}^{(+)}) \cup \sigma_{\text{ess}}(\mathcal{A}^{(-)}), & \sigma_{\text{pp}}(\mathcal{A}) &= \sigma_{\text{pp}}(\mathcal{A}^{(+)}) \cup \sigma_{\text{pp}}(\mathcal{A}^{(-)}), \\ \sigma_{\text{disc}}(\mathcal{A}) &= \{\sigma_{\text{disc}}(\mathcal{A}^{(+)}) \cup \sigma_{\text{disc}}(\mathcal{A}^{(-)})\} \setminus \sigma_{\text{ess}}(\mathcal{A}) \end{aligned}$$

tengliklar yordamida aniqlangan munosabatlar o'rinlidir. Ta'kidlash joizki,  $\mathcal{A}^{(s)}$  operatorli matritsaning biror xos qiymati  $\mathcal{A}^{(-s)}$  operatorli matritsaning muhim spektriga tegishli bo'lishi mumkin. Shu bois  $\sigma_{\text{disc}}(\mathcal{A}) = \sigma_{\text{disc}}(\mathcal{A}^{(+)}) \cup \sigma_{\text{disc}}(\mathcal{A}^{(-)})$  tenglik hamisha ham o'rinli bo'lavermaydi.

$\mathcal{A}$  operatorli matritsaning muhim spektrini aniqlash va uning chegaralarini baholash maqsadida  $\mathcal{F}_{\text{as}}^{(1)}(L_2(\mathbb{T}^d))$  Hilbert fazosida aniqlangan chiziqli, chegaralangan va o'z-o'ziga qo'shma bo'lgan quyidagi  $\mathcal{A}_1^{(s)}$ ,  $s = \pm$  operatorli matritsani qaraymiz:

$$\mathcal{A}_1^{(s)} := \begin{pmatrix} \widehat{A}_{00}^{(s)} & \widehat{A}_{01} \\ \widehat{A}_{01}^* & \widehat{A}_{11}^{(s)} \end{pmatrix}.$$

Chekli o'lchamli qo'zg'alishlarda muhim spektrning o'zgormasligi haqidagi mashhur Veyl teoremasiga ko'ra  $\mathcal{A}_1^{(s)}$  operatorli matritsaning muhim spektri

$$\sigma_{\text{ess}}(\mathcal{A}_1^{(s)}) = [-s\varepsilon + m; -s\varepsilon + M]$$

kabi aniqlanadi. Bu yerda  $m$  va  $M$  sonlari uchun

$$m := \min_{k_1 \in \mathbb{T}^d} w(k_1), \quad M := \max_{k_1 \in \mathbb{T}^d} w(k_1)$$

tengliklar o‘rinlidir.

Endi  $\mathbb{C} \setminus [-s\varepsilon + m; -s\varepsilon + M]$  sohada regulyar funksiya hisoblangan va  $\mathcal{A}_1^{(s)}$ ,  $s = \pm$  operatorli matritsaning Fredholm determinanti deb nomlangan

$$\Delta^{(s)}(z) := s\varepsilon - z - \alpha^2 \int_{\mathbb{T}^d} \frac{v^2(t) dt}{-s\varepsilon + w(t) - z}$$

ko‘rinishidagi funksiyanı kiritamiz.

$\Delta^{(s)}(\cdot)$  funksiya orqali  $\mathcal{A}_1^{(s)}$  operatorli matritsaning diskret spektri

$$\sigma_{\text{disc}}(\mathcal{A}_1^{(s)}) = \{z \in \mathbb{C} \setminus [-s\varepsilon + m; -s\varepsilon + M] : \Delta^{(s)}(z) = 0\} \quad (1)$$

quyidagi tenglik yordamida aniqlanadi.

Navbatdagi izlanishlarda qulaylik uchun quyidagi belgilashlarnı kiritamiz:

$$\sigma^{(s)} := \bigcup_{k_1 \in \mathbb{T}^d} \left\{ w(k_1) + \sigma_{\text{disc}}(\mathcal{A}_1^{(-s)}) \right\}, \quad \Sigma^{(s)} := \sigma^{(s)} \cup [s\varepsilon + 2m; s\varepsilon + 2M].$$

Yuqoridagi belgilashga ko‘ra,  $\sigma_{\text{ess}}(\mathcal{A}^{(s)}) = \Sigma^{(s)}$  tenglik o‘rinli ekanligidan  $\mathcal{A}$  operatorli matritsa muhim spektrining joylashuv o‘rni uchun quyidagi teorema o‘rinli [7].

**1-teorema.**  $\mathcal{A}$  operatorli matritsaning muhim spektri  $\Sigma^{(+)}$  va  $\Sigma^{(-)}$  to‘plamlar birlashmasidan iborat, ya’ni  $\sigma_{\text{ess}}(\mathcal{A}) = \Sigma^{(+)} \cup \Sigma^{(-)}$ . Bundan tashqari,  $\sigma_{\text{ess}}(\mathcal{A})$  to‘plam ko‘pi bilan oltita kesmalar birlashmasidan iborat.

Eslatib o‘tish lozimki, 1-teoremada  $\sigma_{\text{ess}}(\mathcal{A})$  to‘plam necha kesmalar birlashmasidan iborat ekanligi  $\alpha$  ta’sirlashish parametrining qiymatidan bog‘liq.

### 3. Uchinchi tartibli operatorli matritsa muhim spektrining chegaralari uchun baholashlar

Maqolaning ushbu qismida  $\mathcal{A}$  operatorli matritsa muhim spektrining quyi va yuqori chegaralari uchun baholashlar olingan.

Keyingi izlanishlarda qulaylik uchun  $E_{\min}^{(s)} := \min \sigma_{\text{ess}}(\mathcal{A}^{(s)})$  deb belgilash kiritamiz. U holda

$$E_{\min} := \min \sigma_{\text{ess}}(\mathcal{A}) = \min \{E_{\min}^{(+)}; E_{\min}^{(-)}\}$$

tenglik o‘rinli bo‘ladi.

Quyida  $\mathcal{A}$  operatorli matritsa muhim spektrining quyi chegarasi baholangan teoremlarnı keltiramiz.

**2-teorema.** Faraz qilaylik  $m \geq 0$  bo‘lsin. U holda ixtiyoriy  $\alpha > 0$  soni uchun  $E_{\min} < -\varepsilon + 2m$  tengsizlik bajariladi.

**Isbot.** Dastlab  $\Delta^{(s)}(\cdot)$ ,  $s = \pm$  funksiyani tadqiq qilamiz. Bu funksiyaning  $z$  argument bo'yicha differensial uchun barcha  $z \in \mathbb{R} \setminus [-s\varepsilon + m; -s\varepsilon + M]$  sonlari uchun

$$\frac{d}{dz}\Delta^{(s)}(z) = -1 - \alpha^2 \int_{\mathbb{T}^d} \frac{v^2(t)dt}{(-s\varepsilon + w(t) - z)^2} < 0$$

munosabat o'rinli bo'ladi. Bundan esa  $\Delta^{(s)}(\cdot)$  funksiyaning  $z \in \mathbb{R} \setminus [-s\varepsilon + m; -s\varepsilon + M]$  to'plamda monoton kamayuvchi bo'lishi kelib chiqadi. Bundan tashqari,

$$\lim_{z \rightarrow -\infty} \Delta^{(s)}(z) = +\infty. \quad (2)$$

Faraz qilaylik,  $m \geq 0$  bo'lsin. U holda

$$\Delta^{(-)}(-\varepsilon + m) = -m - \alpha^2 \int_{\mathbb{T}^d} \frac{v^2(t)dt}{2\varepsilon + w(t) - m}.$$

Ko'rinib turibdiki, ixtiyoriy  $\alpha > 0$  soni uchun  $\Delta^{(-)}(-\varepsilon + m) \leq 0$  tengsizlik bajariladi.  $\Delta^{(-)}(\cdot)$  funksiyaning monotonlik xossasini va (2) tenglikni inobatga olsak, bu funksiya  $(-\infty; -\varepsilon + m)$  oraliqda yagona  $E_1$  nolga ega bo'lishi kelib chiqadi. U holda  $\mathcal{A}_1^{(-)}$  operatorning diskret spektri uchun keltirilgan (1) tenglikdan topilgan  $E_1$  soni  $\mathcal{A}_1^{(-)}$  operatorli matritsaning oddiy xos qiymati bo'ladi. 1-teorema hamda  $\sigma^{(-)}$  to'plamning aniqlanishiga ko'ra,

$$E_{\min} \leq E_1 + m < -\varepsilon + 2m$$

munosabatni hosil qilamiz. 2-teorema isbotlandi.

Faraz qilaylik,

$$\int_{\mathbb{T}^d} \frac{v^2(t)dt}{w(t) - m} < \infty \quad (3)$$

bo'lsin. (3) shart bajariladigan  $v(\cdot)$  va  $w(\cdot)$  parameter funksiyalarga misol keltiramiz. Agar  $d = 1$ ,  $v(k_1) = \sin(k_1/2)$  va  $w(k_1) = -\cos k_1$  bo'lsa, u holda

$$m = \min_{k_1 \in \mathbb{T}} (-\cos k_1) = -1$$

munosabat o'rinli bo'lib, (3) shart bajariladi, ya'ni

$$\int_{\mathbb{T}} \frac{\sin^2(t/2)dt}{-\cos t + 1} = \int_{\mathbb{T}} \frac{\sin^2(t/2)dt}{2\sin^2(t/2)} = \frac{1}{2} \int_{\mathbb{T}} dt = \pi < \infty.$$

(3) shart bajarilib,  $m < 0$  bo'lgan holda quyidagi miqdorlarni kiritamiz:

$$\alpha_1 := \sqrt{-m} \left( \int_{\mathbb{T}^d} \frac{v^2(t)dt}{2\varepsilon + w(t) - m} \right)^{-1/2}, \quad \alpha_2 := \sqrt{2\varepsilon - m} \left( \int_{\mathbb{T}^d} \frac{v^2(t)dt}{w(t) - m} \right)^{-1/2}.$$

**3-teorema.** Faraz qilaylik, (3) shart bajarilsin va  $m < 0$  bo'lsin.

- 1) Agar  $\alpha \leq \alpha_{\min} := \min\{\alpha_1; \alpha_2\}$  bo'lsa, u holda  $E_{\min} = -\varepsilon + 2m$  tenglik bajariladi.
- 2) Agar  $\alpha > \alpha_{\min}$  bo'lsa, u holda  $E_{\min} < -\varepsilon + 2m$  tengsizlik o'rinlidir.



**Isbot.** Faraz qilaylik  $m < 0$  bo'lsin. U holda  $\Delta^{(s)}(\cdot)$  funksiyaning hamda  $\alpha_1$  va  $\alpha_2$  sonlarining aniqlanishiga ko'ra

$$\begin{aligned}\Delta^{(-)}(-\varepsilon + m) < 0 &\Leftrightarrow \alpha > \alpha_1; \\ \Delta^{(+)}(-\varepsilon + m) < 0 &\Leftrightarrow \alpha > \alpha_2.\end{aligned}$$

1) Agar  $\alpha \leq \alpha_{\min}$  bo'lsa, u holda  $\alpha \leq \alpha_{\min} \leq \alpha_1$ , ya'ni  $\alpha \leq \alpha_1$  va  $\alpha \leq \alpha_{\min} \leq \alpha_2$ , ya'ni  $\alpha \leq \alpha_2$  munosabatlar bajariladi. Bunday shartlarni qanoatlantiruvchi  $\alpha$  sonlari uchun  $\Delta^{(s)}(-\varepsilon + m) \geq 0$ ,  $s = \pm$  tengsizliklar bajariladi.  $\Delta^{(s)}(\cdot)$  funksiyaning  $(-\infty; -\varepsilon + m)$  oraliqda monoton kamayuvchi ekanligidan va (2) tenglikdan bu funksiya  $(-\infty; -\varepsilon + m)$  oraliqda nolga ega emasligi kelib chiqadi. U holda  $\mathcal{A}_1^{(s)}$  operatorli matritsaning diskret spektri uchun keltirilgan (1) tenglikdan  $\mathcal{A}_1^{(s)}$  operatorli matritsaning  $(-\infty; -\varepsilon + m)$  oraliqda xos qiymati mavjud emasligini hosil qilamiz. Shu sababli  $\sigma^{(-)}$  va  $\sigma^{(+)}$  to'plamlarning aniqlanishiga ko'ra,  $E_{\min} = -\varepsilon + 2m$  tenglik o'rinli bo'ladi.

2) Faraz qilaylik  $\alpha > \alpha_{\min}$  bo'lsin. U holda  $\alpha > \alpha_1$  va  $\alpha > \alpha_2$  tengsizliklarning kamida bittasi bajariladi. Aniqlik uchun  $\alpha_1 < \alpha_2$  deb olamiz. Mazkur holda  $\alpha > \alpha_1$  tengsizlik bajariladi hamda bunday  $\alpha$  sonlari uchun  $\Delta^{(-)}(-\varepsilon + m) < 0$  munosabat o'rinli bo'ladi. Ushbu tengsizlikdan,  $\Delta^{(-)}(\cdot)$  funksiyaning  $(-\infty; -\varepsilon + m)$  oraliqda monoton kamayuvchi ekanligidan va (2) tenglikdan  $\Delta^{(-)}(\cdot)$  funksiya  $-\varepsilon + m$  sonidan chapda yagona  $E_1$  nolga ega ekanligi kelib chiqadi. (1) tenglikga ko'ra  $E_1$  soni  $\mathcal{A}_1^{(-)}$  operatorli matritsaning xos qiymati bo'ladi.  $\sigma^{(-)}$  va  $\sigma^{(+)}$  to'plamning aniqlanishiga ko'ra,  $E_{\min} \leq E_1 + m < -\varepsilon + 2m$  tengsizlik o'rinli bo'ladi. 3-teorema isbotlandi.

Faraz qilaylik,

$$\int_{\mathbb{T}^d} \frac{v^2(t)dt}{w(t) - m} = +\infty \quad (4)$$

bo'lsin. (4) shart bajariladigan  $v(\cdot)$  va  $w(\cdot)$  parameter funksiyalarga misol keltiramiz. Agar  $d = 1$ ,  $v(k_1) = 1$  va  $w(k_1) = \cos k_1$  bo'lsa, u holda

$$m = \min_{k_1 \in \mathbb{T}} (\cos k_1) = -1$$

bo'ladi hamda integralning qiymati quyidagicha aniqlanadi:

$$\int_{\mathbb{T}} \frac{dt}{\cos t + 1} = \int_{\mathbb{T}} \frac{dt}{2 \cos^2(t/2)} = \frac{1}{2} \int_{\mathbb{T}} \frac{dt}{\cos^2(t/2)} = \infty.$$

**4-teorema.** Agar (4) shart bajarilsa, u holda ixtiyoriy  $m$  va barcha  $\alpha > 0$  sonlari uchun  $E_{\min} < -\varepsilon + 2m$  tengsizlik o'rinli.

**Isbot.** Faraz qilaylik, (4) shart bajarilsin. U holda  $\Delta^{(+)}(-\varepsilon + m) = -\infty$  tenglik o'rinli bo'ladi.  $\Delta^{(+)}(\cdot)$  funksiyaning monotonlik xossasini va (3) tenglikni inobatga olsak, bu funksiya  $(-\infty; -\varepsilon + m)$  oraliqda yagona  $E_2$  nolga ega bo'ladi. U holda  $\mathcal{A}_1^{(+)}$  operatorli matritsaning diskret spektri uchun keltirilgan (1) tenglikdan  $E_2$  soni  $\mathcal{A}_1^{(+)}$  operatorli matritsaning oddiy xos qiymati bo'ladi. 1-teorema hamda  $\sigma^{(+)}$  to'plamning aniqlanishiga ko'ra,

$$E_{\min} \leq E_2 + m < -\varepsilon + 2m$$

munosabatni hosil qilamiz. 4-teorema isbotlandi.

Maqolaning keyingi qismida  $\mathcal{A}$  operatorli matritsa muhim spektrining yuqori chegarasi uchun baholashlar olamiz. Buning uchun quyidagi belgilashlarni kiritamiz:

$$E_{\max}^{(s)} := \max \sigma_{\text{ess}}(\mathcal{A}^{(s)}), \quad E_{\max} := \max \sigma_{\text{ess}}(\mathcal{A}) = \max\{E_{\max}^{(+)}; E_{\max}^{(-)}\}.$$

**5-teorema.** Agar  $M \leq 0$  bo'lsa, u holda ixtiyoriy  $\alpha > 0$  soni uchun  $E_{\max} > \varepsilon + 2M$  tengsizlik bajariladi.

Faraz qilaylik,

$$\int_{\mathbb{T}^d} \frac{v^2(t)dt}{M - w(t)} < \infty \quad (5)$$

bo'lsin. U holha  $M > 0$  bo'lgan holda quyidagi miqdorlarni kiritamiz:

$$\alpha_3 := \sqrt{M} \left( \int_{\mathbb{T}^d} \frac{v^2(t)dt}{2\varepsilon - w(t) + M} \right)^{-1/2}, \quad \alpha_4 := \sqrt{2\varepsilon + M} \left( \int_{\mathbb{T}^d} \frac{v^2(t)dt}{M - w(t)} \right)^{-1/2}.$$

**6-teorema.** Faraz qilaylik, (5) shart bajarilsin va  $M > 0$  bo'lsin.

- 1) Agar  $\alpha \leq \alpha'_{\min} := \min\{\alpha_3; \alpha_4\}$  bo'lsa, u holda  $E_{\max} = \varepsilon + 2M$  tenglik o'rinlidir.
- 2) Agar  $\alpha > \alpha'_{\min}$  bo'lsa, u holda  $E_{\max} > \varepsilon + 2M$  tengsizlik o'rinlidir.

**7-teorema.** Agar

$$\int_{\mathbb{T}^d} \frac{v^2(t)dt}{M - w(t)} = +\infty$$

shart bajarilsa, u holda ixtiyoriy  $\alpha > 0$  va barcha  $M$  sonlari uchun  $E_{\max} > \varepsilon + 2M$  munosabat o'rinli bo'ladi.

$\mathcal{A}$  operatorli matritsa muhim spektrining yuqori chegarasi uchun keltirilgan natijalar, uning quyi chegarasi uchun keltirilgan natijalar kabi isbotlanadi.

**Xulosa.** Mazkur maqolada fermionli Fok fazosida berilgan operatorli matritsa muhim spektrining quyi va yuqori chegarasi uchun baholashlar olingan. Dastlab operatorli matritsaning spektral xossalari o'rganish maqsadida diskret parametrli yordamchi operatorli matritsalar keltirilgan. Ularning Fredholm determinanti aniqlangan. Qaralayotgan operatorli matritsaning muhim va diskret spektrlari topilgan. Ta'sirlashish parametrlarining ba'zi tabiiy shartlarida muhim spektrining quyi va yuqori chegaralari uchun baholashlar keltirilgan.

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### РЕЗЮМЕ

В этой статье рассматривается линейная, ограниченная и самосопряженная операторная матрица третьего порядка в фермионное пространство Фока. Описано существенный спектр этой операторной матрицы. Получены оценки для нижней и верхней грани существенного спектра относительно значения параметра взаимодействия.

**Ключевые слова:** фермион, операторная матрица, определитель Фредгольма, собственное значение, существенный спектр, нижняя грань, верхняя грань, спектральная оценка.

### RESUME

In this paper, we consider the linear, bounded and self-adjoint operator matrix of order three in the fermionic Fock space. The essential spectrum of the operator matrix is described. The estimates for the lower and upper bounds of the essential spectrum with respect to the values of the coupling constant.

**Key words:** fermion, operator matrix, Fredholm determinant, eigenvalue, essential spectrum, lower bound, upper bound, spectral estimate.

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## SPECTRAL PROPERTIES OF THE ONE-PARTICLE SCHRÖDINGER OPERATOR PERTURBED WITH NON-LOCAL DELTA-POTENTIAL

Ismoilov G.I. \*

### RESUME

The paper considers a one-dimensional Schrödinger Hamiltonian interacted by two identical non-local Delta-function interactions situated symmetrically with respect to the origin at the points  $\pm x_0$ . The corresponding Schrödinger operator is constructed as a self-adjoint extension of the symmetric Laplace operator. An essential spectrum is described and the condition for the existence of the eigenvalue of the Schrödinger operator is studied. The main results of the work are based on the study of the operator extension spectrum of the operator  $\mathbf{h}_\lambda$ . Moreover, for a given parameter  $x_0$ , we find the asymptotics of the eigenvalues  $z_u(\lambda)$  and  $z_v(\lambda)$  as  $\lambda \searrow \lambda_u$  and  $\lambda \searrow \lambda_v$ , respectively.

**Key words:** Schrödinger operators, non-local Delta interactions, eigenvalues, eigenfunctions, eigenvalue asymptotics.

### Introduction

The issues surrounding the point interaction of two and three identical quantum particles interacting via point potentials (also known as contact or singular potentials) have been explored in various scientific studies. Berezina and Faddeeva [1], as well as R.A. Minlosa and L.D. Faddeev [2], [3], were the first to propose a rigorous mathematical framework for describing the point interaction of two and three particles, respectively.

In [2], [3], the Hamiltonian of the system was analyzed using the theory of self-adjoint extensions of symmetric operators, and was presented as a self-adjoint extension of the symmetric Laplace operator, defined on the domain of functions of three variables,  $x_1, x_2, x_3$ ;  $x_j \in \mathbb{R}$ ,  $j = 1, 2, 3$ , which vanishes whenever two arguments  $x_j = x_k$ ,  $j \neq k$ ,  $k = 1, 2, 3$  coincide.

This proposed extension is referred to as the Skornyakov-Ter-Martirosyan expansion. In [4], building on the results of [1], [2], the Hamiltonian of three particles (two fermions and one particle of a different type) with identical masses interacting via point potentials was studied. It was demonstrated that the Skornyakov-Ter-Martirosyan extensions are self-adjoint and semi-bounded.

In this work, we consider a particle moving in one-dimensional space, interacting with external fields through non-local Dirac delta potentials  $\delta'(x \pm x_0)$ , located at  $\pm x_0$  ( $x_0 \in \mathbb{R} \setminus \{0\}$ ), with equal strength  $\lambda \in \mathbb{R}$ . Formally, the Schrödinger operator for such a system can be written as:

$$\widehat{\mathbf{h}}_\lambda := \widehat{\mathbf{h}}_0 - \lambda(\delta'(x + x_0) + \delta'(x - x_0)), \quad (1)$$

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\* **G.I. Ismoilov** – Samarkand State University, University Boulevard 15, Samarkand 140104, Uzbekistan. golibjon.ismoilov.tdtu@gmail.com

where  $\widehat{\mathbf{h}}_0 := -\Delta$  is the Laplace operator and  $\delta'(x)$  is the first derivative of the Dirac delta function. However, the operator in equation (1) does not properly define an operator in the Hilbert space  $L^2(\mathbb{R})$ , because the derivatives of the delta function have singularities, making them ill-defined in this space.

Therefore, in order to give a meaningful definition to the expression (1), we must restrict the domain of the operator. This is done by constructing it as a symmetric self-adjoint extension of the Laplace operator. This process is necessary to ensure that the operator is mathematically well-defined and consistent. By using the theory of self-adjoint extensions, we can define the operator rigorously and study its physical implications in a more precise manner.

In the work [5], a formal non-local Schrödinger operator, as defined in (1), is studied using the renormalization method for the coupling constant  $\lambda$ . This approach leads to the construction of a self-adjoint extension of the operator. It is shown that this self-adjoint extension has two negative eigenvalues. The dependence of these eigenvalues on the coupling constant and separation distances is analyzed. Additionally, the resonances of the self-adjoint extension operator are also examined.

Note that the advantage of one-dimensional models with point perturbations is clear as they are useful for the study of a variety of qualitative properties. For instance, you can see [6, 7, 8, 9, 10, 11, 12] for one-body problems with local and non-local delta potential.

In the momentum representation of the Hamiltonian after reduction of the variables we establish the Skornyakov-Ter-Martirosyan extension  $\mathbf{h}_\lambda$  of the associated Hamiltonian. It is proved that the essential spectrum of the expansion under consideration coincides with the set of the nonnegative real numbers and the condition for the existence of the eigenvalue of the Schrödinger operator is studied. The main results of the work are based on the study of the expansion spectrum of the  $\mathbf{h}_\lambda$  operator. We describe the essential spectrum (cf. Theorem 2.) and explicitly derive (cf. Theorem 3.) the existence of eigenvalues of the operator and their dependence on the parameters  $\lambda$  and  $x_0 \in \mathbb{R}$ . Moreover, for a given parameter  $x_0$ , we find the asymptotics of the eigenvalues  $z_u(\lambda)$  and  $z_v(\lambda)$  as  $\lambda \searrow \lambda_u$  and  $\lambda \searrow \lambda_v$ , respectively.

## Preliminaries

Now, in order to give meaning to a Schrödinger operator with particle interactions involving non-local delta functions as defined in (1), we define it on the set of functions in the  $L^2(\mathbb{R})$  space that satisfy the condition  $\phi'(\pm x_0) = 0$ . Thus, the singular contributions arising from the action of the Laplace operator are canceled out by the delta functions in (1). It should be emphasized that any operator defined in this way is an extension of the symmetric  $\widehat{\mathbf{h}}_0$  operator, which is defined on the following manifold:

$$D(\widehat{\mathbf{h}}_0) = \{ \phi \in L^2(\mathbb{R}) : \Delta\phi \in L^2(\mathbb{R}), \phi'(\pm x_0) = 0, x_0 > 0 \}, \quad (2)$$

where the singular contributions related to the delta functions in (1) disappear.

After the action of the corresponding Fourier transform, the operator  $\widehat{\mathbf{h}}_0$  will go over to the operator

$$(\mathbf{h}_0 f)(p) = p^2 f(p)$$

defined on the set  $D(\mathbf{h}_0) \subset L^2(\mathbb{R})$  of functions  $f(p)$ , satisfying the following conditions:

$$\int_{\mathbb{R}} p^4 |f(p)|^2 dp < \infty, \quad \int_{\mathbb{R}} p e^{\pm i x_0 p} f(p) dp = 0. \quad (3)$$

According to [13] the deficiency subspace  $\mathfrak{R}_z$  of the operator  $\mathbf{h}_0$ , is determined by

$$\mathfrak{R}_z = \{ \Phi \in L^2(\mathbb{R}) : ((\mathbf{h}_0 - zI)f, \Phi) = 0, \quad f \in D(\mathbf{h}_0) \}.$$

**Lemma 1.** *For any  $z \in \Pi_0 = \mathbb{C} \setminus [0, \infty)$  the deficiency subspace  $\mathfrak{R}_z \subset L^2(\mathbb{R})$  of  $\mathbf{h}_0$  consists of functions of the form*

$$\Phi(p) = \frac{c_1 p e^{i x_0 p} + c_2 p e^{-i x_0 p}}{p^2 - \bar{z}}, \quad c_1, c_2 \in \mathbb{C}.$$

**Proof.** Let  $\Phi \in \mathfrak{R}_z$ . Then for any  $f \in D(\mathbf{h}_0)$  the relation

$$((\mathbf{h}_0 - zI)f, g) = \int_{\mathbb{R}} (p^2 - z) f(p) \overline{\Phi(p)} dp = 0$$

holds.

From the last relation and conditions (3), it follows that

$$(p^2 - \bar{z}) p e^{\pm i x_0 p} \Phi(p) = c, \quad c \in \mathbb{C}$$

or

$$\Phi(p) = \frac{c_1 p e^{i x_0 p} + c_2 p e^{-i x_0 p}}{p^2 - \bar{z}} \quad c_1, c_2 \in \mathbb{C}.$$

□

It follows from Lemma 1. that for any  $z \in \Pi_0$  the deficiency subspace  $\mathfrak{R}_z$  is two-dimensional. Therefore,  $\mathbf{h}_0$  is a symmetric operator with defective indices (2,2). Using the general extension theory [13], we find that the operator  $\mathbf{h}_0$  has a two-parameter family of self-adjoint extensions.

Since the operator  $\mathbf{h}_0$  is non-negative, as in [2, 3], we use the theory of extensions of semibounded operators. The deficiency subspace  $\mathfrak{R}_{-1}$  of the operator  $\mathbf{h}_0$  consists of functions of the form

$$\Phi_{-1}(p) = \frac{c_1 p e^{i x_0 p} + c_2 p e^{-i x_0 p}}{p^2 + 1}, \quad c = (c_1, c_2) \in \mathbb{C}^2.$$

Moreover, by the schemes in [2, 3, 4] the adjoint operator  $\mathbf{h}_0^*$  is described using the following lemma.

**Lemma 2.** *Domain of definition  $D(\mathbf{h}_0^*)$  of  $\mathbf{h}_0^*$  consists of functions of the form*

$$g(p) = f(p) + \frac{d_1 p e^{i x_0 p} + d_2 p e^{-i x_0 p}}{p^2 + 1} + \frac{c_1 p e^{i x_0 p} + c_2 p e^{-i x_0 p}}{(p^2 + 1)^2} \quad (4)$$

where  $f \in D(\mathbf{h}_0)$ ,  $c_1, c_2, d_1, d_2 \in \mathbb{C}$ . The operator  $\mathbf{h}_0^*$  acts on an function of the form (4) by the formula

$$\mathbf{h}_0^* g(p) = p^2 g(p) - d_1 p e^{i x_0 p} - d_2 p e^{-i x_0 p},$$

where  $d_1, d_2$ - constants taken from the decomposition (4) of the function  $g$ .

Now select the extensions of the operator  $\mathbf{h}_0^*$ . We define the set  $D(\mathbf{h}_\lambda), D(\mathbf{h}_0) \subset D(\mathbf{h}_\lambda) \subset D(\mathbf{h}_0^*)$ , as follows:

$$D(\mathbf{h}_\lambda) = \left\{ g(p) = f(p) + \lambda \frac{c_1 p e^{ix_0 p} + c_2 p e^{-ix_0 p}}{p^2 + 1} + \frac{c_1 p e^{ix_0 p} + c_2 p e^{-ix_0 p}}{(p^2 + 1)^2}, f \in D(\mathbf{h}_0) \right\}. \quad (5)$$

The restriction of the operator  $\mathbf{h}_0$  to the domain  $D(\mathbf{h}_\lambda)$  is denoted by  $\mathbf{h}_\lambda$  and it has the form

$$(\mathbf{h}_\lambda g)(p) = p^2 g(p) - \lambda (c_1 p e^{ix_0 p} + c_2 p e^{-ix_0 p}). \quad (6)$$

By definition of  $\mathbf{h}_\lambda$ , it is an extension of the operator  $\mathbf{h}_0$ .

**Theorem 1.** *The extension  $\mathbf{h}_\lambda$  is a self-adjoint operator.*

**Proof.** It is straightforward to check that for any  $g_1, g_2 \in D(\mathbf{h}_\lambda)$ , the following relation holds:

$$(\mathbf{h}_\lambda g_1, g_2) = (g_1, \mathbf{h}_\lambda g_2),$$

which shows that  $\mathbf{h}_\lambda$  is a symmetric operator. To prove that it is self-adjoint, it is sufficient to show that the defective indices of  $\mathbf{h}_\lambda$  are equal to  $(0, 0)$ .

Let  $\varphi \in \mathfrak{R}_{-1}(\mathbf{h}_0)$ . Then the function  $\varphi(p)$  has the form

$$\varphi(p) = \frac{b_1 p e^{ix_0 p} + b_2 p e^{-ix_0 p}}{p^2 + 1}, \quad b_1, b_2 \in \mathbb{C}.$$

For any  $g \in D(\mathbf{h}_\lambda)$ , the equality

$$((\mathbf{h}_\lambda + I)g, \varphi) = 0$$

holds. Considering (5), we obtain

$$((\mathbf{h}_\lambda + I)g, \varphi) = ((\mathbf{h}_0 + I)f, \varphi) + \int_{\mathbb{R}} \frac{(c_1 p e^{ix_0 p} + c_2 p e^{-ix_0 p}) \overline{(b_1 p e^{ix_0 p} + b_2 p e^{-ix_0 p})}}{(p^2 + 1)^2} dp.$$

From the relation

$$\int_{\mathbb{R}^3} (p^2 + 1) f(p) \overline{\psi(p)} dp = 0,$$

and choosing  $c_1 = b_1, c_2 = b_2$ , we get

$$\int_{\mathbb{R}} \frac{|b_1 p e^{ix_0 p} + b_2 p e^{-ix_0 p}|^2}{(p^2 + 1)^2} dp = 0,$$

which simplifies to

$$b_1 p e^{ix_0 p} + b_2 p e^{-ix_0 p} = 0,$$

and hence,  $b_1 = b_2 = 0$ . Thus,  $\psi(p) = 0$ , which shows that the defective indices of the operator  $\mathbf{h}_\lambda$  are  $(0, 0)$ .  $\square$

### Spectral properties of the operator $\mathbf{h}_\lambda$

The main results of the paper are the following theorems.

**Theorem 2.** *The essential spectrum of the operator  $\mathbf{h}_\lambda$  coincides with the semiaxis  $[0, \infty)$ .*

**Proof.** For each  $z \geq 0$  consider the sequence of cut-off layers:

$$G_n(z) = \left\{ p \in \mathbb{R} : \sqrt{z} + \frac{1}{n+1} < |p| < \sqrt{z} + \frac{1}{n} \right\}, \quad n = 1, 2, 3, \dots$$

Each layer  $G_n(z)$  is divided into half-layers of type

$$G_n^+(z) = \{p \in G_n(z) : p \geq 0\}, \quad G_n^-(z) = \{p \in G_n(z) : p < 0\}.$$

By construction, the size of these parts are equal, and  $\mu(G_n^+(z)) = \mu(G_n^-(z)) = \frac{1}{2}\mu(G_n(z))$ . Using a simple calculation, we get that the measure of the layer  $G_n(z)$  is equal to

$$V_n = \mu(G_n(z)) = \frac{2}{n(n+1)}.$$

Define a sequence of the test functions  $f_n^{(z)}$ ,  $n \in \mathbb{N}$ ,

$$f_n^{(z)}(p) = \begin{cases} \frac{p \cos(x_0 p)}{\sqrt{V_n}}, & \text{if } p \in G_n^+(z) \\ -\frac{p \cos(x_0 p)}{\sqrt{V_n}}, & \text{if } p \in G_n^-(z) \\ 0, & \text{if } p \in \mathbb{R} \setminus G_n(z). \end{cases}$$

It is easy to verify that  $f_n^{(z)} \in L^2(\mathbb{R})$ ,  $\|f_n^{(z)}\| = 1$ ,  $(f_n^{(z)}, f_m^{(z)}) = 0$  with  $n \neq m$  and  $f_n^{(z)} \in D(\mathbf{h}_0)$ , i.e.

$$\int_{\mathbb{R}} f_n^{(z)}(p) dp = 0, \quad n \in \mathbb{N}.$$

Moreover,

$$\begin{aligned} \|(\mathbf{h}_\lambda - zI)f_n^{(z)}\|^2 &= \int_{\mathbb{R}} |(p^2 - z)f_n^{(z)}(p)|^2 dp = \frac{1}{V_n} \int_{G_n(z)} |(p^2 - z)p \cos(x_0 p)|^2 dp \leq \\ &\leq \frac{1}{V_n} \int_{G_n(z)} |(p^2 - z)p|^2 dp = \frac{2}{V_n} \int_{\sqrt{z+\frac{1}{n+1}}}^{\sqrt{z+\frac{1}{n}}} (p^2 - z)^2 p^2 dp < \\ &< \frac{2}{V_n} \left(2\sqrt{z} + \frac{1}{n}\right)^2 \left(\sqrt{z} + \frac{1}{n}\right)^2 \frac{1}{n^2} \cdot \frac{1}{n(n+1)} = \frac{1}{n^2} \left(2\sqrt{z} + \frac{1}{n}\right)^2 \left(\sqrt{z} + \frac{1}{n}\right)^2. \end{aligned}$$

This shows that

$$\lim_{n \rightarrow \infty} \|(\mathbf{h}_\lambda - zI)f_n^{(z)}\| = 0.$$



This means that if  $z \geq 0$ , then  $z \in \sigma_{ess}(\mathbf{h}_\lambda)$  therefore  $[0; \infty) \subset \sigma_{ess}(\mathbf{h}_\lambda)$ . In order to show the reverse inclusion  $\sigma_{ess}(\mathbf{h}_\lambda) \subset [0; \infty)$ , we construct the resolvent operator of  $\mathbf{h}_\lambda$ .

Let  $\psi \in L^2(\mathbb{R})$ . Then,  $(\mathbf{h}_\lambda - zI)g = \psi$ . Moreover,

$$(p^2 - z)g(p) - \lambda(c_1 p e^{ix_0 p} + c_2 p e^{-ix_0 p}) = \psi(p)$$

or

$$g(p) = \frac{\psi(p)}{p^2 - z} + \lambda \frac{c_1 p e^{ix_0 p} + c_2 p e^{-ix_0 p}}{p^2 - z}. \quad (7)$$

Comparing (5) and (7) we obtain the equation for  $c_1$  and  $c_2$ :

$$f(p) + \lambda \frac{c_1 p e^{ix_0 p} + c_2 p e^{-ix_0 p}}{p^2 + 1} + \frac{c_1 p e^{ix_0 p} + c_2 p e^{-ix_0 p}}{(p^2 + 1)^2} = \frac{\psi(p)}{p^2 - z} + \lambda \frac{c_1 p e^{ix_0 p} + c_2 p e^{-ix_0 p}}{p^2 - z}$$

where  $f \in D(\mathbf{h}_0)$ . Multiplying by  $\varphi_\pm(p) = p e^{\pm ix_0 p}$  the following equations are obtained

$$\begin{aligned} f(p) p e^{ix_0 p} + \lambda \frac{c_1 p^2 e^{2ix_0 p} + c_2 p^2}{p^2 + 1} + \frac{c_1 p^2 e^{2ix_0 p} + c_2 p^2}{(p^2 + 1)^2} &= \frac{\psi(p) p e^{ix_0 p}}{p^2 - z} + \lambda \frac{c_1 p^2 e^{2ix_0 p} + c_2 p^2}{p^2 - z}, \\ f(p) p e^{-ix_0 p} + \lambda \frac{c_1 p^2 + c_2 p^2 e^{-2ix_0 p}}{p^2 + 1} + \frac{c_1 p^2 + c_2 p^2 e^{-2ix_0 p}}{(p^2 + 1)^2} &= \frac{\psi(p) p e^{-ix_0 p}}{p^2 - z} + \lambda \frac{c_1 p^2 + c_2 p^2 e^{-2ix_0 p}}{p^2 - z}. \end{aligned}$$

Integrating the last equalities over  $\mathbb{R}$  we arrive at a system equalized for determining  $c_1$  and  $c_2$ :

$$\begin{cases} a(z)c_1 + b(z)c_2 = \frac{2}{\pi} \int_{\mathbb{R}} \frac{s e^{ix_0 s} \psi(s)}{s^2 - z} ds, \\ b(z)c_1 + a(z)c_2 = \frac{2}{\pi} \int_{\mathbb{R}} \frac{s e^{-ix_0 s} \psi(s)}{s^2 - z} ds, \end{cases} \quad (8)$$

where

$$a(z) = 2\lambda(\sqrt{-z}e^{-2y\sqrt{-z}} - e^{-2y}) + (1 - 2y)e^{-2y}, \quad b(z) = 2\lambda(\sqrt{-z} - 1) + 1, \quad y = |x_0|. \quad (9)$$

Here the following elementary integrals are used

$$\int_{\mathbb{R}} \left( \frac{p^2 e^{ix_0 p}}{p^2 + 1} - \frac{p^2 e^{ix_0 p}}{p^2 - z} \right) dp = -\pi e^{-|x_0|} + \pi \sqrt{-z} e^{-|x_0|\sqrt{-z}}, \quad z < 0, \quad \int_{\mathbb{R}} \frac{p^2 e^{ix_0 p}}{(p^2 + 1)^2} dp = \frac{\pi}{2}(1 - |x_0|)e^{-|x_0|}.$$

Now we find  $c_1, c_2$  in system (8) and get a representation for the resolvent  $R_z(h_\lambda)$ :

$$(R_z \psi)(p) = \frac{\psi(p)}{p^2 - z} + \frac{4\lambda}{\pi(p^2 - z)} \left( \frac{p \cos(x_0 p)}{v(z)} \int_{\mathbb{R}} \frac{s \cos(x_0 s) \psi(s)}{s^2 - z} ds - \frac{p \sin(x_0 p)}{u(z)} \int_{\mathbb{R}} \frac{s \sin(x_0 s) \psi(s)}{s^2 - z} ds \right).$$

Here

$$\begin{aligned} u(z) := u(y, \lambda; z) &= 2\lambda \left( \sqrt{-z}(e^{-2y\sqrt{-z}} - 1) - e^{-2y} + 1 \right) + (1 - 2y)e^{-2y} - 1, \\ v(z) := v(y, \lambda; z) &= 2\lambda \left( \sqrt{-z}(e^{-2y\sqrt{-z}} + 1) - e^{-2y} - 1 \right) + (1 - 2y)e^{-2y} + 1. \end{aligned}$$

This, if  $z < 0$  then  $p^2 - z \neq 0$  and the resolvent of the operator  $\mathbf{h}_\lambda$  is an bounded operator. It follows directly from here that  $\sigma_{ess}(\mathbf{h}_\lambda) = [0; \infty)$ .  $\square$

Since  $\mathbf{h}_\lambda$  is a self-adjoint operator and its essential spectrum coincides with  $[0; \infty)$ , the eigenvalues of the operator  $\mathbf{h}_\lambda$  can lie in the interval  $(-\infty; 0)$ .

The number  $z, z < 0$  is an eigenvalue of the operator  $\mathbf{h}_\lambda$  if and only if the number  $z$  is the zeros of the function  $u(\cdot)$  or  $v(\cdot)$ .

$$\lambda_v(y) = \frac{1 + (1 - 2y)e^{-2y}}{2(1 + e^{-2y})}, \quad \lambda_u(y) = \frac{1 - (1 - 2y)e^{-2y}}{2(1 - e^{-2y})};$$

$$\forall y \in (0, +\infty) \quad 0 < \lambda_v(y) < \lambda_u(y).$$

**Lemma 2.1.**

- (a) If  $\mu \in (-\infty; 0) \cup (\lambda_u(y); +\infty)$ , then the function  $u(\cdot)$  has only simple zero in  $(-\infty, 0)$ .
- (b) If  $\mu \in [0; \lambda_u(y)]$ , then the function  $u(\cdot)$  has no zeros.
- (c) If  $\mu \in (-\infty; 0) \cup (\lambda_v(y); \infty)$ , then the function  $v(\cdot)$  has only simple zero in  $(-\infty, 0)$ .
- (d) If  $\mu \in [0; \lambda_v(y)]$ , then the function  $v(\cdot)$  has no zeros.

**Proof.** First we show the monotonicity of the functions  $u$  and  $v$ . Evaluating the derivatives

$$u'(z) = \lambda \frac{1 - (1 - 2y\sqrt{-z})e^{-2y\sqrt{-z}}}{\sqrt{-z}}, \quad v'(z) = -\lambda \frac{1 + e^{-2y\sqrt{-z}}(1 - 2y\sqrt{-z})}{\sqrt{-z}}.$$

One can see that  $\lambda > 0$ , then

$$u'(z) > 0, \quad v'(z) < 0 \tag{10}$$

and if  $\lambda < 0$ , then

$$u'(z) < 0, \quad v'(z) > 0, \tag{11}$$

i.e., if  $\lambda > 0$ , then the function  $u$  is monotonically increasing and the function  $v$  is monotonically decreasing, and if  $\lambda < 0$  then the function  $u$  is monotonously decreasing and the function  $v$  is monotonously increasing.

In order to look for zeros of the functions  $u$  and  $v$ , we establish the following limits:

$$\begin{aligned} u(-0) &= \lim_{z \nearrow 0} u(z) = 2\lambda(1 - e^{-2y}) + (1 - 2y)e^{-2y} - 1, \\ u(-\infty) &= \lim_{z \searrow -\infty} u(z) = \begin{cases} \text{sign}(\lambda) \cdot -\infty, & \text{if } \lambda \neq 0; \\ (1 - 2y)e^{-2y} - 1, & \text{if } \lambda = 0, \end{cases} \\ v(-0) &= \lim_{z \nearrow 0} v(z) = -2\lambda(e^{-2y} + 1) + (1 - 2y)e^{-2y} + 1, \\ v(-\infty) &= \lim_{z \searrow -\infty} v(z) = \begin{cases} \text{sign}(\lambda) \cdot +\infty & \text{if } \lambda \neq 0; \\ (1 - 2y)e^{-2y} + 1, & \text{if } \lambda = 0. \end{cases} \end{aligned}$$

Note that

$$\forall y \neq 0, \quad e^{2y} > 1 + 2y.$$

Since

$$u(-0) = 2(1 - e^{-2y})(\lambda - \lambda_u), \quad v(-0) = -2(1 + e^{-2y})(\lambda - \lambda_v).$$

Summarising the properties of the functions  $u$  and  $v$  we can arrange the tables 3 and 2, which complete the proof of the lemma.

$\mu \in$	$(-\infty, 0)$	$\{0\}$	$(0, \lambda_v(y))$	$\{\lambda_v(y)\}$	$(\lambda_v(y), \lambda_u(y))$	$\{\lambda_u(y)\}$	$(\lambda_u(y), +\infty)$
Monotonicity	$u \searrow$	const	$u \nearrow$	$u \nearrow$	$u \nearrow$	$u \nearrow$	$u \nearrow$
$u(-0)$	-	-	-	-	-	0	+
$u(-\infty)$	+	-	-	-	-	-	-
Number of zeros	1	0	0	0	0	0	1

Таблица 2: The number of zeros of the function  $u(\cdot)$  in the interval  $(-\infty, 0)$ .

$\mu \in$	$(-\infty, 0)$	$\{0\}$	$(0, \lambda_v(y))$	$\{\lambda_v(y)\}$	$(\lambda_v(y), \lambda_u(y))$	$\{\lambda_u(y)\}$	$(\lambda_u(y), +\infty)$
Monotonicity	$v \nearrow$	const	$v \searrow$	$v \searrow$	$v \searrow$	$v \searrow$	$v \searrow$
$v(-0)$	+	+	+	0	-	-	-
$v(-\infty)$	-	+	+	+	+	+	+
Number of zeros	1	0	0	0	1	1	1

Таблица 3: The number of zeros  $v(z)$  in the interval  $(-\infty, 0)$ .

□

### Theorem 3.

- (a) If  $\lambda \in (-\infty; 0) \cup (\lambda_u(y), +\infty)$ , the operator  $\mathbf{h}_\lambda$  has two negative eigenvalues  $z_u$  and  $z_v$  corresponding to the eigenfunctions (with accuracy up to a constant factor and have the form)

$$g_\lambda(p) = \lambda \frac{p \cos(x_0 p)}{p^2 - z_u} \quad \text{and} \quad f_\lambda(p) = \lambda \frac{p \sin(x_0 p)}{p^2 - z_v},$$

where  $z_u$  and  $z_v$  are zeros of the functions  $u(\cdot)$  and  $v(\cdot)$ , respectively.

- (b) If  $\mu \in (\lambda_v(y); \lambda_u(y)]$ , then the operator  $\mathbf{h}_\lambda$  has a unique simple eigenvalue with the eigenfunction

$$f_\lambda(p) = \lambda \frac{p \sin(x_0 p)}{p^2 - z_v}$$

- (c) If  $\lambda \in [0; \lambda_v(y)]$ , then the operator  $\mathbf{h}_\lambda$  has no eigenvalue.

Moreover, the eigenvalues  $z_u$  and  $z_v$ , have asymptotics of the form

$$\begin{aligned} [1] \quad z_u(y; \lambda) &= - \left( \sqrt{\frac{1 - e^{-2y}}{2y\lambda_u(y)}} (\lambda - \lambda_u(y))^{1/2} + \frac{1 - e^{-2y}}{4\lambda_u(y)} (\lambda - \lambda_u(y)) + O((\lambda - \lambda_u(y))^{3/2}) \right)^2, \\ [2] \quad z_v(y; \lambda) &= - \left( -\frac{1 + e^{-2y}}{2\lambda_v(y)} (\lambda - \lambda_v(y)) + \frac{1 + e^{-2y} + 2y\lambda_v}{\lambda_v^2} (\lambda - \lambda_v(y))^2 + O((\lambda - \lambda_v(y))^3) \right)^2 \end{aligned}$$

in  $\lambda \searrow \lambda_u$  and  $\lambda \searrow \lambda_v$ , respectively.

**Proof.** The proof of the existence of the eigenvalues of  $\mathbf{h}_\lambda$  follows from Lemma 3.

Now we prove that the eigenfunctions of the operator  $\mathbf{h}_\lambda$  have the form

$$g_1(p) = \frac{p \cos(x_0 p)}{p^2 - z_1(x_0)} \quad \text{and} \quad g_2(p) = \frac{p \sin(x_0 p)}{p^2 - z_2(x_0)}.$$

From equation  $(h_\lambda - zI)g(p) = 0$  we receive

$$g(p) = \lambda \frac{c_1 p e^{ix_0 p} + c_2 p e^{-ix_0 p}}{p^2 - z}. \quad (12)$$

Comparing (5) and (12) we take the equality

$$f(p) + \lambda \frac{c_1 p e^{ix_0 p} + c_2 p e^{-ix_0 p}}{p^2 + 1} + \frac{c_1 p e^{ix_0 p} + c_2 p e^{-ix_0 p}}{(p^2 + 1)^2} = \lambda \frac{c_1 p e^{ix_0 p} + c_2 p e^{-ix_0 p}}{p^2 - z}.$$

Then we have the following equation:

$$\begin{aligned} f(p) p e^{ix_0 p} + \lambda \frac{c_1 p^2 e^{2ix_0 p} + c_2 p^2}{p^2 + 1} + \frac{c_1 p^2 e^{2ix_0 p} + c_2 p^2}{(p^2 + 1)^2} &= \lambda \frac{c_1 p^2 e^{2ix_0 p} + c_2 p^2}{p^2 - z}, \\ f(p) p e^{-ix_0 p} + \lambda \frac{c_1 p^2 + c_2 p^2 e^{-2ix_0 p}}{p^2 + 1} + \frac{c_1 p^2 + c_2 p^2 e^{-2ix_0 p}}{(p^2 + 1)^2} &= \lambda \frac{c_1 p^2 + c_2 p^2 e^{-2ix_0 p}}{p^2 - z}. \end{aligned}$$

Integrating the last equalities over  $\mathbb{R}$  we obtain a system of equations for determining  $c_1$  and  $c_2$ ,

$$\begin{cases} a(z)c_1 + b(z)c_2 = 0, \\ b(z)c_1 + a(z)c_2 = 0, \end{cases}$$

where

$$a(z) := a(y, \lambda; z) = 2\lambda(\sqrt{-z}e^{-2y\sqrt{-z}} - e^{-2y}) + (1-2y)e^{-2y}, \quad b(z) := b(\lambda; z) = 2\lambda(\sqrt{-z}-1)+1, \quad y = |x_0|.$$

Hence

$$(a^2(z) - b^2(z)) c_i = 0, \quad i = 1, 2.$$

If  $a(z) = b(z)$  (resp.  $a(z) = -b(z)$ ), then as  $c_i$  we can take any number, in particular,  $c_i = 1$ .

Thus, if the number  $z$  satisfies the equation  $u(z) = a(z) - b(z) = 0$  (resp.  $v(z) = a(z) + b(z) = 0$ ), then  $z$  is an eigenvalue of the operator  $h_\mu$  and of the functions

$$g_\lambda(p) = \lambda \frac{p \cos(x_0 p)}{p^2 - z} \quad \left( \text{resp.} \quad g_\lambda(p) = \lambda \frac{p \sin(x_0 p)}{p^2 - z} \right)$$

corresponding eigenfunctions of the operator  $\mathbf{h}_\lambda$ .

1) Asymptotics for  $z_u$ :

To understand the asymptotics of  $z_u$ , we start with the equation for the eigenvalue  $u(\lambda, z)$ :

$$u(\lambda, z) = 2\lambda \left( \sqrt{-z} e^{-2y\sqrt{-z}} - \sqrt{-z} - e^{-2y} + 1 \right) + (1-2y)e^{-2y} - 1 = 0.$$

Let's introduce substitutions:

$$\sqrt{-z} = \xi, \quad \lambda - \lambda_u = \eta.$$

Using the expansion for  $e^{-2y\xi}$ :

$$e^{-2y\xi} = 1 - 2y\xi + 2y^2\xi^2 - \frac{4}{3}y^3\xi^3 + O(\xi^4),$$

we substitute this expansion into the equation  $u(\eta, \xi)$ :

$$u(\eta, \xi) = (\eta + \lambda_u) \left( -2y\xi^2 + 2y^2\xi^3 - \frac{4}{3}y^3\xi^4 + O(y^4) \right) + (1 - e^{-2y})\eta = 0.$$

Now, let's make the substitutions  $\eta = \theta^2$  and  $\xi = \theta(c_0 + w)$ , where  $w = w(\theta)$ , to get:

$$u(\theta, w) = -2y\theta^2(c_0 + w)^2 + 2y^2\theta^3(c_0 + w)^3 - \frac{4}{3}y^3\theta^4(c_0 + w)^4 + O(\theta^4).$$

Next, we compute the derivatives of  $u(\theta, w)$ :

$$\begin{aligned} u'_\theta(\theta, w) &= -4y\theta(c_0 + w)^2 + 6y^2\theta^2(c_0 + w)^3 + O(\theta^3), \\ u'_w(\theta, w) &= -4y\theta^2(c_0 + w) + 6y^2\theta^3(c_0 + w)^2 + O(\theta^4). \end{aligned} \quad (13)$$

At the point  $(0, 0)$ , we obtain:

$$u'_\theta(0, 0) = 2\lambda_u y^2 c_0^3, \quad u'_w(0, 0) = -4\lambda_u y c_0.$$

Thus, by (13) the derivative  $w'(0)$  is:

$$w'(0) = \frac{y c_0^2}{2}.$$

Therefore, the asymptotic behavior for  $w(\theta)$  is:

$$w(\theta) = \frac{y c_0^2}{2} \theta + O(\theta^2), \quad \theta \searrow 0.$$

Hence, we have the asymptotics for  $\sqrt{-z(\eta)}$ :

$$\sqrt{-z(\eta)} = c_0 \sqrt{\eta} + \frac{y c_0^2}{2} \eta + O(\eta^{3/2}), \quad \eta \searrow 0.$$

Thus, the asymptotic behavior of  $z(\lambda)$  as  $\lambda \searrow \lambda_u$  is:

$$z(\lambda) = - \left( -c_0(\lambda - \lambda_u)^{1/2} + \frac{y c_0^2}{2}(\lambda - \lambda_u) + O((\lambda - \lambda_u)^{3/2}) \right)^2.$$

## 2) Asymptotics for $z_v$ :

Next, we consider the asymptotics of  $z_v$ , starting with the equation for the eigenvalue  $v(\lambda, z)$ :

$$v(\lambda, z) = 2\lambda \left( \sqrt{-z} e^{-2y\sqrt{-z}} + \sqrt{-z} - e^{-2y} - 1 \right) + (1 - 2y)e^{-2y} + 1 = 0.$$

Using similar substitutions:

$$\sqrt{-z} = \xi, \quad \lambda - \lambda_v = \eta,$$

and expanding  $e^{-2y\xi}$  as in the previous case, we write:

$$v(\eta, \xi) = (\eta + \lambda_v) \left( 2\xi - 2y\xi^2 + 2y^2\xi^3 - \frac{4}{3}y^3\xi^4 + O(y^4) \right) - (1 + e^{-2y})\eta = 0.$$

Now, we compute the first and second-order partial derivatives at the point  $(0, 0)$ :

$$v'_\eta(\eta, \xi) = 2\xi - 2y\xi^2 + 2y^2\xi^3 - \frac{4}{3}y^3\xi^4 + O(\xi^5) - (1 + e^{-2y}),$$

$$v'_\xi(\eta, \xi) = \eta \left( 2 - 4y\xi + 6y^2\xi^2 - \frac{16}{3}y^3\xi^3 + O(\xi^4) \right) + \lambda_v (2 - 4y\xi + 6y^2\xi^2 + O(y^4)),$$

$$v'_\eta(0, 0) = -(1 + e^{-2y}), \quad v'_\xi(0, 0) = 2\lambda_v, \quad v''_{\eta\eta}(0, 0) = 0, \quad v''_{\eta\xi}(0, 0) = 2, \quad v''_{\xi\xi}(0, 0) = -4y\lambda_v. \quad (14)$$

Using (14), we get

$$\xi'(0) = \frac{v_\eta(0, 0)}{v_\xi(0, 0)} = \frac{1 + e^{-2y}}{2\lambda_v},$$

$$\xi''(0) = -\frac{v_\eta^2 v_{\eta\eta} - 2v_{\eta\xi} v_\eta v_\xi + v_{\xi\xi} v_\eta^2}{v_\xi^3} = -\frac{2 - y(1 + e^{-2y})}{2\lambda_v^2} (1 + e^{-2y}).$$

Thus, the asymptotic expansion for  $\xi(\eta)$  is:

$$\xi(\eta) = \frac{1 + e^{-2y}}{2\lambda_v} \eta - \frac{2 - y(1 + e^{-2y})}{4\lambda_v^2} (1 + e^{-2y}) \eta^2 + O(\eta^3), \quad \eta \searrow 0.$$

Finally, the asymptotic behavior of  $z(\lambda)$  as  $\lambda \rightarrow \lambda_v$  is:

$$z(\lambda) = - \left( \frac{1 + e^{-2y}}{2\lambda_v} (\lambda - \lambda_v) - \frac{2 - y(1 + e^{-2y})}{4\lambda_v^2} (1 + e^{-2y}) (\lambda - \lambda_v)^2 + O((\lambda - \lambda_v)^3) \right)^2.$$

□

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## REZYUME

Ushbu ishda biz bir o'lchamli Shrödinger Hamiltonianni ko'rib chiqamiz, u ikkita bir xil lokal bo'lmagan Delta-funktsiyali o'zaro ta'sirga ega bo'lib, ular koordinatalar boshiga nisbatan simmetrik tarzda  $\pm x_0$  nuqtalarida joylashgan. Mos ravishda Shrödinger operatori simmetrik Laplas operatorining o'z-o'ziga qo'shma kengaytmasi sifatida qurilgan. Shrödinger operatorining muhim spektri tafsiflangan va xos qiymatlari mavjudligi uchun shartlar o'rganilgan. Ishning asosiy natijalari kengaytma  $\mathbf{h}_\lambda$  operatorning spektrini o'rganishga asoslangan. Bundan tashqari, berilgan parametr  $x_0$  uchun  $z_u(\lambda)$  va  $z_v(\lambda)$  xos qiymatlarining uchun mos ravishda  $\lambda \searrow \lambda_u$  va  $\lambda \searrow \lambda_v$  holatlarda asimptotiklar topilgan.

**Kalit so'zlar:** Shrödinger operatorlari, lokal bo'lmagan Delta o'zaro ta'sirlar, xos qiymatlar, xos funksiyalar, xos qiymat asimptotikalari.

## РЕЗЮМЕ

В работе рассматривается одномерный Гамильтониан Шрёдингера, взаимодействующий с двумя идентичными нелокальными дельта-функциональными взаимодействиями, расположенными симметрично относительно начала координат в точках  $\pm x_0$ . Соответствующий оператор Шрёдингера строится как самосопряжённое расширение симметричного оператора Лапласа. Характеризован существенный спектр, а также исследовано условие существования собственных значений оператора Шрёдингера. Основные результаты работы основаны на изучении спектра расширений оператора  $\mathbf{h}_\lambda$ . Кроме того, для заданного параметра  $x_0$  найдены асимптотики собственных значений  $z_u(\lambda)$  и  $z_v(\lambda)$  при  $\lambda \searrow \lambda_u$  и  $\lambda \searrow \lambda_v$ , соответственно.

**Ключевые слова:** операторы Шрёдингера, нелокальные дельта-взаимодействия, собственные значения, собственные функции, асимптотики собственных значений.



UDC 517.55

 $(m, \psi, \delta)$  – REGULARITY OF COMPACTS IN  $\mathbb{C}^n$ 

Kuldoshev K. K., NarzillaeV N. Kh. \*

## RESUME

It is known that, the  $m$ -subharmonic measure  $\omega^*(z, E, D)$  of a set  $E \subset D$ , related to a domain  $D \subset \mathbb{C}^n$ , is defined by  $m$ -subharmonic functions in  $D$ . This work is devoted to properties of a weighted  $m$ -subharmonic measure  $\omega^*(z, E, D, m, \psi, \delta)$ , in particular,  $(m, \psi, \delta)$ -regularity of a compact set  $K \subset D \subset \mathbb{C}^n$ . We generalize the  $m$ -subharmonic measure and prove that several theorems established in [1] regarding the regularity of the compact set  $K$  also hold in the generalized case.

**Key words:**  $m$ -subharmonic function,  $m$ -subharmonic measure,  $m$ -polar set, globally  $m$ -regular compact, locally  $m$ -regular compact.

**Introduction.** Plurisubharmonic measure and Green function are fundamental concepts in the theory of plurisubharmonic functions. Their applications have provided solutions to many problems in multidimensional complex analysis in a series of fundamental works by A. Sadullaev [4], [5], [6], E. Bedford, A. Taylor [7], J. Siciak [8], V. P. Zaharjuta [9] and others. Further, weighted Green functions and delta-extremal functions, i.e. generalized Green functions are studied in works [10], [11], [12], [13].

One of the important part of the potential theory is theory of  $m$ -subharmonic ( $sh_m$ ) functions. It expands and develops the pluripotential theory, which is the main subject for studying analytic functions of several complex variables and plurisubharmonic functions.

The  $sh_m$  functions are defined by the operators

$$(dd^c u)^k \wedge \beta^{n-k}, \quad 1 \leq k \leq n, \quad (1)$$

where  $d = \partial + \bar{\partial}$ ,  $d^c = \frac{\partial - \bar{\partial}}{4i}$  and  $\beta = dd^c |z|^2 = \frac{i}{2} \sum_{i=1}^n dz_i \wedge d\bar{z}_i$  is standard canonical (1,1) form in  $\mathbb{C}^n$ . Then  $dV_n = \frac{1}{n!} \beta^n$  is volume form in  $\mathbb{C}^n$ . The operator (1) gives the Laplace operator for  $k = 1$  and the Monge-Ampere operator for  $k = n$ . The operator (1) is called the complex Hessians operator, as it can be shown that  $(dd^c u)^k \wedge \beta^{n-k} = k!(n-k)!H_k(u) \beta^n$ , where  $u \in C^2(D)$  and  $H_k(u) = \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} \lambda_{j_1} \cdot \lambda_{j_2} \cdot \dots \cdot \lambda_{j_k}$  – is the Hessian of dimension  $k$  of the eigenvalue vector  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  of the matrix  $(u_{j,\bar{l}})$ ,  $u_{j,\bar{l}} = \frac{\partial^2 u}{\partial z_j \partial \bar{z}_l}$ ,  $j, l = 1, 2, \dots, n$ . The theory related to the class of  $sh_m$  functions was constructed in the works [2], [3], [4]. In their studies, the  $sh_m$  functions in the class of integrable functions  $L_{loc}^1(D)$  were defined as following.

**Definition 1.** Let  $u \in C^2(D)$ , where  $D \subset \mathbb{C}^n$ , is called  $m$ -subharmonic ( $1 \leq m \leq n$ ) at the point  $z^0 \in D$ , if the eigenvalues  $\lambda(u) = (\lambda_1(u), \lambda_2(u), \dots, \lambda_n(u))$  of the matrix  $(u_{j,\bar{k}})|_{z=z^0}$  belong to  $\Gamma_{n-m+1} = \{\lambda : H_1(\lambda) \geq 0, H_2(\lambda) \geq 0, \dots, H_{n-m+1}(\lambda) \geq 0\}$ . A function  $u \in C^2(D)$  is called  $m$ -subharmonic in  $D$  if it is  $m$ -subharmonic at every point of  $z^0 \in D$ .

In other words, a function  $u \in C^2(D)$  is called  $m$ -subharmonic if the conditions  $(dd^c u)^k \wedge \beta^{n-k} \geq 0$ ,  $\forall k = 1, 2, \dots, n - m + 1$  holds.

\*Kuldoshev K. K., NarzillaeV N. Kh. – National University of Uzbekistan, qobiljonmath@gmail.com, narzillaeV.nurbek@mail.ru

It is known that for all twice differentiable  $m$ -subharmonic functions  $u, v_1, v_2, \dots, v_{n-m}$  it is true

$$dd^c u \wedge dd^c v_1 \wedge dd^c v_2 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \geq 0. \quad (2)$$

Moreover, if a twice differentiable function  $u$  satisfies (2) for all twice differentiable  $m$ -subharmonic functions  $v_1, v_2, \dots, v_{n-m}$  then  $u$  is necessarily  $m$ -subharmonic function. Using this, we can define  $m$ -subharmonic functions in the class of the upper semicontinuous functions.

**Definition 2.** A function  $u$  is called  $m$ -subharmonic in the domain  $D \subset \mathbb{C}^n$ , if it is upper semicontinuous and for any twice differentiable  $m$ -subharmonic functions  $v_1, v_2, \dots, v_{n-m}$  the current  $dd^c u \wedge dd^c v_1 \wedge dd^c v_2 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1}$  defined as

$$\begin{aligned} & [dd^c u \wedge dd^c v_1 \wedge dd^c v_2 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1}] (\omega) = \\ & = \int u \wedge dd^c v_1 \wedge dd^c v_2 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega, \quad \omega \in F^{0,0} \text{ is positive, i.e.} \\ & \int u \wedge dd^c v_1 \wedge dd^c v_2 \wedge \dots \wedge dd^c v_{n-m} \wedge \beta^{m-1} \wedge dd^c \omega \geq 0, \quad \forall \omega \geq 0. \end{aligned}$$

Class of  $m$ -subharmonic functions we denote as  $sh_m(D)$ . It is clear, that

$$psh = sh_1 \subset sh_2 \subset sh_m \subset \dots \subset sh_n = sh. \quad (3)$$

(3) follows, that if  $u(z) \in sh_m(D)$ ,  $u(z) \not\equiv -\infty$ , then  $u(z) \in L^1_{loc}(D)$ .

**Definition 3.** A set  $E \subset D$  is called  $m$ -polar in  $D \subset \mathbb{C}^n$  if there exist a function  $u(z) \in sh_m(D)$ ,  $u(z) \not\equiv -\infty$ , such that  $u|_E = -\infty$ .

(3) follows also, that a  $m$ -polar set is polar in the sense of the classical potential theory, so that for  $m$ -polar set  $E \subset D$  the Hausdorff measure  $H_{2n-2+0}(E) = 0$ .

**Definition 4.** A domain  $D \subset \mathbb{C}^n$  is called  $m$ -regular if there exists a  $m$ -subharmonic function  $\rho(z)$  in  $D$  such that  $\rho|_D < 0$ ,  $\lim_{z \rightarrow D} \rho(z) = 0$ , i.e.  $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ .

The  $m$ -subharmonic measure is defined as an extremal function in the class of  $m$ -subharmonic ( $sh_m$ ) functions. Let  $E \subset D$  be some subset of the domain  $D \subset \mathbb{C}^n$ . For the sake of simplicity, we assume that  $D$  is a bounded and  $m$ -regular domain, We denote by  $\mathcal{U}(E, D)$  the class of all functions  $u \in sh_m(D)$ , such that  $u|_E \leq -1$ ,  $u|_D < 0$  and let

$$\omega(z, E, D) = \sup \{u(z) : u(z) \in \mathcal{U}(E, D)\}.$$

**Definition 5.** The regularization

$$\omega^*(z, E, D) = \overline{\lim}_{w \rightarrow z} \omega(w, E, D) = \lim_{\varepsilon \rightarrow 0} \sup_{w \in B(z, \varepsilon)} \omega(w, E, D)$$

is called the  $m$ -subharmonic measure ( $\mathcal{P}_m$ -measure) of  $E$  with respect to  $D$  (see [2], [4]).

Let  $D \subset \mathbb{C}^n$  be a domain and  $K \subset D$  a compact.

**Definition 6.** A point  $z^0 \in K$  is said to be globally  $m$ -regular if  $\omega^*(z^0, K, D) = -1$ . It is said to be locally  $m$ -regular if for any neighborhood  $B, z^0 \in B \subset \mathbb{C}^n$ , the intersection  $K \cap \bar{B}$  is globally  $m$ -regular at the point  $z^0$ , i.e.  $\omega^*(z^0, K \cap \bar{B}, D) = -1$ . If all points of a compact set  $K$  are globally (or locally)  $m$ -regular, then the compact set  $K$  is called a globally (or locally)  $m$ -regular compact. (see [2], [4]).

The following Hartogs and Choquet's lemmas are used repeatedly below.

**Lemma 1.** (Hartogs', see [4]). Suppose that  $g(z)$  is a continuous real valued function in a domain  $D \subset \mathbb{C}^n$  and  $u_j(z)$ ,  $j \in \mathbb{N}$ , is a sequence of locally uniformly upper bounded subharmonic functions such that

$$\overline{\lim}_{j \rightarrow \infty} u_j(z) \leq g(z)$$

at each point  $z \in D$ . Then for any compact set  $K \subset D$ , for any  $\varepsilon > 0$  there exists an integer  $j_0$  such that  $u_j(z) \leq g(z) + \varepsilon$ , for each  $z \in K$  and each  $j > j_0$ .

**Lemma 2.** (Choquet technical lemma). For any family  $\{u_\alpha(x)\}$ ,  $\alpha \in \Lambda$ , of functions of  $x \in D \subset \mathbb{R}^n$ , there is a countable set  $\Lambda_0 \subset \Lambda$  such that if we denote  $u(x) = \sup_{\alpha \in \Lambda} u_\alpha(x)$  and  $v(x) = \sup_{\alpha \in \Lambda_0} u_\alpha(x)$ , then  $\{x \in D : u(x) < u^*(x)\} \subset \{x \in D : v(x) < v^*(x)\}$  and  $u^*(x) \equiv v^*(x)$ , where  $u^*$ ,  $v^*$ —are regularizations.

Let  $u_\alpha$  be a family of upper semicontinuous functions in  $D \subset \mathbb{C}^n$  which is locally uniformly bounded from above. Then the upper envelope  $u(z) = \sup_{\alpha} u_\alpha(z)$  is not always upper semicontinuous. But if we consider the upper semicontinuous regularization  $u^*(z) = \lim_{\varepsilon \rightarrow 0} \sup_{w \in B(z, \varepsilon)} u(w)$ , where  $B(z, \varepsilon) \subset D$  is a ball, then  $u^*$  is upper semicontinuous and holds

**Theorem 1.** (see [4]) Let  $\{u_\alpha(z)\}$ ,  $\alpha \in \Lambda$ , be an arbitrary locally uniformly upper bounded family of  $m$ -subharmonic functions in the domain  $D \subset \mathbb{C}^n$  and  $u(z) = \sup_{\alpha} \{u_\alpha(z)\}$ . Then the regularization  $u^*(z)$  of  $u(z)$  is a  $m$ -subharmonic function in  $D$ .

**1.  $(m, \psi, \delta)$ - subharmonic measure and its properties.** Let  $D \subset \mathbb{C}^n$  be a  $m$ -regular domain,  $E \subset D$  be any fixed set and  $\psi(z)$  be bounded function in  $E$ . We denote by  $\mathcal{U}(E, D, \psi, \delta)$  the class of all functions  $u(z) \in sh_m(D)$ , such that  $u|_E \leq \psi|_E$ ,  $u|_D < \delta$ , where  $\delta \in \mathbb{R}$ . Using this family of functions, we define the function

$$\omega(z, E, D, \psi, \delta) = \sup \{u(z) : u(z) \in \mathcal{U}(E, D, \psi, \delta)\}.$$

**Definition 1.1.** The function

$$\omega^*(z, E, D, \psi, \delta) = \overline{\lim}_{w \rightarrow z} \omega(w, E, D, \psi, \delta)$$

is called a  $(m, \psi, \delta)$ -subharmonic measure ( $(m, \psi, \delta)$ -measure) of the set  $E$  with respect to  $D$ .

Note that  $\omega^*(z, E, D, -1, 0)$ ,  $\psi \equiv -1$ ,  $\delta = 0$ , coincides with the  $m$ -measure of the potential theory in the class of  $u(z) \in sh_m(D)$ , i.e.  $\omega^*(z, E, D, -1, 0) = \omega^*(z, E, D)$ . The weighted  $(m, \psi)$ -measure  $\omega^*(z, E, D, \psi, 0)$ , case  $\delta = 0$ , was considered in our previous work [15].

As can be seen from the definition 1.1., the inequality  $\inf_{z \in E} \psi(z) \leq \omega^*(z, E, D, \psi, \delta) \leq \delta$  holds for all  $z \in D$ . By Theorem 0.1., the function  $\omega^*(z, E, D, \psi, \delta)$  is  $m$ -subharmonic in  $D$ . If  $\delta \leq \inf_{z \in E} \psi(z)$ , then  $\omega^*(z, E, D, \psi, \delta) = \delta$ ,  $\forall z \in D$ . Therefore, in this paper, we will consider the special case where  $\delta > \sup_{z \in E} \psi(z)$  is satisfied.

In this context, we present several characteristics of the  $(m, \psi, \delta)$ -subharmonic measure;

**Proposition 1.1.** a) let  $E_1 \subset E_2 \subset D_1 \subset D_2$ . Then

- $\omega^*(z, E_2, D_2, \psi, \delta) \leq \omega^*(z, E_1, D_2, \psi, \delta) \leq \omega^*(z, E_1, D_1, \psi, \delta)$  for all  $z \in D_1$ .
- b) let  $\psi_1|_E \leq \psi_2|_E$ . Then  $\omega^*(z, E, D, \psi_1, \delta) \leq \omega^*(z, E, D, \psi_2, \delta)$  for all  $z \in D$ .
- c) let  $\sup_{z \in E} \psi(z) < \delta_1 \leq \delta_2, \delta_1, \delta_2 \in \mathbb{R}$ . Then  $\omega^*(z, E, D, \psi, \delta_1) \leq \omega^*(z, E, D, \psi, \delta_2), \forall z \in D$ .
- d) let  $c \geq 1$ . Then  $\omega^*(z, E, D, \frac{\psi}{c}, \delta) = \frac{1}{c} \omega^*(z, E, D, \psi, c\delta), \forall z \in D$ .
- e) let  $c \leq 0$ . Then

$$\omega^*(z, E, D, \psi + c, \delta) = c + \omega^*(z, E, D, \psi, \delta - c), \forall z \in D.$$

The proofs of Proposition 1.1. follow easily from the definition of the  $(m, \psi, \delta)$ -subharmonic measure.

**Proposition 1.2** (On two constants Theorem). If the function  $u(z)$  is  $m$ -subharmonic in the domain  $D \subset \mathbb{C}^n$  and  $u|_D < C, u|_E \leq c$ , where  $E \subset D, c < C$  then the inequality

$$u(z) \leq C \cdot \left( 1 + \frac{\omega^*(z, E, D, \psi, \delta) - \delta}{\delta - \inf_{z \in E} \psi(z)} \right) - c \cdot \frac{\omega^*(z, E, D, \psi, \delta) - \delta}{\delta - \inf_{z \in E} \psi(z)}$$

holds for all  $z \in D$ .

The proof of the proposition 1.2. follows easily from the relation

$$\frac{u(z) - C}{C - c} \left( \delta - \inf_{z \in E} \psi(z) \right) + \delta \in \mathcal{U}(E, D, \psi, \delta).$$

**Proposition 1.3.** The inequality

$$\left( \delta - \inf_{z \in E} \psi(z) \right) \cdot \omega^*(z, E, D) + \delta \leq \omega^*(z, E, D, \psi, \delta) \leq \left( \delta - \sup_{z \in E} \psi(z) \right) \cdot \omega^*(z, E, D) + \delta$$

holds for any set  $E \subset D$  and for all  $z \in D$ .

It follows from proposition 1.3 that the measure  $\omega^*(z, E, D, \psi, \delta)$  is either nowhere  $\delta$  or identically  $\delta$ . The latter holds if and only if  $E$  is  $m$ -polar in  $D$ .

*Proof of proposition 1.3.* Take an arbitrary function  $u(z) \in \mathcal{U}(E, D)$  i.e.  $u(z)|_E \leq -1, u(z)|_D < 0$ . Then  $\left( \delta - \inf_{z \in E} \psi(z) \right) \cdot u(z) + \delta \in sh_m(D)$ . From  $\psi(z)|_E < \delta$ , it follows  $\delta - \inf_{z \in E} \psi(z) > 0$ . Note that

$$\left( \left( \delta - \inf_{z \in E} \psi(z) \right) \cdot u(z) + \delta \right) \Big|_D \leq \delta, \left( \left( \delta - \inf_{z \in E} \psi(z) \right) \cdot u(z) + \delta \right) \Big|_E \leq \inf_{z \in E} \psi(z) - \delta + \delta \leq \psi|_E.$$

Consequently  $\left( \delta - \inf_{z \in E} \psi(z) \right) \cdot u(z) + \delta \in \mathcal{U}(E, D, \psi, \delta)$  and  $\left( \delta - \inf_{z \in E} \psi(z) \right) \cdot u(z) + \delta \leq \omega^*(z, E, D, \psi, \delta)$ .

As the function  $u$  is arbitrary, we get the inequality

$$\left( \delta - \inf_{z \in E} \psi(z) \right) \cdot \omega^*(z, E, D) + \delta \leq \omega^*(z, E, D, \psi, \delta) \text{ for all } z \in D.$$

Now we show that the inequality  $\omega^*(z, E, D, \psi, \delta) \leq \left( \delta - \sup_{z \in E} \psi(z) \right) \cdot \omega^*(z, E, D) + \delta$  holds. Take any function  $u(z) \in \mathcal{U}(E, D, \psi, \delta)$  and consider the function  $\frac{u(z)-\delta}{\delta - \sup_{z \in E} \psi(z)}$ . It can be easily verify that the function  $\frac{u(z)-\delta}{\delta - \sup_{z \in E} \psi(z)}$  is  $m$ -subharmonic in  $D$  and satisfies the following conditions:

$$\frac{u(z)-\delta}{\delta - \sup_{z \in E} \psi(z)} \Big|_D < 0 \text{ and } \frac{u(z)-\delta}{\delta - \sup_{z \in E} \psi(z)} \Big|_E < -1.$$

Thus  $\frac{u(z)-\delta}{\delta - \sup_{z \in E} \psi(z)} \in \mathcal{U}(E, D)$  and  $\frac{u(z)-\delta}{\delta - \sup_{z \in E} \psi(z)} \leq \omega^*(z, E, D)$ . Therefore the inequality  $\omega^*(z, E, D, \psi, \delta) \leq \left( \delta - \sup_{z \in E} \psi(z) \right) \cdot \omega^*(z, E, D) + \delta$  follows from the arbitrariness of the function  $u(z) \in \mathcal{U}(E, D, \psi, \delta)$ . *The proposition 1.3. is proven.*

**Proposition 1.4.** Let  $E = \bigcup_{j=1}^{\infty} E_j$ ,  $E_j \subset D$ ,  $\forall j \in \mathbb{N}$ ,  $\delta \leq 0$ . Then for all  $z \in D$  the inequality

$$\omega^*(z, E, D, \psi, \delta) \geq \sum_{j=1}^{\infty} \omega^*(z, E_j, D, \psi, \delta)$$

holds.

*Proof.* Take  $\forall u_j \in \mathcal{U}(E_j, D, \psi, \delta)$  and consider the class

$$\left\{ \sum_{j=1}^{\infty} u_j(z) : u_j \in \mathcal{U}(E_j, D, \psi, \delta) \right\}.$$

Since  $u_j$  is  $m$ -subharmonic and negative in  $D$ , it follows that the sum  $\sum_{j=1}^{\infty} u_j(z)$  is also  $m$ -subharmonic function in  $D$ . We can easily check that  $\sum_{j=1}^{\infty} u_j(z) \in \mathcal{U}(E, D, \psi, \delta)$ .

Hence,

$$\begin{aligned} \omega(z, E, D, \psi, \delta) &\geq \sup \left\{ \sum_{j=1}^{\infty} u_j(z) : u_j \in \mathcal{U}(E_j, D, \psi, \delta) \right\} = \\ &= \sum_{j=1}^{\infty} \sup \{ u_j(z) : u_j \in \mathcal{U}(E_j, D, \psi, \delta) \} = \sum_{j=1}^{\infty} \omega(E_j, D, \psi, \delta). \end{aligned}$$

Now we investigate the sets  $P_j = \{z \in D : \omega(z, E_j, D, \psi, \delta) < \omega^*(z, E_j, D, \psi, \delta)\}$ ,  $j \in \mathbb{N}$ . We know that the sets  $P_j$  are  $m$ -polar, and their Lebesgue measure is zero. Therefore, the Lebesgue measure of  $P = \bigcup_{j=1}^{\infty} P_j$  is also zero, i.e.  $mes(P) = mes\left(\bigcup_{j=1}^{\infty} P_j\right) = 0$ . If we take an upper regularization, we see

$$\begin{aligned} \omega^*(z, E, D, \psi, \delta) &\geq \overline{\lim}_{w \rightarrow z} \sum_{j=1}^{\infty} \omega(z, E_j, D, \psi, \delta) \geq \\ &\geq \overline{\lim}_{w \rightarrow z, w \in D \setminus P} \sum_{j=1}^{\infty} \omega^*(z, E_j, D, \psi, \delta) = \sum_{j=1}^{\infty} \omega^*(z, E_j, D, \psi, \delta). \end{aligned}$$

The Proposition 1.4. is proven.

**Proposition 1.5.** If  $E \subset\subset D$ , then  $\lim_{z \rightarrow \partial D} \omega^*(z, E, D, \psi, \delta) = \delta$ .

*Proof.* Since  $D$  is a  $m$ -regular domain, there exists a function such that  $\rho(z) \in sh_m(D)$  and  $\lim_{z \rightarrow \partial D} \rho(z) = 0$ . It followed that  $C \cdot \rho(z) + \delta \in \mathcal{U}(E, D, \psi, \delta)$ , where  $C = \frac{\inf_{z \in E} \psi(z) - \delta}{\max_{z \in E} \rho(z)}$ . So, from the relations  $C \cdot \rho + \delta \leq \omega^*(z, E, D, \psi, \delta) \leq \delta$  and  $\lim_{z \rightarrow \partial D} C \cdot \rho = 0$ , we get  $\lim_{z \rightarrow \partial D} \omega^*(z, E, D, \psi, \delta) = \delta$ . The proposition 1.5. is proven.

**Proposition 1.6.** Let  $E \subset\subset D_1$  and  $D_j \subset D_{j+1}$ ,  $\bigcup_{j=1}^{\infty} D_j = D$ ,  $j \in \mathbb{N}$ . Then

$$\lim_{j \rightarrow \infty} \omega^*(z, E, D_j, \psi, \delta) = \omega^*(z, E, D, \psi, \delta).$$

*Proof.* According to the proposition 1.1., the inequality  $\omega^*(z, E, D_j, \psi, \delta) \geq \omega^*(z, E, D_{j+1}, \psi, \delta)$  is valid. It follows that  $\omega^*(z, E, D_j, \psi, \delta)$  is decreasing with respect to  $j$  and  $\lim_{j \rightarrow \infty} \omega^*(z, E, D_j, \psi, \delta) = \omega(z) \in sh_m(D)$ . Therefore the inequality  $\lim_{j \rightarrow \infty} \omega^*(z, E, D_j, \psi, \delta) \geq \omega^*(z, E, D, \psi, \delta)$  holds for all  $z \in D$ .

Now we have to show that  $\lim_{j \rightarrow \infty} \omega^*(z, E, D_j, \psi, \delta) \leq \omega^*(z, E, D, \psi, \delta)$ . We choose  $\varepsilon_j > 0$ ,  $\lim_{j \rightarrow \infty} \varepsilon_j = 0$ , so that the following relationships  $\{z \in D : \rho \leq -\varepsilon_j\} \subset D_j$  hold. We note, that  $\rho \in sh_m(D)$ ,  $\lim_{z \rightarrow \partial D} \rho(z) = 0$ . Take an arbitrary  $u \in \mathcal{U}(E, D_j, \psi, \delta)$  and consider the function

$$v(z) = \begin{cases} \max\{u - C\varepsilon_j, C\rho + \delta\}, & z \in D_j \\ C\rho + \delta, & z \in D \setminus D_j \end{cases},$$

where  $C = \frac{\inf_{z \in E} \psi(z) - \delta}{\max_{z \in E} \rho(z)}$ . It is not difficult to see that  $v \in \mathcal{U}(E, D, \psi, \delta)$  and  $u(z) - C\varepsilon_j \leq v(z) \leq \omega^*(z, E, D, \psi, \delta)$  for all  $z \in D_j$ . Since  $u \in \mathcal{U}(E, D_j, \psi, \delta)$  is arbitrary, the inequality  $\omega^*(z, E, D_j, \psi, \delta) - C\varepsilon_j \leq \omega^*(z, E, D, \psi, \delta)$  holds for arbitrary  $z \in D_j$ . As a result, we obtain the inequality  $\lim_{j \rightarrow \infty} \omega^*(z, E, D_j, \psi, \delta) \leq \omega^*(z, E, D, \psi, \delta)$  for all  $z \in D$ . The proof is over.

**Proposition 1.7.** a) let  $E \subset D$  be an arbitrary set and a function  $\psi(z)$  be a lower semicontinuous in  $V \subset D$ , where  $V$  some neighborhoods of  $E$ . Then there exists a sequence of open sets  $U_j \supset E$ ,  $U_j \supset U_{j+1}$  such that

$$\left( \lim_{j \rightarrow \infty} \omega^*(z, U_j, D, \psi, \delta) \right)^* = \omega^*(z, E, D, \psi, \delta).$$

b) let  $U \subset D$  be an open set and  $U = \bigcup_{j=1}^{\infty} K_j$ , where  $K_j \subset K_{j+1}^0$  are a compact sets and  $\psi(z)$  is an upper semicontinuous function in  $U$ . Then

$$\omega^*(z, K_j, D, \psi) \downarrow \omega^*(z, U, D, \psi).$$

*Proof.* a) By technical Choquet Lemma (see [4]) there exists a class of countable functions  $\{u_k\} \subset \mathcal{U}(E, D, \psi, \delta)$  such that  $\left( \sup_k u_k(z) \right)^* = \omega^*(z, E, D, \psi, \delta)$ . It is evident that the functional sequence  $v_j(z) = \max\{u_1(z), u_2(z), \dots, u_j(z)\}$  is increasing and

$\left(\lim_{j \rightarrow \infty} v_j(z)\right)^* = \omega^*(z, E, D, \psi, \delta)$ . Now take the sets  $U_j = \left\{z \in V : v_j < \psi(z) + \frac{1}{j}\right\}$ ,  $j \in \mathbb{N}$ . The sets  $U_j$  are open because the functions  $v_j(z) - \psi(z)$  are upper semicontinuous in  $V$ . It is easy to see, that  $U_j \supset U_{j+1}$  and  $v_j - \frac{1}{j} \in \mathcal{U}(U_j, D, \psi, \delta)$  for all  $j \in \mathbb{N}$ . By proposition 1.1., we get  $\omega^*(z, U_j, D, \psi, \delta) \leq \omega^*(z, E, D, \psi, \delta)$ ,  $\forall j \in \mathbb{N}$ . On the other hand  $v_j - \frac{1}{j} \leq \omega(z, U_j, D, \psi, \delta)$ ,  $\forall j \in \mathbb{N}$ . As a result,

$$v_j - \frac{1}{j} \leq \omega(z, U_j, D, \psi, \delta) \leq \omega^*(z, E, D, \psi, \delta), \quad \forall j \in \mathbb{N}.$$

Now we take the limit  $j \rightarrow \infty$  and the regularization:

$$\left\{\lim_{j \rightarrow \infty} \left(v_j(z) - \frac{1}{j}\right)\right\}^* \leq \left\{\lim_{j \rightarrow \infty} \omega(z, U_j, D, \psi, \delta)\right\}^* \leq \left\{\lim_{j \rightarrow \infty} \omega(z, E, D, \psi, \delta)\right\}^*.$$

Consequently  $\left\{\lim_{j \rightarrow \infty} \omega(z, U_j, D, \psi, \delta)\right\}^* = \omega^*(z, E, D, \psi, \delta)$ . The proof of a) is over.

b). By proposition 1.1. we have  $\omega^*(z, K_j, D, \psi, \delta) \geq \omega^*(z, K_{j+1}, D, \psi, \delta)$ . The functions  $\omega^*(z, K_j, D, \psi, \delta)$  are decreasing  $m$ -subharmonic, and the limit is  $m$ -subharmonic, i.e.

$$\lim_{j \rightarrow \infty} \omega^*(z, K_j, D, \psi, \delta) = \omega(z), \quad \omega(z) \in sh_m(D).$$

By the monotony  $\omega^*(z, K_j, D, \psi, \delta) \geq \omega^*(z, U, D, \psi, \delta)$ ,  $\forall j \in \mathbb{N}$  and  $\omega(z) \geq \omega^*(z, U, D, \psi, \delta)$  for all  $z \in D$ . Now we have to show that  $\omega(z) \leq \omega^*(z, U, D, \psi, \delta)$ . Since the function  $\psi(z)$  is an upper semicontinuous in  $U$ ,

$$\forall z \in \bigcup_{j=1}^{\infty} K_j = U, \quad \exists j_0, \quad \forall j > j_0, \quad \omega^*(z, K_j, D, \psi, \delta) \leq \psi(z).$$

It follows from this that  $\omega(z)|_U \leq \psi(z)|_U$ . Therefore  $\omega(z) \in \mathcal{U}(U, D, \psi, \delta)$  and  $\omega(z) \leq \omega^*(z, U, D, \psi, \delta)$ . *The proof is over.*

**2.  $(m, \psi, \delta)$ -regularity of compacts.** Let the function  $\psi(z)$  be extended to the domain  $D$  as a function from the class  $\mathcal{U}(E, D, \psi, \delta)$  i.e. if there is a function

$$\tilde{\psi} \in sh_m(D), \quad \tilde{\psi}|_E = \psi|_E, \quad \tilde{\psi}|_D < \delta, \quad (2.1)$$

then it is obvious  $\omega(z, E, D, \psi, \delta) \geq \tilde{\psi}(z)$ ,  $\forall z \in D$  and

$$\omega(z, E, D, \psi, \delta) = \psi(z), \quad \forall z \in E. \quad (2.2)$$

However, if condition (2.1) is not satisfied, then, in general, equality (2.2) does not hold. In the following example, it can be observed that equality (2.2) fails to hold.

**Example 2.1.** Let  $\psi(z) = 1 - |z|^2$ ,  $\delta = 2$ ,  $D = B(0, 2) \subset \mathbb{C}^n$ ,  $E = \bar{B}(0, 1) \subset \mathbb{C}^n$ . According to the maximum principle and by the definition 1.1.,

$$\omega(z, \bar{B}(0, 1), B(0, 2), 1 - |z|^2, 2) = 0 \neq \psi(z), \quad \forall z \in B(0, 1).$$

We assume that condition (2.2) holds in the definitions of  $\omega(z, E, D, \psi, \delta)$ . We also assume, that  $D \subset \mathbb{C}^n$  is a  $m$ -regular domain and  $K \subset D$  is a compact.

**Definition 2.1.** A point  $z^0 \in K$  is said to be globally  $(m, \psi, \delta)$ -regular if  $\omega^*(z^0, K, D, \psi, \delta) = \psi(z^0)$ . It is said to be locally  $(m, \psi, \delta)$ -regular if for any neighborhood  $B$ ,  $z^0 \in B \subset \mathbb{C}^n$ , the intersection  $K \cap \bar{B}$  is globally  $(m, \psi, \delta)$ -regular at the point  $z^0$ , i.e.  $\omega^*(z^0, K \cap \bar{B}, D, \psi, \delta) = \psi(z^0)$ . If all points of a compact set  $K$  are globally (or locally)  $(m, \psi, \delta)$ -regular, then the compact set  $K$  is called a globally (or locally)  $(m, \psi, \delta)$ -regular compact.

**Proposition 2.1.** Let  $\delta_1, \delta_2 \in \mathbb{R}$ ,  $\delta_1 \leq \delta_2$  and  $K$  be a compact set in  $D \subset \mathbb{C}^n$  and condition (2.2) be satisfied, i.e.  $\omega(z, E, D, \psi, \delta_1) = \psi(z)$ ,  $\forall z \in E$ . If  $K$  is  $(m, \psi, \delta_2)$ -regular at  $z_0 \in K$ , then  $K$  is  $(m, \psi, \delta_1)$ -regular at  $z_0 \in K$ . Thusly, if  $K$  is  $(m, \psi, \delta_2)$ -regular at  $z_0 \in K$ , then  $K$  is  $(m, \psi, \delta_1)$ -regular at  $z_0 \in K$ ,  $\delta_1 \leq \delta_2$ .

The proof of proposition 2.1. easily follows from definition 2.1. and proposition 1.1.

**Theorem 2.1.** Let  $K$  be  $(m, \psi, \delta)$ -regular a compact set and  $\psi(z)$  be a continuous in the compact set  $K$ . Then  $\omega^*(z, K, D, \psi) \equiv \omega(z, K, D, \psi) \in C(\bar{D})$  for any  $z \in D$ .

*Proof.* Let  $K$  be  $(m, \psi, \delta)$ -regular a compact i.e.  $\omega^*(z, K, D, \psi, \delta)|_K = \psi|_K$ . It is evident that  $\omega^*(z, K, D, \psi, \delta) \subset \mathcal{U}(K, D, \psi, \delta)$  and  $\omega^*(z, K, D, \psi, \delta) \equiv \omega(z, K, D, \psi, \delta)$ .

Now we prove that  $\omega^*(z, K, D, \psi, \delta)$  is continuous in  $\bar{D}$ . Let us fix  $\varepsilon > 0$  to be sufficiently small and then construct the domain  $G_\varepsilon = \{z \in D : \omega^*(z, K, D, \psi, \delta) < \delta - \varepsilon\}$ , where  $\delta - \varepsilon > \psi|_K$ . It is easy to see that according to the proposition 1.5., the relation  $K \subset G_\varepsilon \subset\subset D$  is valid. There exists a sequence of monotonic functions  $u_j(z) \in sh_m(G) \cap C^\infty(G)$  such that  $u_j(z) \downarrow \omega^*(z, K, D, \psi, \delta)$  holds for any  $z \in G$ , where  $G_\varepsilon \subset\subset G \subset\subset D$ . By applying Hartog's lemma to the sets  $G$  and  $\bar{G}_\varepsilon$ , we establish the relation

$$\exists j_1 \in \mathbb{N}, \forall j > j_1, \forall z \in \bar{G}_\varepsilon : u_j(z) < \delta.$$

On the other hand, since the function  $\psi(z)$  is continuous on the compact set  $K$ , according to Whitney's theorem [14], there exists some continuous function  $\tilde{\psi}(z)$  in  $D$  such that  $\tilde{\psi}(z)|_K = \psi(z)|_K$ . Now we consider open sets  $U_\varepsilon = \{z \in D : \omega^*(z, K, D, \psi, \delta) < \tilde{\psi}(z) + \varepsilon\}$ . It is clear that  $K \subset U_\varepsilon$ . We again apply Hartogs' lemma to the pair of sets  $U_\varepsilon$  and  $K$  and get

$$\exists j_2 \in \mathbb{N}, \forall j > j_2, \forall z \in K : u_j(z) < \psi(z) + 2\varepsilon.$$

Let us consider the function

$$v(z) = \begin{cases} \max\{u_j(z) - 2\varepsilon, \omega^*(z, K, D, \psi, \delta)\}, & z \in G_\varepsilon \\ \omega^*(z, K, D, \psi, \delta), & z \in D \setminus G_\varepsilon \end{cases}.$$

It is obvious that  $v|_K \leq \psi|_K$ ,  $v|_D < \delta$  for  $\forall j > j_3 = \max\{j_1, j_2\}$ . It implies  $v(z) \in \mathcal{U}(K, D, \psi, \delta)$  and  $v(z) \leq \omega^*(z, K, D, \psi, \delta)$ . Consequently

$$u_j(z) - 2\varepsilon \leq \omega^*(z, K, D, \psi, \delta) \leq u_j(z), \forall j > j_3, \forall z \in G_\varepsilon.$$

Therefore,  $\omega^*(z, K, D, \psi, \delta)$  is the uniform limit of the  $u_j(z)$  in  $G_\varepsilon$ . This implies that  $\omega^*(z, K, D, \psi, \delta) \in C(G_\varepsilon)$ . Since  $G_\varepsilon \subset\subset G \subset\subset D$ , and  $\varepsilon > 0$  is arbitrary, then  $\omega^*(z, K, D, \psi, \delta) \in C(D)$ . *The theorem is proven.*



**Theorem 2.2.** Let  $\psi \in C(K)$  and condition (2.2) be satisfied, i.e.  $\omega(z, E, D, \psi, \delta_1) = \psi(z)$ ,  $\forall z \in K$ . A fixed point  $z^0 \in K \subset \mathbb{C}^n$  is locally  $(m, \psi, \delta)$ -regular if and only if it is locally  $m$ -regular,  $\omega^*(z^0, K \cap \overline{B}, D) = -1$ .

*Proof.* To prove this theorem, we show that if the point  $z^0 \in K$  is not local  $m$ -regular, then it is not local  $(m, \psi, \delta)$ -regular and conversely, if point  $z^0 \in K$  is not local  $(m, \psi, \delta)$ -regular, then it is not local  $m$ -regular. Let us assume that the point  $z^0 \in K$  is not a local  $m$ -regular. i.e. there exists a ball  $B$  such that  $z^0 \in B \subset D$  and the equality  $\omega^*(z^0, K \cap \overline{B}, D) \geq -1 + \varepsilon$ ,  $0 < \varepsilon < 1$  is valid. According to monotonicity  $\omega^*(z^0, K \cap \overline{B_1}, D) \geq -1 + \varepsilon$  for any ball  $B_1$ , where  $z^0 \in B_1 \subset B$ . Therefore by proposition 1.3.

$$\begin{aligned} \omega^*(z^0, K \cap \overline{B_1}, D, \psi, \delta) &\geq \left( \delta - \inf_{z \in K \cap \overline{B_1}} \psi(z) \right) \cdot \omega^*(z^0, K \cap \overline{B_1}, D) + \delta \geq \\ &\geq \left( \delta - \inf_{z \in K \cap \overline{B_1}} \psi(z) \right) (-1 + \varepsilon) + \delta = \delta \cdot \varepsilon + \inf_{z \in K \cap \overline{B_1}} \psi(z) \cdot (1 - \varepsilon). \end{aligned}$$

Since  $\psi(z)$  is continuous, by choosing the neighborhood  $B_1$  and  $\varepsilon$  small enough, we obtain

$\inf_{x \in K \cap \overline{B_1}} \psi(z) \geq \frac{\psi(z^0) - \varepsilon \cdot \delta}{1 - \varepsilon}$ . From this inequality we get the relation

$$\omega^*(z^0, K \cap \overline{B_1}, D, \psi) \geq \varepsilon \cdot \delta + \inf_{z \in K \cap \overline{B_1}} \psi(z) \cdot (1 - \varepsilon) > \varepsilon \cdot \delta + \frac{(\psi(z^0) - \varepsilon \cdot \delta) \cdot (1 - \varepsilon)}{1 - \varepsilon} = \psi(z^0).$$

Therefore  $z^0 \in K$  is not local  $(m, \psi, \delta)$ -regular.

Conversely, we have to show that if the point  $z^0 \in K$  is not local  $(m, \psi, \delta)$ -regular, then it is not local  $m$ -regular. Suppose  $z^0 \in K$  is not  $(m, \psi, \delta)$ -regular i.e., there exists the ball  $B$ ,  $z^0 \in B \subset D$ , such that  $\omega^*(z^0, K \cap \overline{B}, D, \psi, \delta) \geq \psi(z^0) + \alpha$ ,  $\alpha > 0$ . By using the previous technique we get  $\omega^*(z^0, K \cap \overline{B_1}, D, \psi, \delta) \geq \psi(z^0) + \alpha$  for any ball  $B_1$ , where  $z^0 \in B_1 \subset B$ . Therefore by proposition 1.3.,

$$\psi(z^0) + \alpha \leq \omega^*(z^0, K \cap \overline{B_1}, D, \psi, \delta) \leq \left( \delta - \sup_{z \in K \cap \overline{B_1}} \psi(z) \right) \cdot \omega^*(z^0, K \cap \overline{B_1}, D) + \delta.$$

Given that  $\psi(z)$  is continuous, we can make  $\sup_{z \in K \cap \overline{B_1}} \psi(z) < \psi(z^0) + \alpha < \delta$  by selecting sufficiently small values for  $\alpha$  and the neighborhood  $B_1$ . Thus,

$$\psi(z^0) + \alpha \leq \omega^*(z^0, K \cap \overline{B_1}, D, \psi, \delta) < (\delta - \psi(z^0) - \alpha) \cdot \omega^*(z^0, K \cap \overline{B_1}, D) + \delta.$$

From the last inequality we get  $\omega^*(z^0, K \cap \overline{B_1}, D) > -1$ . Hence  $z^0 \in K$  is not the  $m$ -regular point. *The theorem is proven.*

**Theorem 2.3.** Let the function  $\psi(z)$  be extended to  $\mathcal{U}(K, D, \psi, \delta)$  as a strictly  $m$ -subharmonic function in some neighbourhood  $D^+ \supset \overline{D}$  of closure  $\overline{D}$ , i.e, there exists a function  $\tilde{\psi}$  that is strictly  $m$ -subharmonic in the domain  $D^+$  and  $\tilde{\psi}|_K = \psi|_K$ ,  $\tilde{\psi}|_D < \delta$ . Then a fixed point  $z^0 \in K \subset D$  is locally  $(m, \psi, \delta)$ -regular if and only if it is globally  $(m, \psi, \delta)$ -regular.

Here, a function  $\psi(z)$  is strictly  $m$ -subharmonic function in the domain  $D$ , if for any compact domain  $G \subset\subset D$ , there exists  $\varepsilon > 0$ :  $\psi(z) - \varepsilon|z|^2 \in sh_m(G)$ .

*Proof.* It is clear that a locally  $(m, \psi, \delta)$ -regular point is also globally  $(m, \psi, \delta)$ -regular. We will prove the converse: if  $z^0 \in K$  is a globally  $(m, \psi, \delta)$ -regular point, then it is also locally  $(m, \psi, \delta)$ -regular point. Let us assume that the point  $z^0 \in K$  is globally  $(m, \psi, \delta)$ -regular, which means that  $\omega^*(z^0, K, D, \psi, \delta) = \psi(z^0)$ . According to the theorem's condition if the function  $\psi(z)$  be extend as strictly  $m$ -subharmonic function in the domain  $D^+$ . Then there exists a constants  $\varepsilon > 0$  such that the function  $\tilde{\psi}(z) - \varepsilon|z - z^0|^2$  is a strictly  $m$ -subharmonic in the domain  $D$ , where  $\tilde{\psi}$  is a strictly  $m$ -subharmonic function in the domain  $D$  and  $\tilde{\psi}|_K = \psi|_K$ ,  $\tilde{\psi}|_D < \delta$ . Now we will fix the function  $u(z)$  that satisfies the condition;

$$u(z) \in sh_m(D), \quad u|_{K \cap \bar{B}_r} \leq -1, \quad u|_D < 0,$$

where  $B_r = B(z^0, r) \subset\subset D$ ,  $\max_{z \in \bar{B}_r} \tilde{\psi} + r^2 < \delta$ . It can be easily seen that the relation  $\varepsilon r^2(u(z) + 1) - \varepsilon|z - z^0|^2 \leq 0$  is appropriate for any  $z \in K$ . Thus, the function  $\varphi(z) = \varepsilon r^2(u(z) + 1) + \tilde{\psi}(z) - \varepsilon|z - z^0|^2$  is  $m$ -subharmonic in the domain  $D$  and  $\varphi|_K \leq \psi|_K$ ,  $\varphi|_D < \delta$ . Hence,  $\varphi \in \mathcal{U}(K, D, \psi, \delta)$  and

$$\varphi(z) \leq \omega^*(z, K, D, \psi, \delta), \quad \forall z \in D.$$

Thus,

$$\varepsilon r^2 \left( \omega^*(z, K \cap \bar{B}_r, D) + 1 \right) + \tilde{\psi}(z) - \varepsilon|z - z^0|^2 \leq \omega^*(z, K, D, \psi, \delta), \quad \forall z \in D.$$

Putting here  $z = z^0$  we have

$$\varepsilon r^2 \left( \omega^*(z^0, K \cap \bar{B}_r, D) + 1 \right) + \psi(z^0) \leq \omega^*(z^0, K, D, \psi, \delta) = \psi(z^0)$$

Hence,

$$\varepsilon r^2 \left( \omega^*(z^0, K \cap \bar{B}_r, D) + 1 \right) \leq 0,$$

which implies  $\omega^*(z^0, K \cap \bar{B}_r, D) = -1$ .

This implies that the point  $z^0 \in K$  is locally  $m$ -regular. According to Theorem 2.2. we conculde the local  $(m, \psi, \delta)$ -regularity of the point  $z^0 \in K$ . *The theorem is proven.*

From the Theorem 2.2. and the Theorem 2.3., we obtain several important corollaries.

**Corollary 2.1.** If the compact set  $K \subset D$  is globally  $(m, \psi, \delta)$ -regular, where the function  $\psi(z)$  is extended to  $\mathcal{U}(K, D, \psi, \delta)$  as a strictly  $m$ -subharmonic function in some neighbourhood  $D^+ \supset \bar{D}$  of closure  $\bar{D}$ , then  $K$  is locally  $m$ -regular.

**Corollary 2.2.** If  $\psi_1$  and  $\psi_2$  are extended to  $\mathcal{U}(K, D, \psi_1, \delta)$  and  $\mathcal{U}(K, D, \psi_2, \delta)$  as strictly  $m$ -subharmonic functions in some neighbourhood  $D^+ \supset \bar{D}$  of closure  $\bar{D}$ , respectively, then the point  $z^0 \in K \subset D$  is  $(m, \psi_1, \delta)$ -regular if and only if it is  $(m, \psi_2, \delta)$ -regular.

**Corollary 2.3.** If the compact set  $K \subset D$  is globally  $(m, \psi, \delta)$ -regular, where  $\psi(z)$  is extended to  $\mathcal{U}(K, D, \psi, \delta)$  as a strictly  $m$ -subharmonic function in some neighbourhood  $D^+ \supset \bar{D}$  of closure  $\bar{D}$ , then  $K$  is not  $m$ -polar at each of its point. It means that for any  $z^0 \in K$  and for any neighborhood  $B \subset D$ ,  $z^0 \in B$  the intersection  $E = B \cap K$  is not  $m$ -polar.

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**REZYUME**

Ma'lumki,  $D \subset \mathbb{C}^n$  sohaga tegishli  $E \subset D$  to'plamining  $m$ -subgarmonik o'lchovi  $\omega^*(z, E, D)$  funksiya  $D$  sohadagi  $m$ -subgarmonik funksiyalar yordamida aniqlanadi. Ushbu ish  $\omega^*(z, E, D, m, \psi, \delta)$  vaznli  $m$ -subgarmonik o'lchov xossalariga, xususan,  $K \subset D \subset \mathbb{C}^n$  kompakt to'plamning  $(m, \psi, \delta)$ -regulyarligiga bag'ishlangan. Biz  $m$ -subgarmonik o'lchovni umumlashtiramiz va [1] maqolada  $K$  kompakt to'plamning regulyarligi bo'yicha keltirilgan bir nechta teoremlar umumlashtirilgan holatda ham bajarilishini isbotlaymiz.

**Kalit so'zlar:**  $m$ -subgarmonik funksiya,  $m$ -subgarmonik o'lchov,  $m$ -polyar to'plam, global  $m$ -regulyar kompakt, lokal  $m$ -regulyar kompakt.

**РЕЗЮМЕ**

Известно, что  $m$ -субгармоническая мера  $\omega^*(z, E, D)$  множества  $E \subset D$ , связанная с областью  $D \subset \mathbb{C}^n$ , определяется  $m$ -субгармоническими функциями в  $D$ . Настоящая работа посвящена изучению свойств  $m$ -субгармонической меры с весом  $\omega^*(z, E, D, m, \psi, \delta)$ , в частности,  $(m, \psi, \delta)$ -регулярности компактного множества  $K \subset D \subset \mathbb{C}^n$ . Обобщается понятие  $m$ -субгармонической меры и доказывается, что некоторые теоремы, доказанные в [1], относительно регулярности компактного множества  $K$ , также верны в обобщенном случае.

**Ключевые слова:**  $m$ -субгармоническая функция,  $m$ -субгармоническая мера,  $m$ -полярное множество, глобально  $m$ -регулярный компакт, локально  $m$ -регулярный компакт.

UDC 517.55

## NUMERICAL METHODS FOR THE DIFFERENTIAL EQUATIONS FRACTIONAL ORDER

Madraximova Z. S., Khayrullaev U. B., Temirova S. B. \*

### RESUME

This work is devoted to applications of a numerical method for partial differential equations of fractional order.

It is well known that in applications of PDEs, it is often required to find an approximate solution of the problem in mathematical physics. In this work, the method of finite differences, which is one of the numerical methods, was used to approximate the solution of fractional order PDEs. Note that the uploaded power technology and software packages enable us to find an approximate solution to the investigated problem. Detailed information on differential, integral, and integro-differential operators, as well as fractional order differential equations, is available in references [1]-[4].

Numerical methods are used to investigate initial-boundary value problems for fractional order wave equations, which involve Caputo derivatives in both single and multiply connected domains. Additionally, by employing the method of finite differences, we can approximately transform a fractional order PDE into a system of linear equations.

**Key words:** derivatives of fractional order, initial-boundary value conditions, finite difference methods.

We consider equation

$${}_C D_{0t}^\alpha u(x, t) - a \cdot u_{xx}(x, t) - b \cdot u(x, t) = f(x, t) \quad (1)$$

in simple connected domain  $\Omega = \{(x, t) : 0 < x < 1, 0 < t < T\}$ , where  $f(x, t)$  is given function,  ${}_C D_{0t}^\alpha u$  is a differential operator Caputo fractional order  $\alpha$ , which has a form [1], [2]:

$${}_C D_{0t}^\alpha u = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - z)^{-\alpha} u_z(x, z) dz, \quad (2)$$

$$\Gamma(\alpha) = \int_0^\infty e^{-t\alpha} t^{\alpha-1} dt, T, a, b, \alpha = const, 0 < \alpha < 1.$$

### Problem formulation.

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\***Madraximova Z. S.** – National university of Uzbekistan, Tashkent, Uzbekistan, Nordic International University, zilolaxonmadrahimova@gmail.com, **Khayrullaev U. B.** – Novosibirsk State University. Novosibirsk (Russian Federation), u.khairullaev@g.nsu.ru, **Temirova S. B.** – National university of Uzbekistan, Tashkent, Uzbekistan, temirovasitora1999@gmail.com

**Problem A.** Find an approximate value of solution of the equation (1), on the fixed points of the domain  $\Omega$ , which satisfies following boundary and initial conditions:

$$u(0, t) = \varphi_1(t), \quad 0 \leq t \leq T; \quad u(l, t) = \varphi_2(t), \quad l \leq t \leq T; \quad (3)$$

$$u(x, 0) = \tau(x), \quad 0 \leq x \leq l; \quad (4)$$

where  $\varphi_i(t) \in C[0, T]$  ( $i = 1, 2$ ),  $\tau(x) \in C[0, l]$ .

On the first step we demonstrate covering of the bounded domain with the network:

For instance, assuming  $(x, t)$  is a point of the Decart orthogonal coordinates,  $x, t$  surface we cover by the square network

$$x = m \cdot h, \quad x = n \cdot h, \quad m, n = 0, 1, 2, \dots \quad (5)$$

where  $h$  is given positive constant. Each vertices of obtained squares we recall NODEs and a numbers  $h$  STEP. Further applying (Phyton) programm package we demonstrate a covering surface of variables  $x, t$  by square network (5):

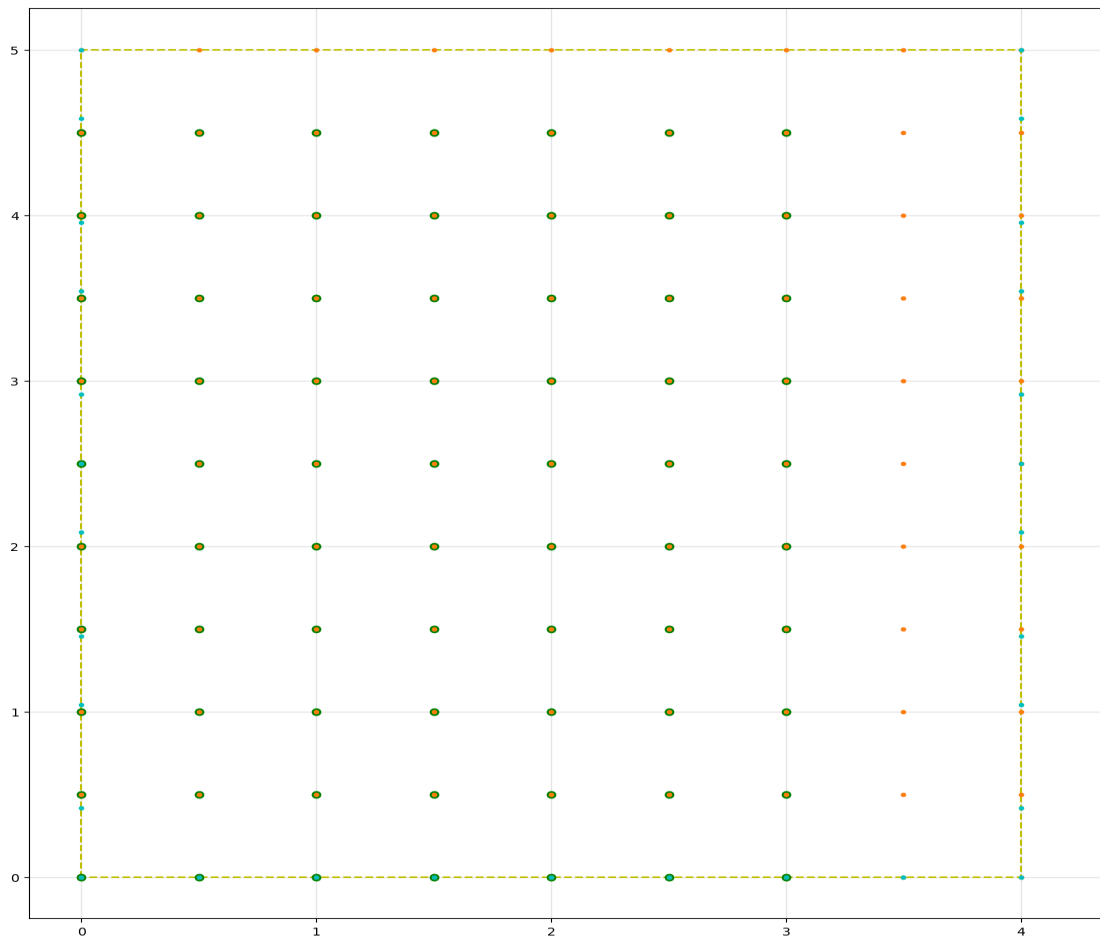
```

1 | ## Import a library of necessities ##
2 | import numpy as np
3 | ## Import a library MATH for base mathematical functions and constants.
4 | import math
5 | # Import a library MATPLOTLIB for create graphs and diagrams.
6 | import matplotlib.pyplot as plt
7 | ## Import a module SYMPY for operation symbolic math
8 | from sympy import *
9 | # Import a function 'PARSE_EXPR' from the module SYMPY
10 | from sympy.parsing.sympy_parser import parse_expr
11 | # Import a function 'MATHML' from the module SYMPY for demonstration objects
    Sympy in the line MathML.
12 | from sympy.printing.mathml import mathml
13 | # Import a function 'DISPLAY' from the library IPYTHON for demonstration objects
    in block notes IPYTHON.
14 | from IPython.display import display

```

Further, we define number of nodes which belongs to the considered domain.

For example, if **m=5 and n=4**, then we get covering as follows:



Now, we make following next operations:

```

1 | # We check, that the given point (a, b) inside of considered domain given by network.
2 | def check(a,b,h):
3 |     if a+h<=N and b+h<=M+h and a-h>=0 and b-h>=0:
4 |         return True
5 |     else :
6 |         return False
7 | # We define nearest point (a, b) of a bounds of the considered domain based on the
Euclidian distance.
8 | def change_n(a,b):
9 |     r=[]
10 |     for i in range(len(f_)):
11 |         r.append(np.sqrt((x_l[i]-a)**2+(f_[i]-b)**2))
12 |     return x_l[np.argmin(r)],f_[np.argmin(r)]
13 | # We check, that the point (a, b) inside of the considered domain.
14 |     if a+2*h<=N and b+h<=M and a>=0 and b>=0:
15 |         return True
16 |     else:

```

17|       **return False**

On the next step we write approximate equations on the each defined nodes. All steps we make using by Python programming package. Due to definitions of partial derivatives, on the each nodes  $(x, t)$  (assuming that all points  $(x, t)$ ,  $(x + h, t)$ ,  $(x + 2h, t)$ ,  $(x, t + h)$  inside of  $\Omega$ ), we can get follows

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-z)^{-\alpha} u_z(x, z) dz \approx \frac{1}{h \cdot \Gamma(1-\alpha)} \int_0^t \frac{u(x, z+h) - u(x, z)}{(t-z)^\alpha} dz$$

$$u_x(x, t) \approx \frac{u(x+h, t) - u(x, t)}{h}$$

$$u_{xx}(x, t) \approx \frac{u(x+2h, t) - 2u(x+h, t) + u(x, t)}{h^2}$$

1| # We restore a system of equations for nodes belongs to the solutions domain.

2| **def** integ(m,n):

3|    x = Symbol('x')

4|    z = Symbol('z')

5|    t = Symbol('t')

6|    h = Symbol('h')

7|    alpha = Symbol('α')

8|    f = Function('f')

9|    U = Function('u')

10|   G = Function('Γ')

11|   **if** n%2 == 0 :

12|         eq = Eq(((h\*\*2)\*\*((n+1)\*h)\*\*(1-alpha))/G(2-alpha)\*(U(m\*h,(n+1)\*h)-  
U(m\*h,n\*h)), U(m\*h+2\*h,(n+1)\*h)-2\*U(m\*h+h,(n+1)\*h)+U(m\*h,(n+1)\*h))

13|    **if** n%2 == 1 :

14|         eq=Eq(((h\*\*2)\*\*((n+1)\*h)\*\*(1-alpha))/G(2-alpha)\*(U(m\*h,n\*h)-  
U(m\*h,n\*h+h)), U(m\*h+2\*h,(n+1)\*h)-2\*U(m\*h+h,(n+1)\*h)+U(m\*h,(n+1)\*h))

15|    **return** eq

Now, we define the system of algebraic equations on each inner nodes, as follows

$$\left\{ \begin{array}{l} \frac{h^2 h^{1-\alpha} (u(0,0)+u(0,h))}{\Gamma(2-\alpha)} = u(0, h) - 2u(h, h) + u(2h, h) \\ \frac{h^2 (2h)^{1-\alpha} (u(0,h)+u(0,2h))}{\Gamma(2-\alpha)} = u(0, 2h) - 2u(h, 2h) + u(2h, 2h) \\ \frac{h^2 (3h)^{1-\alpha} (-u(0,2h)+u(0,3h))}{\Gamma(2-\alpha)} = u(0, 3h) - 2u(h, 3h) + u(2h, 3h) \\ \dots\dots\dots \\ \frac{h^2 ((2n-1)h)^{1-\alpha} (-u(0,(2n-2)h)+u(0,(2n-1)h))}{\Gamma(2-\alpha)} = u(0, (2n-1)h) - 2u(h, (2n-1)h) + u(2h, (2n-1)h) \\ \frac{h^2 (2nh)^{1-\alpha} (u(0,(2n-1)h)-u(0,2nh))}{\Gamma(2-\alpha)} = u(0, 2nh) - 2u(h, 2nh) + u(2h, 2nh) \end{array} \right.$$



$$\left\{ \begin{array}{l} \frac{h^2 h^{1-\alpha} (-u(h,0) + u(h,h))}{\Gamma(2-\alpha)} = u(h, h) - 2u(2h, h) + u(3h, h) \\ \frac{h^2 (2h)^{1-\alpha} (u(h,h) - u(h,2h))}{\Gamma(2-\alpha)} = u(h, 2h) - 2u(2h, 2h) + u(3h, 2h) \\ \frac{h^2 (3h)^{1-\alpha} (-u(h,2h) + u(h,3h))}{\Gamma(2-\alpha)} = u(h, 3h) - 2u(2h, 3h) + u(3h, 3h) \\ \dots\dots\dots \\ \frac{h^2 ((2n-1)h)^{1-\alpha} (-u(h, (2n-2)h) + u(h, (2n-1)h))}{\Gamma(2-\alpha)} = u(h, (2n-1)h) - 2u(2h, (2n-1)h) + u(3h, (2n-1)h) \\ \frac{h^2 (2nh)^{1-\alpha} (u(h, (2n-1)h) - u(h, 2nh))}{\Gamma(2-\alpha)} = u(h, 2nh) - 2u(2h, 2nh) + u(3h, 2nh) \\ \dots\dots\dots \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{h^2 h^{1-\alpha} (-u(nh,0) + u(nh,h))}{\Gamma(2-\alpha)} = u(nh, h) - 2u((n+1)h, h) + u((n+2)h, h) \\ \frac{h^2 (2h)^{1-\alpha} (u(nh,h) - u(nh,2h))}{\Gamma(2-\alpha)} = u(nh, 2h) - 2u((n+1)h, 2h) + u((n+2)h, 2h) \\ \frac{h^2 (3h)^{1-\alpha} (-u(nh,2h) + u(nh,h))}{\Gamma(2-\alpha)} = u(nh, 3h) - 2u((n+1)h, 3h) + u((n+2)h, 3h) \\ \dots\dots\dots \\ \frac{h^2 ((2n-1)h)^{1-\alpha} (-u(nh, (2n-2)h) + u(nh, (2n-1)h))}{\Gamma(2-\alpha)} = \\ = u(nh, (2n-1)h) - 2u((n+1)h, (2n-1)h) + u((n+2)h, (2n-1)h) \\ \frac{h^2 (2nh)^{1-\alpha} (u(nh, (2n-1)h) - u(nh, 2nh))}{\Gamma(2-\alpha)} = u(nh, 2nh) - 2u((n+1)h, 2nh) + u((n+2)h, 2nh) \end{array} \right.$$

We will find approximate solution of the Problem A, which satisfies boundary  $u(0, t) = t$ ,  $0 \leq t \leq 3$ ;  $u(0, t) = t$ ,  $0 \leq t \leq 3$ ; and initial conditions  $u(x, 0) = 0$ ,  $0 \leq x \leq 2$ .

$$\left\{ \begin{array}{l} 0.0997355701 = -2u(0.5, 0.5) + u(1.0, 0.5) + 0.5 \\ -0.1410473958 = -2u(0.5, 1.0) + u(1.0, 1.0) + 1.0 \\ 0.1727470747 = -2u(0.5, 1.5) + u(1.0, 1.5) + 1.5 \\ -0.1994711402 = -2u(0.5, 2.0) + u(1.0, 2.0) + 2.0 \\ 0.2230155145 = -2u(0.5, 2.5) + u(1.0, 2.5) + 2.5 \\ -0.2443012559 = -2u(0.5, 3.0) + u(1.0, 3.0) + 3.0 \\ 0.1994711402u(0.5, 0.5) = u(0.5, 0.5) - 2u(1.0, 0.5) + u(1.5, 0.5) \\ 0.2820947917u(0.5, 0.5) - 0.2820947917u(0.5, 1.0) = u(0.5, 1.0) - 2u(1.0, 1.0) + u(1.5, 1.0) \\ -0.3454941494u(0.5, 1.0) + 0.3454941494u(0.5, 1.5) = u(0.5, 1.5) - 2u(1.0, 1.5) + u(1.5, 1.5) \\ 0.3989422804u(0.5, 1.5) - 0.3989422804u(0.5, 2.0) = u(0.5, 2.0) - 2u(1.0, 2.0) + u(1.5, 2.0) \\ -0.4460310290u(0.5, 2.0) + 0.4460310290u(0.5, 2.5) = u(0.5, 2.5) - 2u(1.0, 2.5) + u(1.5, 2.5) \\ 0.4886025119u(0.5, 2.5) - 0.4886025119u(0.5, 3.0) = u(0.5, 3.0) - 2u(1.0, 3.0) + u(1.5, 3.0) \\ 0.1994711402u(1.0, 0.5) = u(1.0, 0.5) - 2u(1.5, 0.5) + 0.5 \\ 0.2820947917u(1.0, 0.5) - 0.2820947917u(1.0, 1.0) = u(1.0, 1.0) - 2u(1.5, 1.0) + 1.0 \\ -0.3454941494u(1.0, 1.0) + 0.3454941494u(1.0, 1.5) = u(1.0, 1.5) - 2u(1.5, 1.5) + 1.5 \\ 0.3989422804u(1.0, 1.5) - 0.3989422804u(1.0, 2.0) = u(1.0, 2.0) - 2u(1.5, 2.0) + 2.0 \\ -0.4460310290u(1.0, 2.0) + 0.4460310290u(1.0, 2.5) = u(1.0, 2.5) - 2u(1.5, 2.5) + 2.5 \\ 0.4886025119u(1.0, 2.5) - 0.4886025119u(1.0, 3.0) = u(1.0, 3.0) - 2u(1.5, 3.0) + 3.0 \end{array} \right.$$

1 | # We solve the system of algebraic equations

2 | solve(eq\_1, dict=**True**)

Obtained system of algebraic equations (6) has a unique solution. This solution has a form:

$$u(0.5, 0.5) = 0.371129081858283,$$

$$\begin{aligned}u(0.5, 1.0) &= 1.32169150027025, \\u(0.5, 1.5) &= 1.36987559973530, \\u(0.5, 2.0) &= 2.52240480268656, \\u(0.5, 2.5) &= 2.39721862365525, \\u(0.5, 3.0) &= 3.70314793087377, \\u(1.0, 0.5) &= 0.341993733816924, \\u(1.0, 1.0) &= 1.50233560465356, \\u(1.0, 1.5) &= 1.41249827420627, \\u(1.0, 2.0) &= 2.84533846517239, \\u(1.0, 2.5) &= 2.51745276182959, \\u(1.0, 3.0) &= 4.16199460579609, \\u(1.5, 0.5) &= 0.386887926895481, \\u(1.5, 1.0) &= 1.41483100154687, \\u(1.5, 1.5) &= 1.47176827313997, \\u(1.5, 2.0) &= 2.70847949920361, \\u(1.5, 2.5) &= 2.58184997974925, \\u(1.5, 3.0) &= 3.98276094084379\end{aligned}$$

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### REZYUME

Ushbu ish kasr tartibli xususiy hosilali differensial tenglamalar uchun sonli usulni qo'llashga bag'ishlangan.

Ma'lumki, XHDT (xususiy hosilali differensial tenglama) larni qo'llashda ko'pincha matematik-fizika masalalarining taqribiy yechimini topish talab qilinadi. Ushbu ishda kasr tartibli xususiy hosilali differensial tenglamalarning taqribiy yechimini topishda sonli usullardan biri chekli ayirmalar usulidan foydalanildi. Yangilangan

texnologiya va dastur paketlari o'rganilayotgan masalaning tahminiy yechimini topish imkonini beradi. Differensial, integral va integro-differensial operatorlar hamda kasr tartibli differensial tenglamalar haqida batafsilroq ma'lumotlar [1]-[4] da berilgan.

Kasr tartibli to'liq tenglamalari uchun boshlang'ich chegaraviy masalalar Kaputo hosilalarini bir va ko'p bog'lamlil sohalarda sonli usullar yordamida o'rganishni o'z ichiga oladi. Chekli ayirmalar usulidan foydalanib, kasr tartibini XHDT (xususiy hosilali differensial tenglama) larni chiziqli tenglamalar sistemasiga keltiramiz.

**Kalit so'zlar:** kasr tartibli differensial tenglamalar, boshlang'ich chegaraviy masalalar, chekli ayirmalar usuli.

### РЕЗЮМЕ

Данная работа посвящена применению численного метода для решения уравнений в частных производных дробного порядка.

Хорошо известно, что в приложениях УЧП часто требуется найти приближенное решение задачи математической физики. В данной работе был использован метод конечных разностей (один из численных методов) восстановления приближенного решения УЧП дробного порядка. Заметим, что обновленные пакеты технологий и программного обеспечения дают нам возможность найти приближенное решение исследуемой задачи. Более подробные сведения о дифференциальных, интегральных и интегро-дифференциальных операторах и дифференциальных уравнениях дробного порядка приведены в [1]-[4].

Начально-краевые задачи для волнового уравнения дробного порядка предполагают исследование производных Капуто в одно и многосвязных областях с использованием численных методов. Фактически, используя метод конечных разностей, мы приближенно сводим дробный порядок УЧП к системе линейных уравнений.

**Ключевые слова:** дифференциальных уравнениях дробного порядка, начально-краевые задачи, плюрирегулярная точка, численных методов.

UDC 517.55

## TRANSLATION INVARIANT GIBBS MEASURES FOR THREE STATE HARD-CORE MODELS IN THE CASE WAND

Mutalliyev N. N. \*

### RESUME

In this paper, we investigate the translation-invariant Gibbs measures for the fertile three-state Hard-Core (HC) models with the activity parameter  $\lambda_i > 0$ , where  $i = 0, 1, 2$  on a Cayley tree. It is known that there exist four types of such models: wrench, wand, hinge and pipe. In the case wand under some conditions on parameters on a Cayley tree of order one, two and three the exact number of the translation invariant Gibbs measures found.

**Key words:** Cayley tree, configuration, fertile graphs, Hard-Core model, Gibbs measure, translation-invariant Gibbs measures.

### INTRODUCTION

The theory of Gibbs measures is crucial for understanding the thermodynamic behavior of physical and biological systems. In statistical mechanics, Gibbs measures provide a primary method to describe systems made up of many particles. It is well established that the collection of all limit Gibbs measures forms a convex and compact set (see, for example [1]-[3]).

A Cayley tree  $\mathfrak{S}^k$  of order  $k \geq 1$  is an infinite tree, i.e. a graph without cycles with exactly  $k + 1$  edges originate from each vertex. Let  $\mathfrak{S}^k = (V, L, i)$ , where  $V$  is the set of vertices  $\mathfrak{S}^k$ ,  $L$  is the set of its edges and  $i$  is the incidence function, which associates each edge  $l \in L$  with its end points  $x, y \in V$ . If  $i(l) = \{x, y\}$ , then  $x$  and  $y$  are called *nearest neighbors of the vertex* and are denoted by  $l = \langle x, y \rangle$ .

For fixed  $x^0 \in V$  we set

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^n W_m,$$

where  $d(x, y)$  is the distance between vertices  $x$  and  $y$  on the Cayley tree, i.e. the number of edges of the shortest path connecting vertices  $x$  and  $y$ . We will write  $x \prec y$  if the path from  $x^0$  to  $y$  goes through  $x$ . A vertex  $y$  is called a direct successor of vertex  $x$  if  $y \succ x$  and  $x, y$  are the nearest neighbors. Note that in  $\mathfrak{S}^k$  every vertex  $x \neq x^0$  has  $k$  direct successors, and the vertex  $x^0$  has  $k + 1$  successors. We denote the successors of the vertex  $x$  by  $S(x)$ , i.e. if  $x \in W_n$ , then

$$S(x) = \{y_i \in W_{n+1} \mid d(x, y_i) = 1, i = 1, 2, \dots, k\}.$$

### PRELIMINARIES

\*Mutalliyev N. N. – Namangan Institute of Engineering and Technology, nodirbekmutalliyev95@gmail.com

**The *HC*-model.** Unlike other models, the Hard-Core (*HC*) model places restrictions on spin values. Hard constraints arise in fields as diverse as combinatorics, statistical mechanics, and queuing theory. In particular, the *HC*-model arises when studying random independent sets of the graph [5], [6], when studying gas molecules on the lattice [4].

*HC*-model arises as a simple example of a loss network with nearest-neighbor exclusion. The state  $\sigma(x)$  at each node  $x$  of the Cayley tree can be 0, 1 and 2. We have Poisson flow of calls of rate  $\lambda$  at each site  $x$ , each call has an exponential duration of mean 1. If a call finds the node in state 1 or 2 it is lost. If it finds the node in state 0 then things depend on the state of the neighboring sites. If all neighbors are in state 0, the call is accepted and the state of the node becomes 1 or 2 with equal probability 1/2. If at least one neighbor is in state 1, and there is no neighbor in state 2 then the state of the node becomes 1. If at least one neighbor is in state 2 the call is lost.

In [7] A. Mazel and Yu. Suhov introduced and studied the *HC*-model on the  $d$ -dimensional lattice  $\mathbb{Z}^d$ . In the present paper, we study this model with three states on the Cayley tree.

In [8], fertile three-state *HC*-models are identified that correspond to graphs of the hinge, pipe, wand, and wrench types. The works [9]-[18] are devoted to the study of Gibbs measures for *HC*-models with three states on the Cayley tree in the case  $\lambda_1 = \lambda_2$ .

Let  $\Phi = \{0, 1, 2\}$  and  $\sigma \in \Omega = \Phi^V$  be the configuration, i.e.,  $\sigma = \{\sigma(x) \in \Phi : x \in V\}$ . In other words, in this model, each vertex  $x$  is assigned to one of the values  $\sigma(x) \in \Phi = \{0, 1, 2\}$ . The values of  $\sigma(x) \neq 0$  mean that the vertex  $x$  is ‘occupied’, and the value  $\sigma(x) = 0$  means that the vertex  $x$  is ‘vacant’. The set of all configurations on  $V$  ( $V_n$ ) is denoted by  $\Omega$  ( $\Omega_{V_n}$ ).

Consider the set  $\Phi$  as the set of vertices of some graph  $G$ . Using the graph  $G$  we define a  $G$ -admissible configuration in the following way. The  $\sigma$  configuration is called *G-admissible configuration* on the Cayley tree (in  $V_n$ ) if  $\{\sigma(x), \sigma(y)\}$  is edge of the graph  $G$  for any the nearest pair of neighbors  $x, y$  from  $V$  (from  $V_n$ ). Let us denote the set of  $G$ -admissible configurations via  $\Omega^{a,G}$  ( $\Omega_{V_n}^{a,G}$ ).

The activity set [8] for the graph  $G$  is the function  $\lambda : G \rightarrow R_+^3$ . Value  $\lambda_i$  of function  $\lambda$  at vertex  $i \in \{0, 1, 2\}$  is called its ‘activity’.

For given  $G$  and  $\lambda$ , we define the Hamiltonian of the  $G$ -*HC*-model as

$$H_G^\lambda(\sigma) = \begin{cases} - \sum_{x \in V} \log \lambda_{\sigma(x)}, & \text{if } \sigma \in \Omega^{a,G}, \\ +\infty, & \text{if } \sigma \notin \Omega^{a,G}. \end{cases}$$

Let  $z : x \mapsto z_x = (z_{0,x}, z_{1,x}, z_{2,x}) \in R_+^3$  vector-valued function on  $V$ . For  $n \geq 1$  and  $\lambda > 0$  consider the probability measure  $\mu^{(n)}$  on  $\Omega_{V_n}^{a,G}$ , defined as

$$\mu^{(n)}(\sigma_n) = \frac{1}{Z_n} \prod_{x \in V_n} \lambda_{\sigma_n(x)} \prod_{x \in W_n} z_{\sigma_n(x), x}. \quad (1)$$

Here  $Z_n$  is the normalizing divisor:

$$Z_n = \sum_{\varphi_n \in \Omega_{V_n}^{a,G}} \prod_{x \in V_n} \lambda_{\varphi_n(x)} \prod_{x \in W_n} z_{\varphi_n(x), x}.$$

The sequence of probability measures  $\mu^{(n)}$  is said to be consistent if for any  $n \geq 1$  and  $\sigma_{n-1} \in \Omega_{V_{n-1}}^{a,G}$  the equality holds

$$\sum_{\omega_n \in \Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n) \mathbf{1}(\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}^{a,G}) = \mu^{(n-1)}(\sigma_{n-1}). \quad (2)$$

In this case, by the theorem Kolmogorov (see [19]) there is a unique measure  $\mu$  on  $(\Omega^{a,G}, B)$  such that

$$\mu(\sigma : \sigma|V_n = \sigma_n) = \mu^{(n)}(\sigma_n)$$

for all  $n$  and any  $\sigma_n \in \Omega_{V_n}^{a,G}$ .

**Definition 1.** The measure  $\mu$  that is the limit of a sequence  $\mu^{(n)}$  defined by (1) with consistency condition (2) is called the splitting HC-Gibbs measure (SGM) with  $\lambda > 0$  corresponding to the function  $z : x \in V \setminus \{x^0\} \mapsto z_x$ . Moreover, an HC-Gibbs measure corresponding to a constant function  $z_x \equiv z$  is said to be translation-invariant (TI).

**Definition 2.** [8] The graph is called fertile if there is an activity set  $\lambda$  such that the corresponding Hamiltonian has at least two TIGMs.

Let  $L(G)$  be the set of edges of a graph  $G$ . We let  $A \equiv A^G = (a_{ij})_{i,j}$  denote the adjacency matrix of the graph  $G$ , i.e.,

$$a_{ij} = a_{ij}^G = \begin{cases} 1, & \text{if } \{i, j\} \in L(G), \\ 0, & \text{if } \{i, j\} \notin L(G). \end{cases}$$

The following theorem states a condition on  $z_x$  that guarantees consistency of the measure  $\mu^{(n)}$ .

**Theorem 1.** Probability measures  $\mu^{(n)}$ ,  $n = 1, 2, \dots$ , given by the formula (1), are consistent if and only if for any  $x \in V$  the following equation holds:

$$\begin{cases} z'_{1,x} = \lambda'_1 \prod_{y \in S(x)} \frac{a_{10} + a_{11}z'_{1,y} + a_{12}z'_{2,y}}{a_{00} + a_{01}z'_{1,y} + a_{02}z'_{2,y}}, \\ z'_{2,x} = \lambda'_2 \prod_{y \in S(x)} \frac{a_{20} + a_{21}z'_{1,y} + a_{22}z'_{2,y}}{a_{00} + a_{01}z'_{1,y} + a_{02}z'_{2,y}}, \end{cases} \quad (3)$$

where  $z'_{i,x} = \lambda'_i \frac{z_{i,x}}{z_{0,x}}$ ,  $\lambda'_i = \frac{\lambda_i}{\lambda_0}$ ,  $i = 1, 2$ .

**Proof** The proof is similar to the proof of Theorem 1 in [15].

In this paper we consider the case  $\lambda_1 \neq \lambda_2$  and we study TISGMs in the case fertile graph  $G = \text{wand}$ :

$$\text{wand} : \{0, 1\}\{0, 2\}\{1, 1\}\{2, 2\}.$$

And from (3) we obtain

$$\begin{cases} z_1 = \lambda_1 \left( \frac{1+z_1}{z_1+z_2} \right)^k, \\ z_2 = \lambda_2 \left( \frac{1+z_2}{z_1+z_2} \right)^k. \end{cases} \quad (4)$$

In particular, in the case  $G = \text{wand}$  for  $\lambda_1 = \lambda_2$  the following facts are known :

- In the case  $k = 2$  ( $k = 3$ ) for  $\lambda \leq 1$  ( $\lambda \leq \frac{4}{27}$ ), there is a unique TIGM  $\nu_0$  and for  $\lambda > 1$  ( $\lambda > \frac{4}{27}$ ) there are exactly three TIGMs  $\nu_0, \nu_1, \nu_2$  (see [17],[9]).
- In the case  $k \geq 2$ , for  $0 < \lambda \leq \lambda_{cr}$  there is a unique TIGM and for  $\lambda > \lambda_{cr}$  there are exactly three TIGMs where  $\lambda_{cr} = \frac{1}{k-1} \cdot \left(\frac{2}{k}\right)^k$  (see [16]).
- In the case  $k = 2$ , the measure  $\nu_0$  for  $0 < \lambda < \lambda_0$  and the measures  $\nu_1, \nu_2$  for  $1 < \lambda < \lambda_1$  are extreme and the measure  $\nu_0$  for  $\lambda > \lambda_0$  is not extreme, where  $\lambda_0 \approx 2.287572$ ,  $\lambda_1 \approx 1.303094$  (see [18]).

### Translation-invariant splitting Gibbs measures

**The case  $k = 1$ .**

Let  $k = 1$  and  $\lambda_1 \neq \lambda_2$ . In the case  $G = wand$  for TISGMs from (4) we have

$$\begin{cases} z_1 = \lambda_1 \frac{1+z_1}{z_1+z_2}, \\ z_2 = \lambda_2 \frac{1+z_2}{z_1+z_2}. \end{cases} \quad (5)$$

Obviously,  $z_1 \neq z_2$  since  $\lambda_1 \neq \lambda_2$ . For convenience, we introduce the new notations  $z_1 = x$ ,  $z_2 = y$ , then (5) reads:

$$\begin{cases} y = \frac{\lambda_1 + \lambda_1 x - x^2}{x}, \\ x = \frac{\lambda_2 + \lambda_2 y - y^2}{y}. \end{cases} \quad (6)$$

From  $x > 0$ ,  $y > 0$  we obtain

$$x < \frac{\lambda_1 + \sqrt{\lambda_1^2 + 4\lambda_1}}{2}, \quad y < \frac{\lambda_2 + \sqrt{\lambda_2^2 + 4\lambda_2}}{2}.$$

We rewrite (6) as:

$$\begin{cases} y = f(\lambda_1, x), \\ x = f(\lambda_2, y), \end{cases}$$

where

$$f(\lambda_1, x) = \frac{\lambda_1 + \lambda_1 x - x^2}{x}, \quad f(\lambda_2, y) = \frac{\lambda_2 + \lambda_2 y - y^2}{y}.$$

As a result, solving the system of equations (6) reduces to the following problem:

$$\begin{cases} y = f(\lambda_1, f(\lambda_2, y)), \\ x = f(\lambda_2, f(\lambda_1, x)), \end{cases}$$

or

$$\begin{cases} y = \lambda_1 \frac{y + \lambda_2(1+y) - y^2}{\lambda_2(1+y) - y^2} - \lambda_2 \frac{1+y}{y} + y, \\ x = \lambda_2 \frac{x + \lambda_1(1+x) - x^2}{\lambda_1(1+x) - x^2} - \lambda_1 \frac{1+x}{x} + x. \end{cases} \quad (7)$$

From (7) we find

$$\begin{cases} \lambda_1 = \lambda_2 \frac{(1+y)(\lambda_2(1+y)-y^2)}{y(y+\lambda_2(1+y)-y^2)}, \\ \lambda_2 = \lambda_1 \frac{(1+x)(\lambda_1(1+x)-x^2)}{x(x+\lambda_1(1+x)-x^2)}. \end{cases} \quad (8)$$

Using the symmetry of the system of equations (8), introduce the following function:

$$h(\lambda_i, x) = \lambda_i \frac{(1+x)(\lambda_i(1+x)-x^2)}{x(x+\lambda_i(1+x)-x^2)}, \text{ where } i = 1, 2.$$

We calculate the derivative of the function  $h(\lambda_i, x)$  with respect to  $x$ :

$$h'_x(\lambda_i, x) = \frac{-2\lambda_i x^4 + 2\lambda_i^2 x^3 - \lambda_i^3 x^2 - (\lambda_i^3 + 2\lambda_i^2)x - \lambda_i^3}{x^2(\lambda_i x - x^2 + \lambda_i + x)^2}.$$

We look at the roots of the equation  $h'_x(\lambda_i, x) = 0$  with respect to  $x$ . Then for  $x > 0$ ,

$$q(\lambda_i, x) = -2\lambda_i x^4 + 2\lambda_i^2 x^3 - \lambda_i^3 x^2 - (\lambda_i^3 + 2\lambda_i^2)x - \lambda_i^3 = 0. \quad (9)$$

It is easy to see that the number of sign changes in  $q(\lambda_i, x)$  is two, then the number of the positive roots of the equation  $q(\lambda_i, x) = 0$  is at most two by the Descartes theorem. On the other hand,

$$q(\lambda_i, 0) = -\lambda_i^3 < 0, \quad \lim_{x \rightarrow +\infty} q(\lambda_i, x) = -\infty.$$

In addition,

$$q'_x(\lambda_i, x) = -8\lambda_i x^3 + 6\lambda_i^2 x^2 - 2\lambda_i^3 x - \lambda_i^3 - 2\lambda_i^2.$$

a) Let  $x > \lambda_i$ . Then  $-8\lambda_i x^3 + 6\lambda_i^2 x^2 - 2\lambda_i^3 x - \lambda_i^3 - 2\lambda_i^2 < 0$  since  $\lambda_i x^3 > \lambda_i^2 x^2$ .

b) Let  $x < \lambda_i$  and  $8\lambda_i x^3 + 2\lambda_i^3 x > 8\lambda_i^2 x^2$ . Then  $-8\lambda_i x^3 + 6\lambda_i^2 x^2 - 2\lambda_i^3 x - \lambda_i^3 - 2\lambda_i^2 < 0$ . It means that  $q(\lambda_i, x) < 0$ , from this  $h'_x(\lambda_i, x) < 0$  for  $\forall x > 0$ , and

$$\lim_{x \rightarrow +0} h(\lambda_i, x) = +\infty, \text{ and } \lim_{x \rightarrow +\infty} h(\lambda_i, x) = -\infty.$$

From these, it follows that the  $h(\lambda_i, x)$  function is decreasing.

Consequently, the following theorem be obtained:

**Theorem 2.** *Let  $k = 1$   $\lambda_1 \neq \lambda_2$ . Then for the three state Hard-Core model in the case  $G = \text{wand}$ , there exists exactly one TISGM for  $\lambda_i > 0$ ,  $i = 1, 2$ .*

**The case  $k = 2$ .**

Let  $k = 2$  and  $\lambda_1 \neq \lambda_2$ . In the case  $G = \text{wand}$  for TISGMs from (4) we have

$$\begin{cases} z_1 = \lambda_1 \left( \frac{1+z_1}{z_1+z_2} \right)^2, \\ z_2 = \lambda_2 \left( \frac{1+z_2}{z_1+z_2} \right)^2. \end{cases} \quad (10)$$

Obviously,  $z_1 \neq z_2$  since  $\lambda_1 \neq \lambda_2$ . For convenience, we introduce the new notations  $z_1 = x$ ,  $z_2 = y$ , then the system (10) is formed:

$$\begin{cases} y = \sqrt{\frac{\lambda_1}{x}}(1+x) - x, \\ x = \sqrt{\frac{\lambda_2}{y}}(1+y) - y. \end{cases} \quad (11)$$



From  $z_1 > 0, z_2 > 0$ ,

$$x < \left( \frac{1}{6}B(\lambda_1) + \frac{2}{3} \frac{\lambda_1}{B(\lambda_1)} + \frac{1}{3}\sqrt{\lambda_1} \right)^2, \quad y < \left( \frac{1}{6}B(\lambda_2) + \frac{2}{3} \frac{\lambda_2}{B(\lambda_2)} + \frac{1}{3}\sqrt{\lambda_2} \right)^2,$$

where  $B(\lambda_i) = \left( 108\sqrt{\lambda_i} + 8\lambda_i^{\frac{3}{2}} + 12\sqrt{12\lambda_i^2 + 81\lambda_i} \right)^{\frac{1}{3}}$ ,  $i = 1, 2$ .

We rewrite (11) as:

$$\begin{cases} y = g(\lambda_1, x), \\ x = g(\lambda_2, y), \end{cases}$$

where

$$g(\lambda_1, x) = \sqrt{\frac{\lambda_1}{x}}(1+x) - x, \quad g(\lambda_2, y) = \sqrt{\frac{\lambda_2}{y}}(1+y) - y.$$

As a result, solving the system of equations (11) reduces to the following problem:

$$\begin{cases} y = g(\lambda_1, g(\lambda_2, y)), \\ x = g(\lambda_2, g(\lambda_1, x)), \end{cases}$$

or

$$\begin{cases} x = \sqrt{\frac{\lambda_2\sqrt{x}}{\sqrt{\lambda_1(1+x)-x\sqrt{x}}}} \frac{\sqrt{x+\sqrt{\lambda_1(1+x)-x\sqrt{x}}}}{\sqrt{x}} - \sqrt{\frac{\lambda_1}{x}}(1+x) + x, \\ y = \sqrt{\frac{\lambda_1\sqrt{y}}{\sqrt{\lambda_2(1+y)-y\sqrt{y}}}} \frac{\sqrt{y+\sqrt{\lambda_2(1+y)-y\sqrt{y}}}}{\sqrt{y}} - \sqrt{\frac{\lambda_2}{y}}(1+y) + y. \end{cases} \quad (12)$$

From (12) we find

$$\begin{cases} \lambda_1 = \frac{\lambda_2(1+y)^2(\sqrt{\lambda_2(1+y)-y\sqrt{y}})}{\sqrt{y}(\sqrt{y+\sqrt{\lambda_2(1+y)-y\sqrt{y}}})^2}, \\ \lambda_2 = \frac{\lambda_1(1+x)^2(\sqrt{\lambda_1(1+x)-x\sqrt{x}})}{\sqrt{x}(\sqrt{x+\sqrt{\lambda_1(1+x)-x\sqrt{x}}})^2}. \end{cases} \quad (13)$$

Using the symmetry of the system of equations (13), introduce the following function:

$$\beta(\lambda_i, x) = \frac{\lambda_i(1+x)^2(\sqrt{\lambda_i(1+x)-x\sqrt{x}})}{\sqrt{x}(\sqrt{x+\sqrt{\lambda_i(1+x)-x\sqrt{x}}})^2}, \quad \text{where } i = 1, 2.$$

We get the derivative of the function  $\beta(\lambda_i, x)$  with respect to  $x$ :

$$\beta'_x(\lambda_i, x) = \frac{1}{2} \frac{\lambda_i(1+x)(\sqrt{\lambda_i}x^{7/2} - 7\sqrt{\lambda_i}x^{5/2} - 5\sqrt{\lambda_i}x^{3/2} - \lambda_i x^3 - \lambda_i x^2 + 8x^3 + 3\sqrt{\lambda_i}x^{1/2} + \lambda_i x + \lambda_i)}{x^{3/2}(\sqrt{\lambda_i}x + \sqrt{\lambda_i} + \sqrt{x} - x^{3/2})^3}$$

We look at the roots of the equation  $\beta'(\lambda_i, x) = 0$  with respect to  $x$ .  $\beta'(\lambda_i, x) = 0$  is equivalent to

$$p(\lambda_i, x) = \sqrt{\lambda_i}x^{7/2} + (8 - \lambda_i)x^3 - 7\sqrt{\lambda_i}x^{5/2} - \lambda_i x^2 - 5\sqrt{\lambda_i}x^{3/2} + \lambda_i x + 3\sqrt{\lambda_i}\sqrt{x} + \lambda_i = 0.$$

It is easy to see that the number of sign changes in  $p(\lambda_i, x)$  is two or does not, then the number of the positive roots of the equation  $p(\lambda_i, x) = 0$  is at most two by the Descartes theorem. On the other hand,

$$p(\lambda_i, 0) = \lambda_i > 0, \quad \lim_{x \rightarrow +\infty} p(\lambda_i, x) = +\infty.$$

In addition,

$$p(\lambda_i, 1) = 8(1 - \sqrt{\lambda_i}).$$

Then  $\lambda_i$  has a value  $\lambda_0$  such that the number of the positive roots of the equation  $s(\lambda_i, x) = 0$  is two. Let us denote these numbers as  $x_1(\lambda_0)$  and  $x_2(\lambda_0)$ . That is,  $x_1(\lambda_0)$  and  $x_2(\lambda_0)$  are the roots of the following equation:

$$p(\lambda_i, x) = 0.$$

Then the function  $\beta(\lambda_i, x)$  is decreasing in the interval  $0 < x < x_1(\lambda_0)$  or  $x > x_2(\lambda_0)$ , and increasing in the interval  $x_1(\lambda_0) < x < x_2(\lambda_0)$ . Then the function  $\beta(\lambda_i, x)$  reaches a local maximum at  $x = x_2(\lambda_0)$  and reaches a local minimum at  $x = x_1(\lambda_0)$ . As a result, we get the following for the parameter  $\lambda_i$ :

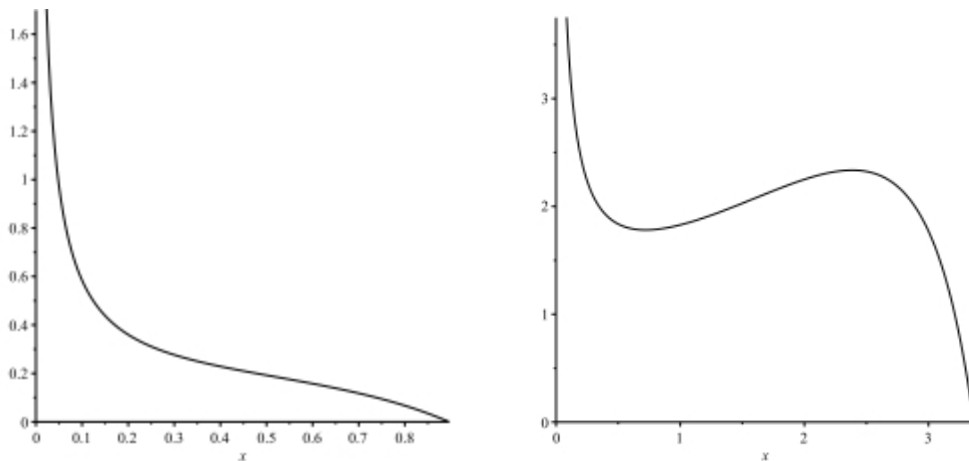


Figure 1. Graph of the function  $\beta(\lambda_i, x)$ . On the left for  $\lambda_i = 0.2$ , on the right for  $\lambda_i = 2$ .

1) If the parameter  $\lambda_i$  varies in the interval  $0 < \lambda_i < \lambda_i(x_1(\lambda_0), \lambda_0)$ , then for each  $\lambda_i$  there is a unique corresponding  $x$ .

2) If  $\lambda_i = \lambda_i(x_1(\lambda_0), \lambda_0)$  or  $\lambda_i = \lambda_i(x_2(\lambda_0), \lambda_0)$ , then for each  $\lambda_i$  there are two corresponding  $x$ .

3) If the parameter  $\lambda_i$  varies in the interval  $\lambda_i(x_1(\lambda_0), \lambda_0) < \lambda_i < \lambda_i(x_2(\lambda_0), \lambda_0)$ , then for each  $\lambda_i$  there are three corresponding  $x$ .

4) If the parameter  $\lambda_i$  varies in the interval  $\lambda_i > \lambda_i(x_2(\lambda_0), \lambda_0)$ , then for each  $\lambda_i$  there is a unique corresponding  $x$ .

Therefore, we have the following result:

**Theorem 3.** *Let  $k = 2$ . Then for the three state Hard-Core model in the case  $G = \text{wand}$ , the number of TISGMs is*

$$N = \begin{cases} 1, & \text{if } 0 < \lambda_i < \nu_1(\lambda_i) \text{ or } \lambda_i > \nu_2(\lambda_i), \\ 2, & \text{if } \lambda_i = \nu_1(\lambda_i) \text{ or } \lambda_i = \nu_2(\lambda_i), \\ 3, & \text{if } \nu_1(\lambda_i) < \lambda_i < \nu_2(\lambda_i). \end{cases}$$

Here  $x_1$  and  $x_2$  are the positive roots of the following equation:

$$\sqrt{\lambda_i}x^{7/2} + (8 - \lambda_i)x^3 - 7\sqrt{\lambda_i}x^{5/2} - \lambda_i x^2 - 5\sqrt{\lambda_i}x^{3/2} + \lambda_i x + 3\sqrt{\lambda_i}\sqrt{x} + \lambda_i = 0,$$

and

$$\nu_1(\lambda_i) = \frac{\lambda_i(1+x_1)^2(\sqrt{\lambda_i}(1+x_1) - x_1\sqrt{x_1})}{\sqrt{x_1}(\sqrt{x_1} + \sqrt{\lambda_i}(1+x_1) - x_1\sqrt{x_1})^2}, \quad \nu_2(\lambda_i) = \frac{\lambda_i(1+x_2)^2(\sqrt{\lambda_i}(1+x_2) - x_2\sqrt{x_2})}{\sqrt{x_2}(\sqrt{x_2} + \sqrt{\lambda_i}(1+x_2) - x_2\sqrt{x_2})^2}.$$

**Remark 1.** If  $\lambda_1 = \lambda_2$ , then  $\nu_1(\lambda_i) = \nu_2(\lambda_i)$  and the results derived from above Theorem 3 correspond with the results obtained from the work [18].

**The case  $k = 3$**

Let  $k = 3$  and  $\lambda_1 \neq \lambda_2$ . In the case  $G = wand$  for TISGMs we write the following from (4)

$$\begin{cases} z_1 = \lambda_1 \left( \frac{1+z_1}{z_1+z_2} \right)^3, \\ z_2 = \lambda_2 \left( \frac{1+z_2}{z_1+z_2} \right)^3. \end{cases} \quad (14)$$

Obviously,  $z_1 \neq z_2$  since  $\lambda_1 \neq \lambda_2$ . For convenience, we introduce the notation  $z_1 = x$ ,  $z_2 = y$  and define the following from (14):

$$\begin{cases} y = (1+x) \sqrt[3]{\frac{\lambda_1}{x}} - x, \\ x = (1+y) \sqrt[3]{\frac{\lambda_2}{y}} - y. \end{cases} \quad (15)$$

From  $z_1 > 0$ ,  $z_2 > 0$

$$\lambda_1 > \frac{x^4}{(1+x)^3}, \quad \text{and} \quad \lambda_2 > \frac{y^4}{(1+y)^3}.$$

Then we can write (15) as:

$$\begin{cases} y = \psi(\lambda_1, x), \\ x = \psi(\lambda_2, y). \end{cases}$$

Where

$$\psi(\lambda_1, x) = (1+x) \sqrt[3]{\frac{\lambda_1}{x}} - x, \quad \psi(\lambda_2, y) = (1+y) \sqrt[3]{\frac{\lambda_2}{y}} - y.$$

As a result, solving the system of (15) equations reduces to the following problem:

$$\begin{cases} y = \psi(\lambda_1, \psi(\lambda_2, y)), \\ x = \psi(\lambda_2, \psi(\lambda_1, x)). \end{cases}$$

That is,

$$\begin{cases} x = \sqrt[3]{\frac{\lambda_2 \sqrt[3]{x}}{\sqrt[3]{\lambda_1(1+x)} - x \sqrt[3]{x}}} \frac{\sqrt[3]{x} + \sqrt[3]{\lambda_1(1+x)} - x \sqrt[3]{x}}{\sqrt[3]{x}} - \sqrt[3]{\frac{\lambda_1}{x}}(1+x) + x, \\ y = \sqrt[3]{\frac{\lambda_1 \sqrt[3]{y}}{\sqrt[3]{\lambda_2(1+y)} - y \sqrt[3]{y}}} \frac{\sqrt[3]{y} + \sqrt[3]{\lambda_2(1+y)} - y \sqrt[3]{y}}{\sqrt[3]{y}} - \sqrt[3]{\frac{\lambda_2}{y}}(1+y) + y. \end{cases} \quad (16)$$

From (16) we find

$$\begin{cases} \lambda_1 = \frac{\lambda_2 (1+y)^3 \left( \sqrt[3]{\lambda_2(1+y)-y} \sqrt[3]{y} \right)}{\sqrt[3]{y} \left( \sqrt[3]{y} + \sqrt[3]{\lambda_1(1+y)-y} \sqrt[3]{y} \right)^3}, \\ \lambda_2 = \frac{\lambda_1 (1+x)^3 \left( \sqrt[3]{\lambda_1(1+x)-x} \sqrt[3]{x} \right)}{\sqrt[3]{x} \left( \sqrt[3]{x} + \sqrt[3]{\lambda_1(1+x)-x} \sqrt[3]{x} \right)^3}. \end{cases} \quad (17)$$

Using the symmetry of the system of equations (17), we introduce the following function:

$$\vartheta(\lambda_i, x) = \frac{\lambda_i (1+x)^3 \left( \sqrt[3]{\lambda_i(1+x)-x} \sqrt[3]{x} \right)}{\sqrt[3]{x} \left( \sqrt[3]{x} + \sqrt[3]{\lambda_i(1+x)-x} \sqrt[3]{x} \right)^3}.$$

We get the derivative of the function  $\vartheta(\lambda_1, x)$  with respect to  $x$ :

$$\begin{aligned} \vartheta'_x(\lambda_i, x) &= \frac{1}{3} \lambda_i (1+x)^2 \times \\ &\times \frac{-2\lambda_i^{1/3} x^{10/3} + 16\lambda_i^{1/3} x^{7/3} - 18x^{8/3} + 2\lambda_i^{2/3} x^3 + 3\lambda_i^{2/3} x^2 + 14\lambda_i^{1/3} x^{4/3} - \lambda_i^{2/3} - 4\lambda_i^{1/3} x^{1/3}}{x^{4/3} (-x^{4/3} + \lambda_i^{1/3} x + \lambda_i^{1/3} + x^{1/3})^4}. \end{aligned}$$

We look the roots of the equation  $\vartheta'_x(\lambda_i, x) = 0$  with respect to  $x$ . Then from  $\vartheta'_x(\lambda_i, x) = 0$  we get

$$\delta(\lambda_i, x) = \lambda_i^{2/3} (2x^3 + 3x^2 - 1) + \lambda_i^{1/3} (-2x^{10/3} + 16x^{7/3} + 14x^{4/3} - 4x^{1/3}) - 18x^{8/3} = 0.$$

We write  $\delta(\lambda_i, x)$  the following:

$$\delta(\lambda_i, x) = -2\lambda_i^{1/3} x^{10/3} + 2\lambda_i^{2/3} x^3 - 18x^{8/3} + 16\lambda_i^{1/3} x^{7/3} + 3\lambda_i^{2/3} x^2 + 14\lambda_i^{1/3} x^{4/3} - 4\lambda_i^{1/3} x^{1/3} - \lambda_i^{2/3}.$$

It is not difficult to see that the number of sign changes in  $\delta(\lambda_i, x)$  is four, then the number of the positive roots of the equation  $\delta(\lambda_i, x) = 0$  is four or two or zero by the Descartes theorem. On the other hand,

$$\delta(\lambda_i, 0) = -\lambda_i^{2/3} < 0, \quad \lim_{x \rightarrow +\infty} \delta(\lambda_i, x) = -\infty.$$

In addition,

$$\delta(\lambda_i, 1) = 4\sqrt[3]{\lambda_i^2} + 24\sqrt[3]{\lambda_i} - 18.$$

Then  $\lambda_i$  has a value  $\lambda_0$  such that the number of the positive roots of the equation  $\delta(\lambda_i, x) = 0$  is two or four. We show that  $\delta(\lambda_i, x) = 0$  equation has at most two solutions in  $x \in (0, \lambda_i)$ ,  $\lambda_i > \lambda_0$ . For this we show that the equation  $\delta'(\lambda_i, x) = 0$  has a unique positive solution.

$$\delta'(\lambda_i, x) = -\frac{20}{3} \lambda_i^{1/3} x^{7/3} + 6\lambda_i^{2/3} x^2 - 48x^{5/3} + \frac{112}{3} \lambda_i^{1/3} x^{4/3} + 6\lambda_i^{2/3} x + \frac{56}{3} \lambda_i^{1/3} x^{1/3} - \frac{4}{3} \lambda_i^{1/3} x^{-2/3} = 0$$

We introduce the  $x = t^3$  substitution, then:

$$\delta'(\lambda_i, t) = -\frac{20}{3} \lambda_i^{1/3} t^7 + 6\lambda_i^{2/3} t^6 - 48t^5 + \frac{112}{3} \lambda_i^{1/3} t^4 + 6\lambda_i^{2/3} t^3 + \frac{56}{3} \lambda_i^{1/3} t - \frac{4}{3} \lambda_i^{1/3} t^{-2} = 0.$$

Using Ferrari formulas for  $\delta'(\lambda_i, t) = 0$ , we find the unique positive  $t(\lambda_i)$  solution. Then the  $\delta(\lambda_i, x)$  function is increasing in  $x < \sqrt[3]{\lambda_i}$  and decreasing in  $x > \sqrt[3]{\lambda_i}$ . As a result, there are at most two solutions of the  $\delta(\lambda_i, x) = 0$  equation in the interval  $x \in (0, l)$  and  $\lambda > \lambda_i$ . Let us denote these numbers as  $x_1(\lambda_0)$  and  $x_2(\lambda_0)$ . That is,  $x_1(\lambda_0)$  and  $x_2(\lambda_0)$  are the roots of the following equation:

$$\delta(\lambda_i, x) = 0.$$

Then the function  $\vartheta(\lambda_i, x)$  is decreasing for the interval  $0 < x < x_1(\lambda_0)$  or  $x > x_2(\lambda_0)$ , and increasing for  $x_1(\lambda_0) < x < x_2(\lambda_0)$ . Then the function  $\vartheta(\lambda_i, x)$  achieves a local maximum at  $x = x_2(\lambda_0)$  and achieves a local minimum at  $x = x_1(\lambda_0)$ . As a result, we get the following for the parameter  $\lambda_i$ :

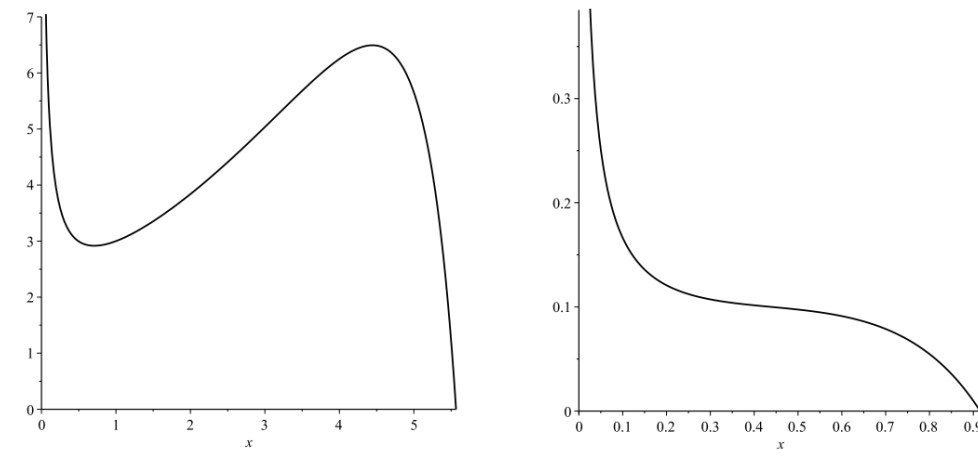


Figure 1. Graph of the function  $\vartheta(\lambda_i, x)$ . On the left for  $\lambda_i = 1$ , on the right for  $\lambda_i = 0.1$ .

- 1) If the parameter  $\lambda_i$  varies in the interval  $0 < \lambda_i < \lambda_i(x_1(\lambda_0), \lambda_0)$ , then for each  $\lambda_i$  there is a unique  $x$  corresponding.
- 2) If  $\lambda_i = \lambda_i(x_1(\lambda_0), \lambda_0)$ , then for each  $\lambda_i$  there are two  $x$  corresponding.
- 3) If the parameter  $\lambda_i$  varies in the interval  $\lambda_i(x_1(\lambda_0), \lambda_0) < \lambda_i < \lambda_i(x_2(\lambda_0), \lambda_0)$ , then for each  $\lambda_i$  there are three  $x$  corresponding.
- 4) If  $\lambda_i = \lambda_i(x_2(\lambda_0), \lambda_0)$ , then for each  $\lambda_i$  there are three  $x$  corresponding.
- 5) If the parameter  $\lambda_i$  varies in the interval  $\lambda_i > \lambda_i(x_2(\lambda_0), \lambda_0)$ , then for each  $\lambda_i$  there is a unique  $x$  corresponding.

We get the following theorem from the above:

**Theorem 4.** *Let  $k = 3$ . Then for the three state Hard-Core model in the case  $G = \text{wand}$ , the number of TISGMs is*

$$N = \begin{cases} 1, & \text{if } 0 < \lambda_i < \eta_1(\lambda_i) \text{ or } \lambda_i > \eta_2(\lambda_i), \\ 2, & \text{if } \lambda_i = \eta_1(\lambda_i) \text{ or } \lambda_i = \eta_2(\lambda_i), \\ 3, & \text{if } \eta_1(\lambda_i) < \lambda_i < \eta_2(\lambda_i). \end{cases}$$

Here  $x_1$  and  $x_2$  are the positive roots of the following equation:

$$-2\sqrt[3]{x^{10}}\sqrt[3]{\lambda_i} + 2x^3\sqrt[3]{\lambda_i^2} - 18\sqrt[3]{x^8} + 16\sqrt[3]{x^7}\sqrt[3]{\lambda_i} + 3x^2\sqrt[3]{\lambda_i^2} + 14\sqrt[3]{\lambda_i}\sqrt[3]{x^4} - 4\sqrt[3]{\lambda_i}\sqrt[3]{x} - \sqrt[3]{\lambda_i^2} = 0,$$

and

$$\eta_1(\lambda_i) = \frac{\lambda_i(1+x_1)^4 \left( \sqrt[3]{\lambda_i}(1+x_1) - x_1 \sqrt[3]{x_1} \right)}{\sqrt[3]{x_1} \left( \sqrt[3]{\lambda_i} + x_1 \sqrt[3]{\lambda_i} - \sqrt[3]{x_1^4} \right)^3},$$

$$\eta_2(\lambda_i) = \frac{\lambda_i(1+x_2)^4 \left( \sqrt[3]{\lambda_i}(1+x_2) - x_2 \sqrt[3]{x_2} \right)}{\sqrt[3]{x_2} \left( \sqrt[3]{\lambda_i} + x_2 \sqrt[3]{\lambda_i} - \sqrt[3]{x_2^4} \right)^3}.$$

**Remark 2.** If  $\lambda_1 = \lambda_2$ , then  $\eta_1(\lambda_i) = \eta_2(\lambda_i)$  and the results derived from above Theorem 4 correspond with the results obtained in the work [16].

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### РЕЗЮМЕ

Ushbu ishda Keli daraxtida unumdor graf aktivlik parametri  $\lambda_i > 0$ ,  $i = 0, 1, 2$  bo'lganda, uch holatli Hard-Core (HC) modeli uchun translyatsion-invariant Gibbs o'lchovlarini o'rganamiz. Ma'lumki, uch holatli HC modelida unumdor graflarning to'rt turi mavjud bo'lib, ular: kalit, jez, sirtmoq va hushtak. Jez holatida birinchi, ikkinchi va uchun tartibli Keli daraxtida  $\lambda_i$  parametrlarning quyidagi ba'zi shartlari asosida translyatsion-invariant Gibbs o'lchovlarining aniq soni aniqlangan.

**Kalit so'zlar:** Cayley tree, configuration, fertile graphs, Hard-Core model, Gibbs measure, translation-invariant Gibbs measure.

### РЕЗЮМЕ

В данной работе мы исследуем трансляционно-инвариантные меры Гиббса для плодородных Hard-Core (HC) моделей с тремя состояниями с параметром активности  $\lambda_i > 0$ ,  $i = 0, 1, 2$ , на дереве Кэли. Известно, что существует четыре типа таких моделей: петля, свисток, жезл и ключ. В случае жезл при некоторых условиях на параметры на дереве Кэли порядка один, два и три, найдено точное количество трансляционно-инвариантных мер Гиббса.

**Ключевые слова:** Дерево Кэли, конфигурация, плодородный граф, модель Hard-core, мера Гиббса, трансляционно-инвариантная мера Гиббса.

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## ON A FAMILY OF VOLTERRA CUBIC STOCHASTIC OPERATORS

Okhunova M. O. \*

## RESUME

In present paper we consider a family of discrete time Kolmogorov systems of three interaction population depending on a parameter  $\theta$ . We show that there is the critic value  $\theta^*$  such that if  $\theta \in (\theta^*, 1]$  this evolution operator is a non-ergodic transformation and it has property being regular when  $\theta \in [0, \theta^*]$ . We give some biological interpretations of our results.

**Key words:** quadratic stochastic operator, cubic stochastic operator, Volterra operator, regular and non-regular operator.

## Introduction

The notion quadratic stochastic operator (QSO) first introduced by Bernstein[1]. Such operators frequently arises in many models of mathematical genetics, namely, theory of heredity and the theory of QSOs developed in many works (see e.g. [9],[14],[17],[18],[21],[22],[27],[28]). Let  $E = \{1, \dots, m\}$  be a finite set and the set of all probability distributions on  $E$

$$S^{m-1} = \left\{ \mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m : x_i \geq 0, \forall i \in E, \sum_{i=1}^m x_i = 1 \right\}$$

be the  $m - 1$ - dimensional simplex.

A *quadratic stochastic operator* is a mapping  $V : S^{m-1} \mapsto S^{m-1}$  of the simplex into itself, of the form  $V(\mathbf{x}) = \mathbf{x}' \in S^{m-1}$ , where

$$x'_k = \sum_{i,j \in E} p_{ij,k} x_i x_j, \quad k \in E, \quad (1)$$

and the coefficients  $p_{ij,k}$  satisfy

$$p_{ij,k} = p_{ji,k} \geq 0, \quad \sum_{k \in E} p_{ij,k} = 1, \quad i, j, k \in E. \quad (2)$$

The *trajectory* (orbit)  $\{\mathbf{x}^{(n)}\}, n = 0, 1, \dots$  of  $V$  for an initial value  $\mathbf{x}^{(0)} \in S^{m-1}$  is defined by

$$\mathbf{x}^{(n+1)} = V(\mathbf{x}^{(n)}) = V^{n+1}(\mathbf{x}^{(0)}), \quad n = 0, 1, 2, \dots$$

For nonlinear (quadratic) dynamical system (1),(2) Ulam [27] suggested an analogue of a measure-theoretic ergodicity in the form of the following ergodic hypothesis: a QSO  $V$  is said to be *ergodic* if the limit

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} V^k(\mathbf{x})$$

\*Okhunova M. O. – Fergana State University, oxunova.muxlisaxon@mail.ru



exists for any  $\mathbf{x} \in S^{m-1}$ .

On the basis of numerical calculations Ulam, in 1960 in [27], conjectured that the ergodic theorem holds for any QSO.

The QSO  $V$  is called *Volterra*, if  $p_{ij,k} = 0$  for any  $k \notin \{i, j\}$ ,  $i, j, k \in E$ . In 1977 in [27], Zakharevich considered the Volterra QSO on  $S^2$

$$V : (x_1^2 + 2x_1x_2, x_2^2 + 2x_2x_3, x_3^2 + 2x_1x_3) \quad (3)$$

and showed that it is a non-ergodic transformation, that is he proved that Ulam's conjecture is false in general. Later in [8] established a necessary condition for a QSO defined on  $S^2$  to be a non-ergodic transformation, that is Zakharevich's result was generalized to a class of Volterra QSOs defined on  $S^2$ . In [9] have shown the correlation between non-ergodicity of Volterra QSOs and rock-paper-scissors games, so Zakharevich's example (3) can be reinterpreted in terms game theory as a paper-rock-scissors game.

The biological treatment of non-ergodicity of a QSO is the following: in the long run the behavior of the distributions of species is unstable. For a recent review on the theory of quadratic stochastic operators see [10].

Note that one can consider triple different sorts of individuals and obtain an other sort which is totally different from previous taken sorts. The genetic engineers found a solution of problem with genetic defects in the mitochondria. The new technique which was developed in Newcastle, uses a modified version of IVF to combine the healthy mitochondria of a donor woman with DNA of the two parents. It results in babies with 0.1% of their DNA from the second woman and is a permanent change that would echo down through the generations<sup>1</sup>. From these biological facts it is natural to consider another class of nonlinear evolution operators which are different from QSOs.

In [19,20,24], a notion cubic stochastic operator (CSO) was introduced and investigated. Regularity of some Volterra CSOs were proved in [12,15,24]. In [13] by authors the results of the works [12,24] were generalised up to a class of constrained Volterra CSOs. A CSO which is a convex combination of a regular and non-ergodic operators was studied in [16]. A family of non-ergodic Volterra CSOs was considered in [23]. Random dynamics of Volterra CSOs was studied in [11]. In [25] by authors were given construction of a cubic stochastic operator. Another class of CSO which is called the class of conditional cubic stochastic operators were studied in [3].

## Main results

The *cubic stochastic operator* is a mapping  $W : S^{m-1} \mapsto S^{m-1}$  of the form

$$W(\mathbf{x})_l = \sum_{i,j,k=1}^m p_{ijk,l} x_i x_j x_k, \quad l = 1, \dots, m, \quad (4)$$

where  $p_{ijk,l}$  are coefficients of heredity such that

$$p_{ijk,l} \geq 0, \quad \sum_{l=1}^m p_{ijk,l} = 1, \quad i, j, k, l = 1, \dots, m. \quad (5)$$

<sup>1</sup><http://www.bbc.com/news/health-31594856>

and we suppose that the coefficients  $p_{ijk,l}$  don't change for any permutation of  $i, j, k$ .

Note that  $W$  (defined by (4)) is a non-linear operator, and its dimension increases with  $m$ . Higher dimensional dynamical systems are important, but only relatively few dynamical phenomena have yet been analyzed (see e.g. [4,5,26]).

For a given  $\mathbf{x}^{(0)} \in S^{m-1}$  the trajectory  $\{\mathbf{x}^{(n)}, n = 0, 1, 2, \dots\}$  of initial point  $\mathbf{x}^{(0)}$  under action of CSO (4) is defined by  $\mathbf{x}^{(n+1)} = W(\mathbf{x}^{(n)})$ , where  $n = 0, 1, 2, \dots$  with  $\mathbf{x} = \mathbf{x}^{(0)}$ .

One of the main problems in mathematical biology consists in the study of the asymptotical behavior of the trajectories. In [19,20,24] this problem was considered for a class of Volterra CSO. A class Volterra CSO is defined by (4), (5) and with additional assumption

$$p_{ijk,l} = 0 \text{ if } l \notin \{i, j, k\}. \quad (6)$$

The biological treatment of conditions (6) is rather clear: the offspring repeats the genotype of one of its parents.

The following notations we will use in below. We let  $\partial S^{m-1}$  denote the boundary of the simplex  $S^{m-1}$ ,  $\partial S^{m-1} = \{\mathbf{x} \in S^{m-1} : x_i = 0 \text{ for at least one } i \in E\}$ ; a face of the simplex  $S^{m-1}$  be the set  $\Gamma_\alpha = \{\mathbf{x} \in S^{m-1} : x_i = 0, \quad i \notin \alpha \subset E\}$ ; the interior of  $S^{m-1}$  be the set  $\int S^{m-1} = \{\mathbf{x} \in S^{m-1} : x_1 x_2 \cdots x_m > 0\}$ ; let  $M_i = (\delta_{1i}, \delta_{2i}, \dots, \delta_{mi}) \in S^{m-1}$ ,  $i = 1, \dots, m$ , denote the vertices of the simplex  $S^{m-1}$ , where  $\delta_{ij}$  is the Kronecker delta; the point  $\mathbf{c} = (1/m, 1/m, \dots, 1/m)$  be the center of  $S^{m-1}$ . Let  $\omega(\mathbf{x}^0)$  be the set of limit points of the trajectory  $\{W^k(\mathbf{x}^0) \in S^{m-1} : k = 0, 1, 2, \dots\}$ . Using Lyapunov functions, one can handle on the set of limit points. Recall the definition of a Lyapunov function.

**Definition 1.** A continuous function  $\varphi: S^{m-1} \rightarrow R$  is called a Lyapunov function for a CSO  $W$  if there exists the limit  $\lim_{n \rightarrow \infty} \varphi^n(\mathbf{x})$  for all  $\mathbf{x} \in S^{m-1}$ .

Note that a Lyapunov function is very helpful to describe an upper estimate of  $\omega(\mathbf{x}^0)$ . However there is no general recipe on how to find such Lyapunov functions.

**Definition 2.** A point  $\mathbf{x} \in S^{m-1}$  is called a fixed point of a CSO  $W$  if  $W(\mathbf{x}) = \mathbf{x}$ .

Let  $DW(\mathbf{x}^*) = (\partial W_i / \partial x_j)(\mathbf{x}^*)$  be a Jacobian of  $W$  at the point  $\mathbf{x}^*$ .

**Definition 3.**[4] A fixed point  $\mathbf{x}^*$  is called hyperbolic if its Jacobian  $DW(\mathbf{x}^*)$  has no eigenvalues on the unit circle in  $\mathbb{C}$ .

**Definition 4.**[4] A hyperbolic fixed point  $\mathbf{x}^*$  is called:

- i) attracting if all the eigenvalues of the Jacobian  $DW(\mathbf{x}^*)$  are in the unit disk;
- ii) repelling if all the eigenvalues of the Jacobian  $DW(\mathbf{x}^*)$  are outside the closed unit disk;
- iii) a saddle otherwise.

In the present work we consider on  $S^2$  the following family of Volterra CSOs defined by

$$W : \begin{cases} x'_1 = x_1(x_1^2 + 3ax_1x_2 + 3dx_1x_3 + 3(1-b)x_2^2 + 3(1-c)x_3^2 + 6e_1x_2x_3), \\ x'_2 = x_2(x_2^2 + 3bx_1x_2 + 3ex_2x_3 + 3(1-a)x_1^2 + 3(1-f)x_3^2 + 6e_2x_1x_3), \\ x'_3 = x_3(x_3^2 + 3cx_1x_3 + 3fx_2x_3 + 3(1-d)x_1^2 + 3(1-e)x_2^2 + 6e_3x_1x_2). \end{cases} \quad (7)$$

where  $a, b, c, d, e, f, e_1, e_2, e_3 \in [0, 1]$  and  $e_1 + e_2 + e_3 = 1$ .

If  $a, b, c, d, e, f = \frac{2}{3}$  then we denote corresponding edges by  $\overline{M_1M_2}, \overline{M_1M_3}, \overline{M_2M_3}$ .

One can rewrite the operator (7) in the following form

$$W : \begin{cases} x'_1 = x_1 (1 + (6e_1 - 2) x_2 x_3), \\ x'_2 = x_2 (1 + (6e_2 - 2) x_1 x_3), \\ x'_3 = x_3 (1 + (6e_3 - 2) x_1 x_2), \end{cases} \quad (8)$$

**Theorem 1.** For the operator  $W$  the followings hold:

- i) If  $e_1 = e_2 = e_3$  then  $W$  is the identical operator;
- ii) Let  $e_i = \max(e_1, e_2, e_3)$  and
  - a) if  $e_i > e_j \geq \frac{1}{3}, e_k < \frac{1}{3}, i, j, k \in \{1, 2, 3\}$  and  $i \neq j, k \neq i$  then all trajectories converge to edge  $\overline{M_iM_j}$ ;
  - b) if  $e_i > \frac{1}{3}, e_j < e_k < \frac{1}{3}, i, j, k \in \{1, 2, 3\}$  and  $i \neq j, k \neq i$  then all trajectories converge to  $M_i$  for any  $\mathbf{x}^{(0)} \in \text{int } S^2$ ;
- iii) If  $e_i = e_j > e_k$  then all trajectories converge to edge  $\overline{M_iM_j}$  for any  $\mathbf{x}^{(0)} \in \text{int } S^2$ .

**Proof.** i) Evidently, from  $e_1 + e_2 + e_3 = 1$  and  $e_1 = e_2 = e_3$  it follows  $e_1 = e_2 = e_3 = \frac{1}{3}$ . Using the latter from (7) one has

$$W : \begin{cases} x'_1 = x_1 \left( 1 + \left( 6 \cdot \frac{1}{3} - 2 \right) x_2 x_3 \right), \\ x'_2 = x_2 \left( 1 + \left( 6 \cdot \frac{1}{3} - 2 \right) x_1 x_3 \right), \\ x'_3 = x_3 \left( 1 + \left( 6 \cdot \frac{1}{3} - 2 \right) x_1 x_2 \right), \end{cases}$$

that is,  $x'_1 = x_1, x'_2 = x_2, x'_3 = x_3$ . Therefore in this case the operator  $W$  (8) is the identity operator.

ii) Let  $e_i = \max(e_1, e_2, e_3)$ . It is easy to see that the point of the edges  $\overline{M_1M_2}, \overline{M_2M_3}$  and  $\overline{M_1M_3}$  are fixed points of  $W$ . For the definiteness we assume that  $i = 1, j = 2$  and  $k = 3$ . The rest cases can be considered in a similar manner.

a) Assume that  $e_1 > e_2 \geq \frac{1}{3}, e_3 < \frac{1}{3}$  and let  $\mathbf{x}^{(0)} \in \text{int } S^2$  be an initial point. Then from (8) we have

$$\begin{aligned} x'_1 &= x_1 (1 + (6e_1 - 2) x_2 x_3) \geq x_1, & x'_2 &= x_2 (1 + (6e_2 - 2) x_1 x_3) \geq x_2, \\ x'_3 &= x_3 (1 + (6 \cdot e_3 - 2) x_1 x_2) \leq x_3, \end{aligned}$$

where we have used  $6e_1 - 2 > 0, 6e_2 - 2 > 0$  and  $6e_3 - 2 < 0$ .

Consequently it follows that

$$x_1^{(n+1)} \geq x_1^{(n)}, \quad x_2^{(n+1)} \geq x_2^{(n)}, \quad x_3^{(n+1)} \leq x_3^{(n)}, \quad n = 0, 1, 2, \dots$$

Therefore we have the sequences  $\{x_1^{(n)}\}_{n=0,1,2,\dots}$  and  $\{x_2^{(n)}\}_{n=0,1,2,\dots}$  are increasing and bounded from the above and the sequence  $\{x_3^{(n)}\}_{n=0,1,2,\dots}$  is decreasing and bounded from the below. Consequently it follows the existence of the limits

$$\lim_{n \rightarrow \infty} x_1^{(n)} = x_1^* > 0, \quad \lim_{n \rightarrow \infty} x_2^{(n)} = x_2^* > 0, \quad \lim_{n \rightarrow \infty} x_3^{(n)} = x_3^* \geq 0.$$

We claim that  $x_3^* = 0$ . Indeed, if we suppose that  $x_3^* > 0$ . Then we have

$$\begin{aligned} 1 &= \lim_{n \rightarrow \infty} \frac{x_3^{(n+1)}}{x_3^{(n)}} = \lim_{n \rightarrow \infty} \frac{x_3^{(n)} \left(1 + (6 \cdot e_3 - 2) x_1^{(n)} x_2^{(n)}\right)}{x_3^{(n)}} \\ &= 1 + (6 \cdot e_3 - 2) \lim_{n \rightarrow \infty} x_1^{(n)} x_2^{(n)} \Rightarrow \lim_{n \rightarrow \infty} x_1^{(n)} x_2^{(n)} = x_1^* x_2^* = 0, \end{aligned}$$

which is a contradiction to  $x_1^* > 0$  and  $x_2^* > 0$ . So it follows that  $x_3^* = 0$ .

Thus in this case we have

$$\lim_{n \rightarrow \infty} W^n(\mathbf{x}^{(0)}) = \mathbf{x}^* = (x_1^*, x_2^*, 0) \in \overline{M_1 M_2}.$$

for all  $\mathbf{x}^{(0)} \in \text{int } S^2$ .

Moreover it follows if  $e_2 = \frac{1}{3}$  then  $\mathbf{x}^* = (1 - x_2^{(0)}, x_2^{(0)}, 0)$  for any  $\mathbf{x}^{(0)} \in \text{int } S^2$ .

b) Let  $e_1 > \frac{1}{3}, e_2 < e_3 < \frac{1}{3}$  and let  $\mathbf{x}^{(0)} \in \text{int } S^2$  be an initial point.

Then from (8) we have

$$\begin{aligned} x'_1 &= x_1 (1 + (6e_1 - 2) x_2 x_3) \geq x_1, \quad x'_2 = x_2 (1 + (6e_2 - 2) x_1 x_3) \leq x_2, \\ x'_3 &= x_3 (1 + (6e_3 - 2) x_1 x_2) \leq x_3, \end{aligned}$$

where we have used  $6e_1 - 2 > 0, 6e_2 - 2 < 0$  and  $6e_3 - 2 < 0$ .

Consequently it follows that

$$x_1^{(n+1)} \geq x_1^{(n)}, \quad x_2^{(n+1)} \leq x_2^{(n)}, \quad x_3^{(n+1)} \leq x_3^{(n)}, \quad n = 0, 1, 2, \dots$$

Therefore we have the sequence  $\{x_1^{(n)}\}_{n=0,1,2,\dots}$  is increasing and bounded from the above and the sequences  $\{x_2^{(n)}\}_{n=0,1,2,\dots}$  and  $\{x_3^{(n)}\}_{n=0,1,2,\dots}$  are decreasing and bounded from the below. Consequently it follows the existence of the limits

$$\lim_{n \rightarrow \infty} x_1^{(n)} = x_1^* > 0, \quad \lim_{n \rightarrow \infty} x_2^{(n)} = x_2^* \geq 0, \quad \lim_{n \rightarrow \infty} x_3^{(n)} = x_3^* \geq 0.$$

We claim that  $x_2^* = x_3^* = 0$ . Suppose on the contrary. Let  $x_2^* > 0, x_3^* > 0$  then it holds  $x_1^* < 1$ . Therefore from the first equation of (8) we have

$$1 = \lim_{n \rightarrow \infty} \frac{1 - x_1^{(n+1)}}{1 - x_1^{(n)}} = \lim_{n \rightarrow \infty} \frac{1 - x_1^{(n)} \left(1 + (6e_1 - 2) x_2^{(n)} x_3^{(n)}\right)}{1 - x_1^{(n)}}$$

$$= 1 - (6e_1 - 2) \lim_{n \rightarrow \infty} \frac{x_1^{(n)} x_2^{(n)} x_3^{(n)}}{1 - x_1^{(n)}} \Rightarrow \lim_{n \rightarrow \infty} x_2^{(n)} x_3^{(n)} = x_2^* x_3^* = 0,$$

which is a contradiction to the assumption  $x_2^* > 0$  and  $x_3^* > 0$ . So it follows that  $x_3^* = 1$ .

Therefore we obtain that all trajectories converge  $M_1$  for all  $\mathbf{x}^{(0)} \in \text{int } S^2$ .

iii) As above we assume  $i = 1$ ,  $j = 2$  and  $k = 3$ . Then we have  $e_1 = e_2 > e_3$  and it holds  $2e_1 + e_3 = 1$ . Therefore it follows

$$2e_3 + e_3 < 2e_1 + e_3 \Rightarrow 3e_3 < 1 \Rightarrow e_3 < \frac{1}{3} \Rightarrow e_1 = e_2 > \frac{1}{3}$$

and we have  $6e_1 - 2 > 0$ ,  $6e_2 - 2 < 0$  and  $6e_3 - 2 < 0$ . Using the latter as in the previous case one has all trajectories converge  $M_1$  for all  $\mathbf{x}^{(0)} \in \text{int } S^2$ .

The proof of theorem is complete.  $\square$

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### REZYUME

Bu maqolada biz  $\theta$  parametriga qarab uchta o'zaro ta'sir populyatsiyasining vaqtli Kolmogorov tizimi oilasini ko'rib chiqamiz. Biz  $\theta$  ning kritik qiymati borligini ko'rsatamiz. Agar  $\theta \in (\theta^*, 1]$  bo'lsa, evolyutsiya operatori noergodik transformatsiya bo'lib, agar  $\theta \in [0, \theta^*]$  bo'lsa, regular bo'ladi.

**Kalit so'zlar:** Kvadratik stoxastik operator, kubik stoxastik operator, Volterra operatori, ergodik va noergodik operator.

### РЕЗЮМЕ

В работе мы рассматриваем семейство дискретных Колмогоровских систем трех взаимодействий зависящих от параметра  $\theta$ . Мы показываем, что существует критическое значение  $\theta$ , такое что, если  $\theta \in (\theta^*, 1]$ , этот оператор эволюции является неэргодическим преобразованием и обладает свойством регулярности, когда  $\theta \in [0, \theta^*]$ .

**Ключевые слова:** Квадратичный стохастический оператор, кубический стохастический оператор, оператор Вольтерра, эргодический и неэргодический оператор.

UDC 519.172, 536.714

**PERIODIC GROUND STATES FOR THE CHUI-WEEKS MODEL ON THE  
CAYLEY TREE OF ORDER TWO \***

**Rahmatullaev M. M. \* Rasulova M. A. \* Hakimova M. A. \***

**RESUME**

In this work, the three-state Chui-Weeks model on the second-order Cayley tree is considered. For this model, all translation-invariant and two-periodic ground states are described.

**Key words:** Cayley tree, Chui-Weeks model, ground state, translation-invariant ground state, periodic ground state.

### 1. Introduction

This work considers the model proposed by S. T. Chui and John D. Weeks in 1981 in the journal Physical Review B. For the Chui-Weeks model in which the transfer matrix has infinite dimension, in [2], the problem of phase transition was studied in 1D systems. So far, the Chui-Weeks model has not been considered on a Cayley tree. In the present work, the three-state Chui-Weeks model is considered on a second-order Cayley tree for the first time and for this model, all translation-invariant and two-periodic ground states are described.

The Cayley tree  $\Gamma^k$  (see, e.g., [1,3]) of order  $k \geq 1$  is an infinite tree, i.e., a graph without cycles, from each vertex of which exactly  $k + 1$  edges issue. Let  $\Gamma^k = (V, L, i)$ , where  $V$  is the set of vertices of  $\Gamma^k$ ,  $L$  is the set of edges of  $\Gamma^k$  and  $i$  is the incidence function associating each edge  $l \in L$  with its endpoints  $x, y \in V$ . If  $i(l) = \{x, y\}$ , then  $x$  and  $y$  are called *nearest neighboring vertices*, and we write  $l = \langle x, y \rangle$ .

It is known (see [1]) that there exists a one-to-one correspondence between the set  $V$  of vertices of the Cayley tree of order  $k \geq 1$  and the group  $G_k$  of the free products of  $k + 1$  cyclic groups  $\{e, a_i\}$ ,  $i = 1, \dots, k + 1$  of the second order (i.e.  $a_i^2 = e$ ,  $a_i^{-1} = a_i$ ) with generators  $a_1, a_2, \dots, a_{k+1}$ .

We consider model where the spin takes values in the set  $\Phi = \{0, 1, 2\}$ . For  $A \subseteq V$  a spin configuration  $\sigma_A$  on  $A$  is defined as a function  $x \in A \mapsto \sigma_A(x) \in \Phi$ ; the set of all configurations coincides with  $\Omega_A = \Phi^A$ . Denote  $\Omega = \Omega_V$  and  $\sigma = \sigma_V$ .

**Definition 1.1.** A configuration  $\sigma \in \Omega$  is called  $G_k^*$ -periodic, if  $\sigma(yx) = \sigma(x)$  for any  $x \in G_k$  and  $y \in G_k^* \subset G_k$ .

For a given periodic configuration the index of the subgroup is called the *period of the configuration*.

**Definition 1.2.** A configuration that is invariant with respect to all shifts is called *translation-invariant*.

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\***Rahmatullaev M. M.** – V.I.Romanovskiy Institute of Mathematics, mrahmatullaev@rambler.ru

\***Rasulova M. A.** – V.I.Romanovskiy Institute of Mathematics, m\_rasulova\_a@rambler.ru

\***Hakimova M. A.** – Namangan State University, hakimovamuslimaxon@gmail.com



## 2. The model

The Chui-Weeks model (see [2]) is defined by the following Hamiltonian

$$H(\sigma) = J \sum_{\langle x,y \rangle \in L} |\sigma(x) - \sigma(y)| + \alpha \sum_{x \in V} \delta_{\sigma(x),0}, \quad (1)$$

where  $J, \alpha \in \mathbb{R}$ ,  $\alpha$  is an external field and  $\sigma \in \Omega$ .

**Remark 2.1.** Recall that model (1) coincides with the SOS model under the condition  $\alpha = 0$  (see, e.g., [3]).

Let  $M$  be the set of all unit balls with vertices in  $V$  and  $S_1(x)$  be the set of all nearest neighboring vertices of  $x \in V$ .

We call the restriction of a configuration  $\sigma$  to the ball  $b \in M$  a *bounded configuration*  $\sigma_b$ . The energy of configuration  $\sigma_b$  on  $b$  is defined by the formula

$$U(\sigma_b) = \frac{J}{2} \sum_{x \in S_1(c_b)} |\sigma(x) - \sigma(c_b)| + \frac{\alpha}{k+2} \sum_{x \in b} \delta_{\sigma(x),0}.$$

where  $J = (J, \alpha) \in \mathbb{R}^2$  and  $c_b$  is the center of the unit ball  $b$ .

The Hamiltonian (1) can be written as

$$H(\sigma) = \sum_{b \in M} U(\sigma_b).$$

## 3. Ground states

In this section we study ground states for the three-state Chui-Weeks model on the Cayley tree of order two.

We have the following lemma.

**Lemma 3.1.** Let  $k = 2$ . Then for each configuration  $\varphi_b$ , we have the following

$$U(\varphi_b) \in \{U_i : i = 1, 2, \dots, 19\},$$

where

$$\begin{aligned} U_1 &= 0; U_2 = \alpha; U_3 = \frac{J}{2}; U_4 = \frac{J}{2} + \frac{\alpha}{4}; U_5 = \frac{J}{2} + \frac{3\alpha}{4}; \\ U_6 &= J; U_7 = J + \frac{\alpha}{4}; U_8 = J + \frac{\alpha}{2}; U_9 = J + \frac{3\alpha}{4}; U_{10} = \frac{3J}{2}; \\ U_{11} &= \frac{3J}{2} + \frac{\alpha}{4}; U_{12} = \frac{3J}{2} + \frac{\alpha}{2}; U_{13} = \frac{3J}{2} + \frac{3\alpha}{4}; U_{14} = 2J + \frac{\alpha}{4}; \\ U_{15} &= 2J + \frac{\alpha}{2}; U_{16} = \frac{5J}{2} + \frac{\alpha}{4}; U_{17} = \frac{5J}{2} + \frac{\alpha}{2}; U_{18} = 3J + \frac{\alpha}{4}; U_{19} = 3J + \frac{3\alpha}{4}. \end{aligned}$$

**Definition 3.1.** A configuration  $\varphi$  is called a ground state for the Hamiltonian (1), if

$$U(\varphi_b) = \min\{U_i : i = 1, 2, \dots, 19\}$$

for any  $b \in M$ .

We denote  $A_\xi = \{(J, \alpha) \in \mathbb{R}^2 : U_\xi = \min\{U_i : i = 1, 2, \dots, 19\}\}$ .

Calculations show that:

$$A_1 = \{(J, \alpha) \in \mathbb{R}^2 : J \geq 0, \alpha \geq 0\};$$

$$A_2 = \{(J, \alpha) \in \mathbb{R}^2 : 12J \geq \alpha, \alpha \leq 0\};$$

$$A_3 = A_6 = \{(J, \alpha) \in \mathbb{R}^2 : J = 0, \alpha \geq 0\};$$

$$A_4 = A_5 = A_7 = A_8 = A_9 = A_{11} = A_{12} = \dots = A_{17} = \{(J, \alpha) \in \mathbb{R}^2 : J = 0, \alpha = 0\};$$

$$A_{10} = \{(J, \alpha) \in \mathbb{R}^2 : -\alpha \leq 6J \leq 0, \alpha \geq 0\};$$

$$A_{18} = \{(J, \alpha) \in \mathbb{R}^2 : J \leq 0, 0 \leq \alpha \leq -6J\};$$

$$A_{19} = \{(J, \alpha) \in \mathbb{R}^2 : J \leq 0, 12J \leq \alpha \leq 0\}$$

and  $\bigcup_{i=1}^{19} A_i = \mathbb{R}^2$  (see Fig. 1).

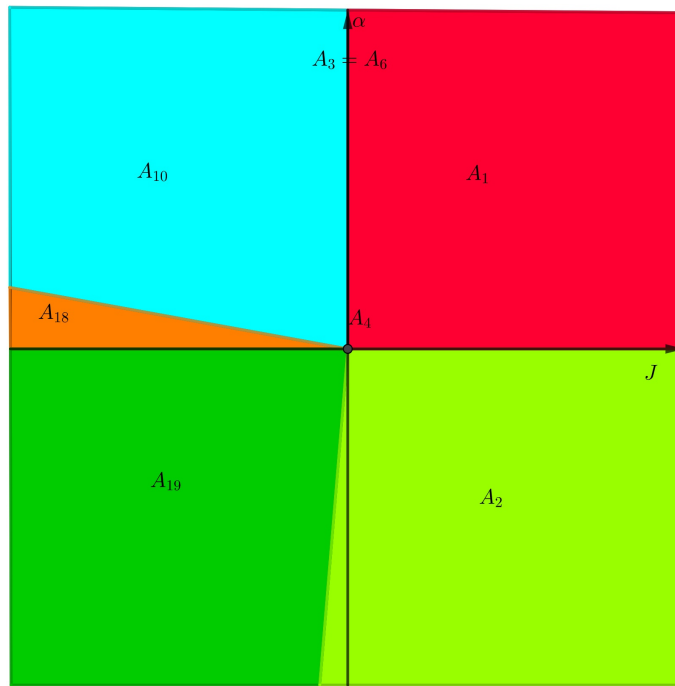


Figure 1. Representation of the sets  $A_i, i = \overline{1, 19}$ .

### 3.1. Translation-invariant ground states

In this subsection we study all translation-invariant ground states for the Chui-Weeks model on the Cayley tree of order two. The following theorem describes all translation-invariant ground states for the three-state Chui-Weeks model.

**Theorem 3.1.** *Let  $k = 2$ . Then for the Chui-Weeks model the following assertions hold*

- i) if  $(J, \alpha) \in A_1$ , then  $\sigma(x) = 1, \forall x \in V$  and  $\sigma(x) = 2, \forall x \in V$  are translation-invariant ground states;*
- ii) if  $(J, \alpha) \in A_2$ , then  $\sigma(x) = 0, \forall x \in V$  is the translation-invariant ground state;*
- iii) if  $(J, \alpha) \in \mathbb{R}^2 \setminus (A_1 \cup A_2)$ , then there is no translation-invariant ground state.*

**Proof.** *i)* Let  $k = 2$ . We consider the configuration  $\sigma(x) = i, i \in \{1, 2\}, \forall x \in V$ . For any  $b \in M$  by Lemma 3.1 we have  $U(\sigma_b) = U_1 = 0$ . Thus the configuration  $\sigma(x) = i, i \in \{1, 2\}, \forall x \in V$  is ground state if  $(J, \alpha) \in A_1$ ;

*ii)* Let  $k = 2$ . We consider the configuration  $\sigma(x) = 0 \forall x \in V$ . For any  $b \in M$  by Lemma 3.1 we have  $U(\sigma_b) = U_2 = \alpha$ . Thus the configuration  $\sigma(x) = 0 \forall x \in V$  is ground state if  $(J, \alpha) \in A_2$ ;

*iii)* It is obvious. Theorem 3.1 is proved.

**Remark 3.1.** *It is known from [4] that for the SOS model with a non-zero external field, the configuration  $\sigma(x) = 1 \forall x \in V$  is not a translation-invariant ground state. From Theorem 3.1, we can see that this configuration is a translation-invariant ground state for the Chui-Weeks model.*

### 3.2. $G_2^{(2)}$ -periodic ground states

In this subsection we study all  $G_2^{(2)}$ -periodic ground states for the Chui-Weeks model on the Cayley tree of order two, where

$$G_2^{(2)} = \{x \in G_2 : |x| \text{ is even}\},$$

where  $|x|$  means length of the word  $x$ .

All  $G_2^{(2)}$ -periodic configurations have the following form:

$$\sigma(x) = \begin{cases} \sigma_0, & \text{if } x \in G_2^{(2)}, \\ \sigma_1, & \text{if } x \in G_2 \setminus G_2^{(2)}, \end{cases}$$

where  $\sigma_0, \sigma_1 \in \Phi$ .

The following theorem describes all  $G_2^{(2)}$ -periodic ground states for the three-state Chui-Weeks's model.

**Theorem 3.2.** *Let  $k = 2$ . Then for the Chui-Weeks model the following assertions hold*

*i) if  $(J, \alpha) \in A_{10}$ , then  $G_2^{(2)}$ -periodic configurations*

$$\sigma(x) = \begin{cases} 1, & \text{if } x \in G_2^{(2)}, \\ 2, & \text{if } x \in G_2 \setminus G_2^{(2)}, \end{cases} \quad \sigma(x) = \begin{cases} 2, & \text{if } x \in G_2^{(2)}, \\ 1, & \text{if } x \in G_2 \setminus G_2^{(2)} \end{cases}$$

*are  $G_2^{(2)}$ -periodic ground states;*

*ii) if  $(J, \alpha) \in A_{18} \cap A_{19}$ , then  $G_2^{(2)}$ -periodic configurations*

$$\sigma(x) = \begin{cases} 0, & \text{if } x \in G_2^{(2)}, \\ 2, & \text{if } x \in G_2 \setminus G_2^{(2)}, \end{cases} \quad \sigma(x) = \begin{cases} 2, & \text{if } x \in G_2^{(2)}, \\ 0, & \text{if } x \in G_2 \setminus G_2^{(2)} \end{cases}$$

*are  $G_2^{(2)}$ -periodic ground states.*

**Proof.** *i)* Let  $k = 2$ . We consider the following  $G_2^{(2)}$ -periodic configuration

$$\sigma(x) = \begin{cases} 1, & \text{if } x \in G_2^{(2)}, \\ 2, & \text{if } x \in G_2 \setminus G_2^{(2)}. \end{cases}$$

Then we have  $\sigma(c_b) = 1$  or  $\sigma(c_b) = 2$ ,  $\forall b \in M$ . If  $\sigma(c_b) = 1$  then  $\forall x \in S_1(c_b)$  we have  $\sigma(x) = 2$ . In this case by Lemma 3.1 we get  $U(\sigma_b) = \frac{3J}{2} = U_{10}$ . If  $\sigma(c_b) = 2$  then  $\forall x \in S_1(c_b)$  we have  $\sigma(x) = 1$ . In this case by Lemma 3.1 we get  $U(\sigma_b) = \frac{3J}{2} = U_{10}$ . From these cases, it follows that the  $G_2^{(2)}$ -periodic configuration we have considered is a ground state, if  $(J, \alpha) \in A_{10}$ .

ii) Let  $k = 2$ . We consider the following  $G_2^{(2)}$ -periodic configuration

$$\sigma(x) = \begin{cases} 0, & \text{if } x \in G_2^{(2)}, \\ 2, & \text{if } x \in G_2 \setminus G_2^{(2)}. \end{cases}$$

Then we have  $\sigma(c_b) = 0$  or  $\sigma(c_b) = 2$ ,  $\forall b \in M$ . If  $\sigma(c_b) = 0$  then  $\forall x \in S_1(c_b)$  we have  $\sigma(x) = 2$ . In this case by Lemma 3.1 we get  $U(\sigma_b) = 3J + \frac{\alpha}{4} = U_{18}$ . If  $\sigma(c_b) = 2$  then  $\forall x \in S_1(c_b)$  we have  $\sigma(x) = 0$ . In this case by Lemma 3.1 we get  $U(\sigma_b) = 3J + \frac{3\alpha}{4} = U_{19}$ . From these cases, it follows that the  $G_2^{(2)}$ -periodic configuration we have considered is a ground state, if

$$(J, \alpha) \in A_{18} \cap A_{19} = \{(J, \alpha) \in \mathbb{R}^2 : J \leq 0, \alpha = 0\}.$$

The remaining cases are proved as above. Theorem 3.2 is proved.

### 3.3. $H_A$ -periodic ground states

In this subsection we study all  $H_A$ -periodic ground states for the Chui-Weeks model on the Cayley tree, where

$$H_A = \{x \in G_2 : \sum_{i \in A} \omega_x(a_i) \text{ is an even number}\},$$

where  $\emptyset \neq A \subseteq N_2 = \{1, 2, 3\}$ , and  $\omega_x(a_i)$  is the number of letters  $a_i$  in a word  $x \in G_2$ . Note that  $|x| = \sum_{j=1}^3 w_j(x)$ . It is known that the sets  $H_A$  and  $G_2^{(2)}$  are normal groups of index two of  $G_2$ , and also any normal group of index two in  $G_2$  is of the form  $H_A$  (see [2]). If  $A = \{1, 2, 3\}$  then the normal subgroup  $H_A$  coincides with the group  $G_2^{(2)}$ .

All  $H_A$ -periodic configurations have the following form:

$$\sigma(x) = \begin{cases} \sigma_0, & \text{if } x \in H_A, \\ \sigma_1, & \text{if } x \in G_2 \setminus H_A, \end{cases}$$

where  $\sigma_0, \sigma_1 \in \Phi$ .

The following theorem describes all  $H_A$ -periodic ground states for the three-state Chui-Weeks model.

**Theorem 3.3.** *Let  $k = 2$ . Then for the Chui-Weeks model the following assertions hold*

i) if  $|A| = 1$  and  $(J, \alpha) \in A_3$ , then the following two  $H_A$ -periodic configurations

$$\sigma(x) = \begin{cases} 1, & \text{if } x \in H_A, \\ 2, & \text{if } x \in G_2 \setminus H_A, \end{cases} \quad \sigma(x) = \begin{cases} 2, & \text{if } x \in H_A, \\ 1, & \text{if } x \in G_2 \setminus H_A \end{cases}$$

are  $H_A$ -periodic ground states;

ii) if  $|A| = 2$  and  $(J, \alpha) \in A_3$ , then the following four  $H_A$ -periodic configurations

$$\sigma(x) = \begin{cases} 0, & \text{if } x \in H_A, \\ 1, & \text{if } x \in G_2 \setminus H_A, \end{cases} \quad \sigma(x) = \begin{cases} 1, & \text{if } x \in H_A, \\ 0, & \text{if } x \in G_2 \setminus H_A, \end{cases}$$

$$\sigma(x) = \begin{cases} 1, & \text{if } x \in H_A, \\ 2, & \text{if } x \in G_2 \setminus H_A, \end{cases} \quad \sigma(x) = \begin{cases} 2, & \text{if } x \in H_A, \\ 1, & \text{if } x \in G_2 \setminus H_A \end{cases}$$

are  $H_A$ -periodic ground states;

iii) all  $H_A$ -periodic ground states except the ground states found in i) and ii) are either translation-invariant or  $G_2^{(2)}$ -periodic.

**Proof.** Let  $B_i = \{x \in S_1(c_b) : \sigma_b(x) = i\}, i \in \Phi$ .

i) Let  $k = 2$  and  $|A| = 1$ . We consider the following  $H_A$ -periodic configuration

$$\sigma(x) = \begin{cases} 1, & \text{if } x \in H_A, \\ 2, & \text{if } x \in G_2 \setminus H_A. \end{cases}$$

If  $c_b \in H_A$ , then we have

$$\sigma(c_b) = 1, \quad |B_1| = 2, \quad |B_2| = 1,$$

thus  $U(\sigma_b) = U_3$ .

If  $c_b \in G_2 \setminus H_A$ , then we have

$$\sigma(c_b) = 2, \quad |B_1| = 1, \quad |B_2| = 2,$$

thus  $U(\sigma_b) = U_3$ .

Consequently, the configuration  $\sigma$  we have considered is an  $H_A$ -periodic ground state if  $(J, \alpha) \in A_3$ .

ii) Let  $k = 2$  and  $|A| = 2$ . We consider the following  $H_A$ -periodic configuration

$$\sigma(x) = \begin{cases} 0, & \text{if } x \in H_A, \\ 1, & \text{if } x \in G_2 \setminus H_A. \end{cases}$$

If  $c_b \in H_A$ , then we have

$$\sigma(c_b) = 0, \quad |B_0| = 1, \quad |B_1| = 2,$$

thus  $U(\sigma_b) = U_3$ .

If  $c_b \in G_2 \setminus H_A$ , then we have

$$\sigma(c_b) = 1, \quad |B_0| = 2, \quad |B_1| = 1,$$

thus  $U(\sigma_b) = U_3$ .

Consequently, the configuration  $\sigma$  we have considered is an  $H_A$ -periodic ground state if  $(J, \alpha) \in A_3$ .

The remaining cases are proved as above. Theorem 3.3 is proved.

**Remark 3.2.** If  $k = 2$  and  $|A| = 2$ , then, as shown for the SOS model with competing interaction, there is no  $H_A$ -periodic ground state [5]. However, according to Theorem 3.3, it can be observed that in this case, there exist four  $H_A$ -periodic ground states for the Chui-Weeks model.

**Remark 3.3.** Recall that when  $k = 2$  and  $|A| = 3$ , all  $H_A$ -periodic ground states are identical to the  $G_2^{(2)}$ -periodic ground states described in Theorem 3.2.

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## REZYUME

Bu ishda ikkinchi tartibli Keli daraxtida uch holatli Chui-Weeks modeli qaraladi. Ushbu model uchun barcha translatsion-invariant va ikki-davriy asosiy holatlar tavsiflanadi.

**Kalit soʻzlar:** Keli daraxti, Chui-Weeks modeli, asosiy holat, translatsion-invariant asosiy holat, davriy asosiy holat.

## РЕЗЮМЕ

В данной работе рассматривается Chui-Weeks модель с тремя состояниями на дереве Кэли второго порядка. Для этой модели описаны все трансляционно-инвариантные и двухпериодические основные состояния.

**Ключевые слова:** Дерево Кэли, Chui-Weeks модель, основное состояние, трансляционно-инвариантное основное состояние, периодическое основное состояние.

UDC 517.55

## INVESTIGATION OF THE ESSENTIAL SPECTRUM OF A $3 \times 3$ OPERATOR MATRIX

Sharipova M. Sh. \*

### RESUME

In the present paper, we precisely describe the location and structure of the essential spectrum of the operator matrix  $\mathcal{A}_\mu$  of order three with spectral parameter  $\mu > 0$ . It is associated to the Hamiltonian of system with non-conserved and no more than three particles on the one-dimensional lattice, in the quasi-momentum representation. Two-particle and three-particle branches of the essential spectrum of  $\mathcal{A}_\mu$  are identified. The number of closed intervals of the essential spectrum of  $\mathcal{A}_\mu$  is studied with respect to the parameter  $\mu > 0$ .

**Key words:** operator matrix, spectral parameter, Fock space, annihilation and creation operators, essential spectrum, branches.

**1. Introduction.** Let  $\mathcal{H}_0$ ,  $\mathcal{H}_1$  and  $\mathcal{H}_2$  be the Hilbert spaces with the norm  $\|\cdot\|_0$ ,  $\|\cdot\|_1$  and  $\|\cdot\|_2$ , respectively. We denote by  $\mathcal{H}$  the direct sum of the Hilbert spaces  $\mathcal{H}_0$ ,  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , that is,  $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$ . Every element  $f$  of the space  $\mathcal{H}$  can be represented as a vector  $f = (f_0, f_1, f_2)$  with coordinates  $f_0 \in \mathcal{H}_0$ ,  $f_1 \in \mathcal{H}_1$  and  $f_2 \in \mathcal{H}_2$ . The norm of the element  $f = (f_0, f_1, f_2) \in \mathcal{H}$  is defined by

$$\|f\| = \sqrt{\|f_0\|_0^2 + \|f_1\|_1^2 + \|f_2\|_2^2}.$$

From the spectral theory of linear bounded operators one can conclude that if the operator  $\mathcal{A}$  is a bounded linear operator in a Hilbert space  $\mathcal{H}$  and a decomposition  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$  is given, then the operator  $\mathcal{A}$  always admits a  $3 \times 3$  operator matrix representation

$$\mathcal{A} = \begin{pmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & A_{12} \\ A_{20} & A_{21} & A_{22} \end{pmatrix} \quad (1)$$

with linear bounded operators  $A_{ij} : \mathcal{H}_j \rightarrow \mathcal{H}_i$ ,  $i, j = 0, 1, 2$ . It clear that the operator matrix  $\mathcal{A}$  is a self-adjoint if and only if  $A_{ij} = A_{ji}^*$ ,  $i, j = 0, 1, 2$ .

It can be seen that the elements of the operator matrix  $\mathcal{A}$  are linearly bounded operators in the corresponding Hilbert spaces. Spectral properties of matrices with operators of this type are studied in many literatures, for example, in the following 2 monographs [1,2]. The order of operator matrices depends on the number of Hilbert spaces. For example, if a direct sum of  $n$  Hilbert spaces is considered, then the resulting operator matrix is of order  $n$ . In the present paper we investigate the case  $n = 3$ .

One of the elements belonging to the class of operator matrices is the Hamiltonian corresponding to a system of non-conserved number of lattice particles. In this system, the

\*Sharipova M. Sh. – Bukhara State University, m.sh.sharipova@buxdu.uz

number of particles can be infinite, as in the spin-boson model [3], or finite, as in the truncated spin-boson model [4-8]. Some spectral properties of the full spin-boson model is studied in [3]. The existence of the wave operators is studied for the truncated standard spin-boson models with at most two photons in [4] and with at most three photons in [5]. The essential and discrete spectrum of the lattice spin-boson model with at most two photons studied in [6-8].

In the present paper we consider the operator matrix  $\mathcal{A}_\mu$  of the form (1) with the spectral parameter  $\mu > 0$ . It is associated to the Hamiltonian of system with non-conserved and no more than three particles on the one-dimensional lattice, in the quasi-momentum representation. The location of the essential spectrum  $\sigma_{\text{ess}}(\mathcal{A}_\mu)$  of the operator matrix  $\mathcal{A}_\mu$  is described. Two-particle and three-particle branches of the  $\sigma_{\text{ess}}(\mathcal{A}_\mu)$  is defined. The number and structure of closed intervals of the essential of  $\mathcal{A}_\mu$  is studied with respect to the parameter  $\mu > 0$ .

The spectrum, the essential spectrum and the discrete spectrum of a bounded self-adjoint operator will be denoted by  $\sigma(\cdot)$ ,  $\sigma_{\text{ess}}(\cdot)$  and  $\sigma_{\text{disc}}(\cdot)$ , respectively.

## 2. Operator matrix of order three and the location of its essential spectrum.

This paper is devoted to the following case:  $\mathcal{H}_0 := \mathbb{C}$  is the field of complex numbers,  $\mathcal{H}_1 := L_2(\mathbb{T})$  is the Hilbert space of square-integrable (complex-valued) functions defined on the one-dimensional torus  $\mathbb{T}$  and  $\mathcal{H}_2 := L_2(\mathbb{T}^2)$  is the Hilbert space of square-integrable (complex-valued) functions defined on the two-dimensional torus  $\mathbb{T}^2$ . The space  $\mathcal{H}_0$  is called a zero-particle subspace of a Fock space, the space  $\mathcal{H}_1$  is called an one-particle subspace of a Fock space, the space  $\mathcal{H}_2$  is called an two-particle subspace of a Fock space and the space  $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$  is called three-particle subspace of the Fock space  $\mathcal{F}(L_2(\mathbb{T}))$  over  $L_2(\mathbb{T})$ , that is,

$$\mathcal{F}(L_2(\mathbb{T})) := \mathbb{C} \oplus L_2(\mathbb{T}) \oplus L_2(\mathbb{T}^2) \oplus \dots$$

The norm of the element  $f = (f_0, f_1, f_2) \in \mathcal{H}$  is defined by

$$\|f\| = \left( |f_0|^2 + \int_{\mathbb{T}} |f_1(x)|^2 dx + \int_{\mathbb{T}^2} |f_2(x, y)|^2 dx dy \right)^{1/2}.$$

The scalar product of the two elements  $f = (f_0, f_1, f_2) \in \mathcal{H}$  and  $g = (g_0, g_1, g_2) \in \mathcal{H}$  is defined by

$$(f, g) = f_0 \cdot \overline{g_0} + \int_{\mathbb{T}} f_1(x) \overline{g_1(x)} dx + \int_{\mathbb{T}^2} f_2(x, y) \overline{g_2(x, y)} dx dy.$$

Let us consider the operator  $\mathcal{A}_\mu$ ,  $\mu > 0$  acting in the Hilbert space  $\mathcal{H}$  as an operator matrix

$$\mathcal{A}_\mu := \begin{pmatrix} A_{00} & \mu A_{01} & 0 \\ \mu A_{01}^* & A_{11} & \mu A_{12} \\ 0 & \mu A_{12}^* & A_{22} \end{pmatrix} \quad (2)$$

with matrix elements  $A_{ij} : \mathcal{H}_j \rightarrow \mathcal{H}_i, i \neq j, i, j = 0, 1, 2$

$$A_{00}f_0 = \varepsilon f_0, \quad (A_{01}f_1)(t) = \int_{\mathbb{T}} \sin(3t)f_1(t)dt,$$

$$(A_{11}f_1(x)) = (\varepsilon + 1 - \cos(3x))f_1(x), \quad (A_{12}f_2)(x, t) = \int_{\mathbb{T}} \sin(3t)f_2(x, t)dt;$$

$$(A_{22}f_2)(x, y) = (\varepsilon + 2 - \cos(3x) - \cos(3y))f_2(x, y), \quad f = (f_0, f_1, f_2) \in \mathcal{H}.$$



Under these assumptions the operator matrix  $\mathcal{A}_\mu$  defined by the formula (2) is linear, bounded and self-adjoint in  $\mathcal{H}$ .

By simple calculations it is possible to make sure that

$$\begin{aligned}(A_{01}^* f_0)(x) &= \mu \sin(3x) f_0; \\ (A_{12}^* f_1)(x, y) &= \mu \sin(3y) f_1(x), \quad (f_0, f_1) \in \mathcal{H}_0 \oplus \mathcal{H}_1.\end{aligned}$$

Usually the off-diagonal elements  $A_{01}$ ,  $A_{12}$  are called annihilation operators and  $A_{01}^*$ ,  $A_{12}^*$  are called creation operators.

In order to study the essential and discrete spectra of the operator matrix  $\mathcal{A}_\mu$  we introduce a generalized Friedrichs model  $h_\mu$ ,  $\mu > 0$  which acts in the Hilbert space  $\mathcal{H}_0 \oplus \mathcal{H}_1$  as

$$h_\mu := \begin{pmatrix} A_{00} & \mu A_{01} \\ \mu A_{01}^* & A_{11} \end{pmatrix}.$$

Its matrix elements  $A_{ij}$ ,  $i \leq j$ ,  $i, j = 0, 1$  are given in the above. It is not difficult to prove that operator  $h_\mu$  is linear, bounded and self-adjoint. We consider operator matrix  $h_0 := h_\mu|_{\mu=0}$  in the Hilbert space  $\mathcal{H}_0 \oplus \mathcal{H}_1$  as

$$h_0 := \begin{pmatrix} A_{00} & 0 \\ 0 & A_{11} \end{pmatrix}.$$

The perturbation  $h_\mu - h_0$  of the operator  $h_0$  is a bounded self-adjoint operator matrix of rank 2. From the definition of  $h_0$  one can conclude that the spectrum of  $h_0$  is equal to

$$\sigma(h_0) = \sigma(A_{00}) \cup \sigma(A_{11}),$$

where

$$\sigma(A_{00}) = \sigma_{\text{disc}}(A_{00}) = \varepsilon; \quad \sigma(A_{11}) = \sigma_{\text{ess}}(A_{11}) = [\varepsilon; \varepsilon + 2].$$

According to the famous Weyl's theorem on the conservation of the essential spectrum under finite rank perturbations implies that the essential spectra of operators  $h_0$  and  $h_\mu$  coincide. Therefore,

$$\sigma_{\text{ess}}(h_\mu) = \sigma_{\text{ess}}(h_0) = [\varepsilon; \varepsilon + 2].$$

One can see that

$$\begin{aligned}\min_{x \in \mathbb{T}^1} (\varepsilon + 1 - (\cos(3x))) &= \varepsilon + 1 - 1 = \varepsilon; \\ \max_{x \in \mathbb{T}^1} (\varepsilon + 1 - (\cos(3x))) &= \varepsilon + 1 + 1 = \varepsilon + 2.\end{aligned}$$

For any fixed  $\mu > 0$ , we define an analytic function  $\Delta_\mu(\cdot)$  in the domain  $\mathbb{C} \setminus [\varepsilon; \varepsilon + 2]$  by

$$\Delta_\mu(z) := \varepsilon - z - \mu^2 \int_{\mathbb{T}} \frac{\sin^2(3t) dt}{\varepsilon + 1 - \cos(3t) - z}.$$

Usually the function  $\Delta_\mu(\cdot)$  is called the Fredholm determinant associated with the operator matrix  $h_\mu$ .

Let us establish a relation between the eigenvalues of  $h_\mu$  and zeros of  $\Delta_\mu(\cdot)$ .

**Lemma 1.** For any fixed  $\mu > 0$  the operator matrix  $h_\mu$  has the eigenvalue  $z_\mu \in \mathbb{C} \setminus [\varepsilon; \varepsilon + 2]$  if and only if  $\Delta_\mu(z_\mu) = 0$ .

From Lemma 1 it follows that for the discrete spectrum of  $h_\mu$  the equality

$$\sigma_{\text{disc}}(h_\mu) = \{z \in \mathbb{C} \setminus [\varepsilon; \varepsilon + 2] : \Delta_\mu(z) = 0\}$$

holds.

**Theorem 1.** For the essential spectrum of operator matrix  $\mathcal{A}_\mu$  the equality

$$\sigma_{\text{ess}}(\mathcal{A}_\mu) = [\varepsilon; \varepsilon + 4] \cup \{\sigma_{\text{disc}}(h_\mu) + [\varepsilon; \varepsilon + 2]\}$$

holds.

The proof of this theorem was done on the paper [9].

**Definition 1.** The sets  $\sigma_{\text{two}}(\mathcal{A}_\mu)$  and  $\sigma_{\text{three}}(\mathcal{A}_\mu)$  are called the two-particle and three-particle branches of the essential spectrum  $\sigma_{\text{ess}}(\mathcal{A}_\mu)$  of  $\mathcal{A}_\mu$ , where

$$\sigma_{\text{two}}(\mathcal{A}_\mu) := \{\sigma_{\text{disc}}(h_\mu) + [\varepsilon; \varepsilon + 2]\};$$

$$\sigma_{\text{three}}(\mathcal{A}_\mu) := [\varepsilon; \varepsilon + 4].$$

In the next section, we will study the position of the two-particle and three-particle branches of the essential spectrum of  $\mathcal{A}_\mu$  with respect to the parameter  $\mu > 0$ .

### 3. The components of the essential spectrum of $3 \times 3$ operator matrix

Denote by  $E_\mu^{(1)}$  the eigenvalue of operator matrix  $h_\mu$  which is located on the left of  $\varepsilon$  and by  $E_\mu^{(2)}$  the eigenvalue of operator matrix  $h_\mu$  (while having an eigenvalue), which is located on the right of  $\varepsilon + 2$ .

The following theorem describes the location of essential spectrum of  $\mathcal{A}_\mu$ .

**Theorem 2.** a) If  $\mu \leq 1/\sqrt{\pi}$ , then the essential spectrum  $\sigma_{\text{ess}}(\mathcal{A}_\mu)$  of  $\mathcal{A}_\mu$  satisfies

$$\sigma_{\text{ess}}(\mathcal{A}_\mu) = [E_\mu^{(1)} + \varepsilon; E_\mu^{(1)} + \varepsilon + 2] \cup [\varepsilon; \varepsilon + 4];$$

b) If  $\mu > 1/\sqrt{\pi}$ , then the essential spectrum  $\sigma_{\text{ess}}(\mathcal{A}_\mu)$  of  $\mathcal{A}_\mu$  satisfies

$$\sigma_{\text{ess}}(\mathcal{A}_\mu) = [E_\mu^{(1)} + \varepsilon; E_\mu^{(1)} + \varepsilon + 2] \cup [\varepsilon; \varepsilon + 4] \cup [E_\mu^{(2)} + \varepsilon; E_\mu^{(2)} + \varepsilon + 4].$$

**Proof.** From Theorem 1 and Lemma 1, we have to study the zeros of the function  $\Delta_\mu(\cdot)$  in order to study the location of the essential spectrum of the operator matrix  $\mathcal{A}_\mu$ . From the definition of the function  $\Delta_\mu(\cdot)$  one can see that the inequality

$$\frac{d}{dz} \Delta_\mu(z) = -\mu^2 \int_{\mathbb{T}} \frac{\sin^2(3t) dt}{(1 - \cos(3t) - z)^2} < 0$$

holds for any  $z \in \mathbb{R} \setminus [0; 2]$ . So, the function  $\Delta_\mu(\cdot)$  is monotonically decreasing on the intervals  $(-\infty; \varepsilon)$  and  $(\varepsilon + 2; +\infty)$ .

For evaluation zeros of the function  $\Delta_\mu(\cdot)$ , let us calculate  $\Delta_\mu(\varepsilon)$ :

$$\Delta_\mu(\varepsilon) = \varepsilon - \varepsilon - \mu^2 \int_{\mathbb{T}} \frac{\sin^2(3t) dt}{\varepsilon + 1 - \cos(3t) - \varepsilon} = -\mu^2 \int_{\mathbb{T}} \frac{\sin^2(3t) dt}{1 - \cos(3t)} =$$

$$= -\mu^2 \int_{\mathbb{T}} \frac{(1 - \cos(3t))(1 + \cos(3t))}{1 - \cos(3t)} dt = -\mu^2 \int_{\mathbb{T}} (1 + \cos(3t)) dt = -2\pi\mu^2.$$

From here we conclude that  $\Delta_\mu(\varepsilon) < 0$  for any  $\mu > 0$ . On the other hand

$$\lim_{z \rightarrow -\infty} \Delta_\mu(z) = +\infty.$$

Since the function  $\Delta_\mu(\cdot)$  is continuous and monotonically decreasing on the interval  $(-\infty; \varepsilon)$ , using last two facts we have that for any  $\mu > 0$  this function has an unique zero  $E_\mu^{(1)} \in (-\infty; \varepsilon)$ . By Lemma 1 for any  $\mu > 0$  the number  $E_\mu^{(1)}$  is an unique eigenvalue of the operator matrix  $h_\mu$ , which is located in the interval  $(-\infty; \varepsilon)$ .

Now we investigate  $\Delta_\mu(\varepsilon + 2)$ :

$$\begin{aligned} \Delta_\mu(\varepsilon + 2) &= \varepsilon - \varepsilon - 2 - \mu^2 \int_{\mathbb{T}} \frac{\sin^2(3t) dt}{\varepsilon + 1 - \cos(3t) - \varepsilon - 2} \\ &= -2 + \mu^2 \int_{\mathbb{T}} \frac{\sin^2(3t) dt}{1 + \cos(3t)} = -2 + \mu^2 \int_{\mathbb{T}} \frac{(1 - \cos(3t))(1 + \cos(3t)) dt}{1 + \cos(3t)} \\ &= -2 + \mu^2 \int_{\mathbb{T}} (1 - \cos(3t)) dt = -2 + \mu^2 \int_{\mathbb{T}} dt = -2 + 2\pi\mu^2. \end{aligned}$$

From here one can see that

$$\Delta_\mu(\varepsilon + 2) > 0 \quad \text{if and only if} \quad \mu > \frac{1}{\sqrt{\pi}};$$

$$\Delta_\mu(\varepsilon + 2) \leq 0 \quad \text{if and only if} \quad \mu \leq \frac{1}{\sqrt{\pi}}.$$

a) Let  $\mu \leq 1/\sqrt{\pi}$ . From the inequality  $\Delta_\mu(\varepsilon + 2) \leq 0$  and the equality

$$\lim_{z \rightarrow +\infty} \Delta_\mu(z) = -\infty$$

and using the continuity and monotonicity of the function  $\Delta_\mu(\cdot)$  in the interval  $(\varepsilon + 2; +\infty)$  we obtain that this function has no zeros in  $(\varepsilon + 2; +\infty)$ . By Lemma 1 for any  $\mu \leq 1/\sqrt{\pi}$  the operator matrix  $h_\mu$  has no eigenvalues in  $(\varepsilon + 2; +\infty)$ . So, for any  $\mu \leq 1/\sqrt{\pi}$  we have

$$\sigma_{\text{disc}}(h_\mu) = \{E_\mu^{(1)}\}.$$

Therefore, using Theorem 1 we obtain

$$\sigma_{\text{ess}}(\mathcal{A}_\mu = [E_\mu^{(1)} + \varepsilon; E_\mu^{(1)} + \varepsilon + 2] \cup [\varepsilon; \varepsilon + 4])$$

for any  $\mu \leq 1/\sqrt{\pi}$ .

b) Let now  $\mu > 1/\sqrt{\pi}$ . Then from the inequality  $\Delta_\mu(\varepsilon + 2) > 0$  and the equality

$$\lim_{z \rightarrow +\infty} \Delta_\mu(z) = -\infty,$$

and using the continuity and monotonicity of the function  $\Delta_\mu(\cdot)$  in the interval  $(\varepsilon + 2; +\infty)$  we obtain that this function has an unique zero  $E_\mu^{(2)} \in (\varepsilon + 2; +\infty)$ . By Lemma 1 for any

$\mu > 1/\sqrt{\pi}$  the number  $E_\mu^{(2)}$  is an unique eigenvalue of the operator matrix  $h_\mu$ , which is located in the interval  $(\varepsilon + 2; +\infty)$ . So, for any  $\mu > 1/\sqrt{\pi}$  we have

$$\sigma_{\text{disc}}(h_\mu) = \{E_\mu^{(1)}; E_\mu^{(2)}\}.$$

Therefore, using Theorem 1 one can conclude that

$$\sigma_{\text{ess}}(\mathcal{A}_\mu) = [E_\mu^{(1)} + \varepsilon; E_\mu^{(1)} + \varepsilon + 2] \cup [\varepsilon; \varepsilon + 4] \cup [E_\mu^{(2)} + \varepsilon; E_\mu^{(2)} + \varepsilon + 4]$$

for any  $\mu > 1/\sqrt{\pi}$ . Theorem 2 is proved.

In the process of proving Theorem 2, it was shown that the following equality

$$\sigma_{\text{disc}}(h_\mu) = \begin{cases} \{E_\mu^{(1)}\}, & \text{if } \mu \leq 1/\sqrt{\pi}; \\ \{E_\mu^{(1)}; E_\mu^{(2)}\}, & \text{if } \mu > 1/\sqrt{\pi} \end{cases}$$

holds for the discrete spectrum of the operator matrix  $h_\mu$ .

**Conclusion.** This paper is devoted to the investigation of the spectral properties of the operator matrix  $\mathcal{A}_\mu$  of order three with spectral parameter  $\mu > 0$ . In this case the operator matrix  $\mathcal{A}_\mu$  is associated to the Hamiltonian of system with non-conserved and no more than three particles on the one-dimensional lattice, in the quasi-momentum representation. Using the spectrum of the corresponding generalized Friedrichs model we describe the location of the essential spectrum of the operator matrix  $\mathcal{A}_\mu$ . Then we identify the two-particle and three-particle branches of the essential spectrum of  $\mathcal{A}_\mu$ . The critical value of the spectral parameter  $\mu$  is found using a special representation of operator matrix elements. The number of closed intervals of the essential spectrum of  $\mathcal{A}_\mu$  is studied with respect to the parameter  $\mu > 0$ .

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### REZYUME

Ushbu maqolada  $\mu > 0$  spektral parametrga ega uchinchi tartibli  $\mathcal{A}_\mu$  operatorli matritsa muhim spektrining joylashuv o'ri va tuzulishi batafsil tavsiflangan. U bir o'lchamli panjaradagi soni saqlanmaydigan va uchtadan oshmaydigan zarrachalar sistemasi Hamiltonianining kvazi-impuls ko'rinishiga mos keladi.  $\mathcal{A}_\mu$  operator muhim spektrining ikki zarrachali va uch zarrachali tarmoqlari aniqlangan.  $\mathcal{A}_\mu$  operator muhim spektrini tashkil qiluvchi kesmalar soni  $\mu > 0$  parametrga nisbatan o'rganilgan.

**Kalit so'zlar:** operatorli matritsa, spektral parametr, Fok fazo, yo'qotish va paydo qilish operatorlari, muhim spektr, tarmoqlar.

### РЕЗЮМЕ

В настоящей работе описываем расположение и структуру существенного спектра операторной матрицы третьего порядка  $\mathcal{A}_\mu$  со спектральным параметром  $\mu > 0$ . Этот оператор связан с гамильтонианом системы, с несохраняющимся и не более трёх частиц на одномерном решетке, в квазиимпульсном представлении. Выделены двух- и трехчастичные ветви существенного спектра оператора  $\mathcal{A}_\mu$ . Изучается число сегментов существенного спектра  $\mathcal{A}_\mu$  относительно параметра  $\mu > 0$ .

**Ключевые слова:** операторная матрица, спектральный параметр, пространство фока, операторы уничтожения и рождения, существенный спектр, ветви.

UDC 517.918

**PANJARADAGI UCH ZARRACHALI SISTEMAGA MOS MODEL  
HAMILTONIAN MUHIM SPEKTRINING TAVSIFI**

Umirqulova G. H. \*

**REZYUME**

Mazkur maqolada  $d$  o'lchamli panjaradagi uchta zarrachalar sistemasiga mos model Hamiltonian Hilbert fazosidagi chiziqli, chegaralangan va o'z-o'ziga qo'shma operator sifatida o'rganilgan. Bu model Hamiltonianga mos ikkita kanal operatorlar aniqlangan va bu operatorlarning spektrlari Fridriks modellari oilasining spektri orqali tavsiflangan. Model Hamiltonianning muhim spektri ikkita kanal operatorlar spektrlari birlashmasiga teng bo'lishi isbotlangan.

**Kalit so'zlar:** panjara, zarrachalar sistemasi, model Hamiltonian, kanal operator, yoyiluvchi operator, Fredgolm determinanti, Weyl mezoni, ortonormal sistema, xos qiymat, spektr.

**Kirish.** Panjaradagi uchta zarrachalar sistemasiga mos model operatorlar bilan bog'liq masalalar qattiq jismlar fizikasi [1], statistik fizika [2], kvant maydon nazariyasi [3] va zamonaviy matematik fizikaning ko'plab sohalarida uchraydi. Bunday operatorlarning muhim va diskret spektrini o'rganish masalasi o'z-o'ziga qo'shma operatorlar spektral nazariyasining keng tadqiq qilinadigan masalalaridan biridir. Muhim spektrni o'rganishda odatda Veyl mezoni, Fredgolmning analitik teoremasi va Faddeyev tenglamasidan foydalaniladi [4,5].

Ushbu maqolada  $d$  o'lchamli panjaradagi uchta zarrachalar sistemasiga mos va Hilbert fazosida chiziqli, chegaralangan va o'z-o'ziga qo'shma model Hamiltonian qaralgan. O'rganilayotgan model Hamiltonian muhim spektrini tadqiq qilishda foydalanish uchun qulay bo'lgan ikkita kanal operatorlar kiritilgan. Aniqlangan kanal operatorlar ta'sir qiluvchi fazolarning to'g'ri integral yig'indiga yoyilishidan foydalanib, kanal operatorlarning ham to'g'ri integral yig'indiga yoyilishi ko'rsatilgan. Bundan tashqari, ikkita Fridriks modellari oilasi hamda ularga mos bo'lgan Fredgolm determinantlari qurilgan. So'ngra kanal operatorlarning spektri Fridriks modellari oilasining spektri orqali tavsiflangan. Model Hamiltonianning muhim spektri ikkita kanal operatorlar spektrlari birlashmasidan iborat bo'lishi isbotlangan.

**Masalaning qo'yilishi.**  $d \in \mathbb{N}$  natural soni uchun  $\mathbb{T}^d := (-\pi; \pi]^d$  orqali  $d$  o'lchamli torni belgilaymiz.  $(\mathbb{T}^d)^2$  to'plamda aniqlangan kvadrati bilan integrallanuvchi (umuman olganda kompleks qiymatlarni qabul qiluvchi) simmetrik funksiyalarning Hilbert fazosi bo'lgan  $L_2^s((\mathbb{T}^d)^2)$  fazoda

$$H_{\mu, \lambda}^{(\gamma)} := H_0^{(\gamma)} - \mu(V_1 + V_2) - \lambda V_3 \quad (1)$$

tenglik orqali aniqlanuvchi model Hamiltonianni qaraymiz. Bu yerda  $H_0^{(\gamma)}$  qo'zg'almas operatori  $E_\gamma(\cdot, \cdot)$  funksiyaga ko'paytirish operatori:

$$(H_0^{(\gamma)} f)(x, y) = E_\gamma(x, y)f(x, y), \quad E_\gamma(x, y) := \varepsilon(x) + \varepsilon(y) + \gamma\varepsilon(x + y),$$

\*Umirqulova G. H. – Buxoro davlat universiteti, g.h.umirqulova@buxdu.uz

$$\varepsilon(x) := \sum_{i=1}^d (1 - \cos(mx_i)), \quad x = (x_1, x_2, \dots, x_d) \in \mathbb{T}^d, \quad m \in \mathbb{N}.$$

$V_\alpha$ ,  $\alpha = 1, 2, 3$  operatorlar lokal bo'lmagan potensial operatorlari bo'lib, quyidagi ko'rinishdagi xususiy integrallari operatorlardir:

$$(V_1 f)(x, y) = v(y) \int_{\mathbb{T}^d} v(t) f(x, t) dt,$$

$$(V_2 f)(x, y) = v(x) \int_{\mathbb{T}^d} v(t) f(t, y) dt,$$

$$(V_3 f)(x, y) = \int_{\mathbb{T}^d} f(t, x + y - t) dt.$$

Bunda  $\mu, \lambda, \gamma > 0$  ta'sirlashish parametrlari,  $V_\alpha$ ,  $\alpha = 1, 2, 3$  operatorlar yadrosida ishtirok etuvchi  $v(\cdot)$  funksiya  $\mathbb{T}^d$  torda aniqlangan haqiqiy qiymatli uzluksiz funksiya.

Yuqoridagi (1) tenglik yordamida ta'sir qiluvchi  $H_{\mu, \lambda}^{(\gamma)}$  model Hamiltonian  $L_2^s((\mathbb{T}^d)^2)$  Hilbert fazosida aniqlangan chiziqli, chegaralangan va o'z-o'ziga qo'shma operator.

**Kanal operatorlar va ularning spektri.** Ushbu bo'limda  $H_{\mu, \lambda}^{(\gamma)}$  model Hamiltonianning muhim spektrini tadqiq qilish maqsadida kanal operatorlar deb ataluvchi ikkita operatorlarni kiritamiz [6]. Bunday operatorlar  $L_2(\mathbb{T}^d)^2$  Hilbert fazosida

$$H_\mu^{(\gamma, 1)} := H_0^{(\gamma)} - \mu V_1,$$

$$H_\lambda^{(\gamma, 2)} := H_0^{(\gamma)} - \lambda V_3$$

tengliklar orqali aniqlanadi. Hosil bo'lgan  $H_\mu^{(\gamma, 1)}$  va  $H_\lambda^{(\gamma, 2)}$  operatorlar ham  $H_{\mu, \lambda}^{(\gamma)}$  model Hamiltonian kabi  $L_2(\mathbb{T}^d)^2$  Hilbert fazosida ta'sir qiluvchi chiziqli, chegaralangan va o'z-o'ziga qo'shma operatorlardir.

$L_2(\mathbb{T}^d)^2$  Hilbert fazosining

$$L_2(\mathbb{T}^d)^2 = \int_{\mathbb{T}^d} \oplus L_2(\mathbb{T}^d) dk$$

to'g'ri integralga yoyilmasidan  $H_\mu^{(\gamma, 1)}$  va  $H_\lambda^{(\gamma, 2)}$  kanal operatorlarning ham

$$H_\mu^{(\gamma, 1)} = \int_{\mathbb{T}^d} \oplus (h_\mu^{(\gamma, 1)}(k) + \varepsilon(k)I) dk,$$

$$H_\lambda^{(\gamma, 2)} = \int_{\mathbb{T}^d} \oplus (h_\lambda^{(\gamma, 2)}(k) + \gamma \varepsilon(k)I) dk$$

to'g'ri integralga yoyilishi kelib chiqadi. Bunda  $I$  orqali  $L_2(\mathbb{T}^d)$  Hilbert fazosidagi birlik operator belgilangan,  $h_\mu^{(\gamma, 1)}(k)$  va  $h_\lambda^{(\gamma, 2)}(k)$  operatorlar esa  $L_2(\mathbb{T}^d)$  Hilbert fazosida mos ravishda

$$h_\mu^{(\gamma, 1)}(k) := h_0^{(\gamma, 1)}(k) - \mu v_1, \quad k \in \mathbb{T}^d,$$

$$h_\lambda^{(\gamma, 2)}(k) := h_0^{(\gamma, 2)}(k) - \lambda v_2, \quad k \in \mathbb{T}^d$$

kabi ta'sir qiluvchi Fridriks modellari oilasi deb ataluvchi operatorlar bo'lib,

$$(h_0^{(\gamma,1)}(k)f)(x) = (\varepsilon(x) + \gamma\varepsilon(k+x))f(x), \quad (v_1f)(x) = v(x) \int_{\mathbb{T}^d} v(t)f(t)dt,$$

$$(h_0^{(2)}(k)f)(x) = (\varepsilon(x) + \varepsilon(k-x))f(x), \quad (v_2f)(x) = \int_{\mathbb{T}^d} f(t)dt.$$

Bu tarzda aniqlangan  $h_\mu^{(\gamma,1)}(k)$  va  $h_\lambda^{(2)}(k)$  Fridriks modellari oilalari  $L_2(\mathbb{T}^d)$  Hilbert fazosida chiziqli, chegaralangan va o'z-o'ziga qo'shma operator bo'ladi.

Chekli o'lchamli qo'g'alishlarda muhim spektrning o'zgarmasligi haqidagi Veyl teoremasiga ko'ra  $h_\mu^{(\gamma,1)}(k)$  operatorning muhim spektri  $h_0^{(\gamma,1)}(k)$  operatorning muhim spektri bilan, xuddi shuningdek,  $h_\lambda^{(2)}(k)$  operatorning muhim spektri  $h_0^{(2)}(k)$  operatorning muhim spektri bilan ustma-ust tushadi hamda quyidagi

$$\sigma_{\text{ess}}(h_\mu^{(\gamma,1)}(k)) = [m_1^{(\gamma)}(k); M_1^{(\gamma)}(k)], \quad \sigma_{\text{ess}}(h_\lambda^{(2)}(k)) = [m_2(k); M_2(k)]$$

tengliklar o'rinlidir, bu yerda

$$m_1^{(\gamma)}(k) := \min_{x \in \mathbb{T}^d} (\varepsilon(x) + \gamma\varepsilon(k+x)), \quad M_1^{(\gamma)}(k) := \max_{x \in \mathbb{T}^d} (\varepsilon(x) + \gamma\varepsilon(k+x)),$$

$$m_2(k) := \min_{x \in \mathbb{T}^d} (\varepsilon(x) + \varepsilon(k-x)), \quad M_2(k) := \max_{x \in \mathbb{T}^d} (\varepsilon(x) + \varepsilon(k-x)).$$

Har bir fiksirlangan  $\mu, \lambda, \gamma > 0$  sonlari va  $k \in \mathbb{T}^d$  element uchun mos ravishda  $\mathbb{C} \setminus [m_1^{(\gamma)}(k); M_1^{(\gamma)}(k)]$  va  $\mathbb{C} \setminus [m_2(k); M_2(k)]$  sohalarda analitik bo'lgan

$$\Delta_\mu^{(\gamma,1)}(k, z) := 1 - \mu \int_{\mathbb{T}^d} \frac{v^2(t)dt}{\varepsilon(t) + \gamma\varepsilon(k+t) - z},$$

$$\Delta_\lambda^{(2)}(k, z) := 1 - \lambda \int_{\mathbb{T}^d} \frac{dt}{\varepsilon(t) + \varepsilon(k-t) - z}$$

yordamchi funksiyalarni kiritamiz.

Xususiyatiga ko'ra  $\Delta_\mu^{(\gamma,1)}(k, \cdot)$  va  $\Delta_\lambda^{(2)}(k, \cdot)$  funksiyalarga mos ravishda  $h_\mu^{(\gamma,1)}(k)$  va  $h_\lambda^{(2)}(k)$  operatorlarga mos Fredgolm determinantlari deyiladi.

**1-lemma.** Har bir fiksirlangan  $\mu, \gamma > 0$  sonlari va  $k \in \mathbb{T}^d$  element uchun  $z \in \mathbb{C} \setminus [m_1^{(\gamma)}(k); M_1^{(\gamma)}(k)]$  soni  $h_\mu^{(\gamma,1)}(k)$  operatorning xos qiymati bo'lishi uchun  $\Delta_\mu^{(\gamma,1)}(k, z) = 0$  bo'lishi zarur va yetarlidir.

Bayon qilingan 1-lemmadan quyidagi natija kelib chiqadi.

**1-natija.**  $h_\mu^{(\gamma,1)}(k)$  operatorning diskret spektri uchun

$$\sigma_{\text{disc}}(h_\mu^{(\gamma,1)}(k)) = \{z \in \mathbb{C} \setminus [m_1^{(\gamma)}(k); M_1^{(\gamma)}(k)] : \Delta_\mu^{(\gamma,1)}(k, z) = 0\}$$

tenglik o'rinlidir.

**2-lemma.** Har bir fiksirlangan  $\lambda > 0$  soni va  $k \in \mathbb{T}^d$  element uchun  $z \in \mathbb{C} \setminus [m_2(k); M_2(k)]$  soni  $h_\lambda^{(2)}(k)$  operatorning xos qiymati bo'lishi uchun  $\Delta_\lambda^{(2)}(k, z) = 0$  bo'lishi zarur va yetarlidir.



Xuddi shuningdek 2-lemmadan  $h_\lambda^{(2)}(k)$  operatorning diskret spektri haqidagi quyidagi 2-natija kelib chiqadi.

**2-natija.**  $h_\lambda^{(2)}(k)$  operatorning diskret spektri uchun

$$\sigma_{\text{disc}}(h_\lambda^{(2)}(k)) = \{z \in \mathbb{C} \setminus [m_2(k); M_2(k)] : \Delta_\lambda^{(2)}(k, z) = 0\}$$

tenglik o‘rinli bo‘ladi.

$H_\mu^{(\gamma,1)}$  va  $H_\lambda^{(\gamma,2)}$  kanal operatorlarning spektri  $h_\mu^{(\gamma,1)}(k)$  va  $h_\lambda^{(2)}(k)$  operatorlarning spektri orqali quyidagicha tavsiflanadi.

**1-teorema.**  $H_\mu^{(\gamma,1)}$  va  $H_\lambda^{(\gamma,2)}$  kanal operatorlar sof muhim spektrga ega hamda quyidagi

$$\sigma(H_\mu^{(\gamma,1)}) = \sigma_{\text{two}}(H_\mu^{(\gamma,1)}) \cup [0; d(3 + 3\gamma/2)],$$

$$\sigma(H_\lambda^{(\gamma,2)}) = \sigma_{\text{two}}(H_\lambda^{(\gamma,2)}) \cup [0; d(3 + 3\gamma/2)]$$

tengliklar o‘rinli, bunda

$$\sigma_{\text{two}}(H_\mu^{(\gamma,1)}) := \bigcup_{k \in \mathbb{T}^d} \{\sigma_{\text{disc}}(h_\mu^{(\gamma,1)}(k)) + \varepsilon(k)\},$$

$$\sigma_{\text{two}}(H_\lambda^{(\gamma,2)}) := \bigcup_{k \in \mathbb{T}^d} \{\sigma_{\text{disc}}(h_\lambda^{(2)}(k)) + \gamma\varepsilon(k)\}.$$

**Isbot.** Aniqlanishiga ko‘ra,  $H_\mu^{(\gamma,1)}$  va  $H_\lambda^{(\gamma,2)}$  kanal operatorlar  $L_2(\mathbb{T}^d)^2$  Hilbert fazosidagi

$$(Af)(x, y) = \omega(x, y)f(x, y)$$

operatori bilan o‘rin almashinish xossasiga ega. Bu yerda  $\omega(\cdot, \cdot)$  funksiya  $(\mathbb{T}^d)^2$  to‘plamda aniqlangan uzluksiz funksiyadir. Shu sababli  $L_2(\mathbb{T}^d)^2$  fazoning

$$L_2(\mathbb{T}^d)^2 = \int_{\mathbb{T}^d} \oplus L_2(\mathbb{T}^d) dk$$

to‘g‘ri integralga yoyilmasidan  $H_\mu^{(\gamma,1)}$  va  $H_\lambda^{(\gamma,2)}$  operatorlarning ham

$$H_\mu^{(\gamma,1)} = \int_{\mathbb{T}^d} \oplus (h_\mu^{(\gamma,1)}(k) + \varepsilon(k)I) dk,$$

$$H_\lambda^{(\gamma,2)} = \int_{\mathbb{T}^d} \oplus (h_\lambda^{(2)}(k) + \gamma\varepsilon(k)I) dk$$

yoyilmalari kelib chiqadi. Mazkur holda to‘g‘ri integralga yoyiluvchi operatorlarning spektri haqidagi teoremadan

$$\sigma(H_\mu^{(\gamma,1)}) = \sigma_{\text{two}}(H_\mu^{(\gamma,1)}) \cup [0; d(3 + 3\gamma/2)],$$

$$\sigma(H_\lambda^{(\gamma,2)}) = \sigma_{\text{two}}(H_\lambda^{(\gamma,2)}) \cup [0; d(3 + 3\gamma/2)]$$

tengliklarni hosil qilamiz. Bunda

$$\sigma_{\text{two}}(H_\mu^{(\gamma,1)}) := \bigcup_{k \in \mathbb{T}^d} \{\sigma_{\text{disc}}(h_\mu^{(\gamma,1)}(k)) + \varepsilon(k)\},$$

$$\sigma_{\text{two}}(H_{\lambda}^{(\gamma,2)}) := \bigcup_{k \in \mathbb{T}^d} \left\{ \sigma_{\text{disc}}(h_{\lambda}^{(2)}(k)) + \gamma \varepsilon(k) \right\}.$$

$\Sigma_{\mu,\lambda}^{(\gamma)}$  to'planning aniqlanishiga ko'ra

$$\Sigma_{\mu,\lambda}^{(\gamma)} = \sigma_{\text{two}}(H_{\mu}^{(\gamma,1)}) \cup \sigma_{\text{two}}(H_{\lambda}^{(\gamma,2)}) \cup [0; d(3 + 3\gamma/2)]$$

tenglik o'rinli. Bu esa teoremani isbotlaydi.

**Faddeyev tenglamasi.** Har bir fiksirlangan  $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(H_{\mu,\lambda}^{(\gamma)})$  soni uchun  $L_2^s((\mathbb{T}^d)^2)$  fazoda ta'sir qiluvchi  $T_{\mu,\lambda}^{(\gamma)}(z)$  blok-operatorli matritsani qaraymiz

$$T_{\mu,\lambda}^{(\gamma)}(z) := \begin{pmatrix} T_{11}(\mu, \lambda, \gamma; z) & T_{12}(\mu, \lambda, \gamma; z) \\ T_{21}(\mu, \lambda, \gamma; z) & 0 \end{pmatrix}.$$

Bu yerda  $T_{ij}(\mu, \lambda, \gamma; z) : L_2(\mathbb{T}^d) \rightarrow L_2(\mathbb{T}^d)$ ,  $i, j = 1, 2$  integral operatorlar:

$$(T_{11}(\mu, \lambda, \gamma; z)\varphi_1)(x) = \frac{\mu v(x)}{\Delta_{\mu}^{(\gamma,1)}(x, z - \varepsilon(x))} \int_{\mathbb{T}^d} \frac{v(t)\varphi_1(t)}{E_{\gamma}(x, t) - z} dt,$$

$$(T_{12}(\mu, \lambda, \gamma; z)\varphi_2)(x) = \frac{\lambda}{\Delta_{\mu}^{(\gamma,1)}(x, z - \varepsilon(x))} \int_{\mathbb{T}^d} \frac{v(t-x)\varphi_2(t)}{E_{\gamma}(x, t-x) - z} dt,$$

$$(T_{21}(\mu, \lambda, \gamma; z)\varphi_1)(x) = \frac{2\mu}{\Delta_{\gamma}^{(2)}(x, z - \gamma\varepsilon(x))} \int_{\mathbb{T}^d} \frac{v(x-t)\varphi_1(t)}{E_{\gamma}(t, x-t) - z} dt.$$

Quyidagi lemma  $H_{\mu,\lambda}^{(\gamma)}$  va  $T_{\mu,\lambda}^{(\gamma)}(z)$  operatorlarning xos qiymatlari orasidagi bog'lanishni ifodalaydi [7].

**3-lemma.**  $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(H_{\mu,\lambda}^{(\gamma)})$  soni  $H_{\mu,\lambda}^{(\gamma)}$  model Hamiltonianning xos qiymati bo'lishi uchun 1 soni  $T_{\mu,\lambda}^{(\gamma)}(z)$  operatorning xos qiymati bo'lishi zarur va yetarlidir.

**1-eslatma.** Odatda  $\varphi = T_{\mu,\lambda}^{(\gamma)}(z)\varphi$  operatorli tenglamaga  $H_{\mu,\lambda}^{(\gamma)}$  model Hamiltonianning xos funksiyalariga mos Faddeyev tenglamasi deyiladi.

Bunda

$$\varphi = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = \begin{pmatrix} (T_{11}(\mu, \lambda, \gamma; z)\varphi_1)(x) + (T_{12}(\mu, \lambda, \gamma; z)\varphi_2)(x) \\ (T_{21}(\mu, \lambda, \gamma; z)\varphi_1)(x) \end{pmatrix}.$$

$H_{\mu,\lambda}^{(\gamma)}$  model Hamiltonianning muhim spektri.  $\|\cdot\|$  va  $(\cdot, \cdot)$  orqali mos Hilbert fazolaridagi elementning normasi va elementlarning skalyar ko'paytmasini belgilaymiz.

Qulaylik uchun  $H_{\mu,\lambda}^{(\gamma)}$  model Hamiltonian muhim spektri uchun Veyl kriteriyasini bayon qilamiz [8].  $z$  soni  $H_{\mu,\lambda}^{(\gamma)}$  model Hamiltonianning muhim spektriga tegishli bo'lishi uchun  $L_2^s((\mathbb{T}^d)^2)$  Hilbert fazosida shunday  $\{F_n\}$  ketma-ketlik topilib,  $\|F_n\| = 1$  va

$$\lim_{n \rightarrow \infty} \|(H_{\mu,\lambda}^{(\gamma)} - zE)F_n\| = 0 \quad (2)$$

tenglikning bajarilishi zarur va yetarlidir. Bu yerda  $E$  orqali  $L_2^s((\mathbb{T}^d)^2)$  Hilbert fazosidagi birlik operator belgilangan.  $z$  soni muhim spektriga tegishli bo'lishi uchun yuqoridagi shartlarni

qanoatlantiruvchi, biroq yaqinlashuvchi qisman ketma-ketlikka ega bo‘lmagan  $\{F_n\}$  ketma-ketlik mavjud bo‘lishi zarur va yetarlidir. Masalan,  $\{F_n\}$  ketma-ketlik ortonormal sistemani hosil qilsa, u holda bu ketma-ketlik o‘zida birorta ham yaqinlashuvchi qisman ketma-ketlikni saqlamaydi. Odatda bunday ketma-ketliklarga singulyar ketma-ketlik deyiladi.

Quyidagicha belgilashlar kiritamiz:

$$\begin{aligned}\Sigma_{\mu,\lambda}^{(\gamma)} &:= \sigma_{\text{two}}(H_{\mu}^{(\gamma,1)}) \cup \sigma_{\text{two}}(H_{\lambda}^{(\gamma,2)}) \cup [0; d(3 + 3\gamma/2)], \\ \Omega_{\mu,\lambda}^{(\gamma)} &:= \sigma_{\text{two}}(H_{\mu}^{(\gamma,1)}) \cup \sigma_{\text{two}}(H_{\lambda}^{(\gamma,2)}).\end{aligned}$$

Quyidagi teorema  $H_{\mu,\lambda}^{(\gamma)}$  model Hamiltonianning muhim spektrini tavsiflaydi.

**2-teorema.**  $H_{\mu,\lambda}^{(\gamma)}$  model Hamiltonianning muhim spektri  $H_{\mu}^{(\gamma,1)}$  va  $H_{\lambda}^{(\gamma,2)}$  kanal operatorlarning spektrlari birlashmasi bilan ustma-ust tushadi, ya’ni

$$\sigma_{\text{ess}}(H_{\mu,\lambda}^{(\gamma)}) = \Sigma_{\mu,\lambda}^{(\gamma)}$$

tenglik o‘rinli bo‘ladi. Bundan tashqari,  $\Sigma_{\mu,\lambda}^{(\gamma)}$  to‘plam ko‘pi bilan uchta kesmalar birlashmasidan iborat bo‘ladi.

**Isbot.** Dastlab  $\Sigma_{\mu,\lambda}^{(\gamma)} \subset \sigma_{\text{ess}}(H_{\mu,\lambda}^{(\gamma)})$  munosabat o‘rinli bo‘lishini ko‘rsatamiz. Aniqlanishiga ko‘ra  $\Sigma_{\mu,\lambda}^{(\gamma)} = \Omega_{\mu,\lambda}^{(\gamma)} \cup [0; d(3 + 3\gamma/2)]$  tenglik o‘rinli.  $[0; d(3 + 3\gamma/2)] \subset \sigma_{\text{ess}}(H_{\mu,\lambda}^{(\gamma)})$  ekanligini isbotlaymiz. Faraz qilaylik,  $z_0 \in [0; d(3 + 3\gamma/2)]$  ixtiyoriy nuqta bo‘lsin.  $z_0 \in \sigma_{\text{ess}}(H_{\mu,\lambda}^{(\gamma)})$  ekanligini ko‘rsatamiz. Buni amalga oshirishda Veyl kriteriyasidan foydalanish qulaydir. Ya’ni (2) shartni qanoatlantiruvchi  $\{F_n\} \subset L_2^s((\mathbb{T}^d)^2)$  ortonormal vektor funksiyalar ketma-ketligini qurish yetarlidir.

$E_{\gamma}(\cdot, \cdot)$  funksiya  $(\mathbb{T}^d)^2$  kompakt to‘plamda uzluksiz bo‘lganligi sababli shunday  $(x_0, y_0) \in (\mathbb{T}^d)^2$  nuqta topilib,  $z_0 = E_{\gamma}(x_0, y_0)$  tenglik o‘rinli bo‘ladi.  $n \in \mathbb{N}$  natural soni uchun  $(x_0, y_0) \in (\mathbb{T}^d)^2$  nuqtaning quyidagi

$$W_n := V_n(x_0) \times V_n(y_0), n \in \mathbb{N}$$

atrofini qaraymiz. Bu yerda

$$V_n(x_0) := \{x \in \mathbb{T}^d : \frac{1}{n + n_0 + 1} < |x - x_0| < \frac{1}{n + n_0}\}$$

to‘plam  $x_0$  nuqtaning o‘yilgan atrofi bo‘lib, undan shunday  $n_0 \in \mathbb{N}$  sonini tanlaymizki natijada barcha  $n \in \mathbb{N}$ ,  $n > n_0$  natural sonlar uchun  $V_n(x_0) \cap V_n(y_0) = \emptyset$  tenglik bajariladi. Bunda  $x_0 \neq y_0$  deb olingan.

$\text{mes}(W_n)$  orqali  $W_n$  to‘planning Lebeg o‘lchovi va  $\chi_{W_n}(\cdot, \cdot)$  orqali  $W_n$  to‘planning xarakteristik funksiyasini belgilaymiz.  $\{F_n\} \subset H$  funksiyalar ketma-ketligini quyidagicha aniqlaymiz:

$$F_n(x, y) := \frac{\chi_{W_n}(x, y) + \chi_{W_n}(y, x)}{\sqrt{2\text{mes}(W_n)}}.$$

Osongina ko‘rsatish mumkinki  $\{F_n\}$  ortonormal ketma-ketlikdir. Istalgan  $n \in \mathbb{N}$  soni uchun  $(H_{\mu,\lambda}^{(\gamma)} - z_0 E)F_n$  elementlarni qaraymiz va uning normasini baholaymiz:

$$\|(H_{\mu,\lambda}^{(\gamma)} - z_0 E)F_n\|^2 \leq 2 \sup_{(x,y) \in W_n} |E_{\gamma}(x, y) - z_0|^2 + [8\mu^2 \max_{x \in \mathbb{T}^d} |v(x)|^2 + 2\lambda^2] \text{mes}(V_n(x_0)).$$

$V_n(x_0)$  to'planning qurilishidan va  $E_\gamma(\cdot, \cdot)$  funksiyaning uzluksizlik xossasidan,  $n \rightarrow \infty$  bo'lganda  $\|(H_{\mu,\lambda}^{(\gamma)} - z_0 E)F_n\| \rightarrow 0$  ekanligi, ya'ni  $z_0 \in \sigma_{\text{ess}}(H_{\mu,\lambda}^{(\gamma)})$  munosabat bajarilishi kelib chiqadi.  $z_0$  nuqtaning ixtiyoriy ekanligidan  $[0; d(3 + 3\gamma/2)] \subset \sigma_{\text{ess}}(H_{\mu,\lambda}^{(\gamma)})$  munosabatni hosil qilamiz.

Endi  $\Omega_{\mu,\lambda}^{(\gamma)} \subset \sigma_{\text{ess}}(H_{\mu,\lambda}^{(\gamma)})$  ekanligini ko'rsatamiz.  $z \in \Omega_{\mu,\lambda}^{(\gamma)}$  ixtiyoriy nuqtani olib,  $z \in \sigma_{\text{ess}}(H_{\mu,\lambda}^{(\gamma)})$  bo'lishini ko'rsatamiz. Ikkita hol ro'y berishi mumkin:  $z \in [0; d(3 + 3\gamma/2)]$  yoki  $z \notin [0; d(3 + 3\gamma/2)]$ .

Agar  $z \in [0; d(3 + 3\gamma/2)]$  bo'lsa, u holda yuqorida isbotlangan tasdiqqa ko'ra  $z \in \sigma_{\text{ess}}(H_{\mu,\lambda}^{(\gamma)})$ .

Faraz qilaylik,

$$\bigcup_{k \in \mathbb{T}^d} \{\sigma_{\text{disc}}(h_\mu^{(\gamma,1)}(k) + \varepsilon(k)) \setminus [0; d(3 + 3\gamma/2)]\}$$

bo'lsin. Ushbu

$$\bigcup_{k \in \mathbb{T}^d} \{\sigma_{\text{disc}}(h_\mu^{(\gamma,1)}(k) + \varepsilon(k))\}$$

to'planning ta'rifi va 1-lemmaga ko'ra shunday  $k_1 \in \mathbb{T}^d$  nuqta topilib,  $\Delta_\mu^{(\gamma,1)}(k_1; z) = 0$  tenglik bajariladi.

$\{\Phi_n\}$  ortogonal funksiyalar ketma-ketligini

$$\Phi_n(x, y) := \frac{v(y)\varphi_n(x) + v(x)\varphi_n(y)}{2(E_\gamma(x, y) - z)}$$

ko'rinishida tanlaymiz. Bu yerda

$$\varphi_n(x) := \frac{c_n(x)\chi_{V_n(x_0)}(x)}{\sqrt{\text{mes}(V_n(x_0))}}.$$

Oxirgi tenglikda  $\{c_n(\cdot)\} \subset L_2(\mathbb{T}^d)$  funksiyalar  $\{\Phi_n\}$  uchun ortonormallik sharti, ya'ni  $n \neq m$  uchun

$$(\Phi_n, \Phi_l) := \frac{1}{2\sqrt{\text{mes}(V_n(x_0))}\sqrt{\text{mes}(V_l(y_0))}} \int_{V_n(x_0)} \int_{V_l(y_0)} \frac{v(s)v(t)c_n(s)c_l(t)}{(E_\gamma(s, t) - z)^2} ds dt = 0 \quad (3)$$

va  $\|\Phi_n\| = 1$  shartlardan tanlangan. Bu yerda  $\text{supp}\{c_n(\cdot)\}$  orqali  $c_n(\cdot)$  funksiyaning tashuvchisi belgilangan.  $\{c_n(\cdot)\}$  ketma-ketlikning mavjudligi quyidagi tasdiqda o'z aksini topgan.

**1-tasdiq.** Shunday  $\{c_n(\cdot)\} \in L_2(\mathbb{T}^d)$  ortonormal sistema mavjud bo'lib,  $\text{supp}\{c_n(\cdot)\} \subset V_n(x_0)$  va (3) shartlar bajariladi.

2-teoremani isbotlashda davom etamiz. Endi

$$\lim_{n \rightarrow \infty} \|(H_{\mu,\lambda}^{(\gamma)} - zE)\Phi_n\| = 0$$

ekanligini ko'rsatamiz. Buni amalga oshirishda  $n \in \mathbb{N}$  uchun  $(H_{\mu,\lambda}^{(\gamma)} - zE)\Phi_n$  elementni qaraymiz va uning normasini quyidagicha baholaymiz:

$$\|(H_{\mu,\lambda}^{(\gamma)} - zE)\Phi_n\|^2 \leq C_{\mu,\lambda}^{(1)} \text{mes}(V_n(x_0)) + C_{\mu,\lambda}^{(2)} \sup_{x \in V_n(x_0)} |(\Delta_\mu^{(\gamma,1)}(x; z))|^2. \quad (4)$$

Bunda  $C_{\mu,\lambda}^{(\alpha)} > 0$ ,  $\alpha = 1, 2$  biror son.  $n \rightarrow \infty$  bo‘lganda  $\text{mes}(V_n(x_0)) \rightarrow 0$  va  $\sup_{x \in V_n(x_0)} |(\Delta_{\mu}^{(\gamma,1)}(x; z))^2| \rightarrow 0$  tasdiqlarga ko‘ra (4) baholashlardan foydalanib,  $\|(H_{\mu,\lambda}^{(\gamma)} - zE)\Phi_n\| \rightarrow 0$

0 ekanligini hosil qilamiz. Bundan  $z \in \sigma_{\text{ess}}(H_{\mu,\lambda}^{(\gamma)})$  bo‘lishi kelib chiqadi.  $z$  nuqta ixtiyoriy nuqta ekanligidan foydalanib ushbu

$$\bigcup_{k \in \mathbb{T}^d} \{\sigma_{\text{disc}}(h_{\mu}^{(\gamma,1)}(k) + \varepsilon(k))\} \subset \sigma_{\text{ess}}(H_{\mu,\lambda}^{(\gamma)})$$

tenglikni hosil qilamiz. Xuddi shuningdek,

$$\bigcup_{k \in \mathbb{T}^d} \{\sigma_{\text{disc}}(h_{\lambda}^{(2)}(k) + \gamma\varepsilon(k))\} \subset \sigma_{\text{ess}}(H_{\mu,\lambda}^{(\gamma)})$$

munosabat isbotlanadi. Shunday qilib,  $\Sigma_{\mu,\lambda}^{(\gamma)} \subset \sigma_{\text{ess}}(H_{\mu,\lambda}^{(\gamma)})$  ekanligi isbotlandi.

Endi teskari munosabatni, ya‘ni  $\sigma_{\text{ess}}(H_{\mu,\lambda}^{(\gamma)}) \subset \Sigma_{\mu,\lambda}^{(\gamma)}$  ekanligini ko‘rsatamiz. Har bir  $\mu, \lambda > 0$  va  $z \in \mathbb{C} \setminus \Sigma_{\mu,\lambda}^{(\gamma)}$  sonlari uchun  $T_{\mu,\lambda}^{(\gamma)}(z)$  operator  $\mathbb{C} \setminus \Sigma_{\mu,\lambda}^{(\gamma)}$  sohada kompakt operator qiymatli funksiya bo‘ladi. U holda  $H_{\mu,\lambda}^{(\gamma)}(z)$  model Hamiltonianning o‘z-o‘ziga qo‘shmaligidan va oldingi teoremdan agar  $z$  haqiqiy son va absolyut qiymati bo‘yicha yetarlicha katta bo‘lsa, u holda  $(I - T_{\mu,\lambda}^{(\gamma)}(z))^{-1}$  operator mavjud bo‘lishi kelib chiqadi. Fredgolmning analitik teoremasiga [8] ko‘ra shunday  $S_{\mu,\lambda}^{(\gamma)} \subset \mathbb{C} \setminus \Sigma_{\mu,\lambda}^{(\gamma)}$  diskret to‘plam topilib  $\mathbb{C} \setminus (S_{\mu,\lambda}^{(\gamma)} \cup \Sigma_{\mu,\lambda}^{(\gamma)})$  sohada analitik bo‘lgan  $(I - T_{\mu,\lambda}^{(\gamma)}(z))^{-1}$  operator qiymatli funksiya mavjud bo‘ladi va  $\mathbb{C} \setminus \Sigma_{\mu,\lambda}^{(\gamma)}$  sohada meromorf chekli rangli qoldiqqa ega bo‘ladi. Bu mulohazadan  $\sigma(H_{\mu,\lambda}^{(\gamma)}) \setminus \Sigma_{\mu,\lambda}^{(\gamma)}$  to‘plam yakkaqat nuqtalardan tashkil topgan va  $\Sigma_{\mu,\lambda}^{(\gamma)}$  to‘plamning quyuvlashish nuqtalari uning chegaraviy nuqtalari bo‘lishi kelib chiqadi. Shunday qilib,

$$\sigma(H_{\mu,\lambda}^{(\gamma)}) \setminus \Sigma_{\mu,\lambda}^{(\gamma)} \subset \sigma_{\text{disc}}(H_{\mu,\lambda}^{(\gamma)}) = \sigma(H_{\mu,\lambda}^{(\gamma)}) \setminus \sigma_{\text{ess}}(H_{\mu,\lambda}^{(\gamma)})$$

ekan.

Demak,  $\sigma_{\text{ess}}(H_{\mu,\lambda}^{(\gamma)}) \subset \Sigma_{\mu,\lambda}^{(\gamma)}$  munosabat o‘rinli ekan. Vanihoyat yuqoridagi mulohazalardan  $\sigma_{\text{ess}}(H_{\mu,\lambda}^{(\gamma)}) = \Sigma_{\mu,\lambda}^{(\gamma)}$  tenglikni hosil qilamiz.

Har bir fiksirlangan  $\mu, \lambda, \gamma > 0$  sonlari va  $k \in \mathbb{T}^d$  element uchun  $\Delta_{\mu}^{(\gamma,1)}(k; \cdot)$  funksiya  $(-\infty; m_1^{(\gamma)}(k))$  oraliqda monoton kamayuvchi,  $\Delta_{\lambda}^{(2)}(k; \cdot)$  funksiya  $(-\infty; m_2(k))$  oraliqda monoton kamayuvchi funksiya bo‘ladi. Ikkinchi tomondan barcha  $z > d(3 + 3\gamma/2)$  lar uchun  $\Delta_{\mu}^{(\gamma,1)}(k; z) > 1$  va  $\Delta_{\lambda}^{(2)}(k; z) > 1$  tengsizliklar o‘rinli bo‘ladi. Shu sababli, har bir fiksirlangan  $\mu, \lambda, \gamma > 0$  sonlari va  $k \in \mathbb{T}^d$  element uchun  $h_{\mu}^{(\gamma,1)}(k)$  va  $h_{\lambda}^{(2)}(k)$  operatorlar 0 dan chapda joylashgan ko‘pi bilan 1 ta oddiy xos qiymatga ega bo‘ladi va bu operatorlar  $d(3 + 3\gamma/2)$  dan o‘ngda yotuvchi xos qiymatlarga ega bo‘lmaydi. U holda yoyiluvchi operatorlarning spektri haqidagi teorema ko‘ra va  $\Omega_{\mu,\lambda}^{(\gamma)}$  to‘plamning aniqlanishiga ko‘ra  $\Omega_{\mu,\lambda}^{(\gamma)}$  to‘plam ko‘pi bilan 2 ta kesmalar birlashmasidan iborat ekanligini hosil qilamiz. Demak,  $\Sigma_{\mu,\lambda}^{(\gamma)}$  to‘plam ko‘pi bilan 3 ta kesmalar birlashmasidan iborat ekan. 3-teorema to‘liq isbotlandi.

**Xulosa.** Mazkur maqolada qattiq jismlar fizikasi, statistik fizika, kvant maydon nazariyasi va zamonaviy matematik fizikaning ko‘plab sohalarida uchraydigan  $d$  o‘lchamli panjaradagi

uchta zarrachalar sistemasiga mos model Hamiltonian qaralgan. Bu Hamiltonian Hilbert fazosidagi chiziqli, chegaralangan va o'z-o'ziga qo'shma operator sifatida o'rganilgan. Dastlab model Hamiltonianga mos ikkita kanal operatorlar aniqlangan. Kanal operatorlarning spektrlari mos Fridrixs modellari oilasining spektri orqali tavsiflangan. Model Hamiltonianning muhim spektri ikkita kanal operatorlar spektrlari birlashmasiga teng bo'lishi isbotlangan hamda muhim spektr ko'pi bilan uchta kesmalar birlashmasidan iborat bo'lishi ko'rsatilgan.

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### РЕЗЮМЕ

В данной статье модельный гамильтониан, соответствующая системе трех частиц на  $d$ -мерной решетке, изучается как линейный, ограниченный и самосопряженный оператор в гильбертовом пространстве. Определены два канальных оператора, соответствующие этому модельному гамильтониану и спектры этих операторов описываются спектром семейства моделей Фридрикса. Доказано, что существенный спектр модельного гамильтониана равен объединению спектров двух канальных операторов.

**Ключевые слова:** решетка, система частиц, модельный гамильтониан, оператор канала, разложимый оператор, определитель Фредгольма, критерий Вейля, ортонормированная система, собственное значение, спектр.

## RESUME

In this article, the model Hamiltonian corresponding to the system of three particles in the  $d$  dimensional lattice is studied as a linear, bounded and self-adjoint operator in the Hilbert space. Two channel operators corresponding to this model Hamiltonian are defined, and the spectra of these operators are described by the spectrum of the family of Friedrichs models. It is proved that the essential spectrum of the model Hamiltonian is equal to the union of the spectra of two channel operators.

**Key words:** lattice, system of particles, model Hamiltonian, channel operator, decomposed operator, Fredholm determinant, Weyl criterion, orthonormal system, eigenvalue, spectrum.

UDC 517.55

## APPLICATION OF MATRICES IN PLANT RECOGNITION AND ARTIFICIAL INTELLIGENCE: A PYNQ-Z2-BASED SOLUTION

Yilihamujiang Yusupu \* Matyakubov A. \*

### RESUME

This paper explores the use of matrix computing in plant identification and artificial intelligence, focusing on the efficient implementation and application of the PYNQ(Python Productivity for Zynq)-Z2 platform in the image acquisition of field expedition plant samples. As a development board based on FPGA (field programmable gate array), PYNQ-Z2 has significant advantages in cost, power consumption and portability. This paper analyzes the role of matrix in convolutional neural network (CNN), demonstrates the potential of PYNQ-Z2 platform in hardware acceleration, and discusses the system architecture designed for plant identification and future improvement directions.

**Key words:** Matrix, Plant Recognition, Artificial Intelligence, PYNQ-Z2, Hardware Acceleration.

### 1 Introduction

With the rapid advancement of ecological and environmental research, plant recognition technology is gaining importance in scientific studies. Artificial intelligence (AI), especially deep learning[1], offers new methods to identify and classify plants, making traditional, labor-intensive plant identification processes more efficient. Convolutional Neural Networks [2] have proven highly effective in image recognition tasks, including plant identification, due to their ability to learn and extract complex patterns in images.

However, existing computational platforms like mobile devices and traditional computers have limitations that affect their suitability for plant recognition tasks. These limitations include high power consumption, limited portability, and the need for consistent network connectivity. The constraints are particularly significant in field environments where power and connectivity may be limited.

Therefore, selecting a hardware platform that can perform plant recognition efficiently, even in offline conditions, becomes essential. This study proposes the use of the PYNQ-Z2[3] platform—a low-power, low-cost FPGA-based board—offering efficient matrix computation for AI applications at the edge. In this paper, we explore the structure and capabilities of PYNQ-Z2, focusing on its role in enhancing matrix computations for CNN-based plant recognition.

### 2 Background and Motivation

With the rapid development of ecological and environmental research, plant identification technology has become increasingly important in scientific research. Artificial intelligence

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\*Yilihamujiang Yusupu – National University of Uzbekistan, alam@asleng.org

\*Matyakubov A. – National University of Uzbekistan, almasa@list.ru





Рис. 1: GMOCA Home page

(AI), especially deep learning [4], provides new methods for identifying and classifying plants, making the traditional labor-intensive plant identification process more efficient. Convolutional neural networks[5] have been shown to be very effective in image recognition tasks, including plant identification, due to their ability to learn and extract complex patterns in images. However, existing computing platforms, such as mobile devices and traditional computers, have several limitations that affect their applicability to plant identification tasks. These limitations include high power consumption, limited portability, and the need for consistent network connectivity. These limitations are particularly evident in field environments where power and connectivity may be limited. Therefore, it becomes critical to select a hardware platform that can effectively perform plant identification even under offline conditions. In this paper, we explore the drawbacks and solutions of cache storage in terms of data integrity, as well as the structure and functionality of PYNQ-Z2, focusing on its role in enhancing the computation of CNN-based plant identification matrices. This study proposes the use of the PYNQ-Z2[5] platform (a low-power, low-cost FPGA-based development board) to provide efficient matrix computing for edge AI applications. When collecting data in edge devices, instead of using cache storage, direct writing to files is used to ensure data integrity.

### 3 *Offline storage and transmission of edge segments*

**3.1 Challenges of Offline Data Processing** Field scientific research and investigation usually requires the system to work in an offline environment or in an environment with relatively weak Internet signals, which brings challenges to data storage and processing. In offline mode, the system needs to cache the collected data locally because it may not be able to connect to the network. A powerful caching mechanism is essential for storing data safely and losslessly. When a device software application failure or hardware failure occurs, the collected data may be lost and the lost data cannot be restored.

**3.2 Ensure data integrity and reduce the risk of data being unrecoverable** Data integrity is critical for plant identification applications because errors in data storage can result in incomplete datasets, which in turn affect the accuracy of research. One solution to ensure data integrity is to embed metadata[6] (e.g., GPS coordinates and timestamps) directly into image files. This approach allows the data to be stored securely,

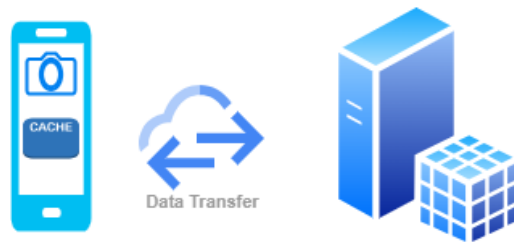


Рис. 2: Stored in Cache

ensures that it can be reliably transmitted when connectivity is restored, and even allows for high-probability data recovery in the event of an edge device failure.



Рис. 3: Stored in image format

#### 4 Role of Matrices in Image Recognition

**4.1 Digital Images as Matrices** In image recognition, digital images are represented as matrices of pixel values. Each pixel in an image corresponds to an element in a matrix. For a grayscale image, each pixel is represented by a single value, while for a color image, each pixel has three values (red, green, and blue channels), resulting in three matrices. Processing image data in matrix form allows AI algorithms to perform calculations and extract features effectively[7].

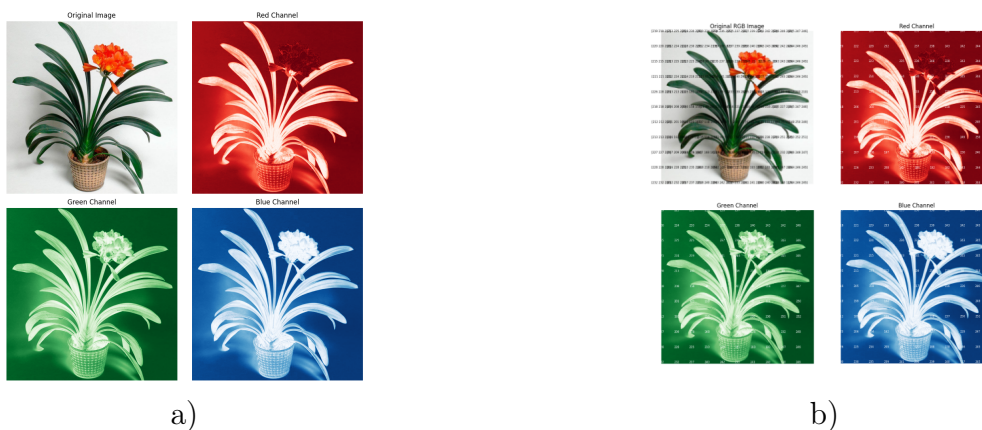


Рис. 4: a) RGB, b) Matrix for RGB channel with photo

**4.2 Convolution Operations in CNNs** A CNN applies convolution operations to images to detect features such as edges and textures. Convolution is a mathematical operation where a small matrix, called a filter or kernel, scans over the image matrix. Each value in the filter is multiplied with the corresponding image matrix value, and the results are summed to form a new matrix. This process is repeated across the image, allowing CNNs to extract spatial features from the image data.

The process of feature extraction can be mathematically represented by applying a specific function  $f$  to the original data  $X$ , which results in the feature representation  $F$ .

$$F = f(X) \quad (1)$$

In this context,  $X$  represents the original data, and  $F$  denotes the extracted feature representation.

In convolutional neural networks, the feature extraction process typically involves several key steps. One of the fundamental operations is the convolution, which extracts local features from various regions by sliding a convolution kernel (filter) across the input data. This operation can be mathematically represented as:

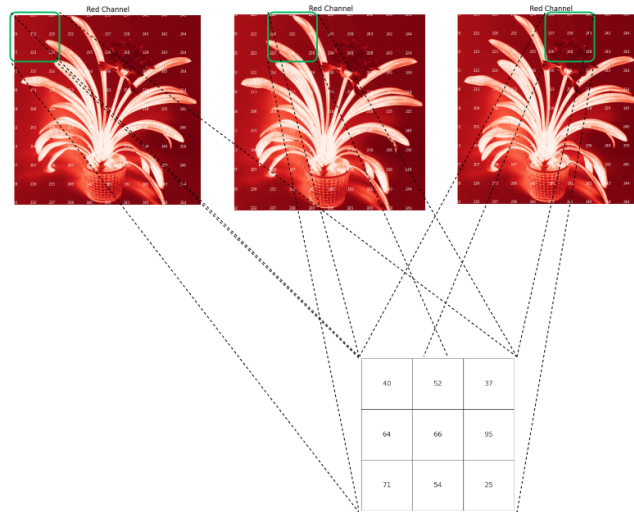


Рис. 5: Extract spatial features from the image data

$$F_{ij}^{(k)} = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \sum_{c=0}^{C-1} X_{i+m, j+n, c} * K_{m, n, c}^{(k)} \quad (2)$$

Here,  $F_{ij}^{(k)}$  represents the value at position  $(i, j)$  in the convolved feature map  $F$ , which corresponds to the output of the  $k$ -th convolution kernel. The double summation sign

$\sum_{m=0}^{M-1} \sum_{n=0}^{N-1}$  used to iterate over each position of the convolution kernel, where  $M$  and  $N$  denote the height and width of the kernel, respectively.

The summation  $\sum_{c=0}^{C-1}$  is used to iterate over each channel of the input image, with  $C$  representing the number of channels in the input image.

$X_{i+m,j+n,c}$  represents the pixel value at position  $(i + m, j + n)$  in the input image  $X$ , corresponding to the value in the  $c$ -th channel. Similarly,  $K_{m,n,c}^{(k)}$  denotes the weight at position  $(m, n)$  in the  $k$ -th convolution kernel, corresponding to the weight in the  $c$ -th channel.

**4.3 Fully Connected Layers and Classification** After feature extraction, CNNs use fully connected layers to transform the extracted features into a classification output. These layers operate on feature vectors derived from the convolution layers and use matrix multiplications to compute a probability distribution over different classes. The final layer, often a Softmax layer[2], generates probabilities that predict the most likely class, such as a specific plant species in this case.

Let the length of the flattened vector  $x$  be  $L$  (where  $L = HWD$ ), The output  $Z_j$  of the  $j$ -th neuron in the fully connected layer can be computed using the following formula[8]:

$$z_j = \sum_{i=1}^L W_{fc,ji} * x_i + b_{fc,j} \quad (3)$$

$z_j$  :Represents the value of the  $j$ -th neuron in the output vector  $z$  of the fully connected layer.

$W_{fc,ji}$  :Represents the value of the  $j$ -th row and  $i$ -th column in the weight matrix  $W_{fc}$  .

$x_i$  :Represents the value of the  $i$ -th element in the flattened vector  $x$ .

$b_{fc,j}$  :Represents the value of the  $j$ -th element in the bias vector  $b_{fc}$  .

Thus, by performing a matrix multiplication between the flattened feature vector  $x$  and the weight matrix  $W^{(fc)}$ , and adding the bias vector  $b^{(fc)}$ , we obtain the output vector  $z$  of the fully connected layer. This output vector can then be nonlinearly transformed through an activation function (e.g., ReLU, sigmoid, tanh, etc.) to produce the activated output  $a$ [8].

**5 Overview of the PYNQ-Z2 Platform, Why PYNQ-Z2?** According to the above artificial intelligence plant image recognition process and calculation formula, it is not difficult to see that the amount of calculation and complexity of this process are relatively high. Therefore, the selection and investment cost of computing equipment are worthy of careful consideration, because the computing speed, power consumption, equipment rental or purchase cost directly affect our subsequent scientific research activities.

**5.1 Why PYNQ-Z2?** The PYNQ-Z2 platform has several advantages that make it an ideal choice for plant recognition tasks:

- **Cost-Effectiveness:** Unlike high-end servers or cloud-based solutions, PYNQ-Z2 provides a more budget-friendly option, making it accessible for small-scale research projects. The price is in the range of 178-195 USD, which is particularly cost-effective compared to renting a server or buying a personal computer.

- **Low Power Consumption:** PYNQ-Z2 consumes less power, suitable for fieldwork where power sources may be limited. The power supply can support micro USB, 5V 2A power consumption, usually 3-10 watts.
- **Hardware Acceleration:** With its FPGA capabilities, PYNQ-Z2 can handle matrix computations in parallel, enhancing the speed and efficiency of AI tasks such as image processing. The board is equipped with an XC7Z020 FPGA, which includes 13,300 logic slices, 220 DSP slices, and 630 KB of block RAM. This enables complex computation and parallel processing, which is critical for hardware acceleration[4].

**5.2 Hardware Architecture** The PYNQ-Z2 development board is based on the Xilinx Zynq-7000 series, featuring an ARM Cortex-A9 processor coupled with FPGA fabric. The FPGA contains thousands of logic units and digital signal processing (DSP) blocks, enabling efficient parallel processing. This configuration makes the PYNQ-Z2 highly suitable for computationally intensive tasks like matrix operations in CNNs.

### 5.3 Advantages of Using PYNQ-Z2

- **Cost and Power Efficiency:** With a cost lower than most high-performance servers and a power requirement of only 5V, PYNQ-Z2 is highly energy efficient.
- **Hardware Acceleration of Matrix Calculations:** PYNQ-Z2's FPGA can perform complex matrix operations, such as convolutions, faster than CPUs by using parallelism.
- **Portability and Field Application:** Its compact design and low power requirements make PYNQ-Z2 ideal for field applications, as it can run on battery power for extended periods.

Platform	Portability	Cost	Power Consumption	Computational Power
Mobile Device	High	Moderate	Moderate	Low
Personal Computer	Low	Moderate	High	High
Server	Very Low	Very High	Very High	Very High
PYNQ-Z2	High	Low	Low	Optimized for AI

**5.4 Comparative Analysis with Other Platforms** Compared with traditional platforms, PYNQ-Z2 combines portability, low cost and energy saving. Although servers provide high computing power, they are not suitable for field tasks. PYNQ-Z2 is optimized for parallel matrix operations, making it an effective choice for plant identification in terms of cost, power consumption, compatibility and scalability, and efficient matrix calculation.

## 6 System Workflow and Architecture



**6.1 Data Capture and Processing Workflow** The plant recognition system workflow begins with image capture. Images are then stored in the matrix with GPS data and time-stamp. The architecture includes data capture modules, matrix processing modules. The results can be stored offline and later uploaded when network connectivity is available.

**6.2 Matrix Operations on PYNQ-Z2** With FPGA-based acceleration, PYNQ-Z2 efficiently handles matrix convolutions and multiplications required by CNNs. This efficiency enables low cost processing, while night donwload the file and batch processing, and allowing immediate feedback for plant identification result, and upload to database.

**6.3 Data Integrity and Offline Mode** In offline mode, data is stored with image format to ensure integrity[9], including metadata like GPS coordinates. This approach ensures the availability and reliability of data, even if the device loses power or network connection unexpectedly.

## 7 Challenges and Limitations

**7.1 Storage Management** High-resolution images and metadata quickly consume storage, especially in offline mode. Effective storage management[10], compression, and periodic data upload strategies are essential to address these challenges.

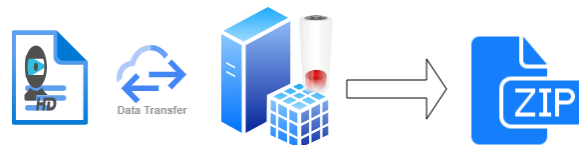


Рис. 6: Storage Management Issue

**7.2 Model Accuracy and Training Data** The accuracy of plant recognition models relies on high-quality training data[11]. Limited datasets can lead to misclassifications. Expanding the dataset and continuously training the model are essential to improve performance.

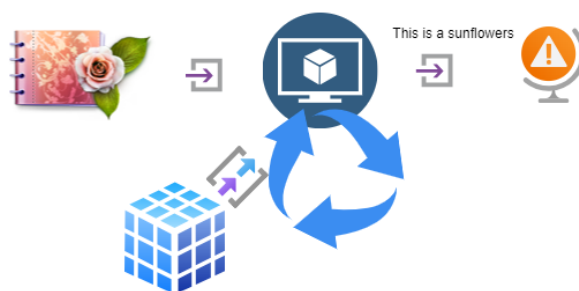


Рис. 7: Recognition process

**7.3 Environmental and Hardware Constraints** Field conditions can expose the system to extreme weather and hardware challenges. Durability of components and redundancies in data storage and processing are necessary to maintain system stability.

## 8 *Future Work*

**8.1 Real-Time Recognition Enhancement** Future developments may include integrating real-time recognition during the upload stage, further improving user experience and system efficiency(Figure 10).

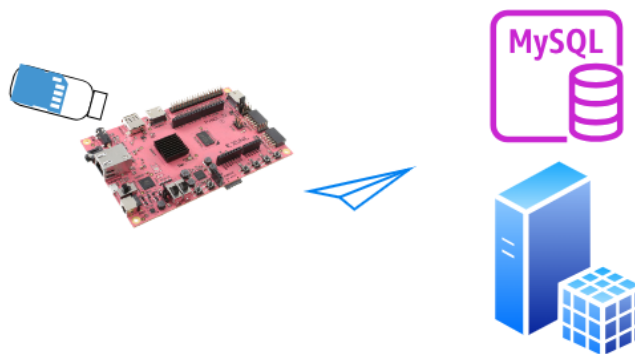


Рис. 8: Insert the memory card and complete the identification and upload immediately

**8.2 Expanding the Recognition Model** Plans are in place to expand the recognition model to cover more plant species, enhancing system applicability across different regions and ecosystems(Figure 11).

**8.3 User Feedback Mechanism** By establishing a user feedback system, users can evaluate recognition results. Feedback data will be used for continuous AI model refinement, improving recognition accuracy.

## 9 *Conclusion*

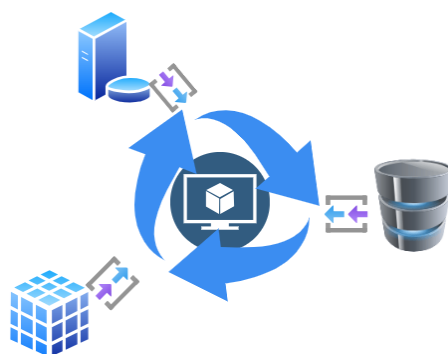


Рис. 9: Training with various datasets

During image processing in our field science expedition platform, the PYNQ-Z2 platform demonstrated great potential for plant recognition-based systems, providing a huge advantage through hardware-accelerated matrix computation. By leveraging its FPGA capabilities, PYNQ-Z2 can efficiently process basic operations such as convolution and matrix multiplication, which are essential for convolutional neural networks (CNNs) and other deep learning models[11]. This ability to efficiently process high-dimensional data is invaluable in environmental and plant research, where accurate and fast plant classification is critical.

Compared with traditional computing platforms, PYNQ-Z2 stands out for its low power consumption, cost-effectiveness, and enhanced portability. These features make it suitable for remote and offline applications, where resource constraints and environmental challenges may limit the use of larger, more power-hungry devices. In addition, the parallel processing capabilities of FPGAs can significantly reduce computation time, enabling real-time response that is critical for field research.

Looking ahead, expanding the recognition model to cover more comprehensive datasets and incorporating feedback mechanisms to continuously improve the accuracy of the model will further improve the applicability of PYNQ-Z2 in different ecological environments[12]. Continued advances in FPGA technology and improvements in AI algorithms are expected to make the platform even more effective, enhancing its role as a valuable tool for plant identification conservation efforts and scientific discovery.

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## REZYUME

Ushbu maqola o'simliklarni aniqlash va sun'iy intellektda matritsalarini hisoblashdan foydalanishni, ayniqsa, dala ekspeditsiyasi davomida olingan o'simlik namunalari tasvirlarini olishda PYNQ-Z2 platformasining samarali qo'llanilishi va tatbiqiga qaratilgan. FPGA asosidagi rivojlantirish platasi sifatida PYNQ-Z2 narx, energiya sarfi va portativlik jihatidan muhim afzalliklarga ega. Ushbu maqolada konvolyutsion neyron tarmoqda matritsaning rolini tahlil qilinadi, PYNQ-Z2 platformasining apparat tezlatishdagi imkoniyatlarini namoyish etadi va o'simliklarni aniqlash uchun mo'ljallangan tizim arxitekturasini hamda kelgusidagi takomillashtirish yo'nalishlarini muhokama qilinadi.

**Kalit so'zlar:** Matritsa, o'simliklarni aniqlash, sun'iy intellekt, PYNQ-Z2, apparat tezlashuvi.

## РЕЗЮМЕ

В этой статье рассматривается использование матричных вычислений в идентификации растений и искусственном интеллекте, с упором на эффективную

реализацию и применение платформы PYNQ(Python Productivity for Zynq)-Z2 при получении изображений образцов растений полевых экспедиций. Как плата разработки на основе FPGA, PYNQ-Z2 имеет значительные преимущества в стоимости, энергопотреблении и портативности. В этой статье анализируется роль матрицы в сверточной нейронной сети, демонстрируется потенциал платформы PYNQ-Z2 в аппаратном ускорении и обсуждается архитектура системы, разработанная для идентификации растений, и направления будущего улучшения.

**Ключевые слова:** Матрица, Распознавание растений, Искусственный интеллект, PYNQ-Z2, Аппаратное ускорение.

УДК 519.6:681.5:539.5

**КОМПЬЮТЕРНОЕ МОДЕЛИРОВАНИЕ ТЕРМОУПРУГОГО  
ДЕФОРМИРОВАНИЯ ТЕЛ С ОТВЕРСТИЕМ**

Икрамов А. М. \*

**РЕЗЮМЕ**

В статье рассматривается моделирование решения задач термоупругости методом конечных элементов. Постановка задачи базируется на законе теплопроводности в форме Фурье и уравнениях механики деформируемого твердого тела. Нелинейность сформулированной задачи обусловлена зависимостью упругих характеристик материала от температуры деформаций. Для непосредственного получения уравнений конечных элементов по имеющемуся набору дифференциальных уравнений в области используется метод Галеркина. Формируя нестационарную задачу теплопроводности, получим систему дифференциальных уравнений, в которой производная по времени заменяется конечной разностью. Таким образом, исходная задача сводится к системе линейных алгебраических уравнений. Решение краевой задачи распадается на две части: определение температурного поля, после чего становится возможным определение поля перемещений и напряжений термоупругой среды. Посредством проведения вычислительного эксперимента исследуется решение задачи о сжатии медной пластины с центральным круглым отверстием. Приведены изотермы температуры при различных моментах времени. С увеличением конечного времени поток тепла от границы отверстия распространяется по всей области пластины. Анализируется распределение значений компоненты нормальных и касательных напряжений в окрестности отверстия под действием приложенных сжимающих внешних нагрузок и температуры.

**Ключевые слова:** термоупругость, несвязанная постановка, нестационарность, метод конечных элементов, конечная разность, отверстие, вычислительный эксперимент.

**Введение**

Применяемые в машиностроении, авиастроении современные материалы, подверженные механическим нагрузкам, часто пребывают в условиях сильных тепловых воздействий. Определение напряженно-деформированного состояния элементов конструкций в силу сложной конфигурации и физической природы материалов составляет актуальную научную задачу.

**Постановка задачи**

Пусть  $\Omega$  – занимаемая телом область с границей  $\Gamma$ . Распределение температуры в области описывается уравнением теплопроводности:

$$\rho c \frac{\partial T}{\partial t} = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right), \quad (1)$$

\*Икрамов А. М. – Национальный университет Узбекистана, ikramovaxmat@gmail.com

где  $T(x, y, t)$ ,  $x, y \in \Omega$ ,  $t \in [0, \bar{T}]$ ,  $\rho$  – плотность материала,  $c$  – теплоемкость,  $k$  – коэффициент теплопроводности. Начальные и граничные условия имеют вид

$$T(x, y)|_{t=0} = T_1, \quad \left. \frac{\partial T(x, y)}{\partial n} \right| = \alpha(T - T_{vnes h}), \quad (2)$$

где  $\alpha$  – коэффициент теплообмена,  $n$  – вектор внешней нормали к границе области,  $T_{vnes h}$  – температура внешней среды. Перемещения, возникающие в области  $\Omega$ , описываются уравнениями:

$$\begin{cases} \frac{E}{2(1-\mu)} \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{E}{2(1+\mu)} \Delta u + \frac{\alpha_T E}{1-\mu} \frac{\partial T}{\partial x} = 0 \\ \frac{E}{2(1-\mu)} \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{E}{2(1+\mu)} \Delta v + \frac{\alpha_T E}{1-\mu} \frac{\partial T}{\partial y} = 0 \end{cases}, \quad (3)$$

где  $u(x, y)$ ,  $v(x, y)$  – функции перемещения по оси  $Ox$  и  $Oy$  соответственно,  $E$  – модуль Юнга,  $\mu$  – коэффициент Пуассона,  $\alpha_T$  – коэффициент линейного температурного расширения. Граничные условия для уравнения термоупругости (3) примут вид:

$$\begin{cases} \sigma_{xx} n_x + \sigma_{xy} n_y = 0 \\ \sigma_{xy} n_x + \sigma_{yy} n_y = p_y \end{cases},$$

где  $\sigma_{ij}$  – тензор напряжений,  $p_y$  – нагрузка по оси  $Oy$ .

Задача (1)-(2) решается методом конечных элементов. Вариационная постановка задачи эквивалентная уравнению теплопроводности (1) записывается в виде:

$$\sum_{i=1}^n \delta q_i \left( \int_{\Omega} \rho c \dot{T} \frac{\partial \dot{T}}{\partial \dot{q}_i} d\Omega + \int_{\Omega} k T, r \left( \frac{\partial T}{\partial q_i} \right)_{,r} d\Omega \right) = - \sum_{i=1}^n \delta q_i \int_{\Gamma} \left( \alpha (T - T_{vnes h}) \frac{\partial T}{\partial q_i} \right) d\Gamma,$$

через запятую обозначены производные по переменным  $x$  и  $y$ ,  $r = 1, 2$ . Интеграл в правой части запишем в виде суммы интегралов:

$$\begin{aligned} & \sum_{i=1}^n \delta q_i \left( \rho \cdot c \dot{T} \frac{\partial \dot{T}}{\partial \dot{q}_i} d\Omega + \int_{\Omega} k \cdot T, r \left( \frac{\partial T}{\partial q_i} \right)_{,r} d\Omega \right) = \\ & = - \sum_{i=0}^n \delta q_i \left( \int_{\Gamma} \alpha \cdot T \frac{\partial T}{\partial q_i} d\Gamma - \int_{\Gamma} \alpha \cdot T_{vnes h} \frac{\partial T}{\partial q_i} d\Gamma \right). \end{aligned} \quad (4)$$

Искомое температурное поле  $T = T(q_1, q_2, \dots, q_n, x, y, t)$  рассматривается как функция, зависящая от  $n$  обобщенных координат  $q_i$ , которые являются неизвестными функциями времени.

### Метод решения

Дискретизация расчетной области  $\Omega$  проводится с помощью треугольных элементов, а приближенное решение вариационной задачи теплопроводности представляется в виде линейной комбинации:

$$\tilde{T} = \sum_{i=1}^n q_i(t) N_i(x, y), \quad (5)$$

где  $q_i(t)$  – неизвестные коэффициенты,  $N_i(x, y)$  – базисные функции.

Подставляя решение (5) в уравнение (4) получим

$$\sum_{i=1}^n \sum_{j=1}^n \left( \int_{\Omega} \rho c \cdot N_i N_j \dot{q}_i d\Omega + \int_{\Omega} k \cdot N_{i,r} N_{j,r} q_i d\Omega + \int_{\Gamma} (\alpha N_i N_j q_i) d \right) = \sum_{j=1}^n \int_{\Gamma} \alpha \cdot T_{vneshe} d\Gamma. \quad (6)$$

Решение вариационной задачи (6) будем искать в линейном приближении. Для каждой вершины элемента строим базисные функции вида

$$N(x, y) = ax + by + c. \quad (7)$$

Уравнение (6) используя базисные функции вида (7) для одного элемента  $\Omega^e$  с вершинами  $i, j, k$ , имеет вид:

$$\rho c \int_{\Omega^e} [N]^T [N] d\Omega^e = \rho c \int_{\Omega^e} \begin{bmatrix} N_i \\ N_j \\ N_k \end{bmatrix} [N_i \quad N_j \quad N_k] d\Omega^e = \rho c \begin{bmatrix} N_i^2 & N_i N_j & N_i N_k \\ N_i N_j & N_j^2 & N_j N_k \\ N_k N_i & N_k N_j & N_k^2 \end{bmatrix} d\Omega^e,$$

матрица производных для одного конечного элемента имеет вид

$$[B] = \begin{bmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial x} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_j}{\partial y} & \frac{\partial N_k}{\partial y} \end{bmatrix} = \begin{bmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \end{bmatrix},$$

следовательно

$$k \int_{\Omega^e} [B]^T [B] d\Omega^e = k \Omega^e \begin{bmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \end{bmatrix} \begin{bmatrix} a_i & b_i \\ a_j & b_j \\ a_k & b_k \end{bmatrix} = k \Omega^e \begin{bmatrix} a_i^2 + b_i^2 & a_i a_j + b_i b_j & a_i a_k + b_i b_k \\ a_j a_i + b_j b_i & a_j^2 + b_j^2 & a_j a_k + b_j b_k \\ a_k a_i + b_k b_i & a_k a_j + b_k b_j & a_k^2 + b_k^2 \end{bmatrix}.$$

Интегралы, учитывающие конвективный теплообмен, вычисляются для узлов элемента, принадлежащих границе области  $\Omega$ :

$$\alpha \int_{\Gamma^e} [N_i]^T [N_j] d\Gamma^e = \alpha \int_{\Gamma^e} \begin{bmatrix} N_i^2 & N_i N_j & 0 \\ N_i N_j & N_j^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} d\Gamma^e.$$

Таким образом, уравнение (6) для одного элемента записывается в виде:

$$\rho c \int_{\Omega^e} [N]^T [N] d\Omega^e + k \int_{\Omega^e} [B]^T [B] d\Omega^e + \alpha \int_{\Gamma^e} [N_i]^T [N_j] d\Gamma^e = \alpha \int_{\Gamma^e} T_{vneshe} d\Gamma^e. \quad (8)$$

Просуммируем по всем конечным элементам области уравнение (8) имеем:

$$\sum_{e=1}^m \left( \rho c \int_{\Omega^e} [N]^T [T] d\Omega^e [\dot{q}] + k \int_{\Omega^e} [B]^T [B] d\Omega^e [q] + \alpha \int_{\Gamma^e} [N_i]^T [N_j] d\Gamma^e [q] \right) = \sum_{e=1}^m \alpha \int_{\Gamma^e} T d\Gamma^e.$$

Обозначим матрицы коэффициентов

$$C = \sum_{e=1}^m \rho c \int_{\Omega^e} [N]^T [N] d\Omega^e,$$

$$B = \sum_{e=1}^m \left( k \int_{\Omega^e} [B]^T [B] d\Omega^e + \alpha \int_{\Gamma^e} [N_i]^T [N_j] d\Gamma^e \right),$$

$$P = \sum_{e=1}^m \alpha \int_{\Gamma^e} T_{vneshe} d\Gamma^e,$$

тогда (8) запишется в виде системы  $C[\dot{q}] + B[q] = P$ .

Получили систему из  $m$  линейных дифференциальных уравнений. Заменяв производную по времени, конечной разностью получаем

$$\left( \frac{C}{\tau} + B \right) q^j = p + \frac{C}{\tau} q^{j-1}, \quad (9)$$

где  $\tau$  — шаг по времени.

Таким образом, исходная задача сводится к системе линейных алгебраических уравнений и на каждом шаге итерации определяется температурное поле.

Задача термоупругости (3) также заменяется эквивалентной вариационной задачей, основанной на принципе виртуальных перемещений. Запишем вариационное уравнение для случая, когда на тело действуют поверхностные силы  $f$  и термические нагрузки, возникающие на неравномерном тепловом поле.

Вариационная задача, равносильная уравнениям (3), имеет вид:

$$\int_{\Omega} (\lambda e \delta e + 2G \varepsilon_{ij} \delta \varepsilon_{ij}) d\Omega - \int_{\Omega} 2(\lambda + G) \alpha_T (T - T_0) \delta e d\Omega = \int_{\Gamma} \sigma_{ij} \delta u_i n_j d\Gamma, \quad (10)$$

где  $e = \varepsilon_{xx} + \varepsilon_{yy}$  — деформация,  $\lambda = \frac{\mu E}{(1-\mu)^2}$ ,  $G = \frac{E}{2(1+\mu)}$  — постоянные Ламе для плоского напряженного состояния,  $u_1 = u$ ,  $u_2 = v$ ,  $i, j = 1, 2$ .

Уравнение (10) описывает изменение полной термомеханической энергии тела, где левая часть этого уравнения выражает изменение энергии термоупругого деформирования, а правая часть изменение работы граничных нагрузок.

В матричной форме, постоянные Ламе  $\lambda$  и  $G$  представляются в виде матрицы констант упругости:

$$D = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & (1-\mu)/2 \end{bmatrix},$$

соотношения Коши, выражающие деформации через перемещения:

$$\varepsilon = [\varepsilon_x \ \varepsilon_y \ \varepsilon_{xy}]^T = \left[ \frac{\partial u}{\partial x} \quad \frac{\partial u}{\partial y} \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right]^T,$$

а перемещения – в виде вектора  $U = [u \ v]^T$ .

Решение задачи будем искать на расчетной сетке с треугольными конечными элементами, которой пользовались для решения задачи теплопроводности. Перемещения аппроксимируем кусочно-линейными функциями вида:

$$u = \sum_{i=1}^n N_i(x, y)u_i, \quad v = \sum_{i=1}^n N_i(x, y)v_i,$$

где  $u_i$  и  $v_i$  – неизвестные коэффициенты.

Соотношения Коши для одного элемента с узлами  $i \ j \ k$  имеет вид:

$$[B] = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & \frac{\partial N_j}{\partial x} & 0 & \frac{\partial N_k}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 & \frac{\partial N_j}{\partial y} & 0 & \frac{\partial N_k}{\partial y} \\ \frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial y} & \frac{\partial N_j}{\partial x} & \frac{\partial N_j}{\partial y} & \frac{\partial N_k}{\partial x} & \frac{\partial N_k}{\partial y} \end{bmatrix}.$$

Для конечного элемента  $e$  уравнение состояния (10) записывается в виде:

$$\int_{\Omega^e} [B]^T [D] \{B\} d\Omega^e = \int_{\Omega^e} [B]^T [D] \{\varepsilon_T\} d\Omega^e + \int_{\Gamma^e} \{P\} \{U\} d\Gamma^e, \quad (11)$$

где  $\varepsilon_T = \alpha_T \Delta T \{1 \ 1 \ 0\}^T$  – вектор температурных деформаций. Суммируя (11) по всем элементам получим систему линейных алгебраических уравнений

$$K \{U\} = F,$$

где  $\{U\}$  – вектор неизвестных коэффициентов для функции перемещений,  $K$  – матрица жесткости,  $F$  – вектор узловых нагрузок. Элементами матрицы жесткости и вектора узловых нагрузок являются

$$k_{ij}^e = \int_{\Omega^e} B_i^T D B_j d\Omega^e, \quad f_i^e = \int_{\Gamma^e} N_i^e p d\Gamma^e.$$

Таким образом, решение краевой задачи распадается на две части: определение температурного поля, после чего становится возможным определение поля перемещений и напряжений термоупругой среды.

### Вычислительный эксперимент и обсуждение результатов

Рассматривается задача о сжатии по оси  $Oy$  медной пластины с центральным круглым отверстием. Внешние границы пластины теплоизолированы. Размер пластины 1м на 2м, начальная температура  $20C^\circ$ . Радиус отверстия 0.25м, на границе задана температура  $100C^\circ$ .

Теплофизические и механические параметры: плотность  $\rho = 8890 \text{ кг/м}^3$ , коэффициент теплообмена  $\alpha_T = 16.7 \cdot 10^{-6} \text{ K}^{-1}$ , коэффициент теплопроводности  $\lambda = 390 \text{ Вт/(мК)}$ , удельная теплоёмкость  $c = 385 \text{ Вт/(мК)}$ ,  $\tau = 60 \text{ сек}$ ,  $t_k = 60 \text{ мин}$ ,  $E = 128.7 \text{ ГПа}$ ,  $\mu = 0.35$ ,  $P_{yy} = 1000 \text{ МПа}$ .

На рис. 1 приведены изотермы распределения температуры при различных моментах времени. С возрастанием времени  $t_k$  поток тепла от границы отверстия распространяется по всей области пластины.

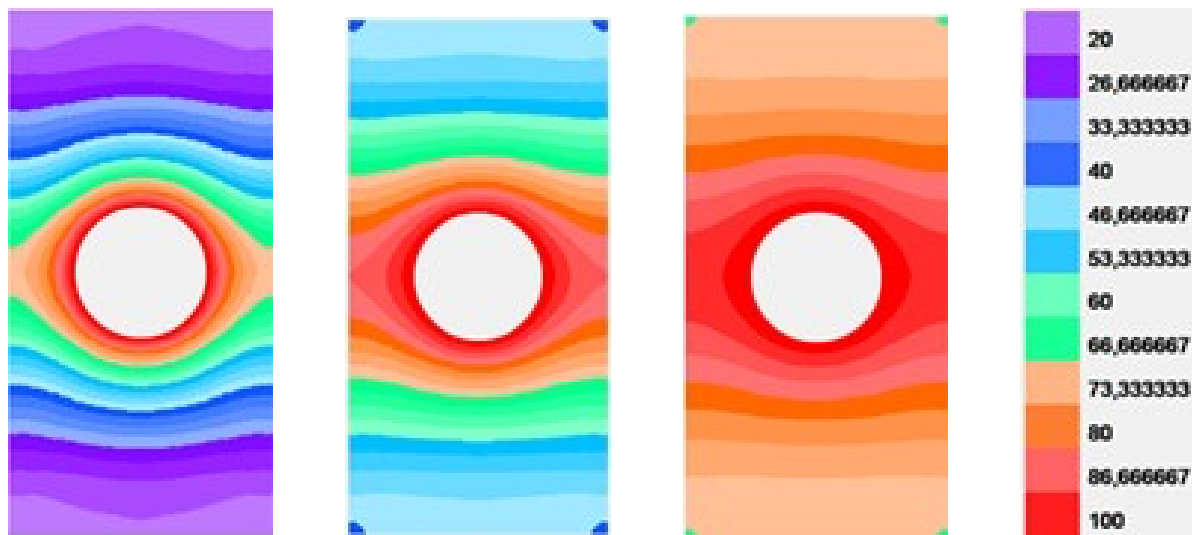


Рис. 10: Изотермы распределения температуры ( $t_k = 10, 30, 60 \text{ мин}$ )

На рис. 2 приведено поле распределения значений нормальных и касательной компоненты напряжений в момент времени  $t_k = 60$ . Анализируя распределение значений компоненты нормальных напряжений  $\sigma_{xx}$ , можно отметить, что по горизонтальному диаметральному сечению окрестности отверстия формируется область сжимающих, а по вертикальной – максимальных растягивающих значений. Относительно значений компоненты нормального напряжения  $\sigma_{yy}$  необходимо отметить, что приложенные сжимающая внешние нагрузки и температура на границе отверстия, приводят к появлению максимальной концентрации сжимающих напряжений в окрестности горизонтального сечения и незначительной области растягивающих – по вертикальному сечению. Интересная картина наблюдается при анализе распределения поля касательных напряжений  $\tau_{xy}$ . В некоторых областях окрестности отверстия, центры которых определяются полярным углом  $\psi \approx \pm 20^\circ$  (и симметричных с ними) сдвиговые напряжения достигают максимального значения, что может привести к деформации конструкции [3-5].

### Выводы

1. Анализируя распределение значений компоненты напряжений  $\sigma_{xx}$ , выявлено, что по горизонтальному диаметральному сечению окрестности отверстия формируется область сжимающих, а по вертикальной – максимальных растягивающих значений.

2. Приложенные сжимающая внешние нагрузки и температура на границе отверстия, приводят к появлению максимальной концентрации сжимающих напряжений  $\sigma_{yy}$



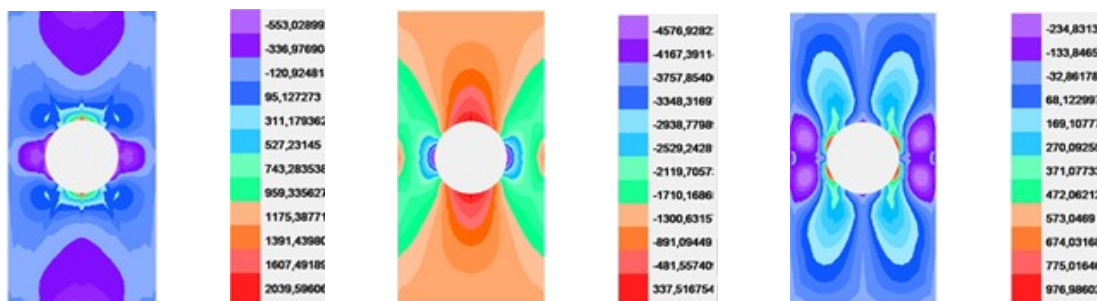


Рис. 11: Значения компонент напряжений  $\sigma_{xx}, \sigma_{yy}, \tau_{xy}$  ( $t_k=60$  мин)

в окрестности горизонтального сечения и незначительной области растягивающих – по вертикальному сечению.

3. В некоторых областях окрестности отверстия, центры которых определяются полярным углом  $\psi \approx \pm 20^\circ$  (и симметричных с ними) значения сдвиговых напряжений  $\tau_{xy}$  достигают максимального значения, что может привести к деформации конструкции [3].

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## REZYUME

Maqolada termoelastiklik masalalarini chekli elementlar usuli yordamida yechishni modellashtirish muhokama qilinadi. Masalani qo'yilishi Furye shaklidagi issiqlik o'tkazuvchanlik qonuni va deformatsiyalanuvchi qattiq jism mexanikasi tenglamalariga asoslanadi. Tuzilgan masalaning noxiziqli bo'lmaganligi materialning elastik xususiyatlarining deformatsiya haroratiga bog'liq. Mavjud differensial tenglamalar to'g'ridan-to'g'ri chekli elementlar tenglamalarini olish uchun Galerkin usuli qo'llaniladi. Nostatsionar issiqlik o'tkazuvchanligi masalasini shakllantirish orqali biz differensial tenglamalar tizimini hosil qilamiz, bunda vaqt hosilasi chekli ayirmalar bilan almashtiriladi. Shunday qilib, dastlabki masala chiziqli algebraik tenglamalar tizimiga keltiriladi. Chegaraviy masala yechimi ikki qismga bo'linadi: harorat maydonini aniqlash, shundan so'ng termoelastik

muhitning siljishlari va kuchlanishlari maydonini aniqlash mumkin bo'ladi. Hisoblash eksperimentini o'tkazish orqali markaziy doirali teshikga ega mis plitani siqish masalasining yechimi o'rganiladi. Turli vaqtlardagi harorat izotermlari ko'rsatilgan. Cheklangan vaqt oshgani sayin, teshik chegarasidan issiqlik oqimi plastinaning butun maydoniga tarqaladi. Tashqi yuklanishlar va harorat ta'sirida teshik yaqinidagi normal va tangensial kuchlanish komponentlar qiymatlarining taqsimlanishi tahlil qilinadi.

**Kalit so'zlar:** termoelastiklik, masalaning bog'lanmagan qo'ilishi, nostatsionarlik, chekli elementlar usuli, chekli ayirma usuli, teshik, hisoblash eksperimenti.

### RESUME

The article deals with the modeling of the solution to non-stationary problems of thermos-elasticity using the finite element method. The statement of the problem is based on the Fourier form of heat conduction and the equations of mechanics of a deformable rigid body. The nonlinearity of the formulated problem is due to the dependence of the elastic characteristics of the material on the deformation temperature. The Galerkin method is used to directly obtain the finite element equations from the available set of differential equations in the domain. Forming a non-stationary heat conduction problem, the authors obtained a system of differential equations in which the time derivative is replaced by a finite difference. Thus, the original problem is reduced to a system of linear algebraic equations, solved by Newton's method. The solution to the boundary value problem is divided into two parts: determination of the temperature field, after which it becomes possible to determine the field of displacements and stresses of the thermos-elastic medium. By means of a computational experiment, the solution to the problem of compression of a copper plate with a central circular hole is investigated. Temperature isotherms are given at different time points. As the finite time increases, the heat flux from the boundary of the hole propagates over the entire area of the plate. The distribution of values of the components of normal and shear stresses in the vicinity of the hole under applied compressive external loads and temperatures is analyzed.

**Key words:** thermos-elasticity, uncoupled formulation, non-stationarity, finite element method, finite difference, hole, computational experiment.

УДК 539.3

**ОБРАТНАЯ ЗАДАЧА О ДИНАМИЧЕСКОМ НАГРУЖЕНИИ СТЕРЖНЯ  
С УЧЕТОМ ОТРАЖЕНИЯ НЕЛИНЕЙНЫХ ВОЛН**

Маматова Н. Т. \*

**РЕЗЮМЕ**

В данной работе рассматривается задача упругопластического нагружения стержня с учетом инерционных продольных сил и при достаточно произвольных криволинейных экспериментально построенных диаграмм нагружений. Для решения задачи применяется обратный метод решения.

**Ключевые слова:** упругопластическое деформирование, сухое трение Кулона, отражение нелинейных волн.

**1. ВВЕДЕНИЕ.**

Если упругопластическое динамическое деформирование стержневой конструкции, взаимодействующей с окружающей ее средой происходит по закону сухого трения Кулона, то нелинейности определяются из нелинейности диаграмм нагружений упругопластического деформирования [1-4]. В случаях отклонения этих диаграмм нагружений от схемы Прандтля, аналитические представления решений становятся невозможными и эти задачи требуют численного метода решения. Кроме того, если заданные деформации или напряжения на торцевых сечениях стержня  $x = 0$  не являются постоянными величинами во времени, то в этих случаях представления аналитических решений в поставленных выше нелинейных задачах остаются нерешенными. Очевидно, что в этих случаях следует воспользоваться численными методами решения нелинейных задач.

В задачах распространения волн в различных моделях сплошных сред используется широкий спектр численных методов решения задач. Особое место среди этих методов решения занимают обратные постановки задач и методы их решения [1,3].

**1. ПОСТАНОВКА ЗАДАЧИ И МЕТОДИКА ЕЕ РЕШЕНИЯ**

Рассмотрим полубесконечный упругопластический стержень, находящийся в упругой среде, который до момента  $t = 0$  времени находился покоящимся и ненапряженным

$$\sigma = 0, u_t = 0, u_x = 0 \text{ при } t = 0,$$

который нагружается с торца стержня.

Уравнение движения стержня в грунтовой среде записывается в виде [1,4]

$$\frac{\partial^2 u}{\partial t^2} = a^2(u_x) \frac{\partial^2 u}{\partial x^2} - \frac{\kappa \tau}{\rho} \quad (1)$$

\*Маматова Н. Т. – Национальный университет Узбекистана,

где  $u$  - перемещение частиц стержня,  $\rho$  - плотность материала стержня,  $\tau$  - касательное напряжение на поверхности стержня со средой,  $\kappa$ - знак скорости относительно среды,  $a^2(u_x) = \frac{1}{\rho} \frac{d\sigma}{d\varepsilon}$  - квадрат скорости распространения волны в упруго - пластическом стержне.

Характеристики квазилинейного гиперболического уравнения (1) и условия вдоль них запишутся в виде

$$dx = \pm a(u_x) dt \quad (2)$$

$$du_t = \pm a(u_x) du_x - \frac{\kappa\tau}{\rho} dt \quad (3)$$

Считаем, что монотонное сжимающее интенсивное динамическое нагружение действует только на торец стержня, в этом случае стержень движется относительно покоящейся среды так, что там нужно положить  $\kappa = +1$ .

Задачу для волны нагрузки решим обратным методом. Предполагая форму фронта волны нагружения  $t = \varphi(x)$ , будем решать задачу Коши в области  $D$ . Полагаем, что на фронте волны нагрузки  $t = \varphi(x)$  известны скорости частиц стержня и их деформации:  $u_t|_{t=\varphi(x)} = 0$  и  $u_x|_{t=\varphi(x)} = 0$ .

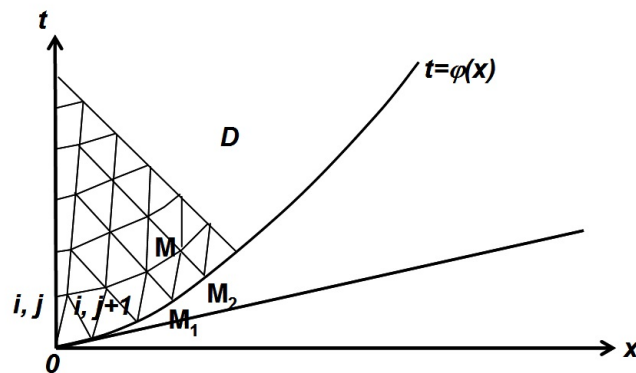


рис.1

Решение в области  $D$  строится численно, методом сеток, образованных характеристиками. В произвольной заданной точке  $M(x, t)$  области  $D$  можно определить решение, проведя из этой точки положительные и отрицательные характеристики до пересечения их с волной нагрузки. Если точка  $M$  достаточно близко расположена к фронту волны  $t = \varphi(x)$ , то соответственно уравнения этих характеристик, проходящих через точки  $M_1(x_1, t_1)$  и  $M_2(x_2, t_2)$  примут вид

$$\begin{cases} x_M - x_1 = a(u_x^1)(t_M - \varphi(x_1)) \\ x_M - x_2 = -a(u_x^2)(t_M - \varphi(x_2)) \end{cases} \quad (4)$$

Из условий на характеристиках (3) получим следующие соотношения

$$\begin{cases} u_t^M - u_t^1 = a(u_x^1)(u_x^M - u_x^1) - \frac{\tau g}{\gamma}(t - t_1) \\ u_t^M - u_t^2 = -a(u_x^2)(u_x^M - u_x^2) - \frac{\tau g}{\gamma}(t - t_2) \end{cases} \quad (5)$$

где  $\gamma = \rho g$ . Здесь  $\gamma$  - удельный вес стержня,  $g = 9.81 < /A^2$ .

соотношения (4) и (5) связывают координаты точки  $M(x, t)$  с координатами точек  $M_1$  и  $M_2$ , а также значениями скоростей и деформаций частиц в этих точках. Задав координаты точки  $M$ , которые достаточно близки к точкам  $M_1$  и  $M_2$  на волне  $t = \varphi(x)$ , мы сможем определить для этой точки (первое пересечение характеристик с фронтом волны нагрузки) формулы для определения значений скоростей и деформаций частиц:

$$\begin{aligned} x_1 &= x_M - a(0)(t_M - \varphi(x_1)), & t_1 &= \varphi(x_1) \\ x_2 &= x_M + a(0)(t_M - \varphi(x_1)), & t_2 &= \varphi(x_2) \\ u_x^M &= \frac{\tau g \cdot (t_2 - t_1)}{2a(0)\gamma}, & u_t^M &= -\frac{\tau g \cdot (2t_M - t_2 - t_1)}{2\gamma} \end{aligned} \quad (6)$$

Далее принимая координаты точки  $M$  за  $M_1(x_1, t_1)$  и  $M_2(x_2, t_2)$  находим решения следующего ряда точек пересечений

$$\begin{aligned} t_M &= \frac{x_2 - x_1 + a(u_x^1) \cdot t_1 + a(u_x^2) \cdot t_2}{a(u_x^1) + a(u_x^2)} \\ x_M &= \frac{x_1 + x_2 + (a(u_x^1) - a(u_x^2)) \cdot t_M - a(u_x^1) \cdot t_1 + a(u_x^2) \cdot t_2}{2} \\ u_x^M &= \frac{u_t^2 - u_t^1 + a(u_x^1) \cdot u_x^1 + a(u_x^2) \cdot u_x^2 + \frac{\tau g}{\gamma} \cdot (t_2 - t_1)}{a(u_x^1) + a(u_x^2)} \\ u_t^M &= \frac{u_t^1 + u_t^2 + (a(u_x^1) - a(u_x^2)) \cdot u_x^M - a(u_x^1) \cdot u_x^1 + a(u_x^2) \cdot u_x^2 - \frac{\tau g}{\gamma} \cdot (2t_M - t_1 - t_2)}{2} \end{aligned} \quad (7)$$

Таким образом, искомые значения деформаций и скоростей частиц за фронтом волны определяются в местах пересечения характеристик в плоскости  $xOt$  согласно формулам (7).

Принимая при  $x=0$   $u_x^{i,j} = u_x^{i,j+1}$  и  $u_t^{i,j} = u_t^{i,j+1}$ , получим решение на конце стержня, т.е. получим такое распределение напряжений на конце, которое соответствует предположенной форме волны нагружения  $t = \varphi(x)$  и условиям на ней. Предполагая поочередно различные формы волны нагрузки, можно получить методом последовательных проб такое распределение деформаций или напряжений на конце, которое было бы близким к заданному.

### ЧИСЛЕННЫЙ ПРИМЕР И ЕЕ АНАЛИЗ

Построенный выше алгоритм решения, о распределении упругопластической волны в обратной постановке, позволяет анализировать решение задачи для различных нелинейных диаграмм связи напряжений от деформаций и сил Кулоновского трения на границах взаимодействия стержня и грунта.

Рассмотрим следующий пример: пусть для  $\sigma = \sigma(\varepsilon)$  имеет место следующая экспериментальная связь  $\sigma = \alpha_1 \varepsilon + \alpha_2 \varepsilon^2$ , а коэффициенты трения могут принимать различные значения.

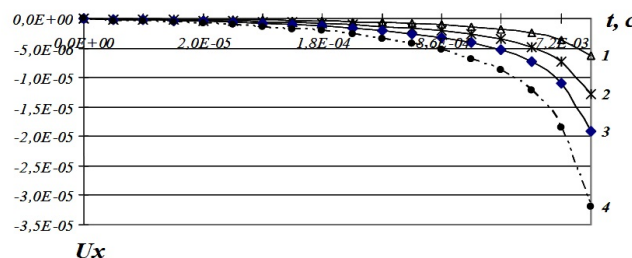
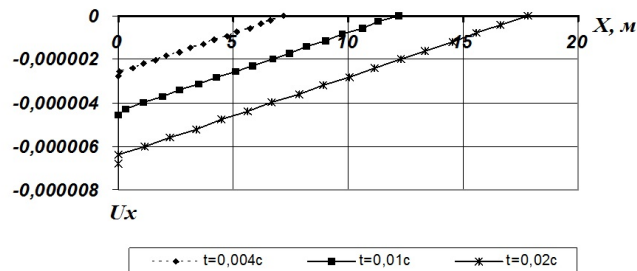


рис.2 Изменение деформации при различных коэффициентах трения в сечении  $x = 0$

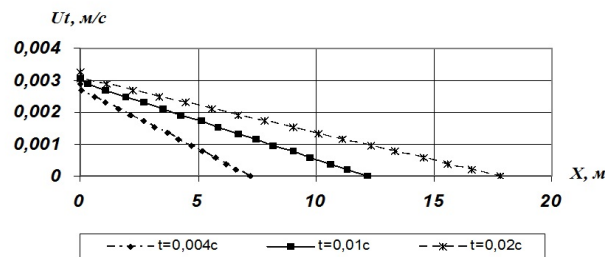
Кривые 1-4 на рис.2 соответствуют деформациям частиц на торце стержня  $x=0$  с коэффициентом трения соответственно  $\nu = 0.1; 0.2; 0.3; 0.5$  и форме волны в виде  $t = \frac{a(0)}{10^8} x^2 + \frac{1}{a(0)} x$ .

Из рис..2 следует, что для одних и тех же фронтов волн увеличение коэффициента трения конструкции и среды приводит к тому, что определяемые значения напряжений по обратному способу на торцевом сечении возрастают. Отметим также, что монотонность изменения во времени напряжений и деформаций на торцевом сечении стержня и конструкции сохраняется для всех рассматриваемых коэффициентов трения. Таким образом, в обратной постановке задачи изменение деформаций могут быть достигнуты заданием фронта волны и условия на нем.

Далее, рассмотрим изменение деформации по длине конструкции (от торцевого сечения) до тех точек, где приходят волны сжатия для различных моментов времени. На рис.3 приведены значения деформаций для фиксированных моментов времени. На рис.4 приведены изменения скоростей частиц стержня по длине конструкции.



**рис.3** Изменение деформаций по длине конструкции при  $\nu = 0.1$  для фиксированных моментов времени



**рис.4** Изменение скоростей частиц по длине конструкции при  $\nu = 0.1$  для фиксированных моментов времени.

Аналогичные графики для  $\nu = 0.2$  и  $\nu = 0.3$  приведены на рис.5-6. Отметим, что с увеличением коэффициентов трения для одних и тех же заданных фронтов волн, деформации и скорости частиц возрастают как на торцевом сечении, так и по всей длине.

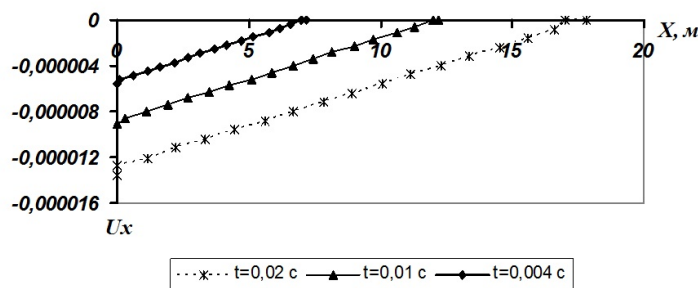


рис.5 Изменение деформаций по длине конструкции при  $\nu = 0.2$  для фиксированных моментов времени

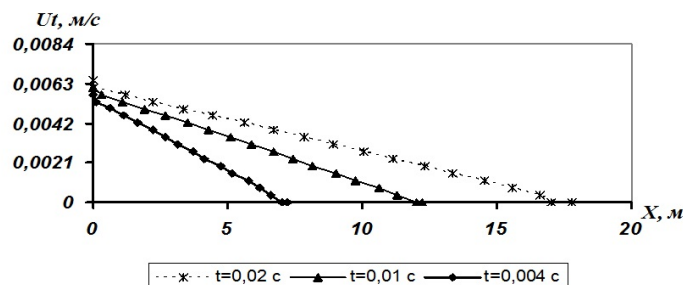


рис.6 Изменение скоростей частиц по длине конструкции при  $\nu = 0.2$  для фиксированных моментов времени

Далее алгоритм построения решения с использованием заданного фронта волн и условия на ней позволяет анализировать изменение напряжения на торцевом сечении, которые зависят от коэффициентов уравнения фронта волны.

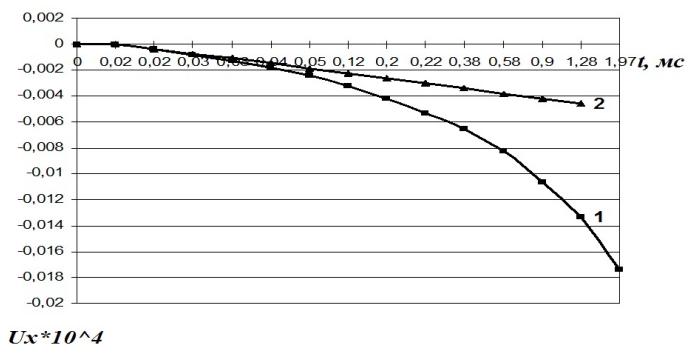


рис.7 Распределение деформаций на конце стержня при различных коэффициентах уравнения фронта волны

На рис.7 приведены изменения деформаций на торцевом сечении для двух значений коэффициентов уравнения параболы, которые соответственно будут в виде  $1 - t = \frac{a(0)}{10^8}x^2 + \frac{1}{a(0)}x$ ,  $2 - t = \frac{a(0)}{10^9}x^2 + \frac{1}{a(0)}x$ . Малые изменения уравнений используемых фронтов волн приводят к достаточно чувствительным изменениям напряжения на торцевом сечении.

Очевидно, путем подбора значений коэффициентов волны можно рассмотреть различные вариации нагружения на торцевом сечении и они могут быть использованы на практике, если какие либо из этих вариаций близки к заданным напряжениям на торцевом сечении при прямой постановке задачи.

Таким образом, предложенный алгоритм решения задачи в обратной постановке позволяет определить напряжения, деформации, скорости частиц при различных значениях коэффициентов трения и нелинейного динамического деформирования конструкции взаимодействующей с окружающей средой по закону сухого трения Кулона.

### **ОБРАТНАЯ ЗАДАЧА О ДИНАМИЧЕСКОМ НАГРУЖЕНИИ СТЕРЖНЯ С УЧЕТОМ ОТРАЖЕНИЯ НЕЛИНЕЙНЫХ ВОЛН**

Известно, что если рассматривается динамически деформируемый упругопластический стержень или нелинейно-упруго деформируемый стержень, процесс активного нагружения, например при сжимающем динамическом нагружении с торцевого сечения, происходит при качественно одинаковых диаграммах связи напряжений от деформаций. Для нелинейно-упругих и упругопластических стержней, взаимодействующих с окружающими их грунтовыми средами в согласии с законом сухого трения Кулона, уравнения распространения волн – квазилинейные дифференциальные уравнения, становятся совершенно одинаковыми для активных процессов нагружения.

Рассмотрим нелинейный стержень, находящийся в упругой грунтовой среде, который до момента  $t = 0$  времени находился покоящимся и ненапряженным. Ось  $x$  направлена вдоль оси стержня.

Пусть на каком-то сечении  $x = l$  от  $x = 0$  находится преграда, т.е. другой конец стержня считаем с заземленным концом.

В момент времени  $t=0$  на конец стержня прикладывается сжимающее напряжение  $\sigma(t)$ , которое мы должны определить в процессе решения. С момента приложения нагрузки по стержню распространяется продольная волна в направлении оси  $x>0$ . Предполагаем форму фронта волны  $t = \varphi(x)$ , полагаем, что на фронте волны нагрузки скорости частиц стержня  $u_t|_{t=\varphi(x)} = 0$  и деформация  $u_x|_{t=\varphi(x)} = 0$ .

Уравнение движения стержня в грунтовой среде записывается в виде (1).

Начальные условия

$$\sigma = 0, u_t = 0, u_x = 0 \text{ при } t = 0$$

Граничные условия

$$u_t|_{x=l} = 0, u_x|_{x=l} = \theta \cdot u_x^O \text{ при } x = l.$$

где  $u_x^O$ - деформация, если бы преграды не было.

Предполагается, что деформация на стенке в момент отражения больше в  $\theta$  раз, чем приложенная на торец стержня  $x = 0$ .

Решение в возмущенной области строится численно, методом сеток, образованных характеристиками. В произвольной заданной точке  $M(x,t)$  области  $D$  можно определить решение, проведя из этой точки положительные и отрицательные характеристики до пересечения их со стенкой  $x=l$  (рис.8).



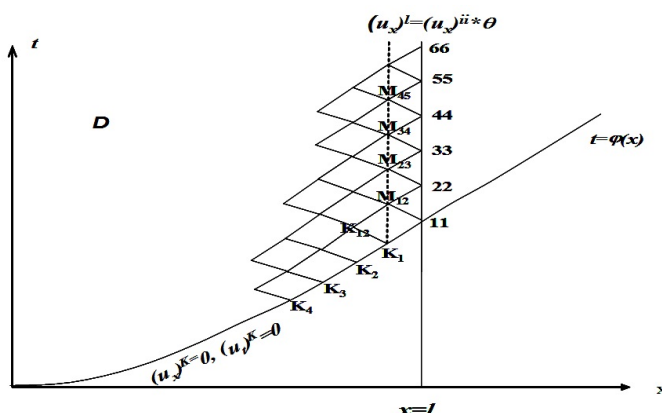


рис.8

Соответственно уравнения этих характеристик, проходящих через точки  $(x_{11}, t_{11})$  и  $(x_{22}, t_{22})$  примут вид

$$\begin{cases} x_{22} - x_M = a(u_x^M)(t_{22} - t_M) \\ x_M - x_1 = -a(u_x^1)(t_M - t_{11}) \end{cases} \quad (8)$$

Из соотношений на характеристиках (3) получим следующие соотношения

$$\begin{cases} u_t^{22} - u_t^M = a(u_x^M)(u_x^{22} - u_x^M) - \frac{\tau g}{\gamma}(t_{22} - t_M) \\ u_t^M - u_t^{11} = -a(u_x^{11})(u_x^M - u_x^{11}) - \frac{\tau g}{\gamma}(t_M - t_{11}) \end{cases} \quad (9)$$

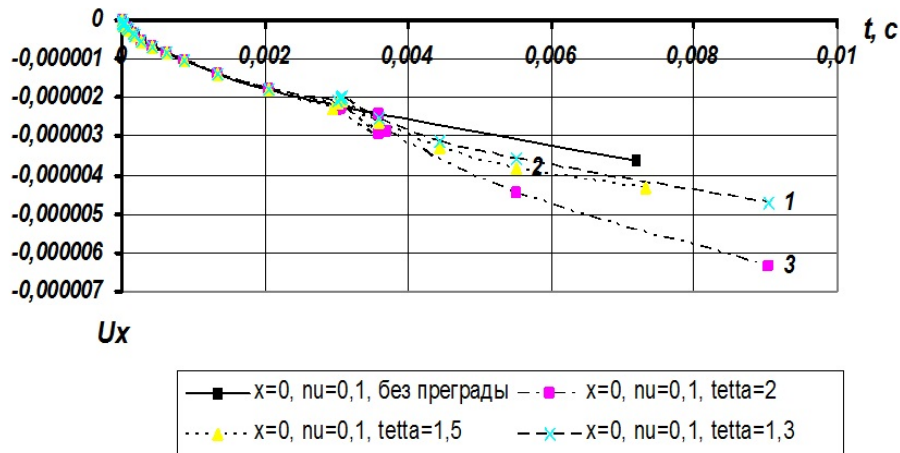
Из соотношений (8) и (9) находим координаты точки  $M_{12}(x, t)$  и значения скоростей и деформаций в этой точке. Далее, по аналогии этими результатами процедур определения искомых параметров, записываем значения деформаций и скоростей частиц в вышерасположенных точках  $M_{12}, M_{23}$  и т.д.

Затем, на заданном фронте волны нагрузки задаем координаты точки  $K_1$  и проводя характеристику положительного наклона из точки  $M_{12}$  и отрицательного наклона через точку  $K_1$ , учитывая, что значения скоростей и деформаций частиц в этой точке равны нулю, определяем решения в точке пересечения.

Далее принимая координаты точек  $K_1$  и  $M_{12}$  соответственно за точки  $M_{11}(x_{11}, t_{11})$  и  $M_{22}(x_{22}, t_{22})$  находим решения для следующего ряда точек пересечений. Продолжая этот процесс построения решений для области  $x \leq l$ , мы можем последовательно определить координаты точек пересечения характеристик положительного и отрицательного наклона, скорости и деформации в этих точках.

$$\begin{aligned} t_{K_{12}} &= \frac{x_{11} - x_{22} + a(u_x^{11}) \cdot t_{11} + a(u_x^M) \cdot t_2}{a(u_x^1) + a(u_x^M)} \\ x_{K_{12}} &= \frac{x_{11} + x_{22} + (a(u_x^M) - a(u_x^{11})) \cdot t_M + a(u_x^{11}) \cdot t_{11} - a(u_x^M) \cdot t_{22}}{2} \\ u_x^{K_{12}} &= \frac{u_t^{11} - u_t^{22} + a(u_x^{11}) \cdot u_x^{11} + a(u_x^M) \cdot u_x^{22} - \frac{\tau g}{\gamma} \cdot (t_{22} - t_{11})}{a(u_x^{11}) + a(u_x^M)} \\ u_t^{K_{12}} &= \frac{u_t^{11} + u_t^{22} - (a(u_x^{11}) - a(u_x^M)) \cdot u_x^M + a(u_x^{11}) \cdot u_x^{11} - a(u_x^M) \cdot u_x^{22} - \frac{\tau g}{\gamma} \cdot (2t_M - t_{11} - t_{22})}{2} \end{aligned} \quad (10)$$

Таким образом, искомые значения деформаций и скоростей частиц за фронтом волны определяются в местах пересечения характеристик в плоскости  $x, t$  согласно формулам (10).



**рис.9** Изменения деформаций на торцевом сечении стержня при коэффициенте трения  $\nu = 0.1$ , если преграда находится на расстоянии  $x = l = 5$  м.

На рис.9 приведены изменения деформаций на торцевом сечении стержня при коэффициенте трения  $\nu = 0.1$ , если преграда находится на расстоянии  $x=l=5$  м.

Отметим, что если  $\theta = 1$ , которое соответствует тому, что при  $x=l=5$  м отсутствует преграда, решение совпадает с результатами в пункте 2 рис. 2. На рис. 9 соответственно 1 -  $\theta = 1.3$ , 2 -  $\theta = 1.5$ , 3 -  $\theta = 2$ .

На основе представленного рис.11, можно установить следующие факторы: в обратной постановке задачи, которая характеризуется заранее заданным уравнением фронта волны и условием на ней, а также условий на границе  $x=l - u_t|_{x=l} = 0, u_x|_{x=l} = \theta \cdot u_x^0$  можно решить определенные задачи об отражении волны от преграды, находящейся на расстоянии  $x=l$  от торцевого сечения. При этом значения  $\theta$  следует задавать исходя из фактических условий в сечении  $x=l$ .

Ясно, что если сечение  $x=l$  свободно от напряжений, то  $\theta = 0$ , что соответствует свободному концу стержня. При выборе значений  $\theta$  следует исходить из того факта, что при отражении линейных или нелинейных упругих волн от жесткой преграды, абсолютные значения напряжения возрастают (в случае упругой задачи в 2 раза, а в некоторых упругопластических задачах в 1-1.7 раз). Если в качестве значения  $\theta$  принять числа в этих диапазонах, то задача становится близким к реальным ситуациям.

Однако в целом, в обратной постановке задачи мы можем априори задаться не только фронтом волны и условием на нем, но и значениями коэффициентов  $\theta$ . В этом случае исходя из приведенной выше численной схемы решения задачи можем определить напряжения и деформации при  $x=0$ , которые формируются, как это следует ожидать от фронта волны  $t = \varphi(x)$ , условий на ней и значения коэффициента  $\theta$ . Изменяя эти величины мы можем получить различные варианты изменения деформации на торцевом сечении и в отдельных ситуациях ими можно пользоваться при решении конкретных задач.

На рис. 9 приведены значения деформаций при  $x=0$  с течением времени для различных значений  $\theta$ .

Видно, что до момента времени, когда исходная волна не достигает преграды зна-

чение деформации при  $x=0$  совпадает с результатами пункта 2. Но с момента времени, когда отраженные волны от торцевого сечения ( $x=l$ ) достигают сечение  $x=0$ , значения деформации существенно изменяются. При этом для отрезка времени  $0.0035 < t < 0.0037 \text{сек}$  значения деформации остаются почти постоянными, но с момента времени  $t > 0.005 \text{сек}$  деформации увеличиваются снова по абсолютной величине. С увеличением значения  $\theta$ , значения деформации на торцевом сечении существенно увеличиваются.

### Выводы

1. Решена обратная задача о динамическом сжимающем ударном нагружении полубесконечного стержня, находящегося в упругой среде и взаимодействующей с ней по закону сухого трения Кулона. Задавая фронт падающих волн нагружений, в нелинейно-деформируемом или упруго-пластическом стержне и условиями на них показана возможность определения граничных условий на торцевом сечении полубесконечного стержня в условиях взаимодействия, происходящих согласно закону сухого трения Кулона.
2. Показана возможность изменения граничных условий задачи, исходя из формы фронта волны, условий на этой волне и коэффициентов Кулонова трения на контакте стержня и среды.
3. В обратной постановке задачи численным методом решения определены законы изменения деформаций на торцевом сечении при  $x=0$  и показано, что при одинаковых формах фронта волны и условий вдоль них, значения соответствующих деформаций, которые должны быть приняты как граничные условия возрастают и имеют монотонный характер роста абсолютных значений деформаций.
4. Решена задача об отражении волн нагружения от закреплений, отстоящих на определенных участках от торцевого сечения конструкции.

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**REZYUME**

Ushbu ishda inersiya bo'ylama kuchlarni e'tiborga olib, tajribadan qurilgan etarlicha ixtiyoriy egri chiziqli yuklanish diagrammalari uchun elastik plastik sterjenni yuklanishi masalasi ko'rilgan. Masalani echishda teskari echish usuli qo'llanadi.

**Kalit so'zlar:** Elastik plastik deformatsiyalanish, Kulon quruq ishqalanish, chiziqsiz to'qinlar qaytishi.

**RESUME**

In this paper, the problem of elastic-plastic loading of a rod is considered, taking into account inertial longitudinal forces and with sufficiently arbitrary curved experimentally constructed loading diagrams. The inverse solution method is used to solve the problem.

**Key words:** elastoplastic deformation, dry Coulomb friction, reflection of nonlinear waves.

УДК 517.98

ДИНАМИКА ВЫРОЖДЕННЫХ ОТОБРАЖЕНИЙ ЛОТКИ-ВОЛЬТЕРРА  
ДЕЙСТВУЮЩИХ НА СИМПЛЕКСЕ  $S^4$ 

Муминов У. Р. \*

## РЕЗЮМЕ

В данной статье рассматривается проблема обучения траекторий точек под действием оператора Лотки-Вольтерра. При этом сигнатуры определяются и делятся на многогранник, соответствующие симплексным сигнатурам, и определяются маршруты между ними.

**Ключевые слова:** Положительные и отрицательные траектории, бассейн неподвижной точки, сигнатура, многогранники данных сигнатур, маршруты траекторий.

Отображение Лотки-Вольтерра на симплексе

$$S^{m-1} = \left\{ x \in R^m : \sum_{i=1}^m x_i = 1, x_i \geq 0 \right\}$$

определяется заданием кососимметрической матрицы  $A = (a_{ij})$  с условием  $|a_{ij}| \leq 1$  и действует равенствами

$$x'_k = x_k \left( 1 + \sum_{i=1}^m a_{ki} x_i \right), k = 1, \dots, m \quad (1)$$

где  $x' = Vx$ ,  $V : S^{m-1} \rightarrow S^{m-1}$ .

**Определение 1.** Кососимметрическая матрица  $A$  называется невырожденной, если все главные миноры четного порядка положительны, в этом случае  $V$  также называется невырожденным.

Известно [1], что множество всех невырожденных матриц открыто и всюду плотно в множестве всех кососимметрических матриц. Асимптотическое поведение траекторий отображения  $V$  в случае малых размерностей изучались в работах [2 – 4] при условии, что  $V : S^{m-1} \rightarrow S^{m-1}$  невырожденное отображение Лотки-Вольтерра. Однако, в дискретных вариантах отображений  $SIR$ [5],[6],  $SEIR$ [4] эволюция системы описываются вырожденным отображением Лотки-Вольтерра. В данной работе изучаются динамические свойства траекторий одного класса отображений Лотки-Вольтерра на симплексе  $S^4$  с вырожденной кососимметрической матрицей

$$A = \begin{pmatrix} 0 & 0 & a_1 & -b_1 & c_1 \\ 0 & 0 & -a_2 & b_2 & -c_2 \\ -a_1 & a_2 & 0 & 0 & 0 \\ b_1 & -b_2 & 0 & 0 & 0 \\ -c_1 & c_2 & 0 & 0 & 0 \end{pmatrix}, 0 < a_1, b_1, c_1, a_2, b_2, c_2 \leq 1.$$

\*Муминов У. Р. – Ферганский государственный университет, ulugbek.muminov.2020@mail.ru

В данном случае отображение Лотки-Вольтерра  $V : S^4 \rightarrow S^4$  имеет вид

$$V : \begin{cases} x'_1 = x_1(1 + a_1x_3 - b_1x_4 + c_1x_5), \\ x'_2 = x_2(1 - a_2x_3 + b_2x_4 - c_2x_5), \\ x'_3 = x_3(1 - a_1x_1 + a_2x_2), \\ x'_4 = x_4(1 + b_1x_1 - b_2x_2), \\ x'_5 = x_5(1 - c_1x_1 + c_2x_2). \end{cases} \quad (2)$$

Пусть  $e_1, \dots, e_5$  вершины симплекса  $S^4$ . На ребре  $\Gamma_{12} = co\{e_1, e_2\}$  отметим точки

$$M_1 \left( \frac{a_2}{a_1 + a_2}, \frac{a_1}{a_1 + a_2}, 0, 0, 0 \right), \quad M_2 \left( \frac{b_2}{b_1 + b_2}, \frac{b_1}{b_1 + b_2}, 0, 0, 0 \right), \\ M_3 \left( \frac{c_2}{c_1 + c_2}, \frac{c_1}{c_1 + c_2}, 0, 0, 0 \right),$$

на  $\Gamma_{34} = co\{e_3, e_4\}$  отметим точки

$$N_1 \left( 0, 0, \frac{b_1}{a_1 + b_1}, \frac{a_1}{a_1 + b_1}, 0 \right), \quad N_2 \left( 0, 0, \frac{b_2}{a_2 + b_2}, \frac{a_2}{a_2 + b_2}, 0 \right),$$

на ребре  $\Gamma_{45} = co\{e_4, e_5\}$  также отметим

$$K_1 \left( 0, 0, 0, \frac{c_1}{b_1 + c_1}, \frac{b_1}{b_1 + c_1} \right), \quad K_2 \left( 0, 0, 0, \frac{c_2}{b_2 + c_2}, \frac{b_2}{b_2 + c_2} \right).$$

Порядок расположения точек  $M_1, M_2, M_3$  на ребре  $\Gamma_{12}$  определяют динамические свойства траекторий отображения  $V$ . Рассмотрим все возможные случаи.

1) Пусть  $M_1 \equiv M_2 \equiv M_3$ , тогда  $a_1b_2 = a_2b_1$ ,  $a_1c_2 = a_2c_1$ ,  $b_1c_2 = b_2c_1$ , следовательно,  $N_1 \equiv N_2$  и  $K_1 \equiv K_2$ . Очевидно, что грани  $\Gamma_{12}$  и  $\Gamma_{345}$  состоят из неподвижных точек. В рассматриваемом  $Fix(V) = \Gamma_{12} \cup \Gamma_{345} \cup M_1N_1K_1$ , т.е. треугольник  $M_1N_1K_1$  также состоит из неподвижных точек, причем  $M_1N_1K_1 = Ker A \cap S^4$ . Рассмотрим функцию  $\varphi_p : S^4 \rightarrow R$  определенную равенством

$$\varphi_p(x) = \prod_{i=1}^5 x_i^{p_i},$$

где  $p = (p_1, p_2, p_3, p_4, p_5) \in S^4$ ,  $x \in S^4$ . Пользуясь неравенством Юнга [7], [8] получаем следующую оценку для  $\varphi_p(Vx)$ :

$$\varphi_p(Vx) = \varphi_p(x) \cdot (1 + a_1x_3 - b_1x_4 + c_1x_5)^{p_1} \cdot (1 - a_2x_3 + b_2x_4 - c_2x_5)^{p_2} \cdot (1 - a_1x_1 + a_2x_2)^{p_3} \cdot \\ \cdot (1 + b_1x_1 - b_2x_2)^{p_4} \cdot (1 - c_1x_1 + c_2x_2)^{p_5} \leq \varphi_p(x) \cdot \left[ \sum_{i=1}^5 p_i - (a_1p_3 - b_1p_4 + c_1p_5) \cdot x_1 - \right. \\ \left. - (-a_2p_3 + b_2p_4 - c_2p_5) \cdot x_2 - (-a_1p_1 + a_2p_2) \cdot x_3 - (b_1p_1 - b_2p_2) \cdot x_4 - (-c_1p_1 + c_2p_2) \cdot x_5 \right]. \quad (3)$$

Если  $p \in M_1N_1K_1$ , то  $Ap = 0$ , поэтому из неравенства (3) получаем

$$\varphi_p(Vx) \leq \varphi_p(x)$$

для любого  $x \in S^4$ , при условии  $p \in M_1N_1K_1$ .

**Следствие.** Если  $p \in M_1N_1K_1$ , то  $\varphi_p(x)$  является функцией Ляпунова для динамической системы (1). Функция  $\varphi_p(x)$  не возрастает вдоль положительных траекторий и не убывает вдоль отрицательных траекторий.

Вычислив якобиан отображения  $V$  в неподвижных точках  $x \in M_1N_1K_1$  получим следующее уравнение для собственных значений

$$J(Vx) = (1 - \lambda)^3 \cdot ((1 - \lambda)^2 + a_1^2x_1x_3 + a_2^2x_2x_3 + b_1^2x_1x_4 + b_2^2x_2x_4 + c_1^2x_1x_5 + c_2^2x_2x_5) = 0$$

Так как в точке  $M_1$  имеем  $x_3 = x_4 = x_5 = 0$ , а на отрезке  $N_1K_1$  координаты  $x_1 = x_2 = 0$ , то в этих неподвижных точках характеристическое уравнение (1) имеет собственное число  $\lambda = 1$  с кратностью 5, а во всех остальных точках из  $M_1N_1K_1$   $\lambda = 1$  имеет кратность 3, а два других собственных числа комплексные и по модулю больше 1. Следовательно, все неподвижные точки из  $riM_1N_1K_1$  репеллеры, т.е. отталкивающие. Сигнатура  $\delta = (+ - + + -)$  - произвольный набор знаков. Рассмотрим однородные неравенства соответствующие, например, данной сигнатуре

$$\begin{aligned} (Ax)_1 &> 0, \\ (Ax)_2 &< 0, \\ (Ax)_3 &> 0, \\ (Ax)_4 &> 0, \\ (Ax)_5 &< 0. \end{aligned} \tag{4}$$

Решение (4) открытое (возможно пустое) множество. Пусть  $F_\delta$  замыкание множества решений неравенств определяемых заданной сигнатурой и принадлежащих симплексу. Очевидно,  $F_\delta$  многогранник причем, если  $F_\delta \neq \emptyset$ , то  $\dim F_\delta = 4$ . Ясно, что  $F_\delta$  определяет какие координаты не убывают, какие не возрастают при отображении  $V$ . Согласно определению  $\bigcup_{\delta} F_\delta = S^4$ , причем пересечение любых двух многогранников либо пусто, либо является общей гранью. Также в любой граничной точке  $x \in F_\delta$  имеем  $x'_k = x_k$  хотя бы для одного индекса  $k$ . В рассматриваемом случае возможны только 4 сигнатуры  $\delta_1 = (+ - - + -)$ ,  $\delta_2 = (+ - + - +)$ ,  $\delta_3 = (- + - + -)$ ,  $\delta_4 = (- + + - +)$  и соответствующие им многогранники

$$F_1 = co\{e_1, e_3, e_5, M_1, N_1, K_1\}, F_2 = co\{e_2, e_3, e_5, M_1, N_1, K_1\},$$

$$F_3 = co\{e_1, e_4, M_1, N_1, K_1\}, F_4 = co\{e_2, e_4, M_1, N_1, K_1\}.$$

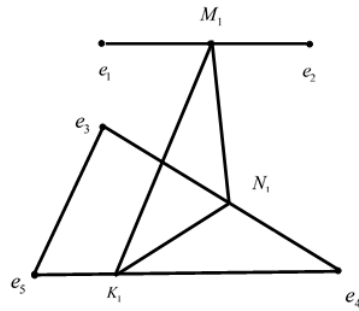


Рис. 12: График множества фиксированных точек  $Fix(V)$ .

Заметим, что 4-мерные многогранники  $F_1, F_2, F_3, F_4$  имеют общую двумерную грань  $M_1N_1K_1$ . Эти многогранники смежные, если они имеют общую грань размерности 3. Итак, получаем следующий граф смежности для этих многогранников.

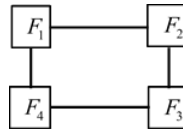


Рис. 13: Граф смежности многогранников.

Если  $x \in F_1 \cap F_2$ , то согласно (2) находим

$$x'_1 > x_1, \quad x'_2 < x_2, \quad x'_3 = x_3, \quad x'_4 = x_4, \quad x'_5 = x_5.$$

Тогда

$$\begin{aligned} x''_3 &= x'_3(1 - a_1x'_1 + a_2x'_2) < x'_3(1 - a_1x_1 + a_2x_2) = x'_3, \\ x''_4 &= x'_4(1 + b_1x'_1 - b_2x'_2) > x'_4(1 + b_1x_1 - b_2x_2) = x'_4, \\ x''_5 &= x'_5(1 - c_1x'_1 + c_2x'_2) < x'_5(1 - c_1x_1 + c_2x_2) = x'_5, \end{aligned}$$

поскольку на  $F_1 \cap F_2$  имеем  $-a_1x_1 + a_2x_2 = b_1x_1 - b_2x_2 = -c_1x_1 + c_2x_2 = 0$ . Таким образом  $V(F_1 \cap F_2) \subset F_1$ . Следовательно, траектория попавшая в  $F_2$  может перейти в  $F_1$ . Аналогично получаем следующие включения:

$$V(F_1 \cap F_3) \subset F_3, \quad V(F_3 \cap F_4) \subset F_4, \quad V(F_4 \cap F_2) \subset F_2.$$

Порядок следования траектории по многогранникам называется маршрутом. В итоге получим следующий результат.

**Теорема 1.** В рассматриваемом случае при  $x^{(0)} \notin Fix(V)$  положительная траектория не сходится, отрицательная траектория сходится к некоторой точке из  $M_1N_1K_1$ . Маршрут траектории  $V : F_1 \rightarrow F_3 \rightarrow F_4 \rightarrow F_2 \rightarrow F_1$  и  $V^{-1} : F_1 \rightarrow F_2 \rightarrow F_4 \rightarrow F_3 \rightarrow F_1$ .



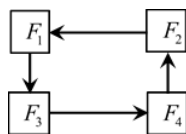


Рис. 14: Маршрут траектории, проходящей через смежные многогранники.

Если  $x^* \in \text{Fix}(V)$ , то бассейн этой точки определяется равенством  $B(x^*) = \{x^{(0)} \in S^4 : x^{(n)} \rightarrow x^*\}$ . Так как  $V$  гомеоморфизм, то различаем положительный бассейн  $B_+(x^*)$  и отрицательный бассейн  $B_-(x^*)$ . В рассматриваемом случае при  $x^* \in \text{ri}M_1N_1K_1$  (относительная внутренность) учитывая якобиан в точке  $x^*$ , согласно теореме Гробмана-Хартмана получаем  $B_+(x^*) = x^*$  и  $\dim B_-(x^*) = 2$ .

1) Пусть  $M_1 \equiv M_3$ , следовательно,  $a_1c_2 = a_2c_1$ . Если  $a_1b_2 > a_2b_1$ , то  $b_1c_2 < b_2c_1$  и получаем следующее расположение точек  $M_1N_1K_1$ :

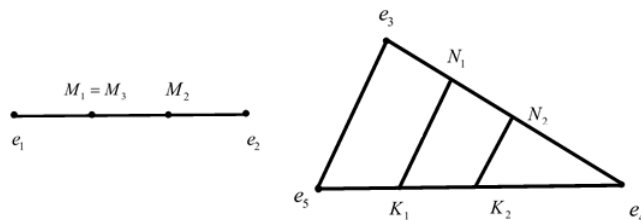


Рис. 15: Расположение точек.

В данном случае  $\text{Fix}(V) = \Gamma_{12} \cup \Gamma_{345}$ , и решив неравенства получим

$$P = \{x \in S^4 : Ax \geq 0\} = \text{co}\{M_1, M_2\}, \quad Q = \{x \in S^4 : Ax \leq 0\} = \text{co}\{N_1, N_2, K_1, K_2\}.$$

Пусть  $p = (p_1, p_2, 0, 0, 0) \in P$  и  $\varphi_p(x) = x_1^{p_1} \cdot x_2^{p_2}$ . По неравенству Юнга имеем

$$\begin{aligned} \varphi_p(Vx) &= \varphi_p(x) \cdot (1 + a_1x_3 - a_2x_4 + a_3x_5)^{p_1} \cdot (1 - b_1x_3 + b_2x_4 - b_3x_5)^{p_2} \leq \\ &\leq \varphi_p(x) \cdot (1 - (-a_1p_1 + b_1p_2)x_3 - (a_2p_1 - b_2p_2)x_4 - (-a_3p_1 + b_3p_2)x_5). \end{aligned}$$

При  $p \in P$  имеем  $-a_1p_1 + b_1p_2 \geq 0$ ,  $a_2p_1 - b_2p_2 \geq 0$ ,  $-a_3p_1 + b_3p_2 \geq 0$ , поэтому  $\varphi_p(Vx) \leq \varphi_p(x)$ . Если  $p \in \text{ri}P$ , то  $\varphi_p(Vx) < \varphi_p(x)$  для любого  $x \in \text{ri}S^4$ . Итак  $\varphi_p(V^{-1}x) > \varphi_p(x)$ , поэтому любая отрицательная траектория сходится к некоторой точке из  $\text{ri}P$ . Вычислив собственные значения якобиана получаем при  $x \in \text{ri}P$   $\lambda_1 = \lambda_2 = 1$ ,  $\lambda_3 > 1$ ,  $\lambda_4 > 1$ ,  $\lambda_5 > 1$ , а при  $x \in \text{ri}Q$  имеем  $\lambda_1 = \lambda_2 = \lambda_3 = 1$  и  $0 < \lambda_4 < 1$ ,  $0 < \lambda_5 < 1$ . Таким образом, положительные траектории сходятся к точке из  $\text{ri}Q$ , а отрицательные траектории к точке из  $\text{ri}P$ . Учитывая спектр якобиана и теорему Гробмана-Хартмана получаем при  $x \in \text{ri}P$  имеем  $\dim B_-(x) = 3$ , если  $y \in \text{ri}Q$ , то  $\dim B_+(y) = 2$ . В данном случае возможны 9 сигнатур:

$$\delta_1 = (+ - - + -), \quad \delta_2 = (+ - + + +), \quad \delta_3 = (+ - + - +),$$

$$\delta_4 = (- - - + -), \delta_5 = (- - + + +), \delta_6 = (- - + - +),$$

$$\delta_7 = (- + - + -), \delta_8 = (- + + + +), \delta_9 = (- + + - +).$$

Соответствующие многогранники:

$$F_1 = co\{e_1, e_3, e_5, M_1, N_1, K_1\}, F_2 = co\{e_3, e_5, M_1, M_2, N_1, K_1\},$$

$$F_3 = co\{e_2, e_3, e_5, M_1, N_1, K_1\}, F_4 = co\{e_1, M_1, N_1, N_2, K_1, K_2\},$$

$$F_5 = co\{M_1, M_2, N_1, N_2, K_1, K_2\}, F_6 = co\{e_2, M_2, N_1, N_2, K_1, K_2\},$$

$$F_7 = co\{e_1, e_4, M_1, N_2, K_2\}, F_8 = co\{e_4, M_1, M_2, N_2, K_2\},$$

$$F_9 = co\{e_2, e_4, M_2, N_2, K_2\}.$$

На рис.5 представлен граф смежности этих многогранников

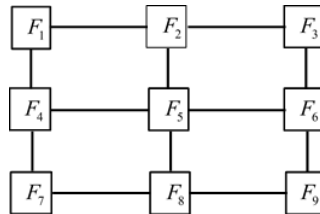


Рис. 16: Граф смежности многогранников.

На грани  $F_1 \cap F_2$  имеем  $x'_3 = x_3$  и  $x'_5 = x_5$ ,  $x'_1 > x_1$ ,  $x'_2 < x_2$ , поэтому

$$x''_3 = x'_3(1 - a_1x'_1 + a_2x'_2) < x_3(1 - a_1x_1 + a_2x_2) = x'_3,$$

$$x''_5 = x'_5(1 - c_1x'_1 + c_2x'_2) < x_5(1 - c_1x_1 + c_2x_2) = x'_5.$$

Следовательно,  $V(F_1 \cap F_2) \subset F_1$ . Однако, на грани  $F_2 \cap F_5$  существуют как точки переходящие в  $F_2$ , так и точки переходящие в  $F_5$ . Определив все возможные переходы траектории получаем следующий ориентированный граф:

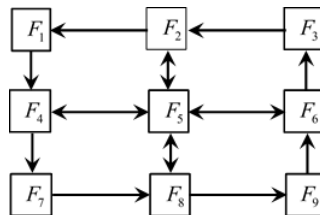


Рис. 17: Маршрут траектории, проходящей через смежные многогранники.

**Теорема 2.** Если  $x^{(0)} \in riS^4$ , то любая положительная траектория сходится, причем начиная с некоторого  $n_0$   $x^{(n)} \in F_4 \cup F_5 \cup F_6$ , отрицательная траектория также сходится и  $x^{(-n)} \in F_2 \cup F_5 \cup F_8$  начиная с некоторого номера.

Маршрут траектории определяется ориентированным графом представленным на рис.6.

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## REZYUME

Ushbu maqolada Lotka-Volterra operatori ta'sirida nuqta traektoriyalarini o'rganish muammosi muhokama qilingan. Shu bilan birga , signaturalar aniqlangan va simpleks signaturalarga mos keladigan ko'pyoqlarga bo'lingan va ular orasida marshrutlar aniqlangan.

**Kalit so'zlar:** Musbat va manfiy traektoriyalar, qo'zg'almas nuqta basseyni, signatura, signaturalarga mos ko'pyoqlar, traektoriya marshrutlari.

## RESUME

This article discusses the problem of learning point trajectories under the influence of the Lotka-Volterra operator. At the same time, the signatures are defined and divided into multiples corresponding to the simplex signatures, and the routes between them are defined.

**Key words:** Positive and negative trajectories, fixed point basin, signature, signature data polyhedra, trajectory routes.

УДК 517.55

**ПОСТРОЕНИЕ П-СТРАТЕГИЙ В ИГРЕ ПРОСТОГО  
ПРЕСЛЕДОВАНИЯ-УБЕГАНИЯ С ИМПУЛЬСНЫМ УПРАВЛЕНИЕМ**

Мустапокулов Х. Я. \* Мамадалиев Н. А. \*

**РЕЗЮМЕ**

В данной статье рассматриваются игры преследования с простым движением, в которых игроки (преследователь, убегающий или оба) используют импульсные управления, воздействия на объект которых осуществляются в заранее заданных моментах времени, и соответствующее управление выражается при помощи дельта-функции Дирака. Для данного класса дифференциальных игр получены достаточные условия разрешимости задачи преследования и убегания.

**Ключевые слова:** разрешающая функция, импульсное управление, дифференциальная игра, задача преследования, задача убегания, П-стратегия.

**I. Постановка задачи**

Пусть в пространстве  $R^n$  управляемый объект  $P$ , называемый преследователем, гонится за другим объектом  $E$ , называемом убегающим. Вектор состояния преследователя обозначим  $x$ , вектор состояния убегающего через  $y$ , соответственно. В настоящей работе рассматривается задача преследования-убегания, динамические возможности которых описываются уравнениями [1, 2, 6]

$$P : \frac{dx}{dt} = u, \quad x(0) = x_0, \quad (1)$$

$$E : \frac{dy}{dt} = v, \quad y(0) = y_0, \quad (2)$$

где  $x, y, u, v \in R^n, n \geq 1$ ;  $x_0, y_0$  – начальные состояния объектов, предполагается  $x_0 \neq y_0$ ;  $u$  и  $v$  – векторы скорости, которые служат параметрами управления.

Предположим, что  $\tau_i = i\Delta, i \in N_0 = N \cup \{0\}$ , где  $\Delta$  – некоторый положительный период.

1. Классом допустимых управлений преследователя и убегающего игроков являются множество импульсных функции, которые выражаются при помощи дельта-функции Дирака [3-5, 7-10]

$$u(t) = \sum_{i=0}^{\infty} u_i \delta(t - i\Delta), \quad u_i \in S_\rho, \quad i \in N_0, \quad t \geq 0; \quad (3)$$

\***Мустапокулов Х.Я.** – Национальный университет Узбекистана им. Мирзо Улугбека, Международный университет Нордик, m\_hamdani@mail.ru

\***Мамадалиев Н.А.** – Национальный университет Узбекистана им. Мирзо Улугбека, Институт математики, им. В.И.Романовского, m\_nuzmana59@mail.ru

$$v(t) = \sum_{i=0}^{\infty} v_i \delta(t - i\Delta), \quad v_i \in S_\sigma, \quad i \in N_0, \quad t \geq 0, \quad (4)$$

где  $S_\rho, S_\sigma$  – шары радиусов  $\rho$  и  $\sigma$  с центрами в начале координат, а  $\rho, \sigma$  – неотрицательные фиксированные числа.

2. Допустимые управления  $u$  должно быть измеримой функцией  $u(\cdot) : [0, \infty) \rightarrow R^n$  и удовлетворять геометрическому ограничению вида:

$$u(t) \in S_\rho, \quad t \geq 0, \quad (5)$$

а допустимые управления убегающего игрока задаются как (4).

3. Допустимые управления преследующего игрока задаются с помощью обобщенной  $\delta$ -функции Дирака, (см. (3)). А допустимые управления убегающего игрока удовлетворяет геометрическому ограничению вида:

$$v(t) \in S_\sigma, \quad t \geq 0. \quad (6)$$

Цель преследователя  $P$  в рассматриваемой игре состоит в осуществлении поимки, т.е. неравенства

$$|x(t) - y(t)| \leq l, \quad (7)$$

где  $x(t), y(t)$  – траектории, порождаемые в процессе управления объектами,  $l$  – заданное неотрицательное число и  $|x(0) - y(0)| > l$ . Убегающий  $E$  стремится уклониться от встречи, а если это невозможно, то как можно дольше отодвинуть момент встречи (7). Для решения задачи преследования введем понятие стратегии преследователя.

**Определение 1.** *Отображение  $u : V \rightarrow U$  называется стратегией преследователя, если выполнены следующие условия:*

(1) Для каждого  $v(\cdot) \in V$  выполнено включение  $u(\cdot) = \mathbf{u}(v(\cdot)) \in U$  в некотором промежутке времени  $[0, T]$ , при этом, функция  $u(t) = \mathbf{u}(v(\cdot))$ ,  $t \geq 0$  называется реализацией стратегии  $\mathbf{u}(v(\cdot))$ ,  $v(\cdot) \in V$ .

(2) Если для  $v_1(\cdot), v_2(\cdot) \in V$  выполнено равенство  $v_1(t) = v_2(t)$  почти всюду на  $[0, T]$ , то  $u_1(t) = u_2(t)$  почти всюду на  $[0, T]$ , где  $u_i(\cdot) = \mathbf{u}(v_i(\cdot))$ ,  $i = 1, 2$ .

В силу уравнений (1)-(2) каждые пары  $(x_0, u(\cdot))$ , где  $u(\cdot) \in U$ , и  $(y_0, v(\cdot))$ , где  $v(\cdot) \in V$ , порождают траектории:

$$x(t) = x_0 + \int_0^t u(s) ds, \quad y(t) = y_0 + \int_0^t v(s) ds$$

соответственно. Тогда  $x(t)$  – называется траекторией движения преследователя, а  $y(t)$  – траекторией движения убегающего.

Пусть  $z(t) = x(t) - y(t)$ ,  $z(0) = z_0$  и  $z_0 = x_0 - y_0$ .

**Определение 2.** *Стратегию  $\mathbf{u}(v(\cdot))$  принято называть стратегией параллельного преследования или П-стратегией, если для каждого  $v(\cdot) \in V$  решение задачи Коши*

$$\frac{dz}{dt} = \mathbf{u}(v(t)) - v(t), \quad (8)$$

$$z(0) = z_0, \quad (9)$$

можно представить в виде

$$z(t) = \Lambda(t, v(\cdot))z_0, \Lambda(0, v(\cdot)) = 1, \quad (10)$$

где  $\Lambda(t, v(\cdot))$  – некоторая скалярная непрерывная функция по  $t$ ,  $t \geq 0$ ; функцию  $\Lambda(t, v(\cdot))$  в дальнейшем назовем функцией сближения в задаче преследования.

**Определение 3.** П-стратегия называется выигрышной для преследователя в промежутке времени  $[0, T]$ , если для любого  $v(\cdot) \in V$ :

а) существует такой момент времени  $t^* \in [0, T]$ , что выполнено неравенство  $|z(t)| \leq l$ ,  $t \in [t^*, T]$ ;

б)  $u(v(\cdot)) \in U$  в промежутке времени  $[0, T]$ ;

при этом, число  $T$  будем называть гарантированным временем преследования или поимки.

Рассмотрим теперь игру  $(U, V)$  с точки зрения убегающего.

**Определение 4.** Управление  $v^*(\cdot) \in V$  называется выигрышным в игре  $(U, V)$  для убегающего, если для любого  $u(\cdot) \in U$  решения  $z(t)$  задачи Коши

$$\dot{z} = u(t) - v^*(t), z(0) = z_0, \quad (11)$$

выполнено неравенство  $|z(t)| > l$  при всех  $t \geq 0$ .

## II. Решение задачи 1

Пусть

$$U_{Imp} = \left\{ u(\cdot) \mid u(t) = \sum_{i=0}^{\infty} u_i \delta(t - i\Delta), u_i \in S_\rho, i \in N_0, t \geq 0 \right\},$$

$$V_{Imp} = \left\{ v(\cdot) \mid v(t) = \sum_{i=0}^{\infty} v_i \delta(t - i\Delta), v_i \in S_\sigma, i \in N_0, t \geq 0 \right\}.$$

После подстановки в правую часть уравнения (8) допустимых управлений игроков получим систему с правой частью с аддитивно входящей обобщенной функцией. Согласно теореме 1 [5, §1, гл.1] эта система имеет решение при любом начальном условии (9) причем оно единственно и абсолютно непрерывно на интервалах  $(\tau_{i-1}, \tau_i)$ ,  $i \in N$ , где  $N$  – множество натуральных чисел, а в моменты времени  $\tau_i$  может иметь разрывы первого рода.

### а) Решение задачи преследования

Для решения задачи преследования предположим, что в текущий момент времени  $t$  преследователю известны начальные параметры  $x_0, y_0$ , текущий момент времени  $t$  и значение управления убегающего  $v(t)$ .

**Определение 5.** Пусть  $\rho \geq \sigma$ . Тогда в игре  $(U_{Imp}, V_{Imp})$  функцию

$$u(z_0, v(t)) = \sum_{i=0}^{\infty} u_i \delta(t - i\Delta), t \geq 0, u_i = v_i - \lambda(z_0, v_i)e_0, i \in N_0, \quad (12)$$

назовем стратегией параллельного преследования (кратко  $\Pi$ -стратегией) преследователя, где

$$\lambda(z_0, v_i) = \langle v_i, e_0 \rangle + \sqrt{\langle v_i, e_0 \rangle^2 + \rho^2 - |v_i|^2}, \quad e_0 = z_0/|z_0|, \quad i \in N_0 \quad (13)$$

$\langle v_i, e_0 \rangle$  – скалярное произведение векторов  $v_i, i \in N_0$  и  $e_0$  в  $R^n$ . Функцию  $\lambda(z_0, v_i), i \in N_0$  обычно называют разрешающей функцией.

Теперь укажем некоторые важные особенности стратегии (12) и разрешающей функции (13).

**Лемма 1.** Стратегия (12) определена для всех  $v_i \in S_\sigma, i \in N_0$  и равенство  $|u_i| = \rho, i \in N_0$  удерживается во время игры преследования.

*Доказательство.* Для всех  $v_i \in S_\sigma, i \in N_0$  мы проверяем, что разрешающая функция  $\lambda(z_0, v_i), i \in N_0$  определена, когда  $\rho > \sigma$ , т.е.

$$D(z_0, v_i) = \langle v_i, e_0 \rangle^2 + \rho^2 - |v_i|^2 \geq \rho^2 - |v_i|^2 \geq \rho^2 - \sigma^2 > 0.$$

Из (12) и (13) легко показать, что  $|u_i| = \rho, i \in N_0$ .

**Лемма 2.** Разрешающая функция (13) определена и неотрицательна для всех  $v_i \in S_\sigma, i \in N_0$ , и эта функция ограничена следующим образом:

$$\rho - \sigma \leq \lambda(z_0, v_i) \leq \rho + \sigma, \quad i \in N_0.$$

*Доказательство.* Максимальное и минимальное значения разрешающей функции (13) определяются для произвольного  $v_i \in S_\sigma, i \in N_0$  следующим образом:

$$\min_{v_i \in S_\sigma} \lambda(z_0, v_i) = \lambda(z_0, v_i) \Big|_{v_i = -\sigma e_0} = \langle -\sigma e_0, e_0 \rangle + \sqrt{\langle -\sigma e_0, e_0 \rangle^2 + \rho^2 - |-\sigma e_0|^2} = \rho - \sigma,$$

$$\max_{v_i \in S_\sigma} \lambda(z_0, v_i) = \lambda(z_0, v_i) \Big|_{v_i = \sigma e_0} = \langle \sigma e_0, e_0 \rangle + \sqrt{\langle \sigma e_0, e_0 \rangle^2 + \rho^2 - |\sigma e_0|^2} = \rho + \sigma, \quad i \in N_0.$$

Следовательно,

$$\rho - \sigma \leq \lambda(z_0, v_i) \leq \rho + \sigma, \quad i \in N_0.$$

**Определение 6.** Если  $\rho > \sigma$ , то скалярная функция

$$\Lambda(t, v(\cdot)) = 1 - \frac{1}{|z_0|} \sum_{i=0}^{\infty} \int_0^t \lambda(z_0, v_i) \delta(s - i\Delta) ds \quad (14)$$

называется функцией сближения игроков в игре  $(U_{Imp}, V_{Imp})$ .

**Лемма 3.** Пусть  $\rho > \sigma$ . То

а) для всех  $v(\cdot) \in V_{Imp}$  функция (14) монотонно убывает по  $t, t \geq 0$ ;

б) функция (14) ограничена для всех  $t \in [0, T]$  следующим образом:

$$1 - \frac{\rho + \sigma}{|z_0|} (N(t) + 1) \leq \Lambda(t, v(\cdot)) \leq 1 - \frac{\rho - \sigma}{|z_0|} (N(t) + 1) \quad (15)$$

где  $N(t) = [\frac{t}{\Delta}]$ .

*Доказательство.* Пусть  $\rho > \sigma$ .

а) Из леммы 2 и формулы (14) следует следующее неравенство:

$$\Lambda(t, v(\cdot)) \Big|_{t \in [0, \Delta]} \geq \Lambda(t, v(\cdot)) \Big|_{t \in [\Delta, 2\Delta]} \geq \Lambda(t, v(\cdot)) \Big|_{t \in [2\Delta, 3\Delta]} \geq \dots$$

Следовательно, для всех  $v(\cdot) \in V_{Imp}$  функция  $\Lambda(t, v(\cdot))$  монотонно убывает по  $t$ ,  $t \geq 0$ .

б) Из формулы (14) получаем следующую оценку:

$$\Lambda(t, v(\cdot)) = 1 - \frac{1}{|z_0|} \sum_{i=0}^{\infty} \int_0^t \lambda(z_0, v_i) \delta(s - i\Delta) ds = 1 - \frac{1}{|z_0|} \sum_{i=0}^{N(t)} \lambda(z_0, v_i).$$

Из этого и из леммы 2, следует двойное неравенство (15). Лемма 3 доказана.

**Теорема 1.** Пусть  $\rho > \sigma$  в игре  $(U_{Imp}, V_{Imp})$ . Тогда П-стратегия (12) является выигрышной для преследователя в промежутке времени  $[0, T]$ , а гарантированное время  $T$  будет равно

$$T = \Delta \left[ \frac{|z_0| - l}{\rho - \sigma} \right], \quad (16)$$

где  $[\ ]$  – целая часть числа.

*Доказательство.* Предположим, что убегающий выбирает некоторое управление  $v(\cdot) \in V_{Imp}$ , а преследователь реализует П-стратегию (12). Тогда согласно (8) и (10) получим следующую функцию:

$$z(t) = z_0 + \int_0^t (\mathbf{u}(z_0, v(s)) - v(s)) ds = z_0 \Lambda(t, v(\cdot)).$$

Из этого, из леммы 3 из (16) вытекает следующая оценка:

$$\begin{aligned} |z(t)| &\leq |z(T)| = |z_0| \cdot \left| \Lambda(T, v(\cdot)) \right| \leq |z_0| \cdot \left| 1 - \frac{\rho - \sigma}{|z_0|} (N(T) + 1) \right| \leq \\ &|z_0| \cdot \left| 1 - \frac{\rho - \sigma}{|z_0|} \left( \left[ \frac{|z_0| - l}{\rho - \sigma} \right] + 1 \right) \right| \leq |z_0| \cdot \left| 1 - \frac{\rho - \sigma}{|z_0|} \cdot \frac{|z_0| - l}{\rho - \sigma} \right| = l, \quad t > T. \end{aligned}$$

Следовательно, П-стратегия (12) гарантирует поимку убегающего на интервале времени  $[0, T]$ . Теорема 1 доказана.

### б) Решение задачи уклонения

В этом разделе мы предложим допустимую стратегию убегающего, гарантирующую в задаче уклонения.

**Определение 7.** Назовем функцию управления

$$v^*(t) = - \sum_{i=0}^{\infty} (\sigma e_0 \delta(t - i\Delta)), \quad t \geq 0 \quad (17)$$

стратегией убегающего в игре  $(U_{Imp}, V_{Imp})$ .



**Теорема 2.** Пусть  $\rho \leq \sigma$ . Тогда управление (17) является выигрышным в игре  $(U_{Imp}, V_{Imp})$  для убегающего и при этом  $|z(t)| \geq |z_0|$  для всех  $t \geq 0$ .

*Доказательство.* Предположим, что  $\rho \leq \sigma$  и преследователь выбирает управление  $u(\cdot) \in U_{Imp}$ , а убегающий применяет стратегию (17). Тогда из уравнения (11) получаем

$$|z(t)| = \left| z_0 + \sum_{i=0}^{N(t)} (\sigma e_0) \right| - \left| \sum_{i=0}^{N(t)} u_i \right| \geq |z_0| + (\sigma - \rho)(N(t) + 1) \geq |z_0| > l$$

при всех  $t \geq 0$ . Теорема 2 доказана.

### III. Решение задачи 2

Пусть

$$U_G = \left\{ u(\cdot) \mid u(t) \in S_\rho, t \geq 0 \right\},$$

$$V_{Imp} = \left\{ v(\cdot) \mid v(t) = \sum_{i=0}^{\infty} v_i \delta(t - i\Delta), v_i \in S_\sigma, i \in N_0, t \geq 0 \right\}.$$

#### а) Решение задачи преследования

**Определение 8.** Пусть  $\rho\Delta \geq \sigma$ . Тогда в игре  $(U_G, V_{Imp})$  функцию  $u = u(z_0, v(t))$ ,  $t \geq 0$  назовем стратегией параллельного преследования преследователя, если для каждого  $v(\cdot) \in V_{Imp}$ :

$$а) \ u(z_0, v(t)) = u(z_0, v_i), t \in [i\Delta, (i+1)\Delta), v_i \in S_\sigma, i \in N_0;$$

$$б) \ |u(z_0, v(t))| = \rho, t \geq 0;$$

$$в) \ u(z_0, v_i) = \frac{1}{\Delta} \left( v_i - \lambda(z_0, v_i) e_0 \right), i \in N_0, где$$

$$\lambda(z_0, v_i) = \langle v_i, e_0 \rangle + \sqrt{\langle v_i, e_0 \rangle^2 + (\rho\Delta)^2 - |v_i|^2}, e_0 = z_0/|z_0|, i \in N_0 \quad (18)$$

$\langle v_i, e_0 \rangle$  – скалярное произведение векторов  $v_i$ ,  $i \in N_0$  и  $e_0$  в  $R^n$ . Функцию  $\lambda(z_0, v_i)$ ,  $i \in N_0$  обычно называют разрешающей функцией.

Теперь укажем некоторые важные особенности разрешающей функции (18).

**Лемма 4.** Разрешающая функция (18) определена и неотрицательна для всех  $v_i \in S_\sigma$ ,  $i \in N_0$ , и эта функция ограничена следующим образом:

$$\rho\Delta - \sigma \leq \lambda(z_0, v_i) \leq \rho\Delta + \sigma, i \in N_0.$$

**Определение 9.** Если  $\rho\Delta > \sigma$ , то скалярная функция

$$\Lambda(t, v(\cdot)) = 1 - \frac{1}{|z_0|} \sum_{i=0}^{N(t)} \lambda(z_0, v_i) \quad (19)$$

называется функцией сближения игроков в игре  $(U_G, V_{Imp})$ .

**Лемма 5.** Пусть  $\rho\Delta > \sigma$ . Тогда

а) для всех  $v(\cdot) \in V_{Imp}$  функция (19) монотонно убывает по  $t$ ,  $t \geq 0$ ;

б) функция (19) ограничена для всех  $t \in [0, T]$  следующим образом:

$$1 - \frac{\rho\Delta + \sigma}{|z_0|} (N(t) + 1) \leq \Lambda(t, v(\cdot)) \leq 1 - \frac{\rho\Delta - \sigma}{|z_0|} (N(t) + 1).$$

Лемма 5 доказывается аналогично лемме 3.

**Теорема 3.** Пусть  $\rho\Delta > \sigma$  в игре  $(U_G, V_{Imp})$ . Тогда  $\Pi$ -стратегия  $u = u(z_0, v(t))$  является выигрышной для преследователя в промежутке времени  $[0, T]$ , а гарантированное время  $T$  будет равно

$$T = \Delta \left[ \frac{|z_0| - l}{\rho\Delta - \sigma} \right],$$

где  $[\cdot]$  – целая часть числа.

Теорема 3 доказывается так же, как и теорема 1.

#### б) Решение задачи уклонения

В этом разделе мы предложим допустимую стратегию убегающего, гарантирующую в задаче уклонения.

**Определение 10.** Назовем функцию управления

$$v^*(t) = - \sum_{i=0}^{\infty} (\sigma e_0 \delta(t - i\Delta)), \quad t \geq 0 \quad (20)$$

стратегией убегающего в игре  $(U_G, V_{Imp})$ .

**Теорема 4.** Пусть  $\rho\Delta \leq \sigma$ . Тогда управление (20) является выигрышным в игре  $(U_G, V_{Imp})$  для убегающего и при этом  $|z(t)| \geq |z_0|$  для всех  $t \geq 0$ .

*Доказательство.* Пусть  $\rho\Delta \leq \sigma$ . Предположим, что преследователь выбирает управление  $u(\cdot) \in U_G$ , а убегающий применяет стратегию (20). Тогда из уравнения (11) получаем

$$|z(t)| = \left| z_0 + \int_0^t u(s) ds + \sigma e_0 (N(t) + 1) \right| \geq$$

$$\left| z_0 + \sigma e_0 (N(t) + 1) \right| - \left| \int_0^t u(s) ds \right| \geq$$

$$|z_0| + \sigma(N(t) + 1) - \rho t \geq |z_0| + (N(t) + 1)(\sigma - \rho\Delta) \geq |z_0| > l$$

при всех  $t \geq 0$ . Теорема 4 доказано.

### IV. Решение задачи 3

Пусть

$$U_{Imp} = \left\{ u(\cdot) \mid u(t) = \sum_{i=0}^{\infty} u_i \delta(t - i\Delta), \quad u_i \in S_\rho, \quad i \in N_0, \quad t \geq 0 \right\},$$

$$V_G = \left\{ v(\cdot) \mid v(t) \in S_\sigma, t \geq 0 \right\}.$$

**а) Решение задачи преследования**

**Определение 11.** Пусть  $\rho \geq \sigma\Delta$ . Тогда в игре  $(U_{Imp}, V_G)$  функцию

$$u(z_0, v(t)) = \sum_{i=0}^{\infty} u_i \delta(t - i\Delta), t \geq 0 \quad (21)$$

назовем стратегией параллельного преследования преследователя, если для каждого  $v(\cdot) \in V_G$ :

а)  $|u_i| = \rho, i \in N_0$ ;

б)  $u_0 = -\rho \cdot e_0, u_i = \bar{v}_i - \lambda(z_0, \bar{v}_i)e_0, i \in N$  где

$$\lambda(z_0, \bar{v}_i) = \langle \bar{v}_i, e_0 \rangle + \sqrt{\langle \bar{v}_i, e_0 \rangle^2 + \rho^2 - |\bar{v}_i|^2}, \bar{v}_i = \int_{(i-1)\Delta}^{i\Delta} v(s)ds, e_0 = z_0/|z_0|, i \in N_0 \quad (22)$$

$\langle \bar{v}_i, e_0 \rangle$  – скалярное произведение векторов  $\bar{v}_i, i \in N_0$  и  $e_0$  в  $R^n$ . Функцию  $\lambda(z_0, \bar{v}_i), i \in N_0$  обычно называют разрешающей функцией [15].

Теперь укажем некоторые важные особенности разрешающей функции (22).

**Лемма 6.** Если  $\rho \geq \sigma\Delta$ , то для произвольного управления  $v(\cdot) \in V_G$  разрешающая функция (22) обладает следующими свойствами: 1)  $\bar{v}_i \in S_{\sigma\Delta}, i \in N_0$ ; 2) определена; 3) имеет место неравенство  $\rho - \sigma\Delta \leq \lambda(z_0, \bar{v}_i) \leq \rho + \sigma\Delta, i \in N_0$ ; 4) разрешающая функция  $\lambda(z_0, \bar{v}_i)$  неотрицательна.

*Доказательство.* Пусть  $\rho \geq \sigma\Delta$  и  $v(\cdot) \in V_G$  т.е.  $v(t) \in S_\sigma, t \geq 0$ .

1)

$$|\bar{v}_i| = \left| \int_{(i-1)\Delta}^{i\Delta} v(s)ds \right| \leq \int_{(i-1)\Delta}^{i\Delta} |v(s)|ds \leq \sigma\Delta, i \in N_0.$$

Следовательно,  $\bar{v}_i \in S_{\sigma\Delta}, i \in N_0$ .

2) Определение разрешающей функции (22) можно показать аналогично лемме 1.

3) Максимальное и минимальное значения разрешающей функции (22) определяются для произвольного  $v(t) \in S_\sigma, t \geq 0$  следующим образом:

$$\min_{\bar{v}_i \in S_{\sigma\Delta}} \lambda(z_0, \bar{v}_i) = \lambda(z_0, \bar{v}_i) \Big|_{\bar{v}_i = -\sigma\Delta e_0} = \langle -\sigma\Delta e_0, e_0 \rangle + \sqrt{\langle -\sigma\Delta e_0, e_0 \rangle^2 + \rho^2 - |-\sigma\Delta e_0|^2} = \rho - \sigma\Delta,$$

$$\max_{\bar{v}_i \in S_{\sigma\Delta}} \lambda(z_0, \bar{v}_i) = \lambda(z_0, \bar{v}_i) \Big|_{\bar{v}_i = \sigma\Delta e_0} = \langle \sigma\Delta e_0, e_0 \rangle + \sqrt{\langle \sigma\Delta e_0, e_0 \rangle^2 + \rho^2 - |\sigma\Delta e_0|^2} = \rho + \sigma\Delta, i \in N_0.$$

Следовательно,  $\rho - \sigma\Delta \leq \lambda(z_0, \bar{v}_i) \leq \rho + \sigma\Delta, i \in N_0$ .

4) С помощью неравенства  $\rho \geq \sigma\Delta$  и свойства 3 легко показать, что разрешающая функция (22) неотрицательна.

**Определение 12.** Если  $\rho > \sigma\Delta$ , то скалярная функция

$$\Lambda(t, v(\cdot)) = 1 - \frac{1}{|z_0|} \left( \rho - \sigma\Delta + \sum_{i=1}^{N(t)} \lambda(z_0, \bar{v}_i) \right) \quad (23)$$

называется функцией сближения игроков в игре  $(U_{Imp}, V_G)$ .

**Лемма 7.** Пусть  $\rho > \sigma\Delta$ . Тогда

а) для всех  $v(\cdot) \in V_G$  функция (23) монотонно убывает по  $t$ ,  $t \geq 0$ ;

б) функция (23) ограничена для всех  $t \in [0, T]$  следующим образом:

$$1 - \frac{\rho + \sigma\Delta}{|z_0|} (N(t) + 1) \leq \Lambda(t, v(\cdot)) \leq 1 - \frac{\rho - \sigma\Delta}{|z_0|} (N(t) + 1).$$

Лемма 7 доказывается аналогично как в лемме 3.

**Теорема 5.** Пусть  $\rho > \sigma\Delta$  в игре  $(U_{Imp}, V_G)$ , тогда  $\Pi$ -стратегия  $\mathbf{u} = \mathbf{u}(z_0, v(t))$  является выигрышной для преследователя в промежутке времени  $[0, T]$ , а гарантированное время  $T$  будет равно

$$T = \Delta \left[ \frac{|z_0| - l}{\rho - \sigma\Delta} \right], \quad (24)$$

где  $[\ ]$  – целая часть числа.

Теорема 5 доказывается так же, как и теорема 1.

### б) Решение задачи уклонения

В этом разделе мы предложим допустимую стратегию убегающего, гарантирующую в задаче уклонения.

**Определение 13.** Назовем

$$v^*(t) = -\sigma e_0 \quad (25)$$

функцию управления стратегии убегающего игрока в игре  $(U_{Imp}, V_G)$ .

**Теорема 6.** Пусть  $\rho \leq \sigma\Delta$  и  $|z_0| > \rho + l$ . Тогда управление (25) является выигрышным в игре  $(U_{Imp}, V_G)$  для убегающего и при этом  $|z(t)| \geq |z_0| - \rho$  для всех  $t \geq 0$ .

*Доказательство.* Пусть  $\rho \leq \sigma\Delta$  и  $|z_0| > \rho$ . Предположим, что преследователь выбирает управление  $u(\cdot) \in U_{Imp}$ , а убегающий применяет стратегию (25). Тогда из уравнения (11) получаем

$$\begin{aligned} |z(t)| &= \left| z_0 + \sigma e_0 t + \sum_{i=0}^{N(t)} u_i \right| \geq \left| z_0 + \sigma e_0 t \right| - \left| \sum_{i=0}^{N(t)} u_i \right| \geq |z_0| + \sigma t - \rho(N(t) + 1) \geq \\ &|z_0| - \rho + (\sigma\Delta - \rho)N(t) \geq |z_0| - \rho > l \end{aligned}$$

при всех  $t \geq 0$ . Теорема 6 доказана.

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### REZYUME

Ushbu maqolada sodda harakatli quvish o'yinlari qaraladi. Bunda o'yinchilar (quvlovchi, qochuvchi yoki ikkalasi ham) impuls boshqaruvlarga ega bo'lib, bu impuls ta'sirlari ob'ektga oldindan belgilangan vaqt momentlarda ta'sir etib, boshqaruv Dirakning delta-funksiyasi yordamida ifodalanadi. Ushbu differentsial o'yinlar sinfi uchun tutish va qochib ketish vazifasining hal etilishi uchun yetarli shartlar olingan.

**Kalit so'zlar:** differentsial o'yin, quvish masalasi, qochish masalasi, P-strategiya, tutish.

### RESUME

This article examines pursuit games with simple motion, in which players (the pursuer, the evader, or both) use impulse controls, meaning that these impulse effects on the object are applied at predetermined moments in time, and the corresponding control is expressed using the Dirac delta function. For this class of

differential games, sufficient conditions for the solvability of the pursuit and evasion problem have been obtained.

**Key words:** differential game, pursuit problem, escape problem, P-strategy, capture.

УДК 517.956.6

**НЕЛОКАЛЬНАЯ ЗАДАЧА С ИНТЕГРАЛЬНЫМ УСЛОВИЕМ  
ДЛЯ УРАВНЕНИЯ ТРЕТЬЕГО ПОРЯДКА**

Сагдуллаева М. М. \*

В данной работе изучается нелокальная задача с интегральным условием для уравнения третьего порядка с оператором теплопроводности в главной части. Доказано существование единственного регулярного решения рассматриваемой задачи методом интегральных уравнений Вольтерра.

**Ключевые слова:** краевая задачи; нелокальное условие; интегральное условие; смешанная задача; параболического уравнения; функция Грина; интегральные уравнения Абеля.

**Введение**

Начально-краевые и нелокальные задачи с интегральными условиями для дифференциальных уравнений третьего порядка возникают при описании конкретных физических явлений. Фильтрации жидкости в пористых средах [1], передачи тепла в гетерогенных средах [2], влагопереноса в почво-грунтах [3] приводят к уравнениям диффузии, которые являются уравнениями в частных производных составного типа.

Смешанные задачи с интегральными условиями для уравнений в частных производных гиперболического типа были рассмотрены в работах [4]-[6], но при этом, в основном исследовались уравнения в частных производных второго порядка.

В настоящей работе рассматривается задача с неклассическими условиями для уравнений в частных производных третьего порядка с оператором теплопроводности в главной части.

**Постановка задачи**

В области  $D = \{(x, t) : 0 < x < l, 0 < t < T\}$  рассмотрим уравнение в частных производных третьего порядка

$$Lu \equiv \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} \right) + c(x, t)u = f(x, t), \quad (1)$$

где  $c(x, t)$ ,  $f(x, t)$  – заданные функции.

Заметим, что уравнение (1) относится к первому каноническому виду относительно старших производных, указанных в работе [7], т. е. уравнение характеристики имеет один интеграл, причём трехкратный. Этот фактор существенно влияет как на корректность постановки задач, так и на их разрешимость.

В работе для уравнения (1) исследуется следующая **задача:** *Найти в области  $D$  решение  $u(x, t)$  уравнения (1), удовлетворяющее начальному*

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq l, \quad (2)$$

\* Сагдуллаева М. М. – Национальный университет Узбекистана, sagdullayevam@mail.ru

и граничным условиям

$$u(0, t) = \mu_1(t) \quad u_x(l, t) = \mu_2(t), \quad 0 \leq t \leq T, \quad (3)$$

и интегральным условиям

$$\int_0^l u(x, t) dx + \int_0^t h(t, \tau) u(l, \tau) d\tau + \mu_1(t), \quad 0 \leq t \leq T, \quad (4)$$

где  $\psi_1(x)$ ,  $\mu_1(t)$ ,  $\mu_2(t)$  и  $h(t, \tau)$  – заданные, непрерывные на  $[0, l]$  и  $[0, T]$  соответственно функции и удовлетворяют условиям согласования: удовлетворяющие условиям согласования:

$$\varphi(0) = \mu_1(0), \quad \varphi'(l) = \mu_2(0); \quad \int_0^l \varphi(x) dx = \mu_3(0).$$

В поставленной задаче в краевых условиях содержится нелокальность по времени, впервые рассмотренные в работе [8] и учеников исследованы разрешимость краевых задач, сочетающих задачи с нелокальными условиями А.А.Самарского и задачи с интегральными условиями.

Через  $C^{k, \ell}(D)$ , обозначен класс функций  $u(x, t)$  непрерывных вместе со своими частными производными порядка  $\partial^{m+n} u(x, t) / \partial x^m \partial t^n$  для всех  $m = \bar{0}, k, n = \bar{0}, l$ ;  $C^{0,0}(D)$ .

Под классом  $C^{k, \nu}(D)$  понимаются определенные в области  $D$  функции, у которых все частные производные порядка  $k$  существует и удовлетворяют условию Гельдера с показателем  $\nu$ ,  $0 < \nu < 1$ .

**Определение.** Регулярным в области  $D$  решением уравнения (1) называется действительная функция  $u(x, t)$ , из класса  $C^{3,1}(D) \cap C^{2,0}(\bar{D})$ , удовлетворяющую условиями (1)-(4) в обычном смысле.

Задачу (1)-(4) исследуем в пространстве  $C^{3,1}(D) \cap C^{2,0}(\bar{D})$ , при этом справедлива следующая теорема о разрешимости нелокальной задачи (1)-(4).

Имеет место следующая теорема о разрешимости нелокальной задачи (1)-(4).

**Теорема.** Пусть коэффициент и правая часть уравнения (1) удовлетворяют условиям  $c(x, t), f(x, t) \in C(\bar{D})$  и заданные функции  $\varphi(x), \psi_i(t), (i = 1, 2, 3)$  и  $h(t, \tau)$  удовлетворяют условиям

$$\varphi(x) \in C^2[0, l]; \mu_1(t), \mu_3(t) \in C^1[0, T]; \mu_2(t) \in C[0, T].$$

Тогда существует единственное непрерывное и ограниченное решение нелокальной задачи (1)-(4).

**Доказательство.** Построим явное решение задачи (1)-(4) с помощью функции Грина для уравнения теплопроводности.

Сложность этой задачи состоит в том, что в обе части граничного условия (4) входит неизвестное решение  $u(x, t)$ . Положим  $u_x(x, t) = v(x, t)$  и обозначим  $v(0, t)$  через  $\mu(t)$  и решим следующую вспомогательную задачу. Найти в области  $D$  решение  $v(x, t)$  уравнения

$$v_t - v_{xx} = F(x, t), \quad (5)$$



удовлетворяющее начальным условиям (4)

$$v(x, 0) = \varphi'(x), \quad 0 \leq x \leq l, \quad (6)$$

и граничным условиям

$$v(0, t) = \mu(t), \quad v(l, t) = \mu_2(t), \quad 0 \leq t \leq T, \quad (7)$$

где  $F(x, t) = f(x, t) - c(x, t)u(x, t)$  и  $\mu(t)$  – пока неизвестная функция, предположим, что  $\mu(t)$ , непрерывно-дифференцируема и интегрируема на  $[0, T]$   $\mu(0) = \varphi'(0)$ .

Решение задачи (5)-(7) известно и представимо в виде [9]

$$v(x, t) = \int_0^l \varphi'(x)G(x, t; \xi, 0)d\xi + \int_0^t \mu(\tau)G_\xi(x, t; 0, \tau)d\tau - \\ - \int_0^t \mu_2(\tau)G_\xi(x, t; 0, \tau)d\tau + \int_0^t \int_0^l G(x, t; \xi, \tau)F(\xi, \tau)d\xi d\tau, \quad (x, t, \xi, \tau) \quad (8)$$

здесь  $G(x, t; \xi, \tau)$  – функция Грина первой краевой задачи для уравнения теплопроводности, которая в рассматриваемом случае имеет в вид

$$G(x, t, \xi; \tau) = \sum_{n=-\infty}^{+\infty} [U(x, t; \xi + 2nl, \tau) - U(x, t; -\xi + 2nl, \tau)], \quad (9)$$

здесь

$$U(x, t; \xi, \tau) = \begin{cases} \frac{1}{2\sqrt{\pi(t-\tau)}} \exp\left[-\frac{(x-\xi)^2}{4(t-\tau)}\right], & t > \tau, \\ 0, & t \leq \tau. \end{cases} \quad (10)$$

Доказательство абсолютной и равномерной сходимости ряда (9) за исключением члена при  $n = 0$  и рядов, полученных из него почленным дифференцированием любое число раз по  $x$  и  $t$  приведено в работе [10].

Учитывая обозначение  $F(x, t)$  из формулы (8) имеем:

$$u_x(x, t) = \Phi_0(x, t) + \int_0^t \mu(\tau)G_\xi(x, t; 0, \tau)d\tau - \int_0^l \int_0^t G(x, t; \xi, \tau)c(\xi, \tau)u(\xi, \tau)d\xi d\tau, \quad (11)$$

где

$$\Phi_0(x, t) = \int_0^l \varphi'(\xi)G(x, t, \xi, 0)d\xi + \int_0^t \mu_2(\tau)G_\xi(x, t; 0, \tau)d\tau + \int_0^t \int_0^l G(x, t; \xi, \tau)f(\xi, \tau)d\xi d\tau.$$

Интегрируя (11) по  $x$ , получим

$$u(x, t) = \Phi(x, t) + \int_0^x ds \int_0^t \mu(\tau)G_\xi(x, t; 0, \tau)d\tau - \int_0^x ds \int_0^l \int_0^t G(x, t; \xi, \tau)c(\xi, \tau)u(\xi, \tau)d\xi d\tau, \quad (12)$$

здесь

$$\Phi(x, t) = \int_0^x \Phi_0(s, t) ds + \mu_1(t).$$

Заметим что,

$$\int_0^x ds \int_0^t \mu(\tau) G_\xi(x, t; 0, \tau) d\tau = \frac{1}{2\sqrt{\pi}} \int_0^t \frac{\mu(\tau) d\tau}{\sqrt{t-\tau}} - \int_0^t G(x, t; 0, \tau) d\tau.$$

Тогда формула (12) примет вид

$$u(x, t) = \Phi(x, t) + \frac{1}{2\sqrt{\pi}} \int_0^t \frac{\mu(\tau) d\tau}{\sqrt{t-\tau}} - \int_0^t G(x, t; 0, \tau) d\tau - \int_0^t \int_0^l K(x, t; \xi, \tau) u(\xi, \tau) d\xi d\tau, \quad (13)$$

где

$$K(x, t; \xi, \tau) = c(\xi, \tau) \int_0^x G(x, t; \xi, \tau) d\xi \xi.$$

Теперь используем условие (4).

Сначала проинтегрируем (13) по  $x$  от 0 до  $l$  и после несложных преобразований получим,

$$\begin{aligned} \int_0^l u(x, t) dx &= \int_0^l \Phi(x, t) dx + \frac{l}{2\sqrt{\pi}} \int_0^t \frac{\mu(\tau) d\tau}{\sqrt{t-\tau}} - \\ &- \int_0^t \left( \int_0^l G(x, t; 0, \tau) dx \right) \mu(\tau) d\tau - \int_0^t \int_0^l \left( \int_0^l K(x, t; \xi, \tau) dx \right) u(\xi, \tau) d\xi d\tau. \end{aligned}$$

В формуле (13) положим  $x = l$  и умножим обе части на  $h(t, \tau)$ , полученное при этом выражение проинтегрируем по  $\tau$  в пределах от 0 до  $t$  и после ряда преобразований, имеем

$$\begin{aligned} \int_0^t h(t, \tau) u(l, \tau) d\tau &= \int_0^t h(t, \tau) \Phi(l, \tau) d\tau + \int_0^t \mu(s) ds \int_s^t \frac{h(t, \tau) d\tau}{\sqrt{\tau-s}} - \\ &\int_0^t \mu(s) ds \int_s^t h(t, \tau) G(l, \tau; 0, s) d\tau - \int_0^t d\xi \int_0^t \left( \int_s^t h(t, \tau) K(l, \tau; \xi, s) d\tau \right) u(\xi, s) ds. \end{aligned}$$

Соберем все слагаемые, отвечающие условию (4) и получим следующее соотношение

$$\frac{l}{2\sqrt{\pi}} \int_0^t \frac{\mu(\tau) d\tau}{\sqrt{t-\tau}} - \int_0^t k_0(t, \tau) d\tau - \int_0^t \int_0^l K_0(t; \xi, \tau) u(\xi, \tau) d\xi d\tau = g_0(t), \quad (14)$$

здесь

$$k_0(t, \tau) = \int_0^l G(x, \tau; 0, \tau) dx + \frac{1}{2\sqrt{\pi}} \int_{\tau}^t \frac{h(t, s) ds}{\sqrt{\tau - s}} + \int_{\tau}^t h(t, s) G(l, \tau; 0, s) ds$$

$$K_0(t; \xi, \tau) = \int_0^l K(x, \tau; \xi, \tau) dx + \int_{\tau}^t K(l, \tau; \xi, s) ds$$

$$g_0(t) = \mu_3(t) + \int_0^t h(l, \tau) \Phi(l, \tau) d\tau - \text{известная функция.}$$

Уравнение (14) перепишем в виде

$$\frac{l}{2\sqrt{\pi}} \int_0^t \frac{\mu(\tau) d\tau}{\sqrt{t - \tau}} = g(t), \quad (15)$$

где

$$g(t) = g_0(t) + \int_0^t k_0(t, \tau) d\tau + \int_0^t \int_0^l K_0(t; \xi, \tau) u(\xi, \tau) d\xi d\tau.$$

Из условия согласования имеем, что  $g(0) = 0$ . Вычислим производную от функции  $g(t)$

$$g'(t) = g'_0(t) + k_0(t, t)\mu(t) + \int_0^t \frac{\partial k_0(t, \tau)}{\partial t} \mu(\tau) d\tau +$$

$$+ \int_0^l K_0(t; \xi, t) u(\xi, t) d\xi + \int_0^t \int_0^l \frac{\partial K_0(t; \xi, \tau)}{\partial t} u(\xi, \tau) d\xi d\tau.$$

Обращая уравнения Абеля (15), получим интегральное уравнение Вольтерра второго рода в виде

$$\mu(t) = \frac{2}{\sqrt{\pi t}} [g_1(t) + \int_0^t k_0(t, \tau) \mu(\tau) d\tau + \int_0^t \int_0^l K_2(t; \xi, \tau) u(\xi, \tau) d\xi d\tau], \quad (16)$$

здесь

$$k_1(t, \tau) = \frac{h(t, t)}{\sqrt{t - \tau}} + h(t, t) G(l, \tau; 0, t) + \int_{\tau}^t \frac{\partial h(t, \tau)}{\partial t} G(l, \tau; 0, s) ds;$$

$$K_2(t; \xi, \tau) = \frac{K_0(t; \xi, \tau)}{\sqrt{t - \tau}} + \int_{\tau}^t \frac{K_0(t; \xi, s)}{\sqrt{\tau - s}};$$

$g_1(t)$  – известная функция.

В силу свойств функции Грина, легко заметим, что

$$|k_1(t, \tau)| \leq \frac{c_1}{\sqrt{t - \tau}}; \quad |K_1(t, \xi, \tau)| \leq \frac{c_2}{\sqrt{t - \tau}}, \quad c_1, c_2 = \text{const} > 0.$$

Как показано в [9], уравнения (16) является интегральным уравнением Вольтерра второго рода со слабой особенностью и правая часть непрерывна, то оно допускает единственное решение. Его решение запишем в виде

$$\mu(t) = g_2(t) - \int_0^t \int_0^l K_3(t; \xi, \tau) u(\xi, \tau) d\xi d\tau,$$

$$K_3(t, \xi, \tau) = K_2(t, \xi, \tau) + \int_{\tau}^t R_1(t, s) K_2(s, \xi, \tau) ds;$$

$$g_2(t) = g_1(t) + \int_0^t R_1(t, \tau) g_1(\tau) d\tau.$$

Подставляя найденное значение  $\mu(t)$  из (17) в (13), после некоторых преобразований получим интегральное уравнение Вольтерра второго рода относительно функции  $u(x, t)$

$$u(x, t) = \Phi_1(x, t) + \int_0^t \int_0^l K_4(x, t; \xi, \tau) u(\xi, \tau) d\xi d\tau, \quad (18)$$

$$K_4(x, t; \xi, \tau) = K(x, t; \xi, \tau) + \frac{1}{\sqrt{\pi}} \int_{\tau}^t \frac{K_3(\tau; \xi, s)}{\sqrt{t-s}} ds - \int_{\tau}^t G(x, t; 0, s) K_3(\tau; \xi, s) ds;$$

$$\Phi_1(x, t) + \frac{1}{2\sqrt{\pi}} \int_0^t \frac{g_2(\tau) d\tau}{\sqrt{t-\tau}} - \int_0^t G(x, t; 0, \tau) g_2(\tau) d\tau.$$

В силу свойств функций  $K_4(x, t; \xi, \tau)$  и  $\Phi_1(x, t)$  заключаем, что уравнение (18) допускает единственное решение в классе непрерывно-дифференцируемых функций. Таким образом, разрешимость нелокальной задачи (1)-(4) доказано.

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#### REZYUME

Bu maqolada bosh qismida issiqlik o'tkasish operatori bo'lgan uchinchi tartibli tenglama uchun nolokal chegaraviy masala o'rganilgan. Qaralayotgan maalaning yagona regular yechimining mavjudligi isbotlangan.

**Kalit so'zlar:** Grin funksiyasi, chegaraviy shart, integral shart, Volter integral tenglamasi.

#### RESUME

In this paper we study non-local boundary-value problem for the third-order equation with heat operator in the main part. We prove the theorem of the existence and uniqueness of regular solution for considered problem.

**Key words:** Green function, boundary condition, integral condition, integral condition of Volterra.

УДК 517.968.7

## НАЧАЛЬНАЯ ЗАДАЧА ДЛЯ ИМПУЛЬСНЫХ СИСТЕМ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

Файзиев А. К. \*

## РЕЗЮМЕ

Исследуется начальная задача для системы обыкновенных интегро-дифференциальных уравнений второго порядка с импульсными воздействиями и максимумами. Существование и единственность решения нелинейной начальной задачи сводятся к однозначной разрешимости системы нелинейных функционалов интегральные уравнения в Банаховом пространстве  $PC([0, T], \mathbb{R}^n)$ . Метод последовательных приближений в сочетании принцип сжимающих отображения используется при доказательстве однозначной разрешимости нелинейного функционала интегральные уравнения.

**Ключевые слова:** начальная задача, импульсный воздействие, интегро-дифференциальное уравнение, банахово пространство, максимумы, нелинейность, метод сжимающего отображения, последовательное приближение.

На интервале  $[0, T]$  для  $t \neq t_i$ ,  $i = 1, 2, \dots, p$  рассматриваются вопросы существования и конструктивные методы вычисления единственного решения системы нелинейных обыкновенных интегро-дифференциальных уравнений второго порядка с импульсными воздействиями и максимумами.

$$x''(t) = f\left(t, x(t), \int_0^T \Theta(t, s) \max \{x(\tau) \mid \tau \in [\lambda_1(s); \lambda_2(s)]\} ds\right), \quad (15)$$

где  $t \neq t_i$ ,  $i = 1, 2, \dots, p$ ,  $0 = t_0 < t_1 < \dots < t_p < t_{p+1} = T$ ,  $x \in X$ ,  $X$  - закрытое множество в пространстве  $\mathbb{R}^n$ ,  $f(t, x, y) \in C([0, T] \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ ,  $0 < \lambda_1(t) < \lambda_2(t) < T$ ,  $\max_{0 \leq t \leq T} \int_0^T |\Theta(t, s)| ds < \infty$ .

Уравнение (15) рассмотрим с нелинейными начальным условиями

$$A_1(t)x(0^+) = B_1 + D_1(t, x(t)), \quad (16)$$

$$A_2(t)x'(0^+) = B_2 + D_2(t, x(t)) \quad (17)$$

и импульсными условиями

$$x(t_i^+) - x(t_i^-) = F_i(x(t_i)), \quad i = 1, 2, \dots, p, \quad (18)$$

$$x'(t_i^+) - x'(t_i^-) = G_i(x(t_i)), \quad i = 1, 2, \dots, p, \quad (19)$$

где  $B_\kappa$  являются  $n$ -мерными постоянными векторами,  $A_\kappa(t) \in C([0, T], \mathbb{R}^n)$  являются  $n \times n$ -мерными функциями-матрицами,  $\det A_\kappa^{-1}(t) \neq 0$ ,  $D_\kappa(t, x(t)) \in C([0, T] \times \mathbb{R}^n, \mathbb{R}^n)$  являются

\*Файзиев А. К. – Ташкентский государственный экономический университет, fayziyev.a@inbox.ru

нелинейными вектор-функциями,  $\kappa = 1, 2, F_i, G_i \in C(\mathbb{R}^n, \mathbb{R}^n), x(t_i^+), x(t_i^-)$  являются правосторонними и левосторонним пределами функции  $x(t)$  в точках  $t = t_i$ , соответственно.

**Постановка задачи.** Найти функцию  $x(t) \in PC([0, T], \mathbb{R}^n)$ , которая для всех  $t \in [0, T], t \neq t_i, i = 1, 2, \dots, p$  удовлетворяет интегро-дифференциальному уравнению второго порядка (15), нелинейному начальному условию (16), (17) и для  $t = t_i, i = 1, 2, \dots, p, 0 < t_1 < t_2 < \dots < t_p < T$  удовлетворяет нелинейным импульсным условиям (18), (19).

Пусть функция  $x(t) \in PC([0, T], \mathbb{R}^n)$  является решением задачи (15)-(19). Тогда, интегрируя интегро-дифференциальное уравнение (15) один раз на интервалах:  $(0, t_1], (t_1, t_2], \dots, (t_p, t_{p+1}]$ , получаем:

$$\begin{aligned} \int_0^{t_1} f(s, x, y) ds &= \int_0^{t_1} x''(s) ds = x'(t_1^-) - x'(0^+), \quad t \in (0, t_1], \\ \int_{t_1}^{t_2} f(s, x, y) ds &= \int_{t_1}^{t_2} x''(s) ds = x'(t_2^-) - x'(t_1^+), \quad t \in (t_1, t_2], \\ &\dots\dots\dots \\ \int_{t_p}^{t_{p+1}} f(s, x, y) ds &= \int_{t_p}^{t_{p+1}} x''(s) ds = x'(t_{p+1}^-) - x'(t_p^+), \quad t \in (t_p, t_{p+1}], \end{aligned}$$

где, для удобства, полагали

$$f(t, x, y) = f\left(t, x(t), \int_0^T \Theta(t, s, \max\{x(\tau) \mid \tau \in [\lambda_1(s), \lambda_2(s)]\}) ds\right).$$

Отсюда, с учетом  $x'(t_{p+1}^-) = x'(t)$ , на интервале  $(0^+, T]$  получаем

$$\begin{aligned} \int_0^t f(s, x, y) ds &= [x'(t_1) - x'(0^+)] + [x'(t_2) - x'(t_1^+)] + \dots + [x'(t) - x'(t_p^+)] = \\ &= -x'(0) - [x'(t_1^+) - x'(t_1)] - [x'(t_2^+) - x'(t_2)] - \dots - [x'(t_p^+) - x'(t_p)] + x'(t). \end{aligned}$$

В силу условия (19), последнее равенство перепишем следующим образом

$$x'(t) = x'(0) + \int_0^t f(s, x, y) ds + \sum_{0 < t_i < t} G_i(x(t_i)). \tag{20}$$

Для функции  $x'(t) \in PC([0, T], \mathbb{R}^n)$  в представлении (20) мы пользуемся нелинейным начальным условием (17). Тогда имеем

$$x'(t) = (B_2 + D_2(t, x(t))) [A_2(t)]^{-1} + \int_0^t f(s, x, y) ds + \sum_{0 < t_i < t} G_i(x(t_i)). \tag{21}$$

Интегрируя интегрально-дифференциальное уравнение (21) один раз на интервале  $(0, t_1], (t_1, t_2], \dots, (t_p, t_{p+1}]$  и учитывая  $x(0^+) = x(0), x(t_{p+1}^-) = x(t)$ , на интервале  $(0, T]$  получим

$$\int_0^t \left[ (B_2 + D_2(s, x(s))) [A_2(s)]^{-1} + \int_0^s f(\theta, x, y) d\theta + \sum_{0 < t_i < s} G_i(x(t_i)) \right] ds =$$

$$\begin{aligned}
&= [x(t_1) - x(0^+)] + [x(t_2) - x(t_1^+)] + \dots + [x(t) - x(t_p^+)] = \\
&= -x(0) - [x(t_1^+) - x(t_1)] - [x(t_2^+) - x(t_2)] - \dots - [x(t_p^+) - x(t_p)] + x(t). \quad (22)
\end{aligned}$$

Принимая во внимание нелинейное импульсное условие (18), из равенства (22) получаем

$$\begin{aligned}
x(t) = x(0) + \int_0^t (B_2 + D_2(s, x(s))) [A_2(s)]^{-1} ds + \int_0^t (t-s)f(s, x, y) ds + \\
+ \int_0^t \sum_{0 < t_i < s} G_i(x(t_i)) ds + \sum_{0 < t_i < t} F_i(x(t_i)). \quad (23)
\end{aligned}$$

Используя начальное условие (16) в уравнении (23), получаем функциональное интегральное уравнение

$$\begin{aligned}
x(t) = J_1(t; x) \equiv (B_1 + D_1(t, x(t))) [A_1(t)]^{-1} + \int_0^t (B_2 + D_2(s, x(s))) [A_2(s)]^{-1} ds + \\
+ \int_0^t (t-s)f \left( s, x(s), \int_0^T \Theta(s, \theta) \max \{x(\tau) \mid \tau \in [\lambda_1(\theta); \lambda_2(\theta)]\} d\theta \right) ds + \\
+ \int_0^t \sum_{0 < t_i < s} G_i(x(t_i)) ds + \sum_{0 < t_i < t} F_i(x(t_i)). \quad (24)
\end{aligned}$$

**Теорема.** Предположим, что следующие условия выполняются:

$$1). M_f = \max_{0 \leq t \leq T} |f(t, x, y)| < \infty; M_{D_\kappa} = \max_{0 \leq t \leq T} |D_\kappa(t, x)| < \infty, \quad \kappa = 1, 2;$$

$$2). M_{F,i} = |F_i(x)| < \infty, \quad M_{G,i} = |G_i(x)| < \infty; \bar{M}_{D_1} = \max_{0 \leq t \leq T} \left| \frac{d}{dt} D_1(t, 0) \right| < \infty;$$

3). Для всех  $t \in [0, T]$ ,  $x, y \in \mathbb{R}^n$  имеет место

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L_1(t) |x_1 - x_2| + L_2(t) |y_1 - y_2|;$$

4). Для всех  $t \in [0, T]$ ,  $x \in \mathbb{R}^n$  имеет место

$$|D_\kappa(t, x_1) - D_\kappa(t, x_2)| \leq L_{3\kappa}(t) |x_1 - x_2|, \quad \kappa = 1, 2;$$

5). Для всех  $t \in [0, T]$ ,  $x \in \mathbb{R}^n$  справедливо

$$\left| \frac{d}{dt} [D_1(t, x_1) - D_1(t, x_2)] \right| \leq \bar{L}_{31}(t) |x_1 - x_2|;$$

6). Для всех  $x \in \mathbb{R}^n$ ,  $i = 0, 1, \dots, p$  справедливы

$$|F_i(x_1) - F_i(x_2)| \leq L_{4,i} |x_1 - x_2|, \quad |G_i(x_1) - G_i(x_2)| \leq L_{5,i} |x_1 - x_2|;$$

7).  $\rho = \max \{ \rho_{1,1} + \rho_{1,2}; \rho_{2,1} + \rho_{2,2} \} < 1$ , где  $\rho_{\kappa,j}$  ( $\kappa, j = 1, 2$ ) определяются из формул (33), (34) и (36), (37) внизу.



Тогда начальная задача (15)-(19) имеет единственное решение  $x(t) \in PC([0, T], \mathbb{R}^n)$ . Это решение может быть найдено из следующего итерационного процесса:

$$x^0(t) = 0, \quad x^k(t) = J_1(t; x^{k-1}), \quad k = 1, 2, 3, \dots \quad (25)$$

**Доказательство.** Продифференцируем уравнения (24) перепишем в виде оператора

$$\begin{aligned} x'(t) = J_2(t; x) \equiv & P(t) + \frac{dD_1(t, x(t))}{dt} A_1^{-1}(t) + \\ & + D_1(t, x(t)) [A_1^{-1}(t)]' + D_2(t, x(t)) A_2^{-1}(t) + \\ & + \int_0^t f\left(s, x(s), \int_0^T \Theta(s, \theta) \max\{x(\tau) \mid \tau \in [\lambda_1(\theta); \lambda_2(\theta)]\} d\theta\right) ds + \\ & + \sum_{0 < t_i < t} [G_i(x(t_i)) + F_i(x(t_i))], \end{aligned} \quad (26)$$

где

$$P(t) = B_1 [A_1^{-1}(t)]' + B_2 A_2^{-1}(t).$$

Наряду с итерационным процессом Пикара (25) мы рассмотрим и для уравнения (26) следующий итерационный процесс:

$$x^{j0}(t) = P(t), \quad x^{jk}(t) = J_2(t; x^{j(k-1)}), \quad k = 1, 2, 3, \dots \quad (27)$$

Рассмотрим следующий оператор

$$J_1 : PC([0, T]; \mathbb{R}^n) \rightarrow PC([0, T]; \mathbb{R}^n),$$

Определенный правой частью уравнения (24). Применяя принцип сжимающего оператора к уравнению (24), мы покажем, что оператор  $J_1$ , определенный уравнением (24), имеет единственную неподвижную точку.

Для первой разности приближений (25) получаем следующую оценку

$$\begin{aligned} \|x^1(t) - x^0(t)\| \leq & \max_{0 \leq t \leq T} |(B_1 + D_1(t, x^0(t)))| | [A_1(t)]^{-1} | + \\ & + \max_{0 \leq t \leq T} \int_0^t |(B_2 + D_2(s, x^0(s)))| | [A_2(s)]^{-1} | ds + \\ & + \max_{0 \leq t \leq T} \int_0^t (t-s) \left| f\left(s, x^0(s), \int_0^T \Theta(s, \theta) \max\{x^0(\tau) \mid \tau \in [\lambda_1(\theta); \lambda_2(\theta)]\} d\theta\right) \right| ds + \\ & + \sum_{0 < t_i < T} [T |G_i(x^0(t_i))| + |F_i(x^0(t_i))|]. \end{aligned} \quad (28)$$

В силу первых двух условий теоремы, из (28) имеем

$$\|x^1(t) - x^0(t)\|_{PC[0, T]} \leq (|B_1| + M_{D_1}) \max_{0 \leq t \leq T} | [A_1(t)]^{-1} | +$$

$$\begin{aligned}
& + (|B_2| + M_{D_2}) \max_{0 \leq t \leq T} \int_0^t |[A_2(s)]^{-1}| ds + \\
& + M_f \max_{0 \leq t \leq T} \int_0^t (t-s) ds + \sum_{i=1}^p [TM_{G,i} + M_{F,i}] < \infty.
\end{aligned} \tag{29}$$

Для итерационного процесса (27), в силу уравнения (26), по аналогии оценки (29) имеем

$$\begin{aligned}
& |x^1(t) - x^0(t)| \leq \left| \frac{d}{dt} D_1(t, x^0(t)) \right| |A_1^{-1}(t)| + \\
& + |D_1(t, x^0(t))| \left| [A_1^{-1}(t)]' \right| + |D_2(t, x^0(t))| |A_2^{-1}(t)| + \\
& + \int_0^t \left| f \left( s, x^0(s), \int_0^T \Theta(s, \theta) \max \{x^0(\tau) | \tau \in [\lambda_1(\theta); \lambda_2(\theta)]\} d\theta \right) \right| ds + \\
& + \sum_{i=1}^p [|G_i(x(t_i))| + |F_i(x(t_i))|].
\end{aligned}$$

или

$$\begin{aligned}
& \|x^1(t) - x^0(t)\|_{PC[0,T]} \leq \bar{M}_{D_1} \max_{0 \leq t \leq T} |A_1^{-1}(t)| + \\
& + \max_{0 \leq t \leq T} \left\{ M_{D_1} \left| [A_1^{-1}(t)]' \right| + M_{D_2} |A_2^{-1}(t)| \right\} + \\
& + M_f T + \sum_{i=1}^p (M_{G,i} + M_{F,i}) < \infty.
\end{aligned} \tag{30}$$

Тогда, в силу условий 3)-6) теоремы, для произвольной последовательной разности приближений, имеем

$$\begin{aligned}
& \|x^{k+1}(t) - x^k(t)\| \leq \\
& \leq \|x^k(t) - x^{k-1}(t)\| \max_{0 \leq t \leq T} \left[ L_{31}(t) |[A_1(t)]^{-1}| + \int_0^t L_{32}(s) |[A_2(s)]^{-1}| ds \right] + \\
& + \max_{0 \leq t \leq T} \int_0^t (t-s) [L_1(s) \|x^k(s) - x^{k-1}(s)\| + \\
& + \int_0^T L_2(s) |\Theta(s, \theta)| \|x^k(\tau) - x^{k-1}(\tau)\| d\theta] ds + \\
& + \sum_{0 < t_i < T} [L_{4,i}T + L_{5,i}] |x^k(t_i) - x^{k-1}(t_i)|,
\end{aligned} \tag{31}$$

где

$$x^k(\tau) = \max \{x^k(\tau) | \tau \in [\lambda_1(\theta); \lambda_2(\theta)]\}.$$

Применяя лемму из (15) к оценке (31), получаем

$$\begin{aligned}
& \|x^{k+1}(t) - x^k(t)\|_{PC[0,T]} \leq \\
& \leq \rho_{11} \|x^k(t) - x^{k-1}(t)\|_{PC[0,T]} + \rho_{12} \|x'^k(t) - x'^{k-1}(t)\|_{PC[0,T]},
\end{aligned} \tag{32}$$

где

$$\begin{aligned} \rho_{11} = & \max_{0 \leq t \leq T} \left[ L_{31}(t) | [A_1(t)]^{-1} | + \int_0^t L_{32}(s) | [A_2(s)]^{-1} | ds \right] + \\ & + \max_{0 \leq t \leq T} \int_0^t (t-s) \left[ L_1(s) + \int_0^T L_2(s) | \Theta(s, \theta) | d\theta \right] ds + \sum_{i=1}^p [L_{4,i}T + L_{5,i}], \end{aligned} \quad (33)$$

$$\rho_{12} = 2(h_1 + h_2) \max_{0 \leq t \leq T} \int_0^t (t-s) \int_0^T L_2(s) | \Theta(s, \theta) | d\theta ds, \quad h_\kappa = \max_{0 \leq t \leq T} \lambda_\kappa(t). \quad (34)$$

Теперь, в силу условий теоремы, из итерации (27) получим

$$\begin{aligned} |x'^{k+1}(t) - x'^k(t)| \leq & \left[ \bar{L}_{31} | A_1^{-1}(t) | + L_{31} | [A_1^{-1}(t)]' | + L_{32} | A_2^{-1}(t) | \right] |x^k(t) - x^{k-1}(t)| + \\ & + \int_0^t \left[ L_1(s) |x^k(s) - x^{k-1}(s)| + L_2(s) \int_0^T | \Theta(s, \theta) | |x^k(\tau) - x^{k-1}(\tau)| d\theta \right] ds + \\ & + \sum_{i=1}^p (L_{4,i} + L_{5,i}) |x^k(t_i) - x^{k-1}(t_i)|. \end{aligned}$$

Отсюда, переходя к норме, получаем, что

$$\begin{aligned} & \|x'^{k+1}(t) - x'^k(t)\|_{PC[0,T]} \leq \\ & \leq \rho_{21} \|x^k(t) - x^{k-1}(t)\|_{PC[0,T]} + \rho_{22} \|x'^k(t) - x'^{k-1}(t)\|_{PC[0,T]}, \end{aligned} \quad (35)$$

где

$$\begin{aligned} \rho_{21} = & \max_{0 \leq t \leq T} \left\{ \bar{L}_{31} | A_1^{-1}(t) | + L_{31} | [A_1^{-1}(t)]' | + L_{32} | A_2^{-1}(t) | \right\} + \\ & + \max_{0 \leq t \leq T} \int_0^t \left[ L_1(s) + L_2(s) \int_0^T | \Theta(s, \theta) | d\theta \right] ds + \sum_{i=1}^p (L_{4,i} + L_{5,i}), \end{aligned} \quad (36)$$

$$\rho_{22} = 2(h_1 + h_2) \max_{0 \leq t \leq T} \int_0^t L_2(s) \int_0^T | \Theta(s, \theta) | d\theta, \quad h_\kappa = \max_{0 \leq t \leq T} \lambda_\kappa(t). \quad (37)$$

Складывая оценок (32) и (35), получаем

$$\begin{aligned} & \|x^{k+1}(t) - x^k(t)\|_{PC[0,T]} + \|x'^{k+1}(t) - x'^k(t)\|_{PC[0,T]} \leq \\ & \leq \rho \cdot \left[ \|x^k(t) - x^{k-1}(t)\|_{PC[0,T]} + \|x'^k(t) - x'^{k-1}(t)\|_{PC[0,T]} \right], \end{aligned} \quad (38)$$

где

$$\rho = \max \{ \rho_{1,1} + \rho_{1,2}; \rho_{2,1} + \rho_{2,2} \}.$$

Согласно последнему условию теоремы, имеем  $\rho < 1$ . Поэтому, из оценки (38) следует что

$$\|x^{k+1}(t) - x^k(t)\|_{PC[0,T]} < \|x^k(t) - x^{k-1}(t)\|_{PC[0,T]}. \quad (39)$$

Из оценки (39) следует что оператор  $J_1$  в правой части уравнения (24) является сжимающим. Согласно принципу неподвижной точки в Банаховом пространстве

$PC([0, T], \mathbb{R}^n)$  и с учетом оценок (29), (30), заключаем что оператор  $J_1$  имеет единственную неподвижную точку. Следовательно, нелинейная начальная задача (15)-(19) имеет единственное решение  $x(t) \in PC([0, T], \mathbb{R}^n)$ .

Теперь покажем непрерывную зависимость решения задачи (15)-(19) от правой части условия (16). Пусть  $B_{11}, B_{12} \in \mathbb{R}^n$  и  $B_{21}, B_{22}$  разные четыре вектора и  $x_1(t), x_2(t) \in PC([0, T], \mathbb{R}^n)$  соответствующие решения задачи (15)-(19). Пусть  $|B_{11} - B_{12}| < \delta_1$ , и  $|B_{21} - B_{22}| < \delta_2$ , где  $0 < \delta_k$  ( $k = 1, 2$ ) малые числа, зависящие от заданного малого положительного числа  $\varepsilon$ . Тогда из уравнения (24) получаем

$$\begin{aligned}
& \|x_1(t) - x_2(t)\|_{PC[0,T]} \leq \\
& \leq |B_{11} - B_{12}| \max_{0 \leq t \leq T} |[A_1(t)]^{-1}| + |B_{21} - B_{22}| \max_{0 \leq t \leq T} \int_0^t |[A_2(s)]^{-1}| ds + \\
& + \|x_1(t) - x_2(t)\|_{PC[0,T]} \max_{0 \leq t \leq T} \left[ L_{31}(t) |[A_1(t)]^{-1}| + \int_0^t L_{32}(s) |[A_2(s)]^{-1}| ds \right] + \\
& + \max_{0 \leq t \leq T} \int_0^t (t-s) [L_1(s) \|x_1(s) - x_2(s)\| + \\
& + \int_0^T L_2(s) |\Theta(s, \theta)| \|x_1(\tau) - x_2(\tau)\| d\theta] ds + \sum_{0 < t_i < T} [L_{4,i}T + L_{5,i}] |x_1(t_i) - x_2(t_i)| \leq \\
& \leq |B_{11} - B_{12}| \max_{0 \leq t \leq T} |A_1^{-1}(t)| + |B_{21} - B_{22}| \max_{0 \leq t \leq T} \int_0^t |A_2^{-1}(s)| ds + \\
& + \rho_{11} \|x_1(t) - x_2(t)\|_{PC[0,T]} + \rho_{12} \|x'_1(t) - x'_2(t)\|_{PC[0,T]}. \tag{40}
\end{aligned}$$

Также из (26) получаем, что

$$\begin{aligned}
& |x'_1(t) - x'_2(t)| \leq |B_{11} - B_{12}| \left| [A_1^{-1}(t)]' \right| + |B_{21} - B_{22}| |A_2^{-1}(t)| + \\
& + \left[ \bar{L}_{31} |A_1^{-1}(t)| + L_{31} \left| [A_1^{-1}(t)]' \right| + L_{32} |A_2^{-1}(t)| \right] + \\
& + \int_0^t \left[ L_1(s) |x_1(s) - x_2(s)| + L_2(s) \int_0^T |\Theta(s, \theta)| |x_1(\theta) - x_2(\theta)| d\theta \right] ds + \\
& + 2(h_1 + h_2) \int_0^t L_2(s) \int_0^T |\Theta(s, \theta)| |x'_1(\theta) - x'_2(\theta)| d\theta ds + \\
& + \sum_{i=1}^p (L_{4,i} + L_{5,i}) |x_1(t) - x_2(t)|.
\end{aligned}$$

Отсюда, переходя к норме, получаем, что

$$\begin{aligned}
& \|x'_1(t) - x'_2(t)\|_{PC[0,T]} \leq |B_{11} - B_{12}| \max_{0 \leq t \leq T} \left| [A_1^{-1}(t)]' \right| + |B_{21} - B_{22}| \max_{0 \leq t \leq T} |A_2^{-1}(t)| + \\
& + \rho_{21} \|x_1(t) - x_2(t)\|_{PC[0,T]} + \rho_{22} \|x'_1(t) - x'_2(t)\|_{PC[0,T]}. \tag{41}
\end{aligned}$$

Из (40) и (41) получаем, что справедлива оценка

$$\begin{aligned} & \|x_1(t) - x_2(t)\|_{PC[0,T]} + \|x'_1(t) - x'_2(t)\|_{PC[0,T]} < \\ & < \delta_1\alpha_1 + \delta_2\alpha_2 + \rho \cdot \left[ \|x_1(t) - x_2(t)\|_{PC[0,T]} + \|x'_1(t) - x'_2(t)\|_{PC[0,T]} \right], \end{aligned} \quad (42)$$

где

$$\alpha_1 = \max_{0 \leq t \leq T} |A_1^{-1}(t)|, \quad \alpha_2 = \max_{0 \leq t \leq T} |A_2^{-1}(t)| + \max_{0 \leq t \leq T} \int_0^t |A_2^{-1}(s)| ds.$$

Поскольку  $\rho < 1$ , то из последнего неравенства (42) следует, что

$$\|x_1(t) - x_2(t)\|_{PC[0,T]} < (\delta_1\alpha_1 + \delta_2\alpha_2)(1 - \rho)^{-1}.$$

Отсюда получаем

$$\|x_1(t) - x_2(t)\|_{PC[0,T]} < \varepsilon,$$

где  $\varepsilon = \delta(\alpha_1 + \alpha_2)(1 - \rho)^{-1}$ ,  $\delta = \max\{\delta_1; \delta_2\}$ . Теорема доказана.

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## REZYUME

Impuls ta'sirli va maksimumi bo'lgan ikkinchi tartibli integro-differensial tenglamalar sistemasi uchun boshlang'ich masala o'rganilgan. Chiziqsiz boshlang'ich masala yechimining mavjudligi va yagonaligi  $PC([0, T], \mathbb{R}^n)$  Banax fazosida chiziqli bo'lmagan funktsional integral tenglamalar sistemasini yechishga keltiriladi. Chiziqli bo'lmagan funktsional integral tenglamalarning yechimi mavjudligi va yagonaligini isbotlash uchun qisqartirib akslantirish prinsipi bilan ketma-ket yaqinlashish usuli

birga qo'llaniladi.

**Kalit so'zlar:** boshlang'ich masala, impulsiv ta'sir, integro-differensial tenglama, Banach fazosi, aralash maksimal, chiziqsiz funksional, qisqartirib akslantirish, ketma-ket yaqinlashish.

### RESUME

An initial problem for a second order system of ordinary integro-differential equations with impulsive effects and maxima investigated. The existence and uniqueness of the solution of the nonlinear initial problem is reduced to the unique solvability of the system of nonlinear functional integral equations in Banach space  $PC([0, T], \mathbb{R}^n)$ . The method of successive approximations in combination with the method of compressing mapping is used in the proof of unique solvability of the nonlinear functional integral equations.

**Key words:** initial problem, impulsive effect, integro-differential equation, Banach space, maxima, nonlinear, method of compressing mapping, successive approximation.

УДК 517.95

## СХОДИМОСТЬ РЕКУРРЕНТНЫХ НЕЙРОННЫХ СЕТЕЙ МНОЖЕСТВЕННЫМИ ВЕСОВЫМИ СВЯЗАМИ СЕТИ

Худайбергенов К. К. \*

### РЕЗЮМЕ

Рекуррентные нейронные сети широко используются для анализа и прогнозирования временных последовательностей. В данной статье рассматривается метод градиентного обучения в режиме онлайн для рекуррентных нейронных сетей с множественными весовыми связями и доказывается его сходимость. В отличие от большинства результатов сходимости, имеющих вероятностную природу в предположении, что доступно большое количество обучающих выборок временных рядов, теорема о сходимости, которую строится здесь, имеет детерминированную природу и основана на предпосылке, что предоставлено лишь ограниченное количество обучающих выборок. Монотонность функции ошибок во время обучающей итерации также гарантируется.

**Ключевые слова:** Рекуррентные нейронные сети, метод онлайн-градиента, монотонность, сходимость.

**Введение.** Рекуррентные нейронные сети (RNN) - это разновидность сетей, имеющих одну или несколько обратных связей. Эти механизмы обратной связи заставляют RNN учиться распознавать и генерировать не только временные паттерны, но и пространственные паттерны ([2]). Было исследовано несколько алгоритмов обучения для RNN, многие из которых основаны на вычислении градиента меры ошибки в весовом пространстве, которые называются градиентными методами. Так же, как и нейронные сети прямого распространения, градиентные методы используются для обучения RNN ([3], [4], [11]) либо в автономном (пакетном), либо в онлайн (инкрементном) режиме. При пакетном обучении изменения веса накапливаются в течение всего представления обучающих данных перед применением, в то время как онлайн-обучение обновляет веса после представления каждого обучающего примера. Глобальный обзор алгоритмов обратного распространения для динамических сетей обобщен в Jesus and Hagan [8]. Чтобы лучше понять поведение сетей, многие исследователи уделили внимание теоретическому анализу сходимости обучающих алгоритмов для RNN. Ку и Ли ([10]) продемонстрировали сходимость Ляпунова онлайн-алгоритма градиентного обучения для диагональной RNN с бесконечным числом обучающих образцов. Некоторые результаты сходимости вероятностного характера для онлайн-метода градиента были установлены Куаном и др. ([6]) с помощью теории стохастических процессов, исходя из предпосылки, что доступно очень большое число обучающих образцов временных последовательностей. Детерминированные сходимости офлайн-алгоритма градиентного спуска для RNN только с конечным числом обучающих образцов были проанализированы в [5] и [9]. Целью данной статьи является исследование детерминированной сходимости онлайн-градиента для RNN, основанной на

\*Худайбергенов К. К. – Ташкентский международный университет Кимё, kabul85@mail.ru

предположении, что доступно только ограниченное число обучающих временных последовательностей. Для простоты сосредоточим внимание на простом рекуррентном нейроне. Доказывается некоторые результаты сходимости детерминированного характера при соответствующих условиях. Монотонность функции ошибки в обучающей итерации также гарантируется. Оставшаяся часть данной статьи организована следующим образом. Архитектура сети и онлайн-алгоритм градиента описаны в следующем разделе. Формула онлайн-обучения для рекуррентных нейронных сетей с множественными весами связей. Рассматривается рекуррентную нейронную сеть с несколькими весами связей, состоящую из  $P$  внешних входных узлов и 1 выходного узла. Обозначим вектор веса сети как  $\mathbf{w}^{(h)} = (w_0^{(h)}, w_1^{(h)}, \dots, w_P^{(h)})^T \in R^{P+1}$ ,  $h = 1, 2, \dots, H$ .  $H$  - размерность весов соединений. Пусть  $\xi^j = (\xi_1^j, \xi_2^j, \dots, \xi_P^j) \in R^P$  будут внешними входными сигналами в момент времени  $j$  ( $1 \leq j \leq J$ ), а  $\zeta^j \in R$  будут выходными сигналами в момент времени  $j$ . Для удобства объединяем  $\zeta^{j-1}$  и  $\xi^j$  для формирования  $(P+1)$ -размерного вектора  $u^j(w)$  следующим образом:

$$u^j(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)}) = \begin{cases} \zeta^{j-1}(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)}), & i = 0 \\ \xi_i^j, & i = 1, 2, \dots, P \end{cases} \quad (43)$$

$$S^j(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)}, \cdot) = w \cdot u^j(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)}) = \sum_{h=1}^H \sum_{i=1}^P \mathbf{w}_i^{(h)} \xi_i^j + w_0^{(h)} \zeta^{j-1}(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)}) \quad (44)$$

быть входом в выходной узел в момент времени. Тогда выходы сети будут

$$\zeta^j(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)}) = g(S^j(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)})), \quad j = 1, 2, \dots, J \quad (45)$$

где  $g$  - заданная функция активации. Начальное условие -

$$\zeta^0(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)}) = 0 \quad (46)$$

Теперь опишем алгоритм обучения для этой сети. Пусть  $\{O^j, \xi^j\}_{j=1}^J \subset R^P \times R$  будет заданным набором обучающих образцов. Обычная функция среднеквадратической ошибки для сети:

$$E(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)}) = \frac{1}{2} \sum_{j=1}^J (O^j - g(S^j(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)})))^2 \equiv \sum_{j=1}^J g_j(S^j(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)})) \quad (47)$$

где  $g_j(S^j(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)})) = \frac{1}{2}(O^j - g(S^j(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)})))^2$ , то его градиент относительно  $\mathbf{w}$  определяется выражением

$$\nabla E(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)}) = \sum_{j=1}^J g'_j(S^j(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)})) \cdot (u^j(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)}) + w_0 p^{j-1}(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)})) \quad (48)$$



где

$$p^j(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)}) = \frac{d\zeta^j(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)})}{dw}$$

$$= g'_j(S^j(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)})) \cdot (u^j(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)}) + w_0^m p^{j-1}(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)})) \quad (49)$$

Начальное условие:

$$p^0(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \dots, \mathbf{w}^{(H)}) = 0$$

Алгоритм градиента онлайн является вариацией стандартного метода градиента, в котором веса сети обновляются немедленно после обработки каждого входного обучающего примера. Теперь для произвольного начального значения  $w^0$  алгоритм градиента онлайн итеративно обновляет веса таким образом, что

$$w^{(h),mJ+j} = w^{(h),mJ+j-1} + \Delta_j w^{(h),mJ+j-1}, \quad (50)$$

$$\Delta_j w^{(h),mJ+j-1} =$$

$$-\eta_m g'_j(S^j(w^{(h),mJ+j-1})) \cdot \left( u^j(w^{(h),mJ+j-1}) + w_0^{(h),mJ+j-1} p^{j-1}(w^{(h),mJ+j-1}) \right) \quad (51)$$

где  $\eta_m$  - скорость обучения в  $m$ -ом цикле обучения,  $j = 1, 2, \dots, J$ ;  $m = 0, 1, \dots$ . Для начального значения  $\eta_0 > 0$ ,  $\eta_m$  изменяется после каждого цикла итерации обучения в соответствии с

$$\frac{1}{\eta_m} = \frac{1}{\eta_{m-1}} + 1, \quad m = 0, 1, 2, \dots \quad (52)$$

Здесь  $l$  константа, значение которой будет зафиксировано позже. По (52,) легко видеть, что

$$\eta_m = \frac{\eta_0}{1 + m\eta_0}. \quad (53)$$

**2. Основной результат.** В данной статье используем для общей положительной константы, которая может быть разной в разных местах, но не зависит от  $j, m, k$ .

В статье приняты следующие допущения.

(A2.1)  $|g(t)|$ ,  $|g'(t)|$  и  $|g''(t)|$  равномерно ограничены для всех  $t \in R$ ;

(A2.2)  $\{w_0^{(h),m}\}$  ( $m = 0, 1, 2, \dots$ ) ограничены;

(A2.3)  $\|\nabla^2 E(\mathbf{w})\| \leq C$ ;

(A2.4)  $\|\nabla p(w)\| \leq C$ ;

**Замечание:** Общим выбором для  $g$  является сигмоидальная функция типа  $g(x) = 1/(1 + e^{-x})$ . Таким образом,  $g$  равномерно ограничена на  $R$ , а также ее производные. Заметим, что из (47) и Предположение (A2.1), что  $|g(t)|$ ,  $|g'(t)|$  и  $|g''(t)|$  также равномерно ограничены для любого  $t \in R$ .

**Теорема 2.1.** Предположим, что (A2.1)-(A2.4) верны, начальное значение  $\eta_0$  выбрано так, чтобы удовлетворялось (4.28). Если вес  $\{w^{(h),mJ}\}$  генерируется алгоритмом (50) и (51), то для  $m = 0, 1, 2, \dots$ , имеется

(1)  $E(w^{(h),(m+1)J}) \leq E(w^{(h),mJ})$ ;

(2)  $\lim_{k \rightarrow \infty} \|\nabla E(w^{(h),k})\| = 0$ ;

(3) Кроме того, если существует замкнутая ограниченная область,  $D \subset R^{P+1}$  удовлетворяющая  $\{w^{(h),m}\}_{m=0}^{\infty} \subset D$ , и множество  $D_0 = \{w \in D | \nabla E(w) = 0\}$  содержит только конечные точки. Тогда существует  $w^* \in D_0$ , такое, что  $\lim_{m \rightarrow \infty} w^{(h),m} = w^*$ .

**3. Доказательства** Сначала нам нужно пройти через несколько обозначений и лемм, прежде чем доказывать основную теорему. Для простоты введем следующие обозначения:

$$w_d^{(h),m} = w^{(h),(m+1)J} - w^{(h),mJ}, \quad S_d^{j,m} = S^j(w^{(h),(m+1)J}) - S^j(w^{(h),mJ}), \quad (54)$$

$$r_{j,m} = \Delta_j w^{(h),mJ+j-1} - \Delta_j w^{(h),mJ}, \quad \Delta_k S_{j,m} = S^k(w^{(h),mJ+j-1}) - S^k(w^{(h),mJ}). \quad (55)$$

$j = 1, 2, \dots, J; k = 1, 2, \dots, j; m = 0, 1, 2, \dots$

Первая лемма показывает свойства переменной ставки. Ее доказательство аналогично лемме 2.1 в [7] и поэтому опускается.

**Лемма 3.1.** Пусть  $\{\eta_m\}$  задано (52), тогда для любого заданного  $\eta_0 > 0$ , справедливы следующие оценки для любого  $m = 1, 2, \dots$

- (1)  $\eta_{m-1} > \eta_m > 0$ ;
- (2)  $\eta_m < \frac{\rho}{m}$ ,  $\rho = \frac{1}{l}$ ;
- (3)  $\eta_m < \frac{\tau}{m}$ , где  $\tau$  - некоторая положительная константа;
- (4)  $\frac{\eta_{m+1}}{\eta_m} > \frac{1}{2}$ .

Следующая лемма может быть доказана непосредственно с помощью индукционных рассуждений, и ее доказательство также опускается.

**Лемма 3.2.** Предположим, что (48-51) удовлетворены, тогда

$$p^k(w^{(h),mJ+j}) = \sum_{i=0}^{k-1} \left( \prod_{l=j-1}^k g'(S^l(w^{(h),mJ+j})) \right) \cdot (w_0^{(h),mJ+j}) \cdot u^{k-i}(w^{(h),mJ+j})$$

Из леммы 3.2 и предположения (A2.1), (A2.2) можно получить, что существует константа  $C > 0$ , и для любого  $k, j, m \in R^+$ , такого, что

$$\|p^k(w^{(h),mJ+j})\| \leq C \quad (56)$$

**Лемма 3.3.** Если предположения (A2.1)-(A2.2) верны, то существует  $C > 0$  такое, что

$$|\Delta_k S_{j,m}|^2 \leq C \|w^{(h),mJ+j-1} - w^{(h),mJ+j}\|^2 \quad (57)$$

И имеет место следующая формула

$$\Delta_k S_{j,m} = (w^{(h),mJ+j-1} - w^{(h),mJ+j}) \left( u^k(w^{(h),mJ}) + w_0^{(h),mJ} p^{k-1}(w^{(h),mJ}) \right) + \delta_{1j,m}^k + \delta_{2j,m}^k \quad (58)$$

где

$$\delta_{1j,m}^k = (w^{(h),mJ+j-1} - w^{(h),mJ}) \times$$

$$\sum_{i=1}^{k-1} \left( \prod_{l=k-i}^{k-1} g'(S^l(w^{(h),mJ})) \right) \cdot \left( (w_0^{(h),mJ+j-1})^i - (w_0^{(h),mJ})^i \right) u^{k-i}(w^{(h),mJ}), \quad (59)$$

$$\delta_{2j,m}^k = \frac{1}{2} \sum_{i=1}^{k-1} \left( \prod_{l=k-i+1}^{k-1} g'(S^l(w^{(h),mJ})) \right) \cdot (w_0^{(h),mJ+1})^i g''(\theta_{2j,m}^{k-i}) (\Delta_{k-1} S_{j,m})^2, \quad (60)$$

и  $\theta_{2j,m}^{k-i}$  это действительное число между  $S^{k-1}(w^{(h),mJ+j-1})$  и  $S^{k-1}(w^{(h),mJ})$ . Более того, у нас есть оценка

$$\delta_{1j,m}^k \leq |\delta_{1j,m}^k| \leq C \|w^{(h),mJ+j-1} - w^{(h),mJ}\|^2, \quad (61)$$

$$\delta_{2j,m}^k \leq |\delta_{2j,m}^k| \leq C \|w^{(h),mJ+j-1} - w^{(h),mJ}\|^2, \quad (62)$$

Особенно

$$|S_d^{j,m}|^2 \leq C \|w_d^{(h),m}\|^2 \quad (63)$$

$$S_d^{j,m} = w_d^{(h),m} \left( u^j(w^{(h),mJ}) + w_0^{(h),mJ} p^{j-1}(w^{(h),mJ}) \right) + \delta_{1j,m}^j + \delta_{2j,m}^j \quad (64)$$

И

$$\delta_{1j,m}^j \leq |\delta_{1j,m}^j| \leq C \|w_d^{(h),m}\|^2, \quad \delta_{2j,m}^j \leq |\delta_{2j,m}^j| \leq C \|w_d^{(h),m}\|^2 \quad (65)$$

**Доказательство.** Из (55), имеем

$$\begin{aligned} \Delta_k S_{j,m} &= S^k(w^{(h),mJ+j-1}) - S^k(w^{(h),mJ}) = \\ &= w^{(h),mJ+j-1} u^k(w^{(h),mJ+j-1}) - w^{(h),mJ} u^k(w^{(h),mJ}) \\ &= (w^{(h),mJ+j-1} - w^{(h),mJ}) \cdot u^k(w^{(h),mJ}) + w^{(h),mJ+j-1} \cdot (u^k(w^{(h),mJ+j-1}) - u^k(w^{(h),mJ})) \end{aligned} \quad (66)$$

$$\begin{aligned} &= (w^{(h),mJ+j-1} - w^{(h),mJ}) \times \\ &\cdot u^k(w^{(h),mJ}) + w_0^{(h),mJ+j-1} \cdot (\zeta^{k-1}(w^{(h),mJ+j-1}) - \zeta^{k-1}(w^{(h),mJ})) \end{aligned} \quad (67)$$

$$\begin{aligned} &= (w^{(h),mJ+j-1} - w^{(h),mJ}) \times \\ &\cdot u^k(w^{(h),mJ}) + w_0^{(h),mJ+j-1} \cdot (g(S^{k-1}(w^{(h),mJ+j-1})) - g(S^{k-1}(w^{(h),mJ}))) \end{aligned} \quad (68)$$

Использование разложения Тейлора первого порядка для (68) демонстрации

$$\Delta_k S_{j,m} = (w^{(h),mJ+j-1} - w^{(h),mJ}) \cdot u^k(w^{(h),mJ}) + w_0^{(h),mJ+j-1} g'(\theta_{1j,m}^{k-1}) \Delta_{k-1} S_{j,m} \quad (69)$$

где  $\theta_{1j,m}^{k-1}$  лежит между  $S^{k-1}(w^{(h),mJ+j-1})$  и  $S^k(w^{(h),mJ})$ . Из (A2.1), (A2.2) и (69), существует константа  $C > 0$ , такая, что

$$\begin{aligned} \|\Delta_k S_{j,m}\|^2 &= \left( (w^{(h),mJ+j-1} - w^{(h),mJ}) \cdot u^k(w^{(h),mJ}) + w_0^{(h),mJ+j-1} g'(\theta_{1j,m}^{k-1}) \Delta_{k-1} S_{j,m} \right)^2 \\ &\leq 2 \left| (w^{(h),mJ+j-1} - w^{(h),mJ}) \cdot u^k(w^{(h),mJ}) \right|^2 + 2 \left| w_0^{(h),mJ+j-1} g'(\theta_{1j,m}^{k-1}) \Delta_{k-1} S_{j,m} \right|^2 \\ &\leq C \|w^{(h),mJ+j-1} - w^{(h),mJ}\|^2 + C |\Delta_{k-1} S_{j,m}|^2 \end{aligned} \quad (70)$$

Заметим, что (70) это рекурсивное неравенство, имеем

$$\|\Delta_k S_{j,m}\|^2 =$$

$$(C + C^2 + \dots + C^{k-1}) \|w^{(h),mJ+j-1} - w^{(h),mJ}\|^2 \leq C \|w^{(h),mJ+j-1} - w^{(h),mJ}\|^2 \quad (71)$$

Итак, неравенство (57) обосновано, и аналогичным образом можем получить (63). Теперь докажем (58) и (60). Применяя разложение Тейлора второго порядка к (68), имеем

$$\Delta_k S_{j,m} = (w^{(h),mJ+j-1} - w^{(h),mJ}) \cdot u^k(w^{(h),mJ}) + \quad (72)$$

$$w_0^{(h),mJ+j-1} g'(S^{k-1}(w^{(h),mJ})) \Delta_{k-1} S_{j,m} + \frac{1}{2} w_0^{(h),mJ+j-1} g''(\theta_{2j,m}^{k-1}) (\Delta_{k-1} S_{j,m})^2, \quad (73)$$

где  $\theta_{2j,m}^{k-1}$  лежит на отрезке между  $S^{k-1}(w^{(h),mJ+j-1})$  и  $S^{k-1}(w^{(h),mJ})$ . Очевидно, это рекурсивная формула для  $\Delta_{k-1} S_{j,m}$ , поэтому получаем

$$\begin{aligned} \Delta_k S_{j,m} &= \\ & (w^{(h),mJ+j-1} - w^{(h),mJ}) \cdot \left[ u^k(w^{(h),mJ}) + w_0^{(h),mJ+j-1} g'(S^{k-1}(w^{(h),mJ})) u^k(w^{(h),mJ}) + \dots + \right. \\ & \left. \left( w_0^{(h),mJ+j-1} \right)^{k-1} g'(S^{k-1}(w^{(h),mJ})) \dots g'(S^1(w^{(h),mJ})) u^1(w^{(h),mJ}) \right] + \\ & \frac{1}{2} w_0^{(h),mJ+j-1} g''(\theta_{2j,m}^{k-1}) (\Delta_{k-1} S_{j,m})^2 + \\ & \frac{1}{2} \left( w_0^{(h),mJ+j-1} \right)^2 g'(S^{k-1}(w^{(h),mJ})) g''(\theta_{2j,m}^{k-2}) (\Delta_{k-1} S_{j,m})^2 + \dots + \\ & \frac{1}{2} w_0^{(h),mJ+j-1} g'(S^{k-1}(w^{(h),mJ})) \dots g'(S^2(w^{(h),mJ})) g''(\theta_{2j,m}^1) (\Delta_1 S_{j,m})^2 \\ & = (w^{(h),mJ+j-1} - w^{(h),mJ}) \left[ u^k(w^{(h),mJ}) + w_0^{(h),mJ} g'(S^{k-1}(w^{(h),mJ})) u^{k-1}(w^{(h),mJ}) + \dots \right. \\ & \left. + \left( w_0^{(h),mJ} \right)^{k-1} g'(S^{k-1}(w^{(h),mJ})) \dots g'(S^1(w^{(h),mJ})) u^1(w^{(h),mJ}) \right] + \\ & = (w^{(h),mJ+j-1} - w^{(h),mJ}) \left[ \left( w_0^{(h),mJ+j-1} - w_0^{(h),mJ} \right) g'(S^{k-1}(w^{(h),mJ})) u^{k-1}(w^{(h),mJ}) + \dots + \right. \\ & = \left( \left( w_0^{(h),mJ+j-1} \right)^{k-1} - \left( w_0^{(h),mJ} \right)^{k-1} \right) g'(S^{k-1}(w^{(h),mJ})) \dots g'(S^1(w^{(h),mJ})) u^1(w^{(h),mJ}) \left. \right] + \\ & \frac{1}{2} \sum_{i=1}^{k-1} \left( \prod_{l=k-i+1}^{k-1} g'(S^l(w^{(h),mJ})) \right) \cdot \left( w_0^{(m+1)J} \right)^i \cdot g''(\theta_{2j,m}^{k-i}) (\Delta_{k-1} S_{j,m})^2 \\ & = (w^{(h),mJ+j-1} - w^{(h),mJ}) \left[ u^k(w^{(h),mJ}) + w_0^{(h),mJ} p^{k-1}(w^{(h),mJ}) \right] + \delta_{1j,m}^k + \delta_{2j,m}^k \quad (74) \end{aligned}$$

Используя (A2.1), (A2.2) и объединяя (59), (60) и (57), получаем

$$\delta_{2j,m}^k \leq |\delta_{2j,m}^k| \leq C |\Delta_{k-1} S_{j,m}|^2 \leq C \|w^{(h),mJ+j-1} - w^{(h),mJ}\|^2 \quad (75)$$

Для  $\delta_{1j,m}^k$ , применяя формулу  $x^i - y^i = (x - y)(x^{i-1} + x^{i-2}y + \dots + y^i)$ , и учитывая (A2.2), имеем

$$\delta_{1j,m}^k \leq C \left| w_0^{(h),mJ+j-1} - w_0^{(h),mJ} \right| \cdot \|w^{(h),mJ+j-1} - w^{(h),mJ}\| \leq C \|w^{(h),mJ+j-1} - w^{(h),mJ}\|^2 \quad (76)$$

Таким же образом можно доказать (64) и (65).  $\square$

**Лемма 3.4.** Если предположение (A2.1) выполняется, то существует константа  $C > 0$  такая, что

$$(1) \quad w^{(h),mJ+j-1} = w^{(h),mJ} + \sum_{k=1}^j (\Delta_k w^{(h),mJ} + r_{k,m}), \quad j = 1, 2, \dots, J;$$

$$(2) \quad \sum_{k=1}^J \|r_{k,m}\| \leq C \eta_m \sum_{k=1}^J \|\Delta_k w^{(h),mJ}\|;$$

$$(3) \quad \|w_d^m\| \leq C \sum_{k=1}^J \|\Delta_k w^{(h),mJ}\|.$$

**Доказательство.** Вывод (1) этой леммы можно вывести непосредственно из (59) и (60). В частности

$$w_d^m = \sum_{k=1}^J (\Delta_k w^{(h),mJ} + r_{k,m}) \quad (77)$$

Из (60) и (55), имеем

$$\begin{aligned} r_{j,m} &= \eta_m g'(S^j(w^{(h),mJ+j-1})) \cdot \left( u^j(w^{(h),mJ+j-1}) + w_0^{(h),mJ+j-1} p^{j-1}(w^{(h),mJ+j-1}) \right) \\ &\quad \eta_m g'(S^j(w^{(h),mJ+j-1})) \cdot \left( u^j(w^{(h),mJ}) + w_0^{(h),mJ} p^{j-1}(w^{(h),mJ}) \right) \\ &= \eta_m \left( g'_j(S^j(w^{(h),mJ+j-1})) u^j(w^{(h),mJ+j-1}) - g'_j(S^j(w^{(h),mJ})) u^j(w^{(h),mJ}) \right) \\ &= \eta_m \left( w_0^{(h),mJ+j-1} g'_j(S^j(w^{(h),mJ+j-1})) p^{j-1}(w^{(h),mJ+j-1}) - w_0^{(h),mJ} g'_j(S^j(w^{(h),mJ})) p^{j-1}(w^{(h),mJ}) \right) \\ &\triangleq \alpha_{1j,m} + \alpha_{2j,m} \end{aligned} \quad (78)$$

Далее дадим оценку около  $\|\alpha_{1j,m}\|$  и  $\|\alpha_{2j,m}\|$  соответственно. По теореме о среднем значении (A2.1) и (56) имеем

$$\begin{aligned} \|\alpha_{1j,m}\| &= \\ &g'_j(S^j(w^{(h),mJ})) u^j(w^{(h),mJ}) \\ &\leq C \eta_m \|\Delta_j S_{j,m}\| + C \eta_m \|\Delta_{j-1} S_{j,m}\| \end{aligned} \quad (79)$$

где  $\theta_{3j,m}^j$  лежит между  $S^j(w^{(h),mJ})$  и  $S^j(w^{(h),mJ+j-1})$ , а из (4.4) и (A2.2) получаем

$$\|\alpha_{1j,m}\| \leq C \eta_m |\Delta_j S_{j,m}| + C \eta_m |\Delta_{j-1} S_{j,m}| \leq C \eta_m \|w^{(h),mJ+j-1} - w^{(h),mJ}\| \quad (80)$$

С другой стороны, разложения Тейлора и комбинации (56) (4.4) A2.1), (A2.2) и (A2.4) приводит к

$$\begin{aligned} \|\alpha_{2j,m}\| &= \\ &\leq C \eta_m \|\Delta_j S_{j,m}\| + C \eta_m \left\| w_0^{(h),mJ+j-1} - w_0^{(h),mJ} \right\| + C \eta_m \|w^{(h),mJ+j-1} - w^{(h),mJ}\| \end{aligned}$$

$$\leq C\eta_m \|w^{(h),mJ+j-1} - w^{(h),mJ}\| \quad (81)$$

где  $w^{(h),mJ}$  лежит между  $w^{(h),mJ+j-1}$  и  $w^{(h),mJ}$ .

Комбинация (36), (39) и (40) леммы 3.4(1) дает

$$\|r_{j,m}\| \leq C\eta_m \|w^{(h),mJ+j-1} - w^{(h),mJ}\| =$$

$$C\eta_m \left\| \sum_{k=1}^{j-1} (\Delta_k w^{(h),mJ} + r_{k,m}) \right\| \leq C\eta_m \left( \sum_{k=1}^{j-1} \|\Delta_k w^{(h),mJ}\| + \sum_{k=1}^{j-1} \|r_{k,m}\| \right) \quad (82)$$

С помощью индукции по  $j$ , используя предыдущую оценку для  $\|r_{k,m}\|$ , получаем неравенство

$$\|r_{k,m}\| \leq C\eta_m \sum_{k=1}^{j-1} \|\Delta_k w^{(h),mJ}\| \quad (83)$$

Ожидаемый вывод (2) этой леммы следует из суммирования (83) по  $j$ . Используя неравенство Коши-Шварца, имеем

$$\left( \sum_{j=1}^J \|\Delta_k w^{(h),mJ}\| \right)^2 \leq J \sum_{j=1}^J \|\Delta_k w^{(h),mJ}\|^2 \quad (84)$$

леммы 3.4(2) и леммы 3.1(1) следует (77), что

$$\|w_d^m\| \leq \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\| + \sum_{j=1}^J \|r^{(h),mJ}\| \leq (C + C\eta_m) \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\| \leq C_1 \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\| \quad (85)$$

Из (42) и (43) получаем

$$\|w_d^m\| \leq \left( C_1 \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\| \right)^2 \leq JC_1 \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\|^2 \leq C \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\|^2$$

Это означает, что вывод (3) имеет место.  $\square$

**Лемма 3.5.** Пусть выполняются предположения (A2.1) и (A2.2), а последовательность  $\{w^{(h),mJ+j}\}$  генерируется алгоритмом (50) и (51), тогда существует положительная константа,  $\gamma$  не зависящая от,  $m$ , такая, что

$$E(w^{(m+1)J}) \leq E(w^{(h),mJ}) - \frac{1}{\eta_m} \left\| \sum_{j=1}^J \Delta_j w^{(h),mJ} \right\|^2 + \gamma \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\|^2$$

**Доказательство.** Используя разложение Тейлора второго порядка и отметив (63), имеем

$$g_j(S^j(w^{(m+1)J})) = g_j(S^j(w^{(h),mJ})) + g'_j(S^j(w^{(h),mJ})) \cdot S_d^{j,m} + \frac{1}{2} g''_j(\theta_{2J,m}^j)(S_d^{j,m})^2 = g_j(S^j(w^{(h),mJ})) +$$

$$g'_j(S^j(w^{(h),mJ})) \cdot \left( w_d^m \left[ u^j(w^{(h),mJ}) + w_0^{(h),mJ} p^{j-1}(w^{(h),mJ}) \right] + \delta_{1J,m}^j + \delta_{2J,m}^j \right) +$$

$$\frac{1}{2} g''_j(\theta_{2J,m}^j)(S_d^{j,m})^2$$

Теперь из (50, 51) и (4.15) следует, что

$$\begin{aligned}
 g_j (S^j (w^{(m+1)J})) &= g_j (S^j (w^{(h),mJ})) - \frac{1}{\eta_m} \Delta_j w^{(h),mJ} \cdot \left( \sum_{k=1}^J (\Delta_k w^{(h),mJ} + r_{k,m}) \right) + \\
 &g_j' (S^j (w^{(h),mJ})) \cdot [\delta_{1J,m}^j + \delta_{2J,m}^j] + \frac{1}{2} g_j'' (\theta_{2J,m}^j) (S_d^{j,m})^2 \\
 \delta_{3J,m}^j &= \frac{1}{2} g_j'' (\theta_{2J,m}^j) (S_d^{j,m})^2
 \end{aligned} \tag{86}$$

Суммируя это уравнение,  $j = 1, 2, \dots, J$  получаем

$$\begin{aligned}
 E (w^{(m+1)J}) &= \\
 &E (w^{(h),mJ}) - \frac{1}{\eta_m} \left\| \sum_{j=1}^J \Delta_j w^{(h),mJ} \right\|^2 - \\
 &\frac{1}{\eta_m} \left( \sum_{j=1}^J \Delta_j w^{(h),mJ} \right) \cdot \left( \sum_{k=1}^J r_{k,m} \right) + \sum_{j=1}^J g_j' (S^j (w^{(h),mJ})) [\delta_{1J,m}^j + \delta_{2J,m}^j] + \sum_{j=1}^J \delta_{3J,m}^j \\
 &\triangleq E (w^{(h),mJ}) - \frac{1}{\eta_m} \left\| \sum_{j=1}^J \Delta_j w^{(h),mJ} \right\|^2 + \alpha_1 + \alpha_2 + \alpha_3 \\
 &\leq E (w^{(h),mJ}) - \frac{1}{\eta_m} \left\| \sum_{j=1}^J \Delta_j w^{(h),mJ} \right\|^2 + |\alpha_1| + |\alpha_2| + |\alpha_3|
 \end{aligned} \tag{87}$$

Теперь оценим  $\alpha_1, \alpha_2, \alpha_3$  соответственно. Во-первых, из Леммы 3.4 (2), Леммы 3.1 (1) и Неравенства Коши-Шварца имеем

$$|\alpha_1| \leq \frac{1}{\eta_m} \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\| \cdot \sum_{k=1}^J \|r_{k,m}\| \leq C \left( \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\| \right)^2 \leq CJ \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\|^2 \tag{88}$$

Далее, из леммы 3.3, леммы 3.4(3) и (A2.1) видим, что

$$|\alpha_2| \leq \sum_{j=1}^J |g_j' (S^j (w^{(h),mJ}))| \cdot [|\delta_{1J,m}^j| + |\delta_{2J,m}^j|] \leq \tag{89}$$

$$C \sum_{j=1}^J (C \|w_d^m\|^2 + C \|w_d^m\|^2) \leq CJ \|w_d^m\|^2 \tag{90}$$

$$\leq CJ \left( \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\| \right)^2 \leq CJ^2 \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\|^2 \leq C \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\|^2 \tag{91}$$

И по (45), (63), (A2.1) и лемме 3.4(3) имеем

$$|\alpha_3| = \left\| \sum_{j=1}^J \delta_{3J,m}^j \right\| = \frac{1}{2} \sum_{j=1}^J |g_j''(\theta_{2J,m}^j)| \cdot |S_d^{j,m}|^2 \leq C \sum_{j=1}^J \|w_d^m\|^2 \leq C \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\|^2 \quad (92)$$

Комбинация (48)-(50) выводит, что существует при условии

$$|\alpha_1| + |\alpha_2| + |\alpha_3| \leq \gamma \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\|^2$$

Эта оценка вместе с уравнением (44) означает, что лемма 3.5 верна.

**Лемма 3.6.** Предположим, что предположения (A2.3) и (A2.4) выполнены, и последовательность  $\{w^{(h),mJ+j}\}$  генерируется алгоритмом (50) и (51) для произвольного начального веса  $w^0 \in R^{P+1}$ , если начальное значение  $\eta_0$  выбрано так, чтобы удовлетворять

$$\frac{1}{\eta_0} \left\| \sum_{j=1}^J \Delta_j w^0 \right\|^2 \geq \gamma \sum_{j=1}^J \|\Delta_j w^0\|^2 \quad (93)$$

Тогда для  $m = 0, 1, 2, \dots$  имеем

$$\frac{1}{\eta_m} \left\| \sum_{j=1}^J \Delta_j w^{(h),mJ} \right\|^2 \geq \gamma \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\|^2 \quad (94)$$

**Доказательство.** Можно подтверждать что этот вывод методом математической индукции. Из (51)  $m = 0$  ясно, что (52) верно для . Предполагая, что (51) выполняется при  $m = 0$ , теперь переходим к подтверждению того, что (51) верно для  $m + 1$ , а именно

$$\frac{1}{\eta_{m+1}} \left\| \sum_{j=1}^J \Delta_j w^{(m+1)J} \right\|^2 \geq \gamma \sum_{j=1}^J \|\Delta_j w^{(m+1)J}\|^2 \quad (95)$$

По ((50) и (51) имеем

$$\begin{aligned} \Delta_j w^{(m+1)J} &= -\eta_{m+1} g_j'(S^j(w^{(m+1)J})) \cdot \left[ w^j(w^{(m+1)J}) + w_0^{(m+1)J} p^{j-1}(w^{(m+1)J}) \right] \\ &= \frac{\eta_{m+1}}{\eta_m} \Delta_j w^{(h),mJ} - \eta_{m+1} \left[ g_j'(S^j(w^{(m+1)J})) \cdot w^j(w^{(m+1)J}) - g_j'(S^j(w^{(h),mJ})) \cdot w^j(w^{(h),mJ}) \right] \\ &\quad + \eta_{m+1} \left[ w_0^{(m+1)J} g_j'(S^j(w^{(m+1)J})) p^{j-1}(w^{(m+1)J}) - w_0^{(h),mJ} g_j'(S^j(w^{(h),mJ})) p^{j-1}(w^{(h),mJ}) \right] \\ &\triangleq \frac{\eta_{m+1}}{\eta_m} \Delta_j w^{(h),mJ} + \alpha_{3j,m} + \alpha_{4j,m} \end{aligned} \quad (96)$$

Аналогично (17)–(19), и учитывая лемму 4.1(1), имеем

$$|\alpha_{3j,m}| \leq C \eta_{m+1} \|w_d^m\| \leq C \eta_m \|w_d^m\|, |\alpha_{4j,m}| \leq C \eta_{m+1} \|w_d^m\| \leq C \eta_m \|w_d^m\| \quad (97)$$



Комбинация (31), (32) и леммы 4.4(3) приводит к

$$\|\Delta_j w^{(m+1)J}\| \leq \quad (98)$$

$$\frac{\eta_{m+1}}{\eta_m} \|\Delta_j w^{(h),mJ}\| + C\eta_m \|w_d^m\| \leq \frac{\eta_{m+1}}{\eta_m} \|\Delta_j w^{(h),mJ}\| + C\eta_m \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\| \quad (99)$$

Возводя в квадрат и суммируя (33) по  $j = 1, 2, \dots, J$ , и объединяя (21), получаем

$$\begin{aligned} \sum_{j=1}^J \|\Delta_j w^{(m+1)J}\|^2 &\leq \frac{\eta_{m+1}^2}{\eta_m^2} \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\|^2 + \\ &2C\eta_m \left( \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\| \right) \cdot \left( \sum_{k=1}^J \|\Delta_k w^{(h),mJ}\| \right) + C^2 J \eta_m^2 \left( \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\| \right)^2 \\ &\leq \frac{\eta_{m+1}^2}{\eta_m^2} \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\|^2 + C\eta_m \left( \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\| \right)^2 \leq \left( \frac{\eta_{m+1}^2}{\eta_m^2} + C J \eta_m \right) \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\|^2 \end{aligned}$$

Это вместе с (29) дает

$$\sum_{j=1}^J \|\Delta_j w^{(m+1)J}\|^2 \leq \frac{1}{\eta_m \gamma} \left( \frac{\eta_{m+1}^2}{\eta_m^2} + C\eta_m \right) \left\| \sum_{j=1}^J \Delta_j w^{(h),mJ} \right\|^2 \quad (100)$$

С другой стороны, согласно (31), (32) и лемме 3.4(3) имеем

$$\begin{aligned} \left\| \sum_{j=1}^J \Delta_j w^{(m+1)J} \right\| &= \left\| \frac{\eta_{m+1}}{\eta_m} \sum_{j=1}^J \Delta_j w^{(h),mJ} + \sum_{j=1}^J \alpha_{3j,m} + \sum_{j=1}^J \alpha_{4j,m} \right\| \geq \\ \frac{\eta_{m+1}}{\eta_m} \left\| \sum_{j=1}^J \Delta_j w^{(h),mJ} \right\| - \left\| \sum_{j=1}^J \alpha_{3j,m} \right\| - \left\| \sum_{j=1}^J \alpha_{4j,m} \right\| &\geq \frac{\eta_{m+1}}{\eta_m} \left\| \sum_{j=1}^J \Delta_j w^{(h),mJ} \right\| - C\eta_m \|w_d^m\| \quad (101) \end{aligned}$$

Применяя неравенство Коши-Шварца и (29), выводим

$$\left( \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\| \right)^2 \leq J \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\|^2 \leq \frac{J}{\eta_m \gamma} \left\| \sum_{j=1}^J \Delta_j w^{(h),mJ} \right\|^2 \quad (102)$$

Кроме того, с помощью (15), леммы 4.4(2) и (36) получаем

$$\begin{aligned} \|w_d^m\| &\leq \left\| \sum_{j=1}^J \Delta_j w^{(h),mJ} \right\| + \left\| \sum_{j=1}^J r_{j,m} \right\| \\ &\leq \left\| \sum_{j=1}^J \Delta_j w^{(h),mJ} \right\| + C\eta_m \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\| \leq C \left\| \sum_{j=1}^J \Delta_j w^{(h),mJ} \right\| \quad (103) \end{aligned}$$

Из (35) и (37) следует, что

$$\left\| \sum_{j=1}^J \Delta_j w^{(m+1)J} \right\| \geq \frac{\eta_{m+1}}{\eta_m} \left\| \sum_{j=1}^J \Delta_j w^{(h),mJ} \right\| - C\eta_m \left\| \sum_{j=1}^J \Delta_j w^{(h),mJ} \right\|$$

Легко видеть, что для  $x, y, z > 0$  и  $x > y - z$  имеет место  $x^2 > y^2 - 2yz$ . Поэтому

$$\left\| \sum_{j=1}^J \Delta_j w^{(m+1)J} \right\|^2 \geq \left( \frac{\eta_{m+1}^2}{\eta_m^2} - C\eta_m \right) \left\| \sum_{j=1}^J \Delta_j w^{(h),mJ} \right\|^2 \quad (104)$$

Из (34) и (38) видно, что неравенство (4.30) выполняется, если

$$\frac{1}{\eta_{m+1}} \left( \frac{\eta_{m+1}^2}{\eta_m^2} - C\eta_{m+1} \right) \geq \frac{1}{\eta_m} \left( \frac{\eta_{m+1}^2}{\eta_m^2} + C\eta_m \right) \quad (105)$$

В силу (2.9), (4.39) эквивалентно

$$l \left( \frac{\eta_{m+1}}{\eta_m} \right)^2 \geq C \quad (106)$$

для подходящей константы  $C$ . Учитывая лемму 3.1(4), Можем выбрать константу  $L > 0$  так, чтобы неравенство (4.40) выполнялось при  $l > L$ .  $\square$

Следующие две леммы имеют решающее значение для результата сходимости. Лемма 3.7 похожа на лемму 2.2 в [7], а лемма 4.8 в основном совпадает с теоремой 14.1.5 в [1]. Поэтому их доказательства опускаются.

**Лемма 3.7.** Предположим, что ряд  $\sum_{n=1}^{\infty} \frac{a_n^2}{n} < \infty$ , что  $a_n > 0$  для  $n = 1, 2, \dots$ , и что существует константа,  $\mu > 0$  удовлетворяющая

$$|a_{n+1} - a_n| < \frac{\mu}{n} (n = 1, 2, \dots).$$

Тогда имеем

$$\lim_{n \rightarrow \infty} a_n = 0.$$

**Лемма 3.8.** Пусть  $F : R^m \rightarrow R^n (m, n \geq 1)$  - непрерывная и дифференцируемая функция на ограниченной замкнутой области  $\Omega \subset R^m$ , и  $\Omega_0 = \{z \in \Omega \mid F(z) = 0\}$  содержит только конечные точки. Если последовательность  $\{z^k\}_{k=1}^{\infty} \subset \Omega$  удовлетворяет

$$\lim_{k \rightarrow \infty} \|z^{k+1} - z^k\| = 0, \quad \lim_{k \rightarrow \infty} \|F(z^k)\| = 0$$

Тогда существует

$$z^* \in \Omega_0,$$

такой что

$$\lim_{k \rightarrow \infty} z^k = z^*$$

Теперь осталось доказать основную теорему о монотонности последовательности ошибок и сходимости  $\{w^m\}$ . Доказательство теоремы 3.1. Вывод (1) является прямым следствием леммы 3.5 и леммы 3.6. По индукции, основанной на лемме 3.5, имеем

$$E(w^{(m+1)J}) \leq \dots \leq E(w^J) - \sum_{k=1}^m \frac{1}{\eta_k} \left\| \sum_{j=1}^J \Delta_j w^{(h),mJ} \right\|^2 + \sum_{k=1}^m \gamma \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\|^2$$

Для  $E(w^{(m+1)J}) \geq 0$ , пусть  $m \rightarrow \infty$ , получим

$$\sum_{m=1}^{\infty} \left( \frac{1}{\eta_m} \left\| \sum_{j=1}^J \Delta_j w^{(h),mJ} \right\|^2 \right) - \sum_{m=1}^{\infty} \left( \gamma \sum_{j=1}^J \|\Delta_j w^{(h),mJ}\|^2 \right) \leq E(w^J) < \infty \quad (107)$$

Комбинация (2.8), (A3.1), (A3.2), (15) и леммы 4.1(2) приводит к

$$\|\Delta_j w^{(h),mJ+j}\| \leq$$

$$\eta_m \left\| g'_j(S^j(w^{(h),mJ+j})) \cdot [w^j(w^{(h),mJ+j}) + w_0^{(h),mJ+j} p^{j-1}(w^{(h),mJ+j})] \right\| \leq C\eta_m < \frac{\rho C}{m}$$

Так

$$\sum_{m=1}^{\infty} \left( \gamma \sum_{j=1}^J \|\Delta_j w^{(h),mJ+j}\|^2 \right) \leq \gamma J \rho^2 C^2 \sum_{m=1}^{\infty} \frac{1}{m^2} < \infty \quad (108)$$

Это вместе с (41) дает

$$\sum_{m=1}^{\infty} \left( \frac{1}{\eta_m} \left\| \sum_{j=1}^J \Delta_j w^{(h),mJ} \right\|^2 \right) < \infty \quad (109)$$

Согласно (26) и (49), градиент функции ошибки можно записать как

$$\nabla E(w^{(h),mJ}) =$$

$$\sum_{j=1}^J g'_j(S^j(w^{(h),mJ})) \cdot [w^j(w^{(h),mJ}) + w_0 p^{j-i}(w^{(h),mJ})] = -\frac{1}{\eta_m} \sum_{j=1}^J \Delta_j w^{(h),mJ}$$

По (44), лемме 4.1 (3) и (66) имеем

$$\begin{aligned} & \sum_{m=1}^{\infty} \frac{1}{m} \|\nabla E(w^{(h),mJ})\|^2 \leq \\ & \sum_{m=1}^{\infty} \frac{1}{m\eta_m} \left( \frac{1}{\eta_m} \left\| \sum_{j=1}^J \Delta_j w^{(h),mJ} \right\|^2 \right) \leq \frac{1}{\tau} \sum_{m=1}^{\infty} \left( \frac{1}{\eta_m} \left\| \sum_{j=1}^J \Delta_j w^{(h),mJ} \right\|^2 \right) < \infty \end{aligned} \quad (110)$$

Из леммы 3.4(3) и () следует, что существует  $C_1 > 0$  такое, что

$$\|w_d^m\| < \frac{C_1}{m} \quad (111)$$

В свете разложения Тейлора и принимая во внимание (46), (A2.3) имеем

$$\|\nabla E(w^{(m+1)J})\| - \|\nabla E(w^{(h),mJ})\| \leq \|\nabla E(w^{(m+1)J})\| - \|\nabla E(w^{(h),mJ})\| \quad (112)$$

$$\leq \|\nabla E(w^{(m+1)J}) - \nabla E(w^{(h),mJ}) - \nabla^2 E(w^{(h),mJ}) w_d^m\| + \|\nabla^2 E(w^{(h),mJ}) w_d^m\| \quad (113)$$

$$\leq o(\|w_d^m\|) + C_1 \|w_d^m\| < C_2 \|w_d^m\| < \frac{C}{m} \quad (114)$$

где  $\nabla^2 E(w) = \left\{ \frac{\partial^2 E}{\partial w_i \partial w_j} \right\}_{0 \leq i, j \leq P}$  матрица Гессе  $E(w)$  Комбинация (70), (71) и леммы 4.7 дает

$$\lim_{m \rightarrow \infty} \|\nabla E(w^{(h),mJ})\| = 0 \quad (115)$$

Аналогично (47) имеем

$$\|\|\nabla(w^{(h),mJ+j})\| - \|\nabla E(w^{(h),mJ})\|\| < \frac{C}{m}, \quad j = 1, 2, \dots, J$$

Следовательно,

$$\|\nabla E(w^{(h),mJ+j})\| \leq \|\nabla E(w^{(h),mJ})\| + \|\nabla E(w^{(h),mJ+j})\| - \|\nabla E(w^{(h),mJ})\| \leq \|\nabla E(w^{(h),mJ})\| + \frac{C}{m}$$

Из (48) имеем

$$\lim_{m \rightarrow \infty} \|\nabla E(w^{(h),mJ+j})\| = 0, \quad j = 1, 2, \dots, J \quad (116)$$

В результате комбинация (48), (49) и леммы 3.6 приводит к

$$\lim_{k \rightarrow \infty} \|\nabla E(w^{(h),k})\| = 0$$

Получаем слабую сходимость, что и является выводом (2). Из (50,51) и (42) следует, что для  $1 \leq j \leq J$ .

$$\lim_{m \rightarrow \infty} \|w^{(h),mJ+j} - w^{(h),mJ+j-1}\| = \lim_{m \rightarrow \infty} \|\Delta_j w^{(h),mJ+j-1}\| \leq \lim_{m \rightarrow \infty} \frac{\rho C}{m} = 0 \quad (117)$$

И по (46)

$$\lim_{m \rightarrow \infty} \|w^{(h),m+1J} - w^{(h),mJ}\| = \lim_{m \rightarrow \infty} \|w_d^{(h),m}\| \leq \frac{C}{m} = 0 \quad (118)$$

Объединяя 117 и 118, видим, что

$$\lim_{i \rightarrow \infty} \|w^{(h),i+1} - w^{(h),i}\| = 0$$

Это вместе с заключением (2) и леммой 4.8 немедленно выводит результат о сильной сходимости  $\lim_{m \rightarrow \infty} w^{(h),m} = w^{(h),*}$  и завершает доказательство.

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## ЛИТЕРАТУРА

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## REZYUME

Takroriy neyron tarmoqlar vaqt ketma-ketligini tahlil qilish va bashorat qilish uchun keng qo'llaniladi. Ushbu maqola ko'p vaznli ulanishlarga ega bo'lgan takroriy neyron tarmoqlar uchun onlayn gradient o'rganish usulini muhokama qiladi va uning konvergensiyasini isbotlaydi. Ko'p sonli vaqtli ketma-ketliklarni o'qitish namunalari mavjud degan faraz ostida tabiatan ehtimollik xususiyatiga ega bo'lgan ko'pgina konvergensiya natijalaridan farqli o'laroq, bu erda tuzilgan konvergensiya teoremasi tabiatan deterministikdir va faqat cheklangan miqdordagi o'quv namunalari taqdim etiladi degan asosga asoslanadi. Treningni takrorlashda xato funksiyasining monotonligi ham kafolatlanadi.

**Kalit so'zlar:** radial asosli funktsiya, radial asosli neyron tarmoq, neurotransmitter, ko'p ulanish, vazn, yashirin qatlam, tasnif.

## RESUME

Recurrent neural networks are widely used for time series analysis and forecasting. In this paper, we present an online gradient learning method for recurrent neural networks with multiple weight connections and prove its convergence. Unlike most convergence results, which are probabilistic in nature under the assumption that a large number of time series training samples are available, the convergence theorem we construct here is deterministic in nature and is based on the assumption that only a limited number of training samples are provided. The monotonicity of the error function during the training iteration is also guaranteed.

***Key words:*** radial basis function, radial basis neural network, neurotransmitter, multiple connection, weight, hidden layer, classification.